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On the positivity of some bilinear functionals for discrete Sobolev orthogonal polynomials

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Riassunto: *Si studia il carattere positivo del funzionale bilineare* $a(f,g)=\sum_{m=0}^{N}\lambda_m c^{(m)}(\Delta^m f,\Delta^m g)$ *in dipendenza dai parametri* λ_m*, essendo* $c^{(0)}$ *il prodotto interno di Charlier o quello di Meixner mentre gli altri* $c^{(m)}$ *sono prodotti interni diversi. Si studiano due diversi casi. Nel primo* p *è la successione di polinomi ortogonali rispetto a* $c^{(0)}$ *e* $\Delta^m p$ *è la successione di polinomi ortogonali rispetto a* $c^{(m)}$*. Nel secondo caso si assume* $c^{(m)}=c^{(0)}$*.*

Abstract: *The positivity of a bilinear functional* $a(f,g)=\sum_{m=0}^{N}\lambda_m\ c^{(m)}(\Delta^m f,\Delta^m g)$ *is studied as a function of coefficients* λ_m*.* $c^{(0)}$ *is or the Charlier inner product either the Meixner inner product.*

Two different cases are considered. If p *is a polynomial orthogonal with respect to* $c^{(0)}$*,* $\Delta^m p$ *is orthogonal with respect to* $c^{(m)}$*,* $m=1,\ldots,N$*, in the first case. In the second case* $c^{(m)}=c^{(0)}$*,* $m=1,\ldots,N$*. As a consequence, when* $N=1$*, the corresponding Markov-Bernstein inequalities are given.*

1 – Introduction

After two papers [4] and [5] devoted to the positivity of some bilinear functionals a linked with some classical continuous inner products (Her-

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mite, Laguerre and Jacobi), it remained to study the discrete case with the Charlier and Meixner measures. The greater part of the results of the continuous case, published in these two papers, remains valid in the discrete case.

In the first paper [4] it was studied the case where the different inner products $c^{(m)}$, $m = 1, \dots, N$, appearing in the definition of a, are such that the derivative of order m of the orthogonal polynomials with respect to $c^{(0)}$ is orthogonal with respect to $c^{(m)}$. The concerned polynomials were the Hermite polynomials with $c^{(m)} = c^{(0)}$, $m = 1, \dots, N$, the Laguerre polynomials with $c^{(m)}(.) = c^{(0)}(x^m\,.)$, $m = 1, \dots, N$, and the Jacobi polynomials with $c^{(m)}(.) = c^{(0)}((1-x^2)^m\,.)$, $m = 1, \dots, N$. By analogy with the continuous case we study in the second section of this paper the Charlier polynomials with $c^{(m)} = c^{(0)}$, $m = 1, \dots, N$, and the Meixner polynomials with $c^{(m)}(.) = c^{(0)}(c^m(\beta + x + 1)^m\,.)$, $m = 1, \dots, N$. Like in [4] the domain $\mathcal{D}$ of positivity is a convex domain obtained from the intersection of all the open half-spaces defined by (7) (see [4] for the detailed proofs of these properties). In the case where $N = 1$, Markov-Bernstein inequalities are obtained in a very natural way, as well as their extremal polynomials which are the Charlier (resp. Meixner) polynomials.

The study of the case where $c^{(0)}$ is the Meixner inner product and $c^{(m)} = c^{(0)}$, $m = 1, \dots, N$, shows oneself to be the most original and interesting one (as it was the case for the study of Laguerre, Gegenbauer and Jacobi polynomials in [5]). In Section 3 the presentation of the discrete analogous case takes one's inspiration from that one in [5]. In the general case where $N > 1$, the domain $\mathcal{D}$ of positivity of a can be deduced from the results given in [5]. They do not need any new proofs. So section 3 is particularly devoted to the case $N = 1$. Indeed we have a new fact with respect to the continuous case: the domain of positivity can contain a non empty interval of negative values of λ_1. Moreover we will give the corresponding Markov-Bernstein inequality.

In [4], [5] and in this paper, we have studied all the cases in which appears a tridiagonal matrix. In the Jacobi case we already had a five diagonal matrix. In more general inner products we would obtain matrices with more diagonals. Then the study of the domain of positivity or of Markov-Berstein inequalities cannot use the classical results of Jacobi matrices. We hope, in spite of this limited number of inner products, but containing the Lebesgue measure, to make easier the study of variational problems defined by (1) or by its continuous analogous. These expressions can be derived from the variational formulation of linear partial differential equations.

$\mathcal{P}$ (resp. $\mathcal{P}_i$) will denote the vector space of polynomials (in one variable) with real coefficients (resp. of degree at most i).

Let us begin to consider the following bilinear functional a:

$$a(f,g) = \sum_{m=0}^{N} \lambda_m c^{(m)}(\Delta^m f(x), \Delta^m g(x)),\ \forall\ f \text{ and } g \in \mathcal{P}\,, \tag{1}$$

where $c^{(m)}, m = 0, \ldots, N$, are symmetric bilinear functionals

$$c^{(m)} : \mathcal{P} \times \mathcal{P} \to \mathbb{R} \text{ for } m = 0, \ldots, N\,,$$

Δ is the difference operator

$$\begin{aligned} \Delta^j f(x) &= \Delta^{j-1} f(x+1) - \Delta^{j-1} f(x)\ \forall j \in \mathbb{N}, j \geq 1\,, \\ \Delta^0 &= I\,, \end{aligned}$$

and where λ_m, $m = 0, \ldots, N$, are $N+1$ fixed real scalars with $\lambda_0 = 1$ and $\lambda_N \neq 0$.

Every functional $c^{(m)}$ is defined from its moments

$$c^{(m)}(x^i, x^j) = \begin{cases} c_{i+j}^{(m)} & \text{if } i \geq 0 \text{ and } j \geq 0\,, \\ 0 & \text{if } i < 0 \text{ or } j < 0\,. \end{cases}$$

We look for the discrete formal orthogonal polynomials with respect to a, that is to say, we look for the polynomials S_n, $n \geq 0$, such that:

$$\begin{aligned} &\deg S_n = n\,, \\ &a(S_n, x^i) = 0 \text{ for } i = 0, \ldots, n-1\,. \end{aligned} \tag{2}$$

These polynomials S_n will be called Sobolev discrete formal orthogonal polynomials.

Setting $\widetilde{M}_n = (a(x^j, x^i))_{i,j=0}^{n-1}$, we have the following obvious result:

THEOREM 1.1. *The discrete formal orthogonal polynomial S_n exists and is unique up to a normalization for the leading coefficient is fixed, if and only if the matrix $\widetilde{M}_n$ is regular.*

Definition 1.2. The bilinear functional a is called quasi-definite if the matrix $\widetilde{M}_n$ is regular $\forall\, n \geq 1$.

The matrix $\widetilde{M}_n$ can be expressed by means of different Hankel matrices.

Let $M_n^{(m)}$ be the Hankel matrices $(c_{i+j}^{(m)})_{i,j=0}^{n-1}$ for $m = 0, \dots, N$ and $\forall\, n \in \mathbb{N}$.

Let θ_n be the $n \times n$ matrix such that $(\theta_n)_{i,j} = \binom{i}{j} = i!/(j!(i-j)!)$ for $j \leq i-1$ and equal to 0 for the other cases. Then we have:

Property 1.3.

$$\widetilde{M}_n = (a(x^j, x^i))_{i,j=0}^{n-1} = \sum_{m=0}^{N} \lambda_m \theta_n^m M_n^{(m)} (\theta_n^m)^T \,. \tag{3}$$

Proof. Since $\Delta x^j = \sum_{k=0}^{j-1} \binom{j}{k} x^k$, we have

$$c^{(m)}(\Delta x^j, \Delta x^i) = \sum_{\ell=0}^{i-1} \sum_{k=0}^{j-1} \binom{i}{\ell} \binom{j}{k} c^{(m)}(x^k, x^\ell) = (\theta_n M_n^{(m)} \theta_n^T)_{i,j} \,.$$

If we assume that

$$(c^{(m)}(\Delta^{r-1} x^j, \Delta^{r-1} x^i))_{i=j=0}^{n-1} = \theta_n^{r-1} M_n^{(m)} (\theta_n^{r-1})^T \,,$$

then

$$\begin{aligned} c^{(m)}(\Delta^r x^j, \Delta^r x^i) &= c^{(m)}(\Delta(\Delta^{r-1} x^j), \Delta(\Delta^{r-1} x^i)) = \\ &= \sum_{\ell=0}^{i-1} \sum_{k=0}^{j-1} \binom{i}{\ell} \binom{j}{k} c^{(m)}(\Delta^{r-1} x^k, \Delta^{r-1} x^\ell) = \\ &= (\theta_n (\theta_n^{r-1} M_n^{(m)} (\theta_n^{r-1})^T) \theta_n^T)_{i,j} \,. \end{aligned}$$

□

We are mainly interested by the positive definite character of a.

Definition 1.4. The bilinear functional a is called positive definite on $\mathcal{P} \times \mathcal{P}$ if, $\forall\, f \in \mathcal{P} - \{0\}$, $a(f, f) > 0$.

A result, given in [5] in the continuous case, can be extended to the discrete case without new proof.

THEOREM 1.5. *The bilinear functional* a *is positive definite on* $\mathcal{P} \times \mathcal{P}$ *if and only if all the formal discrete orthogonal polynomials* S_n, $n \in \mathbb{N}$, *exist with a positive leading coefficient and* $a(S_n, S_n) > 0$, $\forall\, n \in \mathbb{N}$.

2 – Charlier-Sobolev and closely connected orthogonal polynomials

We consider the case where $c^{(0)}$ is the inner product corresponding to Charlier and Meixner polynomials (see [9]).

If d is a classical positive definite discrete linear functional, then d can be characterized by the existence of two polynomials ϕ and ψ, with $\deg \phi \le 2$ and $\deg \psi = 1$, such that $\Delta(\phi d) = \psi d$ ($\Delta(\phi d) = \phi(x-1)\Delta d + \Delta\phi(x-1)d$).

Let $\{P_n\}_n$ be the monic orthogonal polynomial sequence associated to the linear functional d (also denoted by $d^{(0)}$), then $\Delta^{(m)} P_n(x)$ is orthogonal with respect to the (classical) discrete linear functional $d^{(m)} = \phi(x)\phi(x+1)\cdots\phi(x+m-1)d^{(0)}$. In fact, if we denote by $\{P_n^{(m)}\}_n$ the monic orthogonal polynomials sequence associated with $d^{(m)}$, we have

$$\Delta^m P_n(x) = (n)_m P_{n-m}^{(m)}(x)\,, \tag{4}$$

where $(a)_j$, $j \in \mathbb{N}$, denotes the shifted factorial: $(a)_j = a(a-1)\cdots(a-j+1)$.

Moreover, if we denote by $k_n^{(m)} = d^{(m)}((P_n^{(m)})^2)$, we obtain the usual expression

$$\Delta^m P_n(x) = \xi_n^{(m)} \frac{k_n^{(0)}}{k_{n-m}^{(m)}} P_{n-m}^{(m)}(x)\,. \tag{5}$$

All the elements corresponding to the Charlier and Meixner functionals are given in Table 1 in which μ_m denotes the weight function for the inner product $d^{(m)}$ (see [9], [6]). μ_m is a step function with jumps at $x = 0, 1, 2, \ldots$

In this section, since $c^{(m)}(f,g) = c^{(m)}(1, fg)$, we study the case where $c^{(m)}(f,g) = d^{(m)}(fg)$, $m = 0, 1, \ldots, N$. Then we have the obvious relation

$$a(P_n, P_n) = \sum_{m=0}^{N} \lambda_m c^{(m)}(\Delta^m P_n, \Delta^m P_n) = \sum_{m=0}^{N} \lambda_m (n)_m^2 k_{n-m}^{(m)}\,.$$

TABLE 1

Name	**Charlier**	**Meixner**
ϕ	a	$c(\beta+x+1)$
ψ	$-x+a-1$	$\beta\, c-(x+1)(1-c)$
Ω	$]0,+\infty[$	$]0,+\infty[$
μ_m	$\frac{e^{-a}a^{x+m}}{\Gamma(x+1)}$	$\frac{c^{x+m}\,\Gamma(m+\beta+x)}{\Gamma(x+1)\,\Gamma(\beta)}$
$k_n^{(m)}$	$a^n n!$	$\frac{c^n}{(c-1)^{2n}}(1-c)^{-\beta-m}n!(\beta+m+n-1)_n$
$\xi_n^{(m)}$	a^{-m}	$\frac{(1-c)^m}{c^m(\beta+m-1)_m}$
Restrictions	$a>0$	$\beta>0,\ 0<c<1$

From that, all the following results are those presented in [4] in the continuous case for Hermite-Sobolev and closely connected orthogonal polynomials, and they can be obtained with the same proofs. So we only give the results interesting for illustrating our purpose.

$$a(P_n,P_i)=k_n^{(0)}\Big(1+\sum_{m=1}^{N}C_n^{(m)}\lambda_m\Big)\delta_{n,i}\,,$$

where $\delta_{n,i}$ is the Kronecker symbol and

$$C_n^{(m)}=(n)_m^2\frac{k_{n-m}^{(m)}}{k_n^{(0)}}=(n)_m\xi_n^{(m)}$$

with the convention that if $n-m+1\le 0$ for an index m, then $C_n^{(m)}=0$, or in other words this index does not exist in the sum.

Let us recall a result from [4].

THEOREM 2.1 [4]. i) *The bilinear functional a is quasi-definite if and only if $a(P_i,P_i)\neq 0,\ \forall\ i\in\mathbb{N}$, that is to say if and only if the coefficients $\lambda_m,\ m=1,\dots,N$, satisfy the following conditions*

$$1+\sum_{m=1}^{N}C_i^{(m)}\lambda_m\neq 0\ \forall\, i\geq 1\,. \tag{6}$$

ii) *The bilinear functional a is positive definite on $\mathcal{P}\times\mathcal{P}$ if and only if $a(P_i,P_i)>0,\ \forall\ i\in\mathbb{N}$, that is to say if and only if the coefficients*

$\lambda_m,\ m = 1, \dots, N$, *satisfy the following conditions*

$$1 + \sum_{m=1}^{N} C_i^{(m)} \lambda_m > 0 \ \forall\, i \geq 1\,. \tag{7}$$

For a fixed integer i, the set of $\lambda \in \mathrm{I\!R}^N$ of components λ_m, $m = 1, \dots, N$, satisfying relation (7) is an open half-space. Therefore $\mathcal{D}_n$, defined by

$$\mathcal{D}_n = \{\lambda \in \mathrm{I\!R}^N \mid a(p,p) > 0 \ \forall\, p \in \mathcal{P}_n - \{0\}\}\,,$$

is the intersection of these open half-spaces for $i = 1, \dots, n$, and $\mathcal{D}$, defined by

$$\mathcal{D} = \{\lambda \in \mathrm{I\!R}^N \mid a(p,p) > 0 \ \forall\, p \in \mathcal{P} - \{0\}\}\,,$$

is the same intersection for $i \geq 1$.

Like in [4] it can be shown in the same way that, $\forall\, \lambda \in \mathcal{D}$, $\lambda_N > 0$. Thus if $N = 1$, the only positive definite inner products (1) corresponds to $\lambda_1 > 0$.

Let us denote by $\overline{\mathcal{D}}_n^+$ the domain of $\mathrm{I\!R}^N$

$$\overline{\mathcal{D}}_n^+ = \{\lambda \in \mathrm{I\!R}^N, \lambda = (\lambda_1, \dots, \lambda_N) \mid \lambda_i \geq 0,\ i = 1, \dots, N\}\,.$$

If $N > 1$, $\mathcal{D} \setminus \overline{\mathcal{D}}_N^+$ is non empty (the correponding proof remains the same as the one given in [4] (Property 12)). Finally the result concerning the non normal case, given in [4] can also be extended to this particular discrete case without additional elements in the proof.

THEOREM 2.2 [4]. *If $\lambda \in \mathrm{I\!R}^N$ is such that $\lambda \in \mathcal{H}_j\ \forall\, j \in J = \{j_1, \dots, j_p\}$ with $p \leq N$ and $1 \leq j_1 < j_2 < \dots < j_p$, then*

i) *If $n \leq j_1, S_n = \widehat{\alpha}_n^{(n)} P_n$, where $\widehat{\alpha}_n^{(n)}$ is non-zero constant determined by the wanted normalization of S_n.*

ii) *Let J_r be a subset of J: $J_r = \{j_1, \dots, j_r\}$, $r \leq p$.*

For $r = 1, \dots, p-1$, and n such that $j_r < n \leq j_r + 1$, then we have

$$S_n = \widehat{\alpha}_n^{(n)} P_n + \sum_{j \in J_r} \widehat{\alpha}_j^{(n)} P_j\,,$$

where $\widehat{\alpha}_j^{(n)}$, $j \in J_r$, *are* r *arbitrary constants, and* $\widehat{\alpha}_n^{(n)}$ *a non-zero constant.*
iii) *If* $n > j_p$, *then* $S_n = \widehat{\alpha}_n^{(n)} P_n + \sum_{j \in J} \widehat{\alpha}_j^{(n)} P_j$, *where* $\widehat{\alpha}_j^{(n)}$, $j \in J$, *are* p *arbitrary constants, and* $\widehat{\alpha}_n^{(n)}$ *a non-zero constant.*

In addition to [4] we will give the Markov-Bernstein inequalities in the different cases for $N = 1$.

$C_n^{(a)}$ denotes the monic Charlier orthogonal polynomial of degree n, depending on one real parameters $a > 0$, and $m_n^{(\beta,c)}$ denotes the monic Meixner orthogonal polynomial of degree n, depending on two real parameters ($\beta > 0$ and $0 < c < 1$).

Corollary 2.3 Markov-Bernstein inequalities. *Let* $\| \,.\, \|_{(m)}$ *be the norm associated to the inner product* $c^{(m)}$ *respectively for* $m = 0, 1$. *Then* $\forall\, p \in \mathcal{P}_n - \{0\}$, $a(p,p) > 0$ *if and only if* $\lambda_1 > -\frac{1}{C_n^{(1)}}$. *Thus* $\forall\, p \in \mathcal{P}_n$, $\| \Delta p \|_{(1)} \leq \sqrt{C_n^{(1)}} \, \| p \|_{(0)}$.

$$C_n^{(1)} = \begin{cases} \dfrac{n}{a} & \textit{in the Charlier case}\,, \\ \dfrac{n(1-c)}{c\beta} & \textit{in the Meixner case}\,. \end{cases}$$

$\sqrt{C_n^{(1)}}$ *is the best constant.*

An extremal polynomial is

- *a Charlier polynomial* $C_n^{(a)}(x)$ *in the Charlier case,*
- *a Meixner polynomial* $m_n^{(\beta,c)}(x)$ *in the Meixner case.*

3 – Meixner-Sobolev orthogonal polynomials

Like in [5] some matrix relations will be given from the relations (8) to (10) satisfied by the monic Meixner polynomials (see [9], [6]).

$$\Delta m_n^{(\beta,c)}(x) = n\; m_{n-1}^{(\beta+1,c)}(x)\,. \tag{8}$$

$$m_n^{(\beta,c)}(x) = \frac{\Delta m_{n+1}^{(\beta,c)}(x)}{n+1} + e_n^{(\beta,c)} \frac{\Delta m_n^{(\beta,c)}(x)}{n}\,, \tag{9}$$

$$= m_n^{(\beta+1,c)}(x) + e_n^{(\beta,c)} m_{n-1}^{(\beta+1,c)}(x)\,, \tag{10}$$

with $e_n^{(\beta,c)} = \frac{nc}{1-c}$. By convention we complete the set of the previous relations (8) and (9) by the following ones:

$$\left.\begin{array}{ll} m_{-k}^{(\beta,c)} & = m_{-k}^{(\beta+1,c)} = 0\,, \\ \Delta m_{-k+1}^{(\beta,c)} & = m_{-k}^{(\beta+1,c)} \end{array}\right\} \ \forall\, k > 0\,. \tag{11}$$

$\widehat{m}_{-k,n}^{(\beta,c)}$ (resp. $\Delta\widehat{m}_{-k,n}^{(\beta,c)}$) will represent the vector whose the components are $m_j^{(\beta,c)}$(resp. $\Delta m_j^{(\beta,c)}$), $j = -k, \ldots, n$.

Thus, from (8) we have:

$$\widehat{m}_{-k,n}^{(\beta+1,c)} = \widehat{D}_{-k+1,n+1}\, \Delta\widehat{m}_{-k+1,n+1}^{(\beta,c)}\,, \tag{12}$$

where $\widehat{D}_{-k+1,n+1}$ is an $(n+1+k) \times (n+1+k)$ diagonal matrix whose the entries are

$$\widehat{D}_{-k+1,n+1}(i,i) = \begin{cases} 1 & i = 1, \ldots, k\,, \\ \frac{1}{\ell} & \ell = 1, \ldots, n+1 \text{ with } i = k+\ell\,. \end{cases}$$

Moreover, from (10) we have:

$$\widehat{m}_{-k,n}^{(\beta,c)} = E_{-k,n}^{(\beta,c)}\, \widehat{m}_{-k,n}^{(\beta+1,c)} \tag{13}$$

where $E_{-k,n}^{(\beta,c)}$ is the $(n+1+k) \times (n+1+k)$ regular matrix whose the elements are in row i and column j, $i, j = 1, \ldots, n+k+1$:

$$E_{-k,n}^{(\beta,c)}(i,j) = \begin{cases} 1 & \forall\, i = j\,, \\ e_{i-k-1}^{(\beta,c)} & \forall\, i = j+1,\ k+2 \le i \le n+k+1\,, \\ 0 & \text{everywhere else}\,. \end{cases}$$

Finally we get from (12) and (13):

$$\widehat{m}_{-k,n}^{(\beta,c)} = \mathcal{J}_{-k+1,n+1}^{T}\, \Delta\widehat{m}_{-k+1,n+1}^{(\beta,c)}\,, \tag{14}$$

where $\mathcal{J}_{-k+1,n+1}^{T}$ is an $(n+1+k)\times(n+1+k)$ nonsingular matrix obtained by multiplication of $E_{-k,n}^{(\beta,c)}$ on the right by $\widehat{D}_{-k+1,n+1}$.

Let $\{R_i\}_{i\geq 0}$ be a sequence of monic polynomials which satisfies the following conditions:

$$\begin{array}{lll} & \deg R_i = i, & \forall\, i \in \mathbb{N}\,, \\ (15) & \Delta^i R_i = i!\, m_0^{(\beta,c)} = \rho_{i,N}\ m_0^{(\beta,c)}, & \text{for } 0 \leq i \leq N-1\,, \\ (16) & \Delta^N R_i = (i)_N\, m_{i-N}^{(\beta,c)} = \rho_{i,N}\ m_{i-N}^{(\beta,c)}, & \text{for } i \geq N\,. \end{array}$$

Let $\widehat{R}_n$ (resp. $\Delta^j \widehat{R}_n$, $\forall\, j \in \mathbb{N}$) be the vector of components R_i (resp. $\Delta^j R_i$, $\forall\, j \in \mathbb{N}$), $i = 0, \ldots, n$, and $D_{n,N}$ be the $(n+1)\times(n+1)$ diagonal matrix $(\rho_{i,N})_{i=0}^n$.

The matrix interpretation of (15) and (16) is

$$\Delta^N \widehat{R}_n = D_{n,N}\ \widehat{m}_{-N,n-N}^{(\beta,c)}\,. \tag{17}$$

Therefore, using the same technique as in [5], we get a similar relation to relation (24) given in [5], that is to say

$$\widehat{R}_n = D_{n,N}\Big(\prod_{j=0}^{N-1} \mathcal{J}_{-j,n-j}\Big)^T\ \widehat{m}_{0,n}^{(\beta,c)}\,.$$

where $\prod_{j=i}^{N-1} \mathcal{J}_{-j,n-j} = \mathcal{J}_{-i,n-i}\cdots\mathcal{J}_{-N+1,n-N+1}$ in this order. If $i > N-1$, then $\prod_{j=i}^{N-1} \mathcal{J}_{-j,n-j}$ will be taken equal to 1.

From that all the results given in Sections 4 and 5 in [5] remains valid.

The most interesting results concern the case $N = 1$.

Let p and q be two polynomials of degree n. Since $\{R_i\}_{i\geq 0}$ is a basis of $\mathcal{P}$, we can write p and q as:

$$p = (\widehat{R}_n)^T y \text{ and } q = (\widehat{R}_n)^T z\,, \tag{18}$$

where y and z are vectors of $\mathbb{R}^{n+1}$ which contain the coordinates y_i (resp. z_i) of p (resp. q) in the basis $\{R_i\}_{i=0}^n$. Then

$$\begin{array}{ll} (19) & a(p,q) = c^{(0)}(p,q) + \lambda_1 c^{(0)}(\Delta p, \Delta q) \\ (20) & \qquad\quad = z^T D_{n,1}\,(\mathcal{J}_{0,n})^T\, K_{n,0}\, \mathcal{J}_{0,n}\, D_{n,1}\ y + \lambda_1 z^T D_{n,1}\, K_{n,1}\, D_{n,1}\ y\,. \end{array}$$

where $K_{n,j}$ is an $(n+1)\times(n+1)$ matrix such that

$$K_{n,j} = (c^{(0)}(m_i^{(\beta,c)}, m_\ell^{(\beta,c)}))_{i,\ell=-j}^{n-j},\ j=0,1\,.$$

Thus $K_{n,j}$ is a diagonal matrix $(k_i)_{i=-j}^{n-j}$ where $k_i = (c^{(0)}(m_i^{(\beta,c)}, m_i^{(\beta,c)})$, $i \geq 0$, and $k_i = 0$ if $i < 0$. Moreover $(\mathcal{J}_{0,n})^T K_{n,0}\, \mathcal{J}_{0,n}$ is a tridiagonal matrix.

From that $(\mathcal{J}_{0,n})^T K_{n,0}\, \mathcal{J}_{0,n} + \lambda_1 K_{n,1}$ can be written as:

$$(\mathcal{J}_{0,n})^T K_{n,0} \mathcal{J}_{0,n} + \lambda_1 K_{n,1} = \begin{pmatrix} k_0 & 0 \\ 0 & (K_{n-1,0})^{\frac{1}{2}}(J_n + \lambda_1 I)(K_{n-1,0})^{\frac{1}{2}} \end{pmatrix}, \tag{21}$$

since the n last rows and columns of $K_{n,1}$ correspond to $K_{n-1,0}$ which is a definite positive diagonal matrix. $(K_{n-1,0})^{\frac{1}{2}}$ is the diagonal matrix whose entries are the square roots of the entries of $K_{n-1,0}$.

Therefore the matrix J_n is:

$$J_n(i,j) = \begin{cases} \dfrac{k_1}{k_0} = \dfrac{c\beta}{(1-c)^2} & \text{for } i=j=1\,, \\[2ex] \dfrac{c^2}{(1-c)^2} + \dfrac{k_i}{i^2 k_{i-1}} = \dfrac{c}{(1-c)^2}\Big(c + \dfrac{i+\beta-1}{i}\Big) & \\ & \text{for } i=j=2,\dots,n, \\[2ex] \dfrac{c}{(1-c)(i-1)}\sqrt{\dfrac{k_{i-1}}{k_{i-2}}} = \dfrac{c}{(1-c)^2}\sqrt{\dfrac{c(i+\beta-2)}{i-1}} & \\ & \text{for } i=j+1\,, \\[2ex] \text{For } i=j-1 \text{ the matrix is completed by symmetry,} & \\[2ex] 0 \quad \text{everywhere else.} & \end{cases}$$

The three following properties of [5] still remain valid.

Property 3.1 [5]. J_n is a positive definite symmetric matrix.

Property 3.2 [5]. If $\mu_{i,n}$, $i=1,\dots,n$, are the eigenvalues of J_n with $0 < \mu_{1,n} \leq \cdots \leq \mu_{n,n}$, then $\lim_{n\to\infty} \mu_{n,n} < +\infty$.

THEOREM 3.3 [5].

$$\mathcal{D}_n = \{\lambda_1 \in \mathbb{R}, \lambda_1 > -\mu_{1,n}\}\,,$$

where $\mu_{1,n}$ is the smallest eigenvalue of J_n.

Since J_n is a tridiagonal matrix, we immediately get a three-term recurrence relation between the successive monic polynomials $A_\ell(x) = \det(xI - J_\ell)$. Moreover the last coefficient in this recursion being strictly positive, the zeros of all the polynomials A_ℓ, $\ell \geq 1$, are real, positive, distinct.

We summarize these results in the Meixner case in the following theorem:

THEOREM 3.4 (Meixner case). *The eigenvalues $\mu_{i,n}$, $i = 1, \dots, n$, are the zeros of the orthogonal polynomials $A_n(x)$ defined from the following three-term recurrence relation:*

$$\begin{aligned} A_n(x) = \Big(x - \frac{c^2}{(1-c)^2} - \frac{c}{(1-c)^2}\frac{n+\beta-1}{n}\Big)\times \\ \times A_{n-1}(x) - \frac{c^3}{(1-c)^4}\frac{n+\beta-2}{n-1}A_{n-2}(x),\ n \geq 2\,, \end{aligned}$$

with $A_0(x) = 1$ and $A_1(x) = x - \frac{c\beta}{(1-c)^2}$.

These zeros are real, positive, distinct.

From the interlacing property of the zeros of orthogonal polynomials satisfying the previous recursion, $\{\mu_{1,n}\}_{n\geq 1}$ is a decreasing sequence.

Using γ and δ defined by

$$\begin{aligned} \gamma &= \lim_{n\to\infty}\Big(\frac{c^2}{(1-c)^2} + \frac{c}{(1-c)^2}\frac{n+\beta-1}{n}\Big) = \frac{c(1+c)}{(1-c)^2}\,, \\ \delta &= \lim_{n\to\infty}\Big(\frac{c^3}{(1-c)^4}\frac{n+\beta-2}{n-1}\Big) = \frac{c^3}{(1-c)^4}\,, \end{aligned}$$

we get

$$\begin{aligned} \sigma &= \gamma - 2\sqrt{\delta} = \frac{c(1-\sqrt{c})^2}{(1-c)^2} > 0\,, \\ \tau &= \gamma + 2\sqrt{\delta} = \frac{c(1+\sqrt{c})^2}{(1-c)^2}\,. \end{aligned}$$

From Blumenthal's theorem (see CHIHARA [2]) we know that the set of all zeros (contained in $[\sigma, \tau]$) of all polynomials A_n is dense in the interval $[\sigma, \tau]$. However some zeros of A_n can be lain on the outside of $[\sigma, \tau]$. For example, if $c = 1/4$, then $\sigma = 1/9$. If $0 < \beta < 1/4$, then $\mu_{1,1} = \frac{c\beta}{(1-c)^2} < 1/9 = \sigma$.

The new fact is that $\inf_n \mu_{1,n}$ is not necessary equal to zero like in the continuous case (see [4] and [5]). Hence, in the case $N > 1$, if $\lambda \in \mathcal{D}$, λ_N could be negative according to the following result.

THEOREM 3.5 [5]. *Let $\mu_{i,n}$, $i = 1, \dots, n$, be the eigenvalues of J_n, $(0 < \mu_{1,n} \leq \cdots \leq \mu_{n,n})$. $\mu_{n,n}$ is assumed to be bounded $\forall\, n$.*

i) *If $\lim_{n\to\infty} \mu_{1,n} = 0$ and if $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathcal{D}$, then $\lambda_N > 0$.*

ii) *If $\inf_n \mu_{1,n} \geq \kappa > 0$ and if $\lambda_i > 0$, $i = 1, \dots, N-1$, then there exists $\widehat{\kappa} < 0$ depending on the λ_i's, $i = 1, \dots, N-1$ such that $\forall\, \lambda_N > \widehat{\kappa}$, $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathcal{D}$.*

PROPERTY 3.6. i) If $1-\beta > 0$ and $c < \beta$, then there exists a constant $\alpha > 0$, depending only on β and c, such that $\alpha(\beta, c) \leq \inf_n \mu_{1,n}$.
ii) If $1 - \beta < 0$ and $c < \min(\inf_i \xi(i), 1)$, where $\xi(i)$ is the smallest zero of

$$c - \sqrt{c}\Big(\sqrt{\frac{\beta + i - 1}{i}} + \sqrt{\frac{\beta + i - 2}{i - 1}}\Big) + \frac{\beta + i - 1}{i}\,, \tag{22}$$

then there exists a constant $\widehat{\alpha} > 0$, depending only on β and c, such that $\widehat{\alpha}(\beta, c) \leq \inf_n \mu_{1,n}$.
iii) If $\beta = 1$, then $\frac{c}{(1+\sqrt{c})^2} = \inf_n \mu_{1,n}$.

PROOF. The Gerschgorin disks will be used for obtaining a location of the eigenvalues of J_n. Let us denote by $\varphi(i)$ the difference between the center and the radius of the Gerschgorin disks corresponding to the i^{th} row of J_n.

$$\text{If } i = 1, \varphi(1) = \frac{c\sqrt{\beta}}{(1-c)^2}(\sqrt{\beta} - \sqrt{c})$$

$$\text{If } i \geq 2, \varphi(i) = \frac{c}{(1-c)^2}\Big(c - \sqrt{c}\Big(\sqrt{\frac{\beta+i-1}{i}} + \sqrt{\frac{\beta+i-2}{i-1}}\Big) + \frac{\beta+i-1}{i}\Big).$$

In this last relation, the factor of $\frac{c}{(1-c)^2}$ which is exactly (22), has a discriminant equal to

$$\delta = \frac{\beta+i-2}{i-1} + 2\sqrt{\frac{(\beta+i-1)(\beta+i-2)}{i(i-1)}} - 3\frac{\beta+i-1}{i}\,.$$

The quantity $3\frac{\beta+i-1}{i} - \frac{\beta+i-2}{i-1} = \frac{(2i-3)(\beta+i-2)+3(i-1)}{i(i-1)} > 0\ \forall\,\beta > 0$ and $i \geq 2$.

Therefore for studying the sign of δ, it is sufficient to give the sign of

$$\begin{aligned} &4\frac{(\beta+i-1)(\beta+i-2)}{i(i-1)} - \Big(3\frac{\beta+i-1}{i} - \frac{\beta+i-2}{i-1}\Big)^2 = \\ &= -9\Big(\frac{\beta+i-1}{i} - \frac{\beta+i-2}{i-1}\Big)\Big(\frac{\beta+i-1}{i} - \frac{\beta+i-2}{9(i-1)}\Big)\,. \end{aligned} \tag{23}$$

If $1-\beta < 0$, then the function $\frac{\beta+i-1}{i}$ decreases strictly. Thus (23) is positive and Δ too. If $1-\beta > 0$, then the function $\frac{\beta+i-1}{i}$ is strictly increasing. Thus (23) is negative and Δ too.

i) If $1-\beta > 0$, then $\Delta < 0$ and thus $\varphi(i) > 0$ for $i \geq 2$. $\varphi(1) > 0$ if $c < \beta$.

For i fixed, the minimum of $\varphi(i)$ is obtained for $c_m = \frac{1}{2}(\sqrt{\frac{\beta+i-1}{i}} + \sqrt{\frac{\beta+i-2}{i-1}})$ and it is equal to $-\Delta/4$. When i tends to infinity, c_m tends to 1 and Δ tends to 0. Therefore result i) holds.

ii) If $1-\beta < 0$, then $\Delta > 0$ and $\frac{\beta+i-1}{i} > 1\ \forall\, i > 0$. Therefore the two real zeros $\xi(i)$ and $\zeta(i)$ of $\varphi(i)$ in c are such that

$$0 < \xi(i) < \zeta(i) < +\infty, \text{ for } i \geq 2\,. \tag{24}$$

Indeed $1 < \xi(i)\zeta(i) = \frac{\beta+i-1}{i} < +\infty$ and $\xi(i)+\zeta(i) = \sqrt{\frac{\beta+i-1}{i}} + \sqrt{\frac{\beta+i-2}{i-1}} > 0$. Depending on β, the smallest zero $\xi(i)$ can be greater than 1 or less than 1. Due to the bounded character of the product of the zeros, $\inf_i \xi(i) > 0$. Moreover $\varphi(1) > 0$. Thus if $c < \min(\inf_i \xi(i), 1)$, then the result ii) holds.

iii) If $\beta = 1$, then $\varphi(1) = \frac{c(1-\sqrt{c})}{(1-c)^2} > 0$ and $\varphi(i) = \frac{c}{(1-c)^2}(\sqrt{c}-1)^2 > 0$ for $i \geq 2$. In the interval $[\frac{c}{(1+\sqrt{c})^2}, \frac{c}{(1-\sqrt{c})^2}]$ the zeros of all polynomials A_n are dense (see above-mentioned). Thus $\inf_n \mu_{1,n} = \frac{c(\sqrt{c}-1)^2}{(1-c)^2}$. ∎

Corollary 3.7. *In the Meixner case the following Markov-Bernstein inequality is satisfied*

$$\forall\, p \in \mathcal{P}_n,\ \parallel \Delta p \parallel \leq \frac{1}{\sqrt{\mu_{1,n}}} \parallel p \parallel,\ n \geq 1\,,$$

where $\parallel . \parallel$ *denotes the norm corresponding to the Meixner inner product.* $\frac{1}{\sqrt{\mu_{1,n}}}$ *is the best constant.*

An extremal polynomial is

$$p = \sum_{i=1}^{n} \frac{w_i^{(1,n)}}{i\sqrt{k_{i-1}}} R_i\,, \tag{25}$$

where $w^{(1,n)} = (w_1^{(1,n)}, \dots, w_n^{(1,n)})^T$ *is an eigenvector of* J_n *corresponding to the eigenvalue* $\mu_{1,n}$*.*

In the cases i) *and* ii) *of Property* 3.6*, there exists a positive constant* α *depending only on* β *and* c *such that*

$$\forall\, p \in \mathcal{P}, \parallel \Delta p \parallel \leq \frac{1}{\sqrt{\alpha(\beta, c)}} \parallel p \parallel,\ n \geq 1\,.$$

In particular if $\beta = 1$*, then* $\forall\, p \in \mathcal{P}, \parallel \Delta p \parallel \leq \dfrac{1+\sqrt{c}}{\sqrt{c}} \parallel p \parallel$.

Proof. An extremal polynomial is an eigenfunction associated to the eigenvalue $\mu_{1,n}$ for the problem

$$a(p,p) = \mu_{1,n} c^{(0)}(p,p)\,.$$

Therefore, if $w^{(1,n)}$ is an eigenvector of J_n corresponding to the eigenvalue $\mu_{1,n}$, then relations (18) to (21) give the final expression (25) of an extremal polynomial. □

Concerning the case $N > 1$, the greater part of the results given in [5], in particular all the properties of Section 6.2, remains valid in the Meixner case. We give below the most important results.

Let us denote by $\tilde{k}_{i,N}$ the square norm $a(S_i, S_i)$, $\forall\, i \in \mathbb{N}$.

THEOREM 3.8 [5].

$$\widetilde{k}_{0,N} = k_0 = Q_{0,N}\,, \tag{26}$$
$$\widetilde{k}_{n,N} = \frac{Q_{n,N}(\lambda_1,\dots,\lambda_{b(n)})}{Q_{n-1,N}(\lambda_1,\dots,\lambda_{b(n-1)})} \text{ for } n \geq 1\,,$$

where $b(n) = \min(n,N)$. $Q_{n,N}(\lambda_1,\dots,\lambda_{b(n)})$ *is a polynomial in* $b(n)$ *variables* $\lambda_1,\dots,\lambda_{b(n)}$ *of total degree equal to* n.

We think useful to mention that, in Theorem 6.13 in [5], $C_{n,n-2,n}(n,n-1) = n!(n-1)!c/(1-c)k_1$. Thus the result of this theorem, as well as Theorem 6.14 and Corollary 6.15 hold in the Meixner case. Conjecture 6.16 can also be extended to the Meixner case.

On the other hand $Q_{n,N}(\lambda_1,\dots,\lambda_{b(n)}) = 0$ is the equation of an algebraic hypersurface in $\mathbb{R}^N$. Let us denote by $\mathcal{F}_n(\lambda_1,\dots,\lambda_{b(n)})$ the nappe corresponding to the largest zero of $Q_{n,N}(\lambda_{b(n)})$ when $(\lambda_1,\dots,\lambda_{b(n)-1}) \in \mathcal{D}_{b(n)-1}$. If $\lambda_{b(n)}$ is greater than (resp. greater than or equal to) this largest zero, we will denote the corresponding domain by the notation $\mathcal{F}_n(\lambda_1,\dots,\lambda_{b(n)}) > 0$ (resp. $\mathcal{F}_n(\lambda_1,\dots,\lambda_{b(n)}) \geq 0$). In the Meixner case we also have, $\forall\, n \geq 1$

$$\mathcal{D}_n = \{\lambda \in \mathbb{R}^{b(n)} \mid \mathcal{F}_n(\lambda_1,\dots,\lambda_{b(n)}) > 0\}\,,$$

and the nappe $\mathcal{F}_n(\lambda_1,\dots,\lambda_{b(n)}) = 0$ is in $\mathcal{F}_{n-1}(\lambda_1,\dots,\lambda_{b(n-1)}) \geq 0$.

COROLLARY 3.9 [5].

$$\mathcal{D} = \{\lambda \in \mathbb{R}^N \mid \lim_{n\to\infty} \mathcal{F}_n(\lambda_1,\dots,\lambda_N) > 0\}\,.$$

PROPERTY 3.10 [5]. $\mathcal{D}_n$ is a convex domain.

Let us denote by $\mathcal{D}_n^+$ (resp. $\overline{\mathcal{D}}_n^+$) the domain of $\mathbb{R}^{b(n)}$

$$\mathcal{D}_n^+ = \{\lambda \in \mathbb{R}^{b(n)}, \lambda = (\lambda_1,\dots,\lambda_{b(n)}) \mid \lambda_i > 0,\ i = 1,\dots,b(n)\}\,,$$

$$\text{(resp. } \overline{\mathcal{D}}_n^+ = \{\lambda \in \mathbb{R}^{b(n)}, \lambda = (\lambda_1,\dots,\lambda_{b(n)}) \mid \lambda_i \geq 0,\ i = 1,\dots,b(n)\})\,.$$

Of course, if $\lambda \in \mathcal{D}_n^+$, $Q_{n,N}(\lambda_1,\dots,\lambda_{b(n)}) > 0$.

THEOREM 3.11 [5]. $\mathcal{D} \setminus \overline{\mathcal{D}}_N^+$ *is non empty.*

Moreover when Property 3.6 is satisfied, the domain $\mathcal{D}$ contains a region of $\mathbb{R}^N$ in which λ_N is negative (see Theorem 3.5 ii)).

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