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Fluctuations of a droplet in the one dimensional stochastic Ginzburg Landau equation

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RIASSUNTO: *In questo lavoro consideriamo la PDE stocastica studiata in* [1], *ossia l'equazione di Ginzburg-Landau nell'intervallo* $[-\epsilon^{-1}, \epsilon^{-1}]$, $\epsilon > 0$ *perturbata da un rumore bianco additivo di intensità* $\sqrt{\epsilon}$ *e con condizioni di Neumann al bordo. Il dato iniziale è vicino ad una goccia. Proviamo che per* $\epsilon \to 0$ *la soluzione é vicina ad una goccia i cui centri si muovono come due moti Browniani indipendenti.*

ABSTRACT: *In this paper we consider the stochastic PDE considered in* [1], *namely the Ginzburg-Landau equation in the interval* $[-\epsilon^{-1}, \epsilon^{-1}]$, $\epsilon > 0$ *perturbed by an additive white noise of strength* $\sqrt{\epsilon}$ *and Newmann boundary conditions. The initial datum is close to a droplet. We prove that as* $\epsilon \to 0$ *the solution is close to a droplet whose centers move as two independent Brownian Motions.*

1 – Introduction

We consider the one dimensional stochastic Ginzburg-Landau equation considered by BRASSESCO, DE MASI and PRESUTTI, [1]:

$$\tag{1} \begin{cases} \dfrac{\partial}{\partial t} m(x,t) = \dfrac{1}{2} \dfrac{\partial^2}{\partial x^2} m(x,t) - V'(m(x,t)) + \sqrt{\epsilon}\dot{w}_t(x), \\ \qquad\qquad\qquad\qquad\qquad x \in \Lambda_\epsilon := [-\epsilon^{-1}, \epsilon^{-1}], t \geq 0 \\ \text{Neumann Boundary Conditions} \end{cases}$$

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where $V'(m)$ is the derivative of a double well potential $V(m) = \frac{1}{4}m^4 - \frac{1}{2}m^2$ and $\dot{w}_t(x)$ is a white noise in space and time.

This equation appears in the literature as a model equation for phase separation and interface dynamics in systems with non conserved order parameter. In this context the minima ± 1 of the double well potential are the pure phases and the standing wave $\overline{m}(x) = \tanh x$ is called the interface. Thus an interface is a stationary solution of the deterministic equation (that is the equation without noise) in the whole line ($x \in R$) that connects the pure phases at $\pm\infty$. Since all the translations of $\overline{m}$ (as well $-\overline{m}$) are interfaces, there is a one dimensional manifold of stationary solutions of the deterministic equation in R. Brassesco, De Masi and Presutti, [1], have considered the Cauchy problem for (1) with an initial datum close to an interface $\overline{m}_{x_0}(x) = \tanh(x - x_0)$. The number $x_0 \in \Lambda_\epsilon$ is called the *center* of the interface. They prove that in the limit $\epsilon \to 0$, the solution approaches an interface with a center that moves like a Brownian motion. Stronger results are obtained in [2]. The case in which the limiting brownian motion has a drift due to a spacial dependence of the noise strength is studied by Funaki, [3]. Moreover Brassesco and Buttà, [4], have considered a non symmetric double well potential with equal depth well at the minima proving that also in this case the limiting brownian motion has a drift.

In this paper we consider an initial datum close to a droplet that is a function $q_{x_0,y_0}(x)$, $x_0 > 0$, $y_0 < 0$ given by

$$\begin{aligned} q_{x_0,y_0}(x) &= \tanh(x_0 - x) \text{ for all } x > 0 \\ q_{x_0,y_0}(x) &= \tanh(x - y_0) \text{ for all } x \leq 0 \end{aligned} \tag{2}$$

and we prove that in the limit $\epsilon \to 0$, the solution approaches a droplet with centers that move like two independent Brownian motions. Precise definitions and statement of the results will be given in the next section.

Our motivation for this analysis comes from the study of the phase separation in stochastic spin dynamics. It is known, see [6] and references therein, that the deterministic Ginzburg-Landau equation can be derived as a suitable limit of the Glauber + Kawasaki process in the lattice. De Masi, Pellegrinotti, Presutti, Vares [7] have completely characterized the escape from the unstable state ($m \equiv 0$) for the spin dymanics in $d = 1$ dimensions. They prove that there is a sharp and non random phase separation time. At this time the space is divided into large clusters where the spin magnetization typically takes values alternatively equal to ± 1, the clusters are separated by interfaces. The location of the centers of the interfaces is random with known distribution. The problem left out from their results is the study of the successive motion. They conjecture, see [2], that at first the motion is deterministic, that is that the centers will move following the deterministic equation according to the results of [9], [10]. Thus at the beginning the shorter clusters disappear and after some time the clusters that have survived are so long and the deterministic mechanism so slow that fluctuations become relevant. The analysis that we carry on in this paper is a first step in studying this last regime: in fact, as we explain below, we hope to implement in the spin dynamics the techniques used in the present paper.

The paper is organized as follows: in Section 2 we give the main definitions and results that will be proven in Section 3. Similarly to [1], the whole analysis is based on the study of the evolution of the centers of functions close to the droplet, see Definition 2.0.3. First we prove that the Ginzburgh Landau process starting close to a droplet, stays close to a droplet up to times of order ϵ^{-1}. The proof of this property is a simple corollary of the results of [1]. On the other hand the proof of the invariance principle for the centers that we give here is quite different from [1] since does not use the coupling of two processes with same noise and different initial data. This last fact gives us the hope to study this problem in the spin dynamics. In fact in the Glauber + Kawasaki stochastic evolution it is not clear how to give meaning to a coupling with same noise, while it seems feasible the argument based on approximate centers that we use here.

2 – Definitions and main results

We use the same construction of the process as the one given in [1] that we briefly recall. Given any continuous function m in Λ_ϵ, define its extension $\check{m}$ to R by reflecting m through ϵ^{-1} and then extending to R with period $4\epsilon^{-1}$. Then define the process $m_t(x)$ that satisfies the Cauchy problem (1) with initial datum $m_0 \in C(\Lambda_\epsilon)$, as the unique continuous solution of the following integral equation, (see Proposition 2.3 of [1])

$$m_t = H_t \check{m}_0 - \int_0^t ds H^{(\epsilon)}_{t-s}(m_s^3 - m_s) + \sqrt{\epsilon} Z_t \tag{3}$$

with H_t equal to the Green operator in the whole line and

$$Z_t = \check{Z}^{(\epsilon)}_t, \qquad Z^{(\epsilon)}_t(x) = \int_0^t d\dot{w}_t(x) H^{(\epsilon)}_{t-s}(x,y)\,, \tag{4}$$

$H^{(\epsilon)}$ is the Green operator for the Heat equation with Neumann boundary conditions in Λ_ϵ.

We will use an equivalent realization of the process given in Proposition 2.5 of [1]. Given any $x_0 \in \Lambda_\epsilon$, we let

$$\overline{m}_{x_0}(x) = \tanh(x - x_0) \tag{5}$$

and we denote by L_{x_0} the linearizated operator around $\overline{m}_{x_0}$:

$$(L_{x_0}\phi)(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x) - V''(\overline{m}(x_0))\phi(x) \tag{6}$$

L_{x_0} is a self-adjoint operator in $L^2(\mathbb{R}, dx)$ and $\overline{m}'_{x_0}$ is an eigenvector of L_{x_0} with eigenvalue 0. The remaining part of the spectrum is in the negative axis at non zero distance from the origin. This holds also in $C^0(\mathbb{R})$, namely (see Theorem 2.4 of [1]) there are $\alpha > 0$ and c so that for any $\phi \in C^0(\mathbb{R})$ and $x_0 \in \mathbb{R}$

$$\|e^{L_{x_0}t}[\phi - N\widetilde{w}'_{x_0}]\|_\infty \le c e^{-\alpha t}\|\phi - N\widetilde{w}'_{x_0}\|_\infty \tag{7}$$

where

$$\widetilde{w}'_{x_0} = \frac{\sqrt{3}}{2}\overline{m}'_{x_0}, \qquad N = \int dx \widetilde{w}'_{x_0}(x)\phi(x)\,. \tag{8}$$

Denoting by $g_{t,x_0} = e^{L_{x_0}t}$ the semigroup generated by L_{x_0}, in Proposition 2.5 of [1] it has been proven that m_t solves (3) with initial datum

$m_0 = \overline{m}_{x_0} + u_0$ if and only if for all $t \geq 0$, $m_t = u_t + \overline{m}_{x_0}$ where u_t is the unique solution of

$$u_t = g_{t,x_0} u_0 - \int_0^t ds g_{t-s,x_0}(3\overline{m}_{x_0} u_s^2 + u_s^3) + \epsilon^{1/2} \widehat{Z}_{t,x_0} \tag{9}$$

where

$$\widehat{Z}_{t,x_0} = Z_t + \int_0^t ds g_{t-s,x_0} V''(\overline{m}_{x_0}) Z_s \tag{10}$$

In Proposition 5.4 of [1], it has been proved the following result:

Proposition 2.0.1. *Given any $\zeta > 0$, for any $\epsilon > 0$ and any $|x_0| \leq (1-\zeta)\epsilon^{-1}$, the process $\widehat{Z}_{t,x_0}$ has the representation:*

$$\widehat{Z}_{t,x_0} =: B_t \widetilde{m}'_{x_0} + R_{t,x_0} \tag{11}$$

with the following properties.

B_t is a process adapted to Z_t, its law is the law of a Brownian motion with diffusion coefficient D_ϵ, with

$$|D_\epsilon - 1| \leq c e^{-\zeta \epsilon^{-1}}$$

for a suitable constant c. For any $a > 0$ let

$$\mathcal{G}_\epsilon(a, x_0) := \{\|\widehat{Z}_{t,x_0}\|_\infty \leq \epsilon^{-a}(t \vee 1)^{1/2}, \|R_{t,x_0}\|_\infty \leq \epsilon^{-a}, \forall t \leq \epsilon^{-2}\} \tag{12}$$

then for any $n \geq 1$ there is c_n so that

$$P^\epsilon(\mathcal{G}_\epsilon(a, x_0)) \geq 1 - c_n \epsilon^n$$

In the next definition we give the class of initial data to which our results apply.

Definition 2.0.2. Recalling the definition (2) of the droplet, for any $\epsilon > 0$ and $\zeta \in (0,1)$ we set

$$\begin{aligned} C(\epsilon,\zeta) = \{ & m \in C^0(\Lambda_\epsilon) : \|m\|_\epsilon \leq 2, \\ & \text{and there are } x_0 \in [\epsilon^{-1}\zeta, \epsilon^{-1}(1-\zeta)], \\ & y_0 \in [-\epsilon^{-1}(1-\zeta), -\epsilon^{-1}\zeta], \text{ such that } \|m - q_{x_0,y_0}\|_\epsilon \leq \epsilon^{1/4}\} \end{aligned} \tag{14}$$

where $\|\cdot\|_\epsilon$ denotes the sup norm in Λ_ϵ. We say that a function m is an admissible function if there are ϵ and ζ such that $m \in C(\epsilon, \zeta)$.

As in Brassesco *et al.*, the notion of the centers of a function plays a crucial role in our analysis.

DEFINITION 2.0.3. Let $m \in C^0(\mathbb{R})$.We say that $\xi(m) \equiv x_0$, $x_0 > 0$, is the *positive center* of m if

$$\int_{-\infty}^{+\infty} [m(x) + \overline{m}_{x_0}(x)]\widetilde{m}'(x_0 - x)dx = 0 \tag{15}$$

We say that $\eta(m) \equiv y_0$, $y_0 < 0$, is the *negative center* of m if

$$\int_{-\infty}^{+\infty} [m(x) - \overline{m}_{y_0}(x)]\widetilde{m}'(x - y_0)dx = 0 \tag{16}$$

where $\widetilde{m}'_{x_0}$ is the normalized version of $\overline{m}'_{x_0}$, see (8). If m has not centers we set $\xi(m) = \eta(m) = 0$.

The content of the next theorem is that an admissible function has positive and negative center, the proof is the same as the one of Proposition 3.2 of [1] and it is therefore omitted.

THEOREM 2.0.4. *For any $0 < \zeta' < \zeta < 1$ there is ϵ_0 such that for all $\epsilon < \epsilon_0$, any $m \in C(\epsilon, \zeta)$ has a positive center $\xi(m) \in [\epsilon^{-1}\zeta', \epsilon^{-1}(1-\zeta')]$ and a negative center $\eta(m) \in [-\epsilon^{-1}(1-\zeta'), -\epsilon^{-1}\zeta']$. Furthermore there is a suitable constant $c > 0$ such that*

$$|x_0 - \xi(m)| \le c(\|m + \overline{m}_{x_0}\|_{\epsilon,+} + e^{-\epsilon^{-1}\zeta'}) \tag{17}$$

$$|y_0 - \eta(m)| \le c(\|m - \overline{m}_{y_0}\|_{\epsilon,-} + e^{-\epsilon^{-1}\zeta'}) \tag{18}$$

where $\|\cdot\|_{\epsilon,\pm}$ denotes respectively the sup *on $[0, \epsilon^{-1}]$ and on $[-\epsilon^{-1}, 0)$.*

Let $m^\star$ be a continuous function such that $\|m^\star\|_\infty \le 2$ and

$$\|m^\star - m\|_\epsilon \le \delta$$

Then $m^\star$has a unique positive (negative) center $\xi^\star$ ($\eta^\star$) in $[\epsilon^{-1}\zeta', \epsilon^{-1}(1 - \zeta')]$ ($[-\epsilon^{-1}(1-\zeta'), -\epsilon^{-1}\zeta']$) and

$$|\xi(m) - \xi^\star| \le c \int dx \overline{m}'_{x_0} |m^\star(x) - m(x)| \tag{19}$$

$$|\eta(m) - \eta^\star| \le c \int dx \overline{m}'_{y_0} |m^\star(x) - m(x)| \tag{20}$$

In the next two theorems we give our main results. As in Brassesco *et al.* we prove that after a first time layer of order ϵ^{-b}, b arbitrarily small, the process m_t with initial datum $m_0 \in C(\epsilon,\zeta)$, for all times $t \in [\epsilon^{-b}, \epsilon^{-1}T]$, $T > 0$, stays close to a droplet by $\epsilon^{1/2-a}$ with a arbitrarily small. Since this result is essentially a corollary of Proposition 3.4 and Lemma 3.5 of [1] we give an outline of its proof at the end of the next section.

THEOREM 2.0.5. *For any $T > 0$, $0 < \zeta' < \zeta < 1$ and $0 < a < 1/4$, there are $c > 0$ and $b < \min(2a, 1/10)$, and given an integer n there is c_n so that the following holds. Let m_t be the process with initial datum $m_0 \in C(\epsilon,\zeta)$ that satisfies Neumann Boundary conditions. Let $\xi_0 > 0$, and $\eta_0 < 0$ be the centers of m_0. Let $s_k = k\epsilon^{-b}$, and $\xi_k > 0$, $\eta_k < 0$ be the centers of m_{s_k}. Then*

$$P^\epsilon\Big(\sup_{t\le\epsilon^{-b}} \|m_t - q_{\xi_0,\eta_0}\|_\epsilon < c\epsilon^{1/4}, \sup_{\epsilon^{-b}\le s_k\le\epsilon^{-1}T} \|m_{s_k} - q_{\xi_k,\eta_k}\|_\epsilon \le \epsilon^{\frac12-a}\Big) \ge \ge 1 - c_n\epsilon^n . \tag{21}$$

In the next theorem we state our main result, i.e. an invariance principle for the centers.

THEOREM 2.0.6. *Given any $\zeta > 0$ and $\epsilon > 0$, let m_t be the process with initial datum $m_0 \in C(\zeta,\epsilon)$ that satisfies Neumann Boundary conditions. Let $x_0 > 0$, and $y_0 < 0$ be the centers of m_0. For any $t > 0$, define*

$$Y_t^{(1)} = \xi_{\epsilon^{-1}t} - x_0, \qquad Y_t^{(2)} = \eta_{\epsilon^{-1}t} - y_0 \tag{22}$$

where $\xi_{\epsilon^{-1}t}$ and $\eta_{\epsilon^{-1}t}$ are the centers of $m_{\epsilon^{-1}t}$. Let $\mathcal{P}^\epsilon$ be the law on $C([0,T],\mathcal{R}^2)$ of the two dimensional variable $(Y^{(1)}, Y^{(2)})$. Then $\mathcal{P}^\epsilon$, converges weakly to the law of two independent Brownian motions starting from 0 with diffusion coefficient D,

$$D = \int((\tanh x)')^2 dx . \tag{23}$$

3 – Proofs

We start with the proof of Theorem 2.0.6 using Theorem 2.0.5.

Given any $T > 0$ and $b < 1/10$, we decompose the time interval $[0, T]$ in T/T_ϵ intervals $[t_n, t_{n+1}]$, where $t_n = nT_\epsilon$ and

$$T_\epsilon = n_\epsilon \epsilon^{-b}, \quad n_\epsilon = [\epsilon^{-1/10+b}], \qquad \text{so that } T_\epsilon > \epsilon^{-\frac{1}{10}}(1 - \epsilon^{\frac{1}{10}-b})\,.$$

We denote by $\xi_n = \xi(m_{t_n})$ and $\eta_n = \eta(m_{t_n})$ the positive and negative centers of m_{t_n} and we define

$$\begin{aligned} v_n^+(x,t) &= m(x, t_n + t) + \overline{m}_{\xi_n}(x) \qquad t \in [t_n, t_{n+1}) \\ v_n^-(x,t) &= m(x, t_n + t) - \overline{m}_{\eta_n}(x) \qquad t \in [t_n, t_{n+1})\,. \end{aligned} \tag{24}$$

In the next lemma we give some estimates on $v_n(x,t)$ that help us in the rest of the proof.

Lemma 3.0.7. *Let a be as in Theorem 2.0.5 and $v_n^\pm$ as in (24). Then for any n there is a c_n so that the following holds.*

$$P^\epsilon(\|v_n^\pm(x, T_\epsilon)\|_\epsilon \le 2\epsilon^{1/2-a}\sqrt{T_\epsilon}) \ge 1 - c_n \epsilon^n \tag{25}$$

$$P^\epsilon(\|v_n^\pm(x,t) - \sqrt{\epsilon}\widehat{Z}_{t,\beta_n}\|_\infty \le 4\epsilon^{1/2-a}, \quad t \le T_\epsilon) \ge 1 - c_n\epsilon^n \tag{26}$$

$$P^\epsilon\Big(\int_{-\infty}^{+\infty} dx v_n^\pm(x, T_\epsilon)\widetilde{m}'_{\beta_n}(x) \le c\epsilon^{1/2-a}\sqrt{T_\epsilon}\Big) \ge 1 - c_n\epsilon^n \tag{27}$$

$$P^\epsilon\Big(\int_{-\infty}^{+\infty} dx v_n^\pm(x, T_\epsilon)\widetilde{m}''_{\beta_n}(x) \le c\epsilon^{1/2-a}\Big) \ge 1 - c_n\epsilon^n \tag{28}$$

where $\beta_n = \xi_n$ for v_n^+ and it is equal to η_n for v_n^-.

Proof. We prove the Lemma for v_n^+. Using the integral representation (9) and (12) with the positive semigroup g^+_{t,x_0} we have that

$$\|v_n^+(x, T_\epsilon)\|_\epsilon \le c\|v_n^+(x, 0)\|_\epsilon + c\int_0^{T_\epsilon} ds |v_n^+(x,s)|^2 + \epsilon^{1/2-a}\sqrt{T_\epsilon}\,.$$

Now for $t \in [0, T_\epsilon]$ we can use Theorem 2.0.5 and we obtain:

$$\|v_n^+(x, T_\epsilon)\|_\epsilon \le \epsilon^{1/2-a}\sqrt{T_\epsilon}(1 + \epsilon^{1/2-a}\sqrt{T_\epsilon} + cT_\epsilon^{-1/2}) \le 2\epsilon^{1/2-a}\sqrt{T_\epsilon}$$

which proves (25). For the second estimate we have

$$\|v_n^+(x,t) - \sqrt{\epsilon}\widehat{Z}_{t,\xi_n}\|_\infty \le c\epsilon^{1/2-a} + c\epsilon^{1-2a}T_\epsilon \le c\epsilon^{1/2-a}(1+\epsilon^{1/2-a}T_\epsilon) \le 4\epsilon^{1/2-a}$$

which proves (26). (27) is a consequence of (25). The last estimate follows from the fact that $\widehat{Z}_{t,\xi_n}$ has the representation, see (11),

$$\widehat{Z}_{t,\xi_n} = B_t \widetilde{m}'_{\xi_n} + R_{t,\xi_n}\,.$$

Then, being $\widetilde{m}'_{\xi_n}$ orthogonal to $\widetilde{m}''_{\xi_n}$, we have that

$$\int_{-\infty}^{+\infty} dx v_n(x,T_\epsilon)\widetilde{m}''_{\xi_n}(x) \le 4\epsilon^{1/2-a} + c\sqrt{\epsilon}R_{T_\epsilon,\xi_n}$$

hence, from (12), (28). ∎

In the next lemma we give an apriori estimate on the increments of the positive and negative centers in the time intervals $[t_n, t_{n+1}]$.

LEMMA 3.0.8. *Let a and b be as in Theorem 2.0.5. Then for any n there is c_n such that*

$$P^\epsilon(|\xi_{n+1}-\xi_n| \le \epsilon^{\frac{1}{2}-a}\sqrt{T_\epsilon}) \ge 1 - c_n\epsilon^n \tag{29}$$

$$P^\epsilon(|\eta_{n+1}-\eta_n| \le \epsilon^{\frac{1}{2}-a}\sqrt{T_\epsilon}) \ge 1 - c_n\epsilon^n \tag{30}$$

PROOF. We prove (29); (30) is analogous.

From Lemma 3.0.7 $v_n^+(x,T_\epsilon) = m(x,t_{n+1}) + \overline{m}_{\xi_n}(x)$ is such that with large probability

$$\|v_n^+(x,T_\epsilon)\|_\epsilon \le \epsilon^{1/2-a}\sqrt{T_\epsilon}\,.$$

From (21) it follows that we can apply Theorem 2.0.4 therefore (29) follows from (19). ∎

In the next definition we introduce two random variables that we call *approximate centers* and in Lemma 3.0.10 we prove that they are close to the true centers with large probability. This approach is different from the one used by Brassesco *et al.* and indeed it can be used to prove their invariance principle as well.

DEFINITION 3.0.9. We define the linear *approximate positive center* as:

$$X(t) = X(t_n) + \int_{-\infty}^{\infty} dx[m_t(x) + \overline{m}_{\xi_n}(x)]\widetilde{m}'_{\xi_n} \quad \text{if} \quad t_n \leq t < t_{n+1}\,. \tag{31}$$

Analogously we define the linear *approximate negative center* as:

$$W(t) = W(t_n) + \int_{-\infty}^{\infty} dx[m_t(x) - \overline{m}_{\eta_n}(x)]\widetilde{m}'_{\eta_n} \quad \text{if} \quad t_n \leq t < t_{n+1} \tag{32}$$

where ξ_n and η_n are respectively the positive (negative) center of m_{t_n}.

The following holds.

LEMMA 3.0.10. *Let a and b be as in Theorem* 2.0.5 *and such that there is $\gamma > 0$ so that $\sqrt{T_\epsilon}\epsilon^{2a} = \epsilon^{-\gamma}$. Then for any n there is c_n such that*

$$\lim_{\epsilon \to 0} P^\epsilon(|X(t_n) - \xi_n| \leq \epsilon^\gamma) \geq 1 - c_n\epsilon^n \tag{33}$$

$$\lim_{\epsilon \to 0} P^\epsilon(|W(t_n) - \eta_n| \leq \epsilon^\gamma) \geq 1 - c_n\epsilon^n\,. \tag{34}$$

PROOF. We prove (33); (34) is analogous. As usual for $t \in [t_n, t_{n+1}]$ we consider $v_n^+(x,t) = m_{t_n+t}(x) + \overline{m}_{\xi_n}(x)$ and for brevity we drop $+$. Recalling the definition (15) we have

$$\begin{aligned}
0 = &\int_{-\infty}^{+\infty} [m_{t_{n+1}}(x) + \overline{m}_{\xi_{n+1}}(x)]\widetilde{m}'(\xi_{n+1} - x)dx = \\
= &\int_{-\infty}^{+\infty} [m_{t_{n+1}}(x) + \overline{m}_{\xi_n}(x)]\widetilde{m}'(\xi_n - x)dx + \\
&+ \int_{-\infty}^{+\infty} [m_{t_{n+1}}(x) + \overline{m}_{\xi_n}(x)](\widetilde{m}'(\xi_{n+1} - x) - \widetilde{m}'(\xi_n - x))dx + \\
&+ \int_{-\infty}^{+\infty} [\overline{m}_{\xi_{n+1}}(x) - \overline{m}_{\xi_n}(x)]\widetilde{m}'(\xi_n - x)dx + \\
&+ \int_{-\infty}^{+\infty} [\overline{m}_{\xi_{n+1}}(x) - \overline{m}_{\xi_n}(x)](\widetilde{m}'(\xi_{n+1} - x) - \widetilde{m}'(\xi_{n+1} - x))dx\,.
\end{aligned}$$

Using the fact that ξ_{n+1} is close to ξ_n, see Lemma 3.0.8, we consider the Taylor expansion of $\overline{m}_{\xi_{n+1}}$ around $\overline{m}_{\xi_n}$ and we obtain

$$0 = \int_{-\infty}^{+\infty} [v_n(T_\epsilon, x)] \widetilde{m}'(\xi_n - x) dx + \\ + (\xi_{n+1} - \xi_n) \Big[\int_{-\infty}^{+\infty} v_n(T_\epsilon, x) \widetilde{m}''_{\xi_n} dx + 1 \Big] + o((\xi_{n+1} - \xi_n)^3) \,.$$

Using the notation

$$(v_n, w) = \int_{-\infty}^{+\infty} v_n(T_\epsilon, x) w(x) dx \,,$$

we can rewrite the last equation as

$$0 = -(v_n, \widetilde{m}'_{\xi_n}) + (\xi_{n+1} - \xi_n)(1 + (v_n, \widetilde{m}''_{\xi_n})) + o((\xi_{n+1} - \xi_n)^3) \,. \tag{35}$$

Using (29) and (25) we then get

$$\xi_{n+1} - \xi_n = (v_n, \widetilde{m}'_{\xi_n}) - (v_n, \widetilde{m}'_{\xi_n})(v_n, \widetilde{m}''_{\xi_n}) + A(\epsilon, T_\epsilon) \tag{36}$$

where

$$P^\epsilon(|A(\epsilon, T_\epsilon)| \le c\epsilon^{3/2-3a} T_\epsilon^{3/2}) \ge 1 - c_n \epsilon^n \,.$$

Since $X(t_{n+1}) = X(t_n) + (v_n, \widetilde{m}'_{\xi_n})$ we get

$$X(t_{n+1}) - \xi_{n+1} = X(t_n) - \xi_n + (v_n, \widetilde{m}'_{\xi_n})(v_n, \widetilde{m}''_{\xi_n}) + A(\epsilon, T_\epsilon) \,.$$

If we iterate this equation, recalling that $X(t_0) = \xi_0$ we have that

$$X(t_{n+1}) - \xi_{n+1} = \sum_{k=1}^{n} (v_k, \widetilde{m}'_{\xi_k})(v_k, \widetilde{m}''_{\xi_k}) + nA(\epsilon, T_\epsilon) \,.$$

Then using (27) and (28) of Lemma 3.0.7 we have that

$$\lim_{\epsilon \to 0} P^\epsilon(|X(t_{n+1}) - \xi_{n+1}| \le Cn\epsilon^{1-2a}\sqrt{T_\epsilon} + nc\epsilon^{3/2-3a} T_\epsilon^{3/2}) \ge 1 - c_n \epsilon^n \,.$$

Finally, being $n = \epsilon^{-1}/T_\epsilon$ we have

$$\lim_{\epsilon \to 0} P^\epsilon(|X(t_n) - \xi_n| \le \epsilon^\gamma) \ge 1 - c_n \epsilon^n$$

which proves the lemma. ∎

Given any $\tau > 0$, we set

$$X^\epsilon_\tau = X_{\epsilon^{-1}\tau} - x_0 \qquad W^\epsilon_\tau = W_{\epsilon^{-1}\tau} + x_0 \tag{37}$$

and we call $\mathcal{P}^\epsilon_i$, $i = 1, 2$, the laws on $D(\mathbb{R}_+, \mathbb{R})$ of X^ϵ_τ and W^ϵ_τ. We denote by $\mathcal{F}_t$ the σ-algebra generated by the process $\{Z_s, s \le t\}$, recalling that Z_s is adapted to m_t. In the next Proposition we state a criterion of convergence to Brownian Motion that we use to prove the convergence of the marginals $\mathcal{P}^\epsilon_i$, $i = 1, 2$.

Proposition 3.0.11. *Given any $T > 0$, the family $\{\mathcal{P}^\epsilon, \epsilon > 0\}$, on $D([0, T], \mathbb{R})$, is tight if there is c so that for all ϵ*

$$\sup_{t_n \le \epsilon^{-1} t} E^\epsilon(\gamma_i(t_n)^2) \le c, \quad i = 1, 2 \tag{38}$$

where, denoting by $(Y_t, t \in [0, T])$ the canonical variables in $D([0, T], \mathbb{R})$,

$$\gamma_1(t_n) = (\epsilon T_\epsilon)^{-1} E^\epsilon(Y_{t_{n+1}} - Y_{t_n} | \mathcal{F}_{t_n}) \tag{39}$$

$$\gamma_2(t_n) = (\epsilon T_\epsilon)^{-1} E^\epsilon(Y^2_{t_{n+1}} - Y^2_{t_n} | \mathcal{F}_{t_n}) + - (\epsilon T_\epsilon)^{-1} 2 Y^0_{t_n} E^\epsilon(Y_{t_{n+1}} - Y_{t_n} | \mathcal{F}_{t_n}) \tag{40}$$

with

$$Y^0_{t_n} = \frac{1}{2}[Y_{t_n} + E^\epsilon(Y_{t_{n+1}} | \mathcal{F}_{t_n})]\,. \tag{41}$$

If (38) *holds and if*

$$\lim_{\epsilon \to 0} \sup_{t_n \le \epsilon^{-1} T} (\epsilon T_\epsilon)^{-1} E^\epsilon([Y_{t_{n+1}} - Y_{t_n}]^4) = 0 \tag{42}$$

then any limit point $\mathcal{P}$ of $\mathcal{P}^\epsilon$ is supported by $C([0, T], \mathbb{R})$. Finally, if (38) *and* (42) *hold and if*

$$\lim_{\epsilon \to 0} \sup_{t_n \le \epsilon^{-1} T} (\epsilon T_\epsilon)^{-1} E^\epsilon(|\gamma_1(t_n)|) = 0 \tag{43}$$

$$\lim_{\epsilon \to 0} \sup_{t_n \le \epsilon^{-1} T} (\epsilon T_\epsilon)^{-1} E^\epsilon(|D - (\epsilon T_\epsilon)^{-1} E^\epsilon(Y^2_{t_{n+1}} - Y^2_{t_n} | \mathcal{F}_{t_n})|) = 0 \tag{44}$$

then any limit point $\mathcal{P}$ is equal to P, the law of the brownian motion with diffusion D that starts from 0.

The proof of this proposition can be found in [1].

Using the above proposition we prove the invariance principle for the approximate centers.

Theorem 3.0.12. *For any $T > 0$, the laws $\mathcal{P}_i^\epsilon$, $i = 1, 2$, in $D([0, T], \mathbb{R})$ converge to the law of a the brownian motion starting from 0 and with diffusion coefficient D given by* (23).

Proof. We start by proving (38)-(44) for X_t^ϵ. For $t \in [t_n, t_{n+1}]$ we consider $v_n(x, t)$ as usual. We use the integral representation (9) observing that $v_n(x, 0)$ is orthogonal to $\overline{m}'_{\xi_n}$. From (31) we then get

$$X(t_{n+1}) - X(t_n) = -\int dx \widetilde{m}'_{\xi_n}(x) \int_{t_n}^{t_{n+1}} ds g_{t_{n+1}-s,\xi_n}[3\overline{m}_{\xi_n} v_n^2 + v_n^3] + \\ + \sqrt{\epsilon} \int dx \widetilde{m}'_{\xi_n} \widehat{Z}_{t_{n+1},\xi_n} \tag{45}$$

with $v_n = v_n(x, s)$. By Fubini's theorem and observing that

$$\int dx g_{t-s,\xi_n}(x, y) \widetilde{m}'_{\xi_n}(x) = \widetilde{m}'_{\xi_n}(y)$$

we have that

$$X(t_{n+1}) - X(t_n) = -\int_{t_n}^{t_{n+1}} ds \int dy \widetilde{m}'_{\xi_n} [3\overline{m}_{\xi_n} v_n^2 + v_n^3] + \\ + \sqrt{\epsilon} \int dx \widetilde{m}'_{\xi_n} \widehat{Z}_{t_{n+1},\xi_n} \tag{46}$$

using (9) and (25), we obtain that with large probability

$$X(t_{n+1}) - X(t_n) = -3 \int_{t_n}^{t_{n+1}} ds \int dy \widetilde{m}'_{\xi_n} \overline{m}_{\xi_n} v_n^2 + \\ + \sqrt{\epsilon} \int dx \widetilde{m}'_{\xi_n} \widehat{Z}_{t_{n+1},\xi_n} + o((\epsilon^{1/2-a} \sqrt{T_\epsilon})^3) \tag{47}$$

now from (26) and from the representation (10) of $\widehat{Z}_{t_{n+1},\xi_n}$ we get

$$X(t_{n+1}) - X(t_n) = -\sum_{i=1}^{4} I_i + o((\epsilon^{1/2-a} \sqrt{T_\epsilon})^3) \tag{48}$$

where

$$I_1 = 3\epsilon \int_{t_n}^{t_{n+1}} ds \int dy \widetilde{m}'_{\xi_n} \overline{m}_{\xi_n} \int_0^s ds_1 \int_0^s ds_2 \int dy_1 \times$$
$$\times \int dy_2 H^\epsilon_{t-s_1}(y,y_1) H^\epsilon_{t-s_2}(y,y_2) d\dot{w}_{s_1} d\dot{w}_{s_2}\,;$$

$$I_2 = 3\epsilon \int_{t_n}^{t_{n+1}} ds \int dy \widetilde{m}'_{\xi_n} \overline{m}_{\xi_n} \int ds_1 g^2_{t-s_1,\xi_n} V''(\overline{m}_{\xi_n})^2 \int_0^{s_1} ds_2 \int_0^{s_1} ds_3 \times$$
$$\times \int dy_2 \int dy_3 H^\epsilon_{t-s_2}(y,y_2) H^\epsilon_{t-s_3}(y,y_3) d\dot{w}_{s_2} d\dot{w}_{s_3}\,;$$

$$I_3 = 6\epsilon \int_{t_n}^{t_{n+1}} ds \int dy \widetilde{m}'_{\xi_n} \overline{m}_{\xi_n} \int_0^s ds_1 \int dy_1 H^\epsilon_{s-s_1}(y,y_1) d\dot{w}_{s_1} \times$$
$$\times \int_0^s ds_2 g_{s-s_2,\xi_n} V''(\overline{m}_{\xi_n}) \int_0^{s_2} ds_3 \int dy_3 H^\epsilon_{s_2-s_3} d\dot{w}_{s_3}\,;$$

$$I_4 = -\sqrt{\epsilon} \int dx \widetilde{m}'_{\xi_n} \widehat{Z}_{t_{n+1},\xi_n}\,.$$

Now we calculate $\gamma_1(t_n) = (\epsilon T_\epsilon)^{-1} E^\epsilon(X_{t_{n+1}} - X_{t_n} | \mathcal{F}_{t_n})$ and we prove that (43) holds. For I_1 we have that:

$$(\epsilon T_\epsilon)^{-1} E^\epsilon(I_1|\mathcal{F}_{t_n}) = 3(T_\epsilon)^{-1} \int_{t_n}^{t_{n+1}} ds \int dy \widetilde{m}'_{\xi_n} \overline{m}_{\xi_n} \times$$
$$\times \int_0^s ds_1 \int dy_1 H^\epsilon_{t-s_1}(y,y_1)^2 =$$
$$= 3(T_\epsilon)^{-1} \int_{t_n}^{t_{n+1}} ds A(t-s) \int dy \widetilde{m}'_{\xi_n} \overline{m}_{\xi_n}\,.$$

Where $A(t-s) = \int_0^s ds_1 \int dy_1 H^\epsilon_{t-s_1}(y,y_1)^2$ is a function which not depends by y. Then the last integral is zero, being $\overline{m}_{\xi_n}$ an odd function ans its derivative an even one. For I_2 and I_3 the idea is the same: now we use the fact that $V''(\overline{m}_{\xi_n})$ and $V''(\overline{m}_{\xi_n})^2$ are even functions, then we always have an odd integrand function and therefore (43) holds also for I_2 and I_3. The average of I_4 is zero, hence (43) is completely proved.

To prove the remaining estimates we observe that by definition, see Definition 2.0.3, $|\xi_t| \le \epsilon^{-1}$ for all t, and that by Theorem 2.0.5 and the definition of T_ϵ

$$P^\epsilon(|\xi_{t_k}| \le (1-\zeta/2)\epsilon^{-1}; \|m_{t_k} - q_{\xi_{t_k},\eta_{t_k}}\|_\epsilon \le \epsilon^{1/2-a} \forall t_k \le \epsilon^{-1}T) \ge 1 - c_n\epsilon^n .$$

Hence it suffices to prove:

$$E^\epsilon(\gamma_i(T_\epsilon)^2) \le c \quad i = 1,2 \tag{49}$$

$$\lim_{\epsilon\to 0} E^\epsilon((\epsilon T_\epsilon)^{-1}[X_{T_\epsilon} - X_0]^4 + |D - (\epsilon T_\epsilon)^{-1}[X_{T_\epsilon} - X_0]^2|) = 0 \tag{50}$$

which are a consequence of the following lemma.

Lemma 3.0.13. *Given $\eta > 0$, for any $\epsilon > 0$, let $x_0 \in (\zeta\epsilon^{-1}, (1-\zeta)\epsilon^{-1}$ and let m_t be the Ginzburg-Landau process. Set*

$$X^\star = X_{T_\epsilon} - x_0 .$$

Then for any n there is c_n so that

$$|E^\epsilon(X^\star)| \le c_n\epsilon^n \tag{51}$$

and for any positive p there is C_p so that

$$|E^\epsilon(X^\star)^p| \le C_p[\epsilon T_\epsilon]^{p/2} . \tag{52}$$

The proof of this lemma is similar to the proof of Lemma 4.2 of [1] and then it is omitted.

To conclude the proof of (49) and (50) we observe that (49) follows from (51). The first term of (50) vanishes by (52), the second by the following argument. From (26) and (11) we have that

$$\|m_{T_\epsilon} - (-\overline{m}_{x_0} + \epsilon^{1/2}B_{T_\epsilon}\widetilde{w}'_{\xi_n})\|_\infty \le \|\epsilon^{1/2}R_{T_\epsilon,x_0}\|_\infty + c\epsilon^{1/2-a}$$

since $\widetilde{w}'_{x_0} = \sqrt{D}\overline{m}'_{x_0}$, $D = 3/4$, there is a $\bar{c}$ so that

$$\|-\overline{m}_{x_0} + \epsilon^{1/2}B_{T_\epsilon}\widetilde{w}'_{x_0} + \overline{m}_{x_0+\epsilon^{1/2}\sqrt{D}B_{T_\epsilon}}\|_\infty \le \bar{c}\epsilon B^2_{T_\epsilon} .$$

Then by (19) the centers ξ_{T_ϵ} of m_{T_ϵ} and $x_0+\epsilon^{1/2}\sqrt{D}B_{T_\epsilon}$ of $\overline{m}_{x_0+\epsilon^{1/2}\sqrt{D}B_{T_\epsilon}}$ satisfy the following inequality

$$|\xi_{T_\epsilon} - x_0 - \epsilon^{1/2}\sqrt{D}B_{T_\epsilon}| \le c(\|\epsilon^{1/2}R_{T_\epsilon,x_0}\|_\infty + \epsilon B^2_{T_\epsilon} + \epsilon^{1/2-a})$$

for a suitable constant c. Recalling (12),

$$|(\epsilon T_\epsilon)^{-1/2}(\xi_{T_\epsilon} - x_0) - T_\epsilon^{-1/2}\sqrt{D}B_{T_\epsilon}| \le c(2T_\epsilon^{-1/2}\epsilon^{-a} + \epsilon^{1/2}T_\epsilon^{-1/2}B^2_{T_\epsilon})\,.$$

Now, recalling that $X^\star = X_{T_\epsilon} - x_0$, we have that the last term of (50) vanishes observing that the distribution of $T_\epsilon^{-1/2}B_{T_\epsilon}$ is a normal with 0 average and variance D given in Proposition 2.0.1. For W_t^ϵ the proofs are analogous and then theorem is completely proved. □

Now we consider the bidimensional process on $D(\mathbb{R}_+,\mathbb{R}^2)$ $Y_t^\epsilon = (X_t^\epsilon, W_t^\epsilon)$ and let $\mathcal{P}^\epsilon$ its law. Since the arguments to prove the tightness for $\mathcal{P}^\epsilon$ are the same used to see the tightness of its marginals, for brevity we omit the details. Therefore any limit point is supported by $C([0,T],\mathbb{R})$. To prove that any limit point is the law of two independent Brownian motions we use the criterion given by the following theorem.

THEOREM 3.0.14. *Let $X_t = (X_t^1, X_t^2)$, $\mathcal{F}_t$, $0 \le t < \infty$ be a continuous, adapted process in $\mathbb{R}^2$, such that every component:*

$$M_t^k = X_t^k - X_0^k \qquad k = 1, 2$$

is a continuous local martingale relative to $\mathcal{F}_t$ and the cross-variations

$$\langle M_t^k, M_t^j\rangle = \delta_{k,j}t\,.$$

Then X_t is a bidimensional brownian motion.

The proof of this theorem can be found in [5] (see Th. 3.16 page. 157). The cross-variations $\langle M_t^k, M_t^j\rangle$ are defined as

$$\langle X, Y\rangle_t + \frac{1}{4}(\langle X+Y\rangle_t - \langle X-Y\rangle_t)$$

and they are such that $XY - \langle X, Y\rangle$ is a martingale.

Proof of Theroem 2.0.6 (conclusion). From Definition 3.0.9, Lemma 3.0.10 and Theorem 3.0.12 it follows the convergence of each marginal to a Brownian motion. To conclude the proof of Theorem 2.0.6 it remains to prove the independence of two two limiting Brownian motions. From Theorem 3.0.14 it is sufficient to prove that

$$\lim_{\epsilon\to 0} \sup_{t_n \le \epsilon^{-1} T_\epsilon} E^\epsilon(X_{t_{n+1}} W_{t_{n+1}} - X_{t_n} W_{t_n} | \mathcal{F}_{t_n}) = 0 \,. \tag{53}$$

To see this, we write

$$X_{t_{n+1}} W_{t_{n+1}} - X_{t_n} W_{t_n} = X_{t_{n+1}}(W_{t_{n+1}} - W_{t_n}) + W_{t_n}(X_{t_{n+1}} - X_{t_n})$$

and from (31) we get

$$X_{t_{n+1}} W_{t_{n+1}} - X_{t_n} W_{t_n} = \sum_{i=1}^{3} J_i$$

where

$$J_1 = X_{t_n}(W_{t_{n+1}} - W_{t_n})$$
$$J_2 = W_{t_n}(X_{t_{n+1}} - X_{t_n})$$
$$J_3 = \int_{-\infty}^{+\infty} dx \overline{m}'_{\xi_n} v_n(x, t_{n+1})(W_{t_{n+1}} - W_{t_n}) \,.$$

For J_1 and J_2, (53) is a straightforward consequence of Theorem 3.0.12. For J_3 we use the representation (48) for W_t:

$$J_3 = \sum_{j=1}^{4} \int_{-\infty}^{+\infty} dx \widetilde{m}'_{\xi_n} \widehat{Z}_{T_\epsilon,\xi_n} I_j + R_\epsilon(\epsilon, T_\epsilon)$$

with

$$P^\epsilon(|R_\epsilon(\epsilon, T_\epsilon)| \le (\epsilon^{1/2-a} T_\epsilon)^3) \ge 1 - c_n \epsilon^n$$

from the definition of I_j, $j = 1 \dots 4$, and the fact that the third-moment of white noise is zero, it follows that $\forall j \in \{1, \dots, 4\}$

$$E^\epsilon\Big(\int_{-\infty}^{+\infty} dx \widetilde{m}'_{\xi_n} \widehat{Z}_{T_\epsilon,\xi_n} I_j \quad |\mathcal{F}_{t_n}\Big) = 0$$

and then (53) is completely proved. ∎

Proof of Theorem 2.0.5. Being $m_0 \in C(\epsilon,\zeta)$ we have from (17) that $|\xi_0 - x_0| \le c\epsilon^{/1/4}$ and $|\eta_0 - y_0| \le c\epsilon^{/1/4}$. Obviously it is also true that $|q_{x,y} - q_{z,w}| \le c \max\{|x-y|, |z-w|\}$. Hence

$$\|m_0 - q_{\xi_0,\eta_0}\|_\epsilon \le \|m_0 - q_{x_0,y_0}\|_\epsilon + \|q_{x_0,y_0} - q_{\xi_0,\eta_0}\|_\epsilon \le (1+c)\epsilon^{1/4} = C_0\epsilon^{1/4}\,.$$

We next prove (21) only for $x \ge 0$; for the negative x the proof is analogous.

We define

$$\widetilde{w}_0^+(x) = \begin{cases} 1 & \text{if} \quad x - \xi_0 \le -10^{-4}\zeta'\epsilon^{-1} - 1 \\ m_0(x) & \text{if} \quad |x - \xi_0| \le 10^{-4}\zeta'\epsilon^{-1} \\ -1 & \text{if} \quad x - \xi_0 \ge 10^{-4}\zeta'\epsilon^{-1} + 1 \end{cases}$$

and a linear interpolation in the missing intervals completes the definition of $\widetilde{w}_0(x)$. Let $\widetilde{w}_t^+$ and m_t the solution of (3) with initial data respectively $\widetilde{w}_0^+$ and m_0. Using the Barrier lemma, see Proposition 5.3 of [1], for any n, there is a c_n so that

$$P^\epsilon\Big(\sup_{t\le\epsilon^{-b}} \sup_{|x-\xi_0|\le 10^{-5}\zeta'\epsilon^{-1}} |m_t(x) - \widetilde{w}_t^+(x)| \le c_n\epsilon^n\Big) \ge 1 - c_n\epsilon^n\,. \tag{54}$$

Given any $a < 1/4$, we will prove that there is $\hat{a} < a$ and, for any n, c_n so that

$$\begin{aligned} P^\epsilon\Big(&\sup_{t\le\epsilon^{-b}} \sup_{|x-\xi_0|\le 10^{-5}\zeta'\epsilon^{-1}} |\widetilde{w}_t^+(x) + \overline{m}_{\widetilde{x}_0}(x)| \le c\epsilon^{1/4}; \\ &\sup_{|x-\xi_0|\le 10^{-5}\zeta'\epsilon^{-1}} |\widetilde{w}_{\epsilon^{-b}}^+(x) + \overline{m}_{\widetilde{x}_0}(x)| \le \epsilon^{1/2-\hat{a}}\Big) \ge 1 - c_n\epsilon^n \end{aligned} \tag{55}$$

with $\widetilde{x}_0$ positive center of $\widetilde{w}_0^+$. Observe that by (19) and by (20), there is a suitable constant c so that

$$|\widetilde{x}_0 - \xi_0| \le c\epsilon^{1/4}\,. \tag{56}$$

Then from (54), (55), (56) we have:

$$\begin{aligned} P^\epsilon\Big(&\sup_{t\le\epsilon^{-b}} \sup_{|x-\xi_0|\le 10^{-5}\zeta'\epsilon^{-1}} |m_t(x) + \overline{m}_{\xi_0}(x)| \le 2c\epsilon^{1/4}; \\ &\sup_{|x-\xi_0|\le 10^{-5}\zeta'\epsilon^{-1}} |m_{\epsilon^{-b}}(x) + \overline{m}_{\xi_0}(x)| \le 2\epsilon^{1/2-\hat{a}}\Big) \ge 1 - c_n\epsilon^n\,. \end{aligned} \tag{57}$$

After proving (55) and consequently (57) we will extend the sup over the whole $[0, \epsilon^{-1}]$ and with the help of Theorem 2.0.4 we will conclude the proof. Given $a \leq 1/4$, let

$$a' < a/4 \qquad b < 1/4 - 2a' \qquad \tilde{a} = a' + b/2 < a$$

and let

$$\mathcal{B}_\epsilon = \mathcal{G}_\epsilon(a', \widetilde{x}_0) \cap \Big\{ \sup_{t \leq \epsilon^{-b}} \|\widetilde{w}_t^+\|_\infty \leq 2 \Big\}$$

with $\mathcal{G}_\epsilon(a', \widetilde{x}_0)$ as in (12). For any n there is c_n so that

$$P^\epsilon(\mathcal{B}_\epsilon) \geq 1 - c_n \epsilon^n \,.$$

Let $u_t = \widetilde{w}_t^+ + \overline{m}_{\widetilde{x}_0}$.

Then, by (9) there are constants C_1 and C_2 such that, in $\mathcal{B}_\epsilon$, and for x such that $|x - \xi_0| \leq 10^{-5}\zeta'\epsilon^{-1}$,

$$\begin{aligned} |u_t(x)| &\leq C_1 \|u_0\|_\epsilon + C_2 \int_0^t ds \|u_s\|_\epsilon^2 + \epsilon^{1/2} \|\widehat{Z}_{t,\widetilde{x}_0}\|_\epsilon \leq \\ &\leq C_1 \|u_0\|_\epsilon + C_2 \int_0^t ds \|u_s\|_\epsilon^2 + \epsilon^{1/2 - a' - b/2} \end{aligned} \tag{58}$$

for all $t \leq \epsilon^{-b}$. The last term is bounded by $\epsilon^{1/4}$. Now

$$\|u_0(x)\|_\epsilon \leq 2e^{10^{-5}\zeta'\epsilon^{-1}} + c\epsilon^{1/4} \leq C/C_1 \epsilon^{1/4} \tag{59}$$

for a suitable constant C. Then for $|x - \xi_0| \leq 10^{-5}\zeta'\epsilon^{-1}$

$$|u_t(x)| \leq 2C\epsilon^{1/4} + \int_0^t ds \|u_s\|_\epsilon^2 \qquad \text{for all t} \leq \epsilon^{-b} \,. \tag{60}$$

Now we extend the above estimates on the whole Λ_ϵ. Using the Barrier Lemma in the region $\{|x - \xi_0| > 10^{-5}\zeta'\epsilon^{-1}\}$ we reduce to the case with initial datum close to a function identically equal either to 1 or -1. We then obtain an estimate similar to (55) by considering the equation linearized around $m \equiv \pm 1$. The analysis is very similar to the previous one since $m \equiv \pm 1$ is linearly stable for the deterministic evolution. We then obtain that (60) holds for all $x \in \Lambda_\epsilon$. Now let

$$T = \inf\{t \geq 0 \; s.t. \; \|u_t\|_\infty \geq 3C\epsilon^{1/4}\} \,.$$

We next prove by contradiction that $\epsilon^{-b} \le T$, in $\mathcal{B}_\epsilon$. We thus suppose that $T < \epsilon^{-b}$, then

$$\|u_t\|_\infty \le 3C\epsilon^{1/4} \quad \text{and, by continuity of } \|u_t\|_\epsilon, \quad \|u_T\|_\infty = 3C\epsilon^{1/4}\,.$$

Hence

$$3C\epsilon^{1/4} \le 2C\epsilon^{1/4} + C_2T(3C\epsilon^{1/4})^2$$

that is

$$C\epsilon^{1/4} \le [9CC_2\epsilon^{1/4-b}]C\epsilon^{1/4}$$

which cannot hold for all ϵ small enough because $b < 1/4$. Hence $\epsilon^{-b} \le T$. Hence:

$$\|u_{\epsilon^{-b}}\|_\infty \le Ce^{-\alpha\epsilon^{-b}}\epsilon^{1/4} + C_2\epsilon^{-b}(3C)^2\epsilon^{1/2} + \epsilon^{1/2-\hat{a}} \le \widetilde{C}\epsilon^{1/2-\hat{a}}\,.$$

We have thus completed the proof of (55) hence also of (57).

Repeating same arguments for $x < 0$ we have finally that

$$P^\epsilon\Big(\sup_{t\le\epsilon^{-b}} \|m_t(x) - q_{\xi_0,\eta_0}\|_\epsilon \le 2c\epsilon^{1/4}; \|m_{\epsilon^{-b}}(x) - q_{\xi_0,\eta_0}\|_\epsilon \le 2\epsilon^{1/2-\hat{a}}\Big) \ge \\ \ge 1 - c_n\epsilon^n\,. \tag{61}$$

For the second term in (61) we can use the Theorem 2.0.4 to conclude that

$$\|m_{\epsilon^{-b}}(x) - q_{\xi_0,\eta_0}\|_\epsilon \le 2\epsilon^{1/2-\hat{a}} \text{ implies } |\xi_{\epsilon^{-b}} - \xi_0| \le \\ \le c2\epsilon^{1/2-\hat{a}} \text{ and } |\eta_{\epsilon^{-b}} - \eta_0| \le c2\epsilon^{1/2-\hat{a}}\,.$$

Hence we have proved that

$$P^\epsilon\Big(\sup_{t\le\epsilon^{-b}} \|m_t(x) - q_{\xi_0,\eta_0}\|_\epsilon \le c\epsilon^{1/4}; \\ \|m_{\epsilon^{-b}}(x) - q_{\xi_{\epsilon^{-b}},\eta_{\epsilon^{-b}}}\|_\epsilon \le \widetilde{c}\epsilon^{1/2-\hat{a}}\Big) \ge 1 - c_n\epsilon^n \tag{62}$$

for suitable constants c and $\widetilde{c}$. To conclude the proof of the theorem we state the following lemma:

LEMMA 3.0.15. *Let ζ, a, b and m_0 as in Theorem* 2.0.5. *Then there is c' and, given n, c_n so that, setting $s_k = k\epsilon^{-b}$,*

$$P^\epsilon\Big(\sup_{\epsilon^{-b}\le s_k \le \epsilon^{-1-1/8}} \|m_{s_k} - q_{\xi_{s_k},\eta_{s_k}}\|_\epsilon \le \epsilon^{1/2-a}\Big) \ge 1 - c_n\epsilon^n \tag{63}$$

$$P^\epsilon(|\xi_t - x_0| \le c'(1\vee t)\epsilon^{1/4} \quad \text{for all} \quad t < \epsilon^{-1-1/8}) \ge 1 - c_n\epsilon^n \tag{64}$$

$$P^\epsilon(|\eta_t + x_0| \le c'(1\vee t)\epsilon^{1/4} \quad \text{for all} \quad t < \epsilon^{-1-1/8}) \ge 1 - c_n\epsilon^n\,. \tag{65}$$

The proof of this lemma is the same as that of Lemma 3.5 of [1], we omit details. Using (62) and this Lemma we have finally proved the Theorem 2.0.5, namely that the Ginzburg-Landau process is close to a droplet as $\epsilon \to 0$, for all times $t \le \epsilon^{-1-1/8}$. ∎

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