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Hydrodynamic scale for a driven tracer particle
Rigorous results

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To my parents

Riassunto: *Noi consideriamo la dinamica di una particella avente massa e carica interagente con un ambiente omogeneo in equilibrio, consistente di particelle classiche identiche. Noi discutiamo in modo rigoroso i risultati sulla transizione al limite per lo spostamento della particella carica. In qualche caso speciale noi rappresentiamo una conseguenza del Teorema del Limte Centrale e la relazione di Einstein.*

Abstract: *We consider dynamics of a massive charged particle interacting with a homogeneous equilibrium evironment consisting of identical classical particles. We discuss rigorous results on the limit transition for the displacement of the massive particle. In some special cases we represent a derivation of the Central Limit Theorem and the Einstein relation*

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0 – Introduction

0.1 – Physical motivation

We begin with the fundamental Physical description of driven *Brownian motion*, that is the motion of a charged massive particle in a neutral gas in presence of a constant electrical field. We follow the approach developed by Einstein and Smoluchowski. In modern and slightly simplified terms the theory of Brownian motion accepted in Physics may be exposed as follows. Consider a massive charged ball moving in the Euclidean space $\mathbb{R}^d$ that is filled by a neutral gas of identical particles. We assume that the ball undergoes on action of an external electrical field f. The ball interacts with particles of the gas by a suitable repulsive potential. (The reader may think of a gas of point-like material particles colliding elastically with the ball). It is assumed that the system is in thermodynamic equilibrium. Namely, we set $f = 0$ and fix a Gibbs measure on the space of all possible configurations of the system. This Gibbs measure corresponds as usual to certain values of the inverse temperature β and the average particle density. In the case of $f = 0$ this measure is invariant with respect to the dynamics. Denote by $Q(t)$ the position of the ball at time t. The following fundamental relations are accepted as main assumptions of the theory of Brownian motion. They were discovered almost 90 years ago and agree with numerous physical experiments.

1. Drift and diffusion.

(0.1.1) $$Q(t) = D(f)t + \Sigma(f)\mathcal{W}_t + o(\sqrt{t}),\ \text{as } t \to \infty\,.$$

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Here $D(f)$ is a vector called the mean drift and $\Sigma(f)$ is a positive defined operator; Σ^2 is called the diffusivity of the Brownian particle. $\mathcal{W}_t$ denotes the standard Wiener process.

2. Einstein relation.

$$D(f) = \frac{\beta}{2}\Sigma^2(0)f + o(f), \text{ as } f \to 0 \tag{0.1.2}$$

TERMINOLOGICAL REMARK. The principles indicated above are known as “Einstein-Smoluchowski” theory. There exists an alternative approach, called “Ornstein-Uhlenbeck” theory, which postulates that the basic random process describing the *velocity* of the Brownian particle should be the stationary Ornstein-Uhlenbeck process $\mathcal{U}_t$ satisfying the stochastic equation:

$$d\mathcal{U}_t = -\gamma\mathcal{U}_t + \sigma d\mathcal{W}_t \,.$$

If so, the Einstein-Smoluchowski theory for the *displacement* of the particle appears as a limiting case of the Ornstein-Uhlenbeck one. For a detailed discussion of connections between the two theories see Nelson's classical notes [18]. An attempt to obtain a unified theory is contained in [29]. It seems that in physical literature the term “Brownian motion” is more often associated to “Ornstein-Uhlenbeck” than to “Einstein-Smoluchowski”. In the present text we deal with the Einstein-Smoluchovsky theory only.

The main subject of our interest is a rigorous mathematical justification of the physical picture described above. In more precise terms this means the following.

1. We must consider a mechanical system of the type “classical test object + constant external force acting on the test object + gas of classical particles”. The underlying dynamics of the system have to be deterministic and Hamiltonian. No resorting to stochastic evolution is allowed.
2. Randomness may enter only through initial data. A measure on the space of initial data is of the Gibbs type and allow to determine correctly the temperature of the system. This measure has to be invariant under the dynamics corresponding to the trivial external field $f = 0$.

3. The only "large parameter" is time. All physical parameters, for instance, density of particles, temperature, masses of particles etc. should be constant.
4. The Einstein relation should be considered in the physically correct (*strong*) form. That means, the limit transition with respect to the force value should follow after the limit transition in time.

In other words we have to consider the system in the *hydrodynamic* limit. The problem of justification is difficult even if one resorts to stochastic evolution, introduces additional parameter scaling or considers the Einstein relation in the so called "weak" form, when the limit transitions in time and in force are coupled so that $f = f_0/\sqrt{t}$. (See for example [29], [8], [6], [15].) In the present text we consider the "pure mechanical" case, described above, only. Our aim is to outline the progress achieved recently in this area emphasizing mathematical methods motivated by the problem.

0.2 – Description of the main models

So far, there have been no rigorous mathematical models based only on the above physical postulates. A significant progress has been achieved only under some "special" geometrical assumptions. Below we describe the main models we shall deal with.

i) *One-dimensional Rayleigh gas.* This model consists of a massive (charged) particle (M.P.) of mass M on a line immersed in an ideal gas of identical point-like particles of equal mass $m \leq M$. Gas particles do not interact and move according to the free dynamics until a collision with the massive particle occurs. Collisions of M.P. with gas particles are elastic. More precisely, let us define the *extended phase space* of the system by $\widehat{\Omega} = \mathbb{R}^2 \times \mathbf{Y}$, where

$$\mathbf{Y} = \{Y \subset \mathbb{R}^2 : \mathrm{card}(Y \cap K) < \infty \text{ for any compact } K \subset \mathbb{R}^2\}.$$

Any point of Y is called a particle of the ideal gas and denoted by $(q, v) \in Y \in \mathbf{Y}$. Here q and v are the position and the velocity of the particle respectively.

We will denote a point $\hat{\omega} \in \widehat{\Omega}$ by $\hat{\omega} = ((Q, V), Y)$, where Q, V are the position and the velocity of the massive particle. Let us define the action of the one-dimensional translation group on $\widehat{\Omega}$ by the formula:

$$R_a(\hat{\omega}) = \hat{\omega}_1 : (q,v) \in Y(\hat{\omega}_1) \Leftrightarrow (q-a,v) \in Y(\hat{\omega}),$$
$$Q(\hat{\omega}_1) = Q(\hat{\omega}) + a, V(\hat{\omega}_1) = V(\hat{\omega}).$$

The *phase space* Ω is defined as the quotient of $\widehat{\Omega}$ with respect to this action. We may and will identify Ω with the subset of $\widehat{\Omega}$ corresponding to $Q = 0$. A point $\omega \in \Omega$ is written as a pair $\omega = (V, Y)$. We endow Ω (and $\widehat{\Omega}$) with the standard topology and the corresponding sigma-algebra of Borel sets. For this purpose we define a fundamental set of neighborhoods of the point $Y_0 \in \mathbf{Y}$ by $U_S(Y_0) = \{Y \subset \mathbb{R}^2 : \mathrm{card}(Y \cap S) = \mathrm{card}(Y_0 \cap S)\}$, where $S \subset \mathbb{R}^2$ is bounded and open. The space $\mathbf{Y}$ with the given topology is a polish space. The topology on both Ω and $\hat{\Omega}$ is the product one. We denote by $\mathcal{B}$ (and $\widehat{\mathcal{B}}$ respectively), the corresponding Borel σ-algebras. Introduce the basic reference measure on $(\Omega, \mathcal{B})$:

$$\mu(d\omega) = \sqrt{\frac{\beta M}{2\pi}} \exp\Big(-\beta M \frac{V^2}{2}\Big) dV\ \mathcal{P}(dY),$$

where $\mathcal{P}$ denotes the Poisson field on the one-particle phase space $\mathcal{M} = \mathbb{R}^2$ with the intensity measure

$$n(dq\ dv) = \rho \sqrt{\frac{\beta m}{2\pi}} \exp\Big(\frac{-\beta m v^2}{2}\Big) dv\ dq.$$

Here ρ and β are positive constants corresponding to the density of particles and the inverse temperature of the system. Note that any $\mathcal{P}$-typical configuration of particles is locally finite with respect to the q coordinate. The main reference measure for the extended phase space is

$$\hat{\mu} = \mu \times \exp(\beta f Q) dQ.$$

The dynamics $\{\widehat{T}^t\}$ on $\widehat{\Omega}$ corresponds to elastic collisions. All particles keep their velocities until they collide with M.P., which moves between collisions with constant acceleration $\frac{f}{M}$, where f is the force acting on M.P. Since $\widehat{T}^t R_a = R_a \widehat{T}^t$ we may correctly define the corresponding dynamics $\{T^t\}$ on the quotient space Ω. Note that μ is invariant under $\{T^t\}$ only by $f = 0$. Our main object is the displacement of M.P. defined on Ω by

$$Q(t, \omega) = \int_0^t V(T^s(\omega))\, ds.$$

ii) *Modified Rayleigh gas.* This system is a modification of one-dimensional Rayleigh gas proposed by J. L. Lebowitz. The system lives in the plane $\mathbb{R}^2$, the points of which will be denoted by $q = (q_, q_2)$. By convention, we use the adjectives "horizontal" and "vertical" for the first and the second coordinate axis, respectively. The system is two-dimensional and consists of a rod of mass M and of a gas of infinitely many point-like particles with equal masses $m < M$. The rod is supposed to be of length ℓ, and infinitely thin. Its center point is constrained to move on the horizontal q_1-axis, and the rod keeps a vertical position, i.e. its orientation is fixed and orthogonal to the q_1-axis. The horizontal velocity of the rod will be denoted by V and its coordinate on the q_1-axis by Q. The ideal gas in this case is described by a locally (in q) finite subset Y of the one-particle phase space $\mathcal{M} = \mathbb{R}^2 \times \mathbb{R}^2$. A configuration of the whole system is described by a point in the *extended phase space* $\widehat{\Omega}$. A point $\hat{\omega} \in \widehat{\Omega}$ will be written as $\hat{\omega} = ((Q, V), Y)$, where Q, V denote the position and the velocity of the rod, respectively, and Y is the particle configuration. In the same manner as above, we define Ω as the quotient of $\widehat{\Omega}$ with respect to the group of horizontal space translations. A point $\omega \in \Omega$ is written in obvious notations as a pair $\omega = (V, Y)$. One can introduce the Borel sigma-algebras $\widehat{\mathcal{B}}$ and $\mathcal{B}$ on $\widehat{\Omega}$ and Ω respectively, as above. The main reference measure on $(\Omega, \mathcal{B})$ in this case is

$$\mu(d\omega) = \sqrt{\frac{\beta M}{2\pi}} \exp\Big(- \frac{\beta M V^2}{2}\Big) dV \; \mathcal{P}(dY) \,,$$

where $\mathcal{P}$ is the Poisson field on the one particle phase space $\mathcal{M}$ with the intensity measure

$$n(dq\; dv) = \rho \sqrt{\frac{\beta m}{2\pi}} \exp\Big(- \beta m \frac{{v_1}^2}{2}\Big) h(dv_2) dv_1 \; dq \,.$$

$h(.)$ is a distribution of the vertical velocity. We assume that:

$$\int_{-\infty}^{+\infty} h(dw)|w| < \infty$$

and that there exists $a > 0$ such that $h([-a \;,\; a]) = 0$.

The last condition ensures that there are no particles whose vertical velocities are too small and thus that any particle which is inside the region available to the rod will leave it after a renewal time. (This property is crucial since it removes, at least partially, the conservation of memory provided by recollisions of particles with the rod. We shall discuss it in details somewhat later.) The dynamics on $\widehat{\Omega}$ (and on Ω) is the usual dynamics of elastic collisions. All particles keep their velocities until they collide with the rod, and, upon collisions, the vertical velocities do not change. The rod moves between collisions with constant acceleration $\frac{f}{M}$, where f is the force acting on the rod. The main problem is to investigate the displacement of the rod in the horizontal direction.

Let us emphasize that our list of appropriate models is far from being complete. For instance, we do not consider hard rods systems and *Lorentz gas* [7]. In the last system the absorption of energy by the medium is simulated by the "Gaussian dynamics". It is assumed that the Gaussian dynamics can mimic the absorption of energy by the scatters when they are not fixed. For the corresponding arguments see the discussion in [7]. In case the reader is interested in obtaining a more complete picture of the subject we strongly recommend to make use of the references indicated at the end of the text.

0.3 – Formulation of the results. Discussion

0.3.1 – Existence of the Dynamics

Let us consider the dynamics associated with the above models. We have chosen elastic collisions as a convenient idealization. On the other hand this assumption leads to certain mathematical difficulties. There are examples of initial configurations ω for which the dynamics $\{T^t(\omega)\}$ can be constructed only up to some finite time τ. A more detailed analysis reveals the following reasons for that:

a) there are infinitely many particles coming into some bounded neighborhood of the M.P. (rod) in a finite time;
b) a non-transversal collision may occur. This means that the collision is multiple (two or more gas particles and M.P. (or the rod) collide simultaneously) or M.P. (or the rod) and the colliding particle have equal (horizontal) velocities at the time of collision;
c) infinitely many collisions may occur during a finite interval of time.

In order to construct the dynamics mathematically rigorously we have to show that all these situations occur only for a "small" set of initial conditions. For our purposes it is sufficient to understand the adjective "small" as "of μ-measure zero". The difficulty indicated in (a) is typical for all infinite-particle systems. In our case one may easily overcome it making use of the Gaussian distribution of velocities, which guarantees good spatial localization of the particles placed far away from M.P. (or the rod). The cases (b) and (c) concern actually the dynamics of a closed subsystem containing finite number of particles. The difficulty here is specific for systems with singular (elastic) interaction. It is easy to see that the initial conditions leading to the case (b) correspond to a subset of codimension at least 1. Thus the main problem is to estimate the set Z_∞ of initial conditions possessing a finite accumulation point of collision moments. It turns out that this set is also of codimension at least 1 and by $f = 0$ is empty. In Section 1 we consider the problem in full generality. The techniques developed there is of independent interest and has application to many systems important for mathematical physics.

0.3.2 – One-dimensional Rayleigh gas

Assuming that the desired dynamics is correctly constructed, we get down to the main business of the present text. Consider the one-dimensional Rayleigh gas. The first problem we deal with is the lack of a suitable stationary distribution when the external field f is not equal to zero. The natural way to overcome this difficulty is to construct this distribution as the limit of the main reference measure under the time evolution. One can do this rigorously at the moment only for modified Rayleigh gas. For one-dimensional Rayleigh gas the problem is open and we have to restrict our attention to the equilibrium case ($f = 0$). We are interested in proving the central limit theorem for the displacement $Q(t, \omega)$ of M.P. The first result was obtained for the degenerated case $M = m$. The dynamics in this case is relatively simple since the motion of the M.P. just copies the trajectory of the last collided gas particle. This observation was used in the paper [26] (1969) where the displacement of the M.P. was explicitly represented as a functional of the gas configuration.

THEOREM 0.3.1 [26]. *Assume that $M = m$. Then the random process*

$$\frac{1}{\sqrt{N}} Q(tN, .)$$

converges (*as* $N \to \infty$) *in the sense of week convergence of path measures to the Wiener process* $\mathcal{W}_{\sigma^2 t}$. *The limit variance*

$$\sigma^2 = \sqrt{\frac{2}{\pi}} \frac{1}{\rho\sqrt{m\beta}}\,.$$

The case $M > m$ is much harder. The way of reasoning here is based on "balance equations" which, roughly speaking, combine a suitable conservation low with the stationarity of the reference measure. Balance arguments relied on the conservation of the number of particles were used in [22] to establish that the distributions of $\frac{Q(t)}{\sqrt{t}}$ are tight. Moreover, it was shown that any limit distribution of this tight family has the following properties:

1. It may be represented as a distribution of a sum of two identically distributed Gaussian variables with variance
$$\frac{1}{\sqrt{2\pi}} \frac{1}{\rho\sqrt{m\beta}}\,.$$
2. It may be represented as a distribution of the random variable $\gamma + \xi$, where γ is the Gaussian variable with the expectation 0 and the variance
$$\sigma_+^2 = \sqrt{\frac{\pi}{8}} \frac{1}{\rho\sqrt{m\beta}}\,,$$
and ξ is a random variable independent of γ;
3. It is absolutely continuous and its density is analytic;
4. The variance σ_∞^2 of this limit distribution satisfies the inequalities:

$$\sigma_-^2 = \sqrt{\frac{\pi}{8}} \frac{1}{\rho\sqrt{m\beta}} \le \sigma_\infty^2 \le \sigma_+^2 = \sqrt{\frac{8}{\pi}} \frac{1}{\rho\sqrt{m\beta}}\,. \tag{0.3.2}$$

These properties are not sufficient for to conclude that each limit distribution is Gaussian. The inequality (0.3.2) is remarkable, since the boundary values of the asymptotic variance do not depend on M. D. Szasz and B. Toth [28], by a completely different idea, obtained the same lower bound for the limit variance and improved the upper bound:

$$\sigma_\infty^2 \leq \sigma_+^2 = \sqrt{\frac{2}{\pi}} \frac{1}{\rho\sqrt{m\beta}} \,.$$

The upper bound above is sharp and coincides with the limit variance in the case of $M = m$. Intensive numerical simulations [1] show that the limit variance strongly depends on M. Moreover, as seen from the simulations, the lower bound given above is also sharp and occurs when $M \to \infty$. Existence of the boundary values for the limit variance independent of M is perhaps a pure one-dimensional phenomenon. No similar bounds are known for the modified Rayleigh gas. Note that the results concerning the one-dimensional Reyleigh gas are obtained without use of ergodic properties of the system. In fact there are no results of this kind. Let us formulate the following *open problem.* Prove that in the case of the Reyleigh gas and $M \neq m$ the dynamical system is ergodic, mixing, has K-property, etc.

The main reason why we expect this system to be ergodic and the process $V(t)$ to be "close to markovian" is based on the following observation: fresh particles which, after a long time, achieve M.P. become statistically almost independent of its past trajectory since they come from well separated regions of the space. The best situation is: when a "new", particle comes-"old" ones escape for ever and give no contribution to the velocity of M.P. Of course it is not the case because recollisions are possible. Moreover, we have no apriori restrictions on the number of possible recollisions between a given gas particle and M.P. This number may be infinite as well. The following result shows that the situation is not so hopeless: *For μ-almost every ω each particle has finitely many collisions with M.P.* Let particle $x \in Y(\omega)$. Denote by $\Delta(x)$ the time interval between the first and the last collision of x with M.P. This interval is not empty and due to what we said above is finite. Set $D(\omega) = \cup_{x \in Y(\omega)} \Delta(x)$. We say that ω provides a cluster decomposition if $D(\omega) = \cup J_k$, where each J_k is a finite interval and $J_k \cap J_i = \emptyset$ provided $i \neq k$. An abundance of ergodic properties of the system is due to the fact that cluster decomposition is a typical property. This fact was established for slightly different systems, where M.P. was localized by an external potential increasing in infinity or by elastic barriers. It may be used to prove K and Bernoulli regularity. We refer the reader to [13], [2], [24], [25].

In Section 2 we consider the one-dimensional Rayleigh gas and derive the balance equations as in [22]. We do not prove all the known results referring to the original papers indicated above.

In the case of Rayleigh gas, the problem of recollisions seems too difficult.

0.3.3 – Modified Rayleigh gas

We return to the general non-equilibrium case and consider the modified version of Rayleigh gas described above. Introduce the part of one-particle configuration and phase space available to the rod:

$$\mathcal{S} = \{q \in \mathbb{R}^2 : |q| < \ell/2\}, \mathcal{M}_{\mathcal{S}} = \{(q, v) : q \in \mathfrak{S}\}.$$

The subsystem in $\mathcal{S}$ is described by

$$X(\omega) = (V(\omega), Y(\omega) \cap \mathcal{M}_{\mathcal{S}}).$$

The evolution of the configuration inside the strip $\mathcal{S}$ in terms of entire configuration is given by

$$X_t(\omega) = X(T^t(\omega)),$$

where the process X_t is *markovian.* The transition probabilities are given in terms of the Poisson measure:

$$P^t(X, A) = \mathcal{P}\{Y : T^t(X \cup (Y \setminus \mathcal{M}_{\mathfrak{S}}) \in A\}. \tag{0.3.3}$$

This reduction of a similar system to a Markov process appeared in a different context in [13] and then later in [9]. The strategy of proving (0.1.1) consists of establishing strong ergodic properties of the process $X(t)$ which imply a central limit theorem for the velocity V. We emphasize the lack of a natural invariant measure to $X(t)$ given in advance (unless $f = 0$). The required values of $\sigma(f)$ and $d(f)$ arise as asymptotic parameters in the central limit theorem and already their existence presents a difficult problem.

The next question we are dealing with is the validity of the Einstein relation. The crucial observation is the following

PROPOSITION 0.3.4 [3]. *Let* $\mu_t = \mu(T^{-t}(.))$ *be the family of the measures generated by the dynamics. Then for each* t *the measure* μ_t *is equivalent to the measure* μ *and the Radon Nikodym derivative is equal to*

$$\frac{d\mu_t}{d\mu} = \exp\Big(\beta f \int_0^t V(T^{-t+s}(\omega))\, ds\Big)\,.$$

This proposition implies the following *summation rule*:

$$\int_\Omega \mu(d\omega) \exp\Big(- \beta f \int_0^t V(T^s(\omega))\, ds\Big) = 1\,. \tag{0.3.5}$$

independently of t. This equality is an important technical tool and gives the basic argument for the following heuristic derivation of the Einstein relation. Suppose for a minute that the "drift + diffusion" representation (0.1.1) is valid exactly:

$$\int_0^t V(T^s(\omega))\, ds = td(f) + \sigma(f) W_t\,.$$

Substituting this equation into the summation rule we obtain that

$$E(\exp(-\beta f(td(f) + \sigma(f)W_t)) = 1\,.$$

Thus

$$d(f) = \frac{\sigma^2(f)}{2}\beta f = \frac{\sigma^2(0)}{2}\beta f + o(f)$$

under the assumption that the variance is continuous with respect to the value of the external field. In fact, the equality (0.1.1) is far from being explicitly valid. We may understand it in an approximative form only. To make the above arguments work we need a control of convergence in the central limit theorem *uniform in* f.

Section 3 contains the proof of the fact that the modified Rayleigh gas in the hydrodynamic limit admits a complete diffusion theory including the Einstein relation. This section is based on the results of papers [3],

[4], though some proofs differ from those of [3] and [4]. Section 3 is the center of the present text. Let us formulate the results rigorously:

I. Existence of the drift. *The limit*

$$d(f) \equiv \lim_{t\to\infty} \frac{Q_f(t,\omega)}{t} = \nu_f(V)$$

exists, is finite, and does not depend on ω *for* μ *-a.a.* $\omega \in \Omega$. *Moreover* $f d(f) > 0$ *for* $f \neq 0$.

II. Diffusion. *The process*

$$\xi_t^\epsilon = \sqrt{\epsilon}\Big(\, Q_f\Big(\frac{t}{\epsilon},\, \omega\Big) - d(f)\frac{t}{\epsilon}\,\Big)$$

converges weakly, as $\epsilon \to 0$ *in the space of continuous functions of* t *to the Wiener process* $\mathcal{W}_{\sigma^2(f)t}$ *with nondegenerate diffusion constant* $\sigma^2(f) > 0$.

III. Einstein relation. *The drift* $d(f)$ *and the diffusivity* $\sigma^2(f)$ *are continuous functions of* f. *Moreover, the Einstein relation holds, i.e.,*

$$\lim_{f\to 0} \frac{d(f)}{f} = \frac{\beta}{2}\sigma^2(o)\,.$$

1 – Dynamics of particles with elastic collisions

1.1 – Finite systems

1.1.1 – Preliminary observations

Consider a mechanical system consisting of finitely many classical point-like particles on a line colliding elastically with each other. Assume in addition that between collisions the particles are subject to force fields of a general type. We allow long-range interaction forces as well as external forces acting on each particle. To be precise, denote by $Q^1 \le \ldots \le Q^N$ and $V^1, \ldots, V^N$ the positions and the velocities of the particles respectively. Let $m_1, \ldots, m_N$ be their masses. Suppose that

$$\dot V^i = \frac{1}{m_i} \cdot F^{(i)}(Q^1, \ldots, Q^N)\,.$$

In coordinates $q^i = Q^i \cdot \sqrt{m_i}, v^i = V^i \cdot \sqrt{m_i}$ we have:

$$\dot{q} = v\,,\ \dot{v} = f(q) : v, q \in \mathbb{R}^N\,.$$

Moreover, elastic collisions ensure that the system is contained in the (coordinate) polyhedral angle defined by

$$Q^i \le Q^{i+1}$$

or, equivalently,

$$q^i/\sqrt{m_i} \le q^{i+1}/\sqrt{m_{i+1}}$$

Trajectories of the system undergo elastic reflections at the boundary of the angle, provided that the corresponding intersections are transversal. From now on we deal with the problem posed for a conservative system inside a polyhedral angle or more generally inside a polyhedron. The particular case of interacting particles we have started with will be referred to as *particle system.*

We say that the evolution is well defined for a given initial condition x if in any finite time interval at most finitely many collisions with the boundary may occur and each collision is transversal. In the present section we study the set of configurations for which the evolution (or dynamics) is well defined.

1.1.2 – Conservative system in a polyhedron

The *extended phase space* of the system is $X = \mathbb{R}^{2N}$. Each point $x \in X$ is represented as follows:

$$x = (q, v),\ \text{ where } q \in \mathbb{R}^N\,,\ v \in \mathbb{R}^N\,; q = (q^1, \dots, q^N)\,,\ v = (v^1, \dots, v^N)\,.$$

We endow $X = \mathbb{R}^{2N}$ with the usual topology, sigma-algebra of Borel subsets and Lebesgue measure $m(dx) = \prod_1^N dq^i dv^i$. Set

$$||q||^2 = \sum_{j=1}^N (q^j)^2,\ \ ||v||^2 = \sum_{j=1}^N (v^j)^2$$

and

$$||x||^2 = ||q||^2 + ||v||^2\,.$$

Let

$$B_R = \{x : ||x|| \le R\}\,.$$

By $\langle .,. \rangle$ we denote the standard scalar product in $\mathbb{R}^N$. Let $\theta_1, \dots, \theta_K \in \mathbb{R}^N$ be given and satisfy:

$$||\theta_i|| = 1,\ i = 1, \dots, K\,.$$

We introduce the linear forms

$$\vartheta_i(q) = \langle q, \theta_i \rangle - d_i$$

and define

$$\begin{aligned}\Omega &= \{x \in X :\ \vartheta_i(q(x)) \ge 0 \text{ for all } i = 1, \dots, K\}\,,\\ \Xi &= \{q \in \mathbb{R}^N :\ \vartheta_i(q) \ge 0 \ \text{ for all } i = 1, \dots, K\}\,.\end{aligned}$$

Denote by $\partial\Omega$ the boundary of Ω:

$$\partial\Omega = \{x \in X :\ \vartheta_i(q(x)) = 0 \text{ for some } i = 1, \dots, K\}\,.$$

We assume from the very beginning that Ω is not empty and does not coincide with its boundary. This means that there exists $\zeta \in \mathbb{R}^N$ such that

$$\vartheta_i(\zeta) > 0 \ \text{ for all } j = 1, \dots, K\,. \tag{1.1.1}$$

We also assume that none of the forms $\vartheta_1, \dots, \vartheta_K$ may be omitted. In other words, for any $k = 1, \dots, K$

$$\vartheta_k \notin \Big\{ \sum_{j \ne k} p_j \vartheta_j : p_j \ge 0 \Big\}\,.$$

We say that Ξ is a polyhedral angel if $d_j = 0$, $j = 1, \dots, K$ or, in other words, $\vartheta_j(0) = 0$ for any j. Define

$$\begin{aligned}\Gamma_i^{\pm} &= \{x \in \Omega : \vartheta_i(q(x)) = 0, \pm\langle v(x), \theta_i\rangle > 0; \vartheta_j(q(x)) \ne 0 \text{ for all } j \ne i\}\,;\\ \Gamma^{\pm} &= \bigcup_{i=1,\dots,K} \Gamma_i^{\pm}\,;\\ \Gamma &= \Gamma^+ \cup \Gamma^-, \Gamma_i = \Gamma_i^+ \cup \Gamma_i^-\,;\\ \partial\Omega_{\mathrm{sing}} &= \partial\Omega \setminus \Gamma\,.\end{aligned}$$

The *phase space* of the system is

$$\Omega_0 = (\Omega \setminus \partial\Omega) \ \cup \ \Gamma^+ .$$

Notice that

$$\partial\Omega_{\text{sing}} \subset \Big(\bigcup_{i\neq j} R_{i,j}\Big) \cup \Big(\bigcup_i \Pi_i\Big) , \tag{1.1.2}$$

where

$$\begin{aligned} R_{i,j} &= \{x \in X : \vartheta_i(q(x)) = \vartheta_j(q(x)) = 0\} ; \\ \Pi_i &= \{x \in X : \vartheta_i(q(x)) = \langle v(x), \theta_i\rangle = 0\} . \end{aligned}$$

Each $R_{i,j}$, $i \neq j$, as well as Π_i is a linear submanifold in $\mathbb{R}^{2N}$ of codimension 2.

The force field $F = F(q)$ is assumed to be defined on the whole configuration space $\mathbb{R}^N$. We also assume that F is sufficiently smooth (at least C^1). Set

$$\xi(x) = \sum_{i=1}^{N} v^i(x)\frac{\partial}{\partial q^i} + F^{(i)}(q(x))\frac{\partial}{\partial q^i} . \tag{1.2}$$

We assume that the vector field ξ is complete. That is, there exists a one - parameter group of transformations $\{S^t\}_{t\in\mathbb{R}}$ such that

$$\frac{d}{dt}(S^t(x)) = \xi(x) .$$

For $x \in \Gamma$ we define $\Phi(x)$ called a boundary reflection transformation. To this end for $x \in \Gamma_i$ set:

$$\begin{aligned} q(\Phi(x)) &= q(x) ; \\ v(\Phi(x)) &= v(x) - 2\langle v(x), \theta_i\rangle \cdot \theta_i . \end{aligned}$$

Notice that $\Phi(\Phi(x)) = x$ and $\Phi(\Gamma^{\pm}) = \Gamma^{\mp}$. It will be convenient in the sequel to define $\Phi(x)$ for $x \notin \Gamma$ just putting $\Phi(x) = x$. We shall also use the notation $\overline{x}$ for $(q(x), -v(x))$.

Definition. For $x \in \Omega_0$ a positive semitrajectory is said to be defined up to time $\tau > 0$ if there exists a right continuous mapping $t \to T^t(x) \in \Omega_0$ of the closed interval $[0, \tau]$ satisfying the following conditions:

a) $T^0(x) = x$
b) $\mathrm{Card}\{t \in [0, \tau] : T^t(x) \in \Gamma\} < \infty$
c) If $\{t \in [0, \tau] : T^t(x) \in \Gamma\} = \{\tau_1, \dots, \tau_k\}, \tau_j < \tau_{j+1}$ then for any $t \in [\tau_j\,, \tau_{j+1})$ $T^t(x) = S^{t-\tau_j}(T^{\tau_j}(x))$.
d) $T^{\tau_j}(x) = \Phi(T^{\tau_j -}(x))$, where $T^{\tau_j -} = \lim_{t \uparrow \tau_j} T^t(x)$.
e) If $\{t \in [0, \tau] : T^t(x) \in \Gamma\} = \emptyset$ then $T^t(x) = S^t(x)$.

Denote by $\Omega_\tau, \tau > 0$ the subset of phase points $x \in \Omega_0$ for which a positive semitrajectory is well defined up to time τ. Ω_τ is an open subset of the phase space. Observe that

$$\Omega_\tau = \bigcap_{\varepsilon > 0} \Omega_{\tau+\varepsilon},\ T^t(\Omega_\tau) \subseteq \Omega_{\tau - t}\,.$$

A negative semitrajectory is said to be well defined for $x \in \Omega_0$ up to time $\tau < 0$ if $\overline{\Phi(x)} \in \Omega_{|\tau|}$. In this case we put:

$$T^t(x) = \overline{\Phi(T^{|t|}(\overline{\Phi(x)}))}\,.$$

Ω_τ is defined for $\tau < 0$ by $\Omega_\tau = \overline{\Phi(\Omega_{|\tau|})}$.

It is easy to see that $T^{t+s}(x) = T^t(T^s(x))$ provided that $T^{t+s}(x)$ and $T^s(x)$ are correctly defined. For $x \in \Omega_0$ define:

$$\begin{aligned}
\tau_+(x) &= \sup\{t > 0 :\ x \in \Omega_\tau\}\,;\\
\tau_-(x) &= \inf\{t < 0 :\ x \in \Omega_\tau\}\,;\\
D(x) &= (\tau_-(x)\,,\ \tau_+(x))\,;\\
D_c(x) &= \{t \in D(x) : T^t(x) \in \Gamma^+\}\,.
\end{aligned}$$

Notice that $D_c(x)$ is a discrete subset of $D(x)$. Set

$$\begin{aligned}
\Omega_\infty &= \bigcap_{\tau \in \mathbf{R}} \Omega_\tau = \bigcap_{\tau \in \mathbf{Z}} \Omega_\tau\,,\\
\Omega_{\pm\infty} &= \bigcap_{\tau \in \mathbb{R}_\pm} \Omega_\tau = \bigcap_{\tau \in \mathbb{Z}_\pm} \Omega_\tau\,.
\end{aligned}$$

Clearly Ω_∞ is an invariant (under $\{T^t\}$) subset of the phase space. A fundamental property of the dynamics described above is contained in the following assertion: $\{T^t\}$ *preserves the Lebesgue measure.* More precisely, assume that $t \in D(x)$ for any x contained in a measurable subset $A \subseteq \Omega_0$, then:

$$m(T^t(A)) = m(A)\,. \tag{1.1.3}$$

Henceforth we accept the following

ASSUMPTION. For any given $R > 0, \tau > 0$ one may find a constant $\overline{R} = \overline{R}(R, \tau)$ such that for any $t : 0 \leq t \leq \tau$ and any $x \in B_R \cap \Omega_t$ holds

$$||T_t(x)|| \leq \overline{R}\,. \tag{1.1.4}$$

This assumption is valid under usual requirements concerning the force field.
Namely, it is sufficient to suppose that

$$||F(q)|| < C(||q|| + 1)\,.$$

In the Hamiltonian case where

$$F(q) = -\operatorname{grad}(U(q))$$

we may require that

$$U(q) > -C||q||^2, C > 0\,.$$

In the forthcoming sections we will prove the following

THEOREM 1.1.5.

$$m(\Omega_0 \setminus \Omega_\infty) = 0\,.$$

Since $\Omega_\infty = \bigcap_{\tau \in \mathbb{Z}} \Omega_\tau$ and each Ω_τ is open in the phase space, we get

COROLLARY 1.1.6. *$\Omega_0 \setminus \Omega_\infty$ is a first category subset in Ω_0.*

In some particular cases (of a constant or linear force field, for instance) we give much more strength topological characterization of $\Omega_0 \setminus \Omega_\infty$. It is worth mentioning the following strong result of Sinai [21] and Galperin [12] which concerns the case $F = 0$.

Theorem 1.1.7. *Assume that $F = 0$ and Ξ is a polyhedral angle. Then there exists a constant $C > 0$ such that* $\mathrm{Card}(D_c(x)) < C$ *for any x. Under the same assumption*

$$m(\Omega_0 \setminus \Omega_\infty) = 0\,.$$

1.1.3 – Proof of Theorem 1.1.5

It is sufficient to prove that

$$m(\Omega_0 \setminus \Omega_{+\infty}) = 0\,.$$

That is,

$$m(\{x \in \Omega_0 : \tau_+(x) < \infty\}) = 0\,.$$

Set

$$\begin{aligned} Z_\infty &= \{x \in \Omega_0 : \tau_+(x) < \infty,\ \mathrm{Card}(D_c(x) \cap \mathbb{R}_+) = \infty\}\,, \\ Z_{\text{fin}} &= \{x \in \Omega_0 : \tau_+(x) < \infty,\ \mathrm{Card}(D_c(x) \cap \mathbb{R}_+) < \infty\}\,. \end{aligned}$$

Clearly

$$\Omega_0 \setminus \Omega_{+\infty} = Z_\infty \cup Z_{\text{fin}}\,.$$

Lemma 1.1.8. *The set Z_{fin} is measurable and $m(Z_{\text{fin}}) = 0$.*

Proof. Set

$$\Delta = \bigcup_{t>0} S^{-t}(\partial\Omega_{\text{sing}})\,.$$

Clearly Δ is a measurable subset of X. Due to (1.1.2) $m(\Delta) = 0$. Define for $x \in \Omega_0$

$$\sigma(x) = \inf\{t > 0 :\ S^t(x) \in \partial\Omega\}\,.$$

Let

$$A = \{x :\ S^{\sigma(x)} \in \partial\Omega_{\text{sing}}\}\,.$$

Notice that σ is a measurable function of x and that

$$\Omega_0 \setminus A = \{x:\ \sigma(x) = \infty\} \cup \{x:\ S^{\sigma(x)} \in \Gamma\}\,. \tag{1.1.9}$$

The first set in the r.h.s. of (1.1.9) is measurable, the second one is open in the phase space. Hence A is a measurable set. Since $A \subset \Delta$, $m(A) = 0$. Consider $x \in Z_{\text{fin}}$. There exists

$$y = \lim_{t \uparrow \tau_+(x)} T^t(x) = \lim_{t \uparrow \tau_+(x) - \eta(x)} S^t(T^\eta(x))\,,$$

where $\eta(x) = \sup\{t \in D_c(x)\}$. Clearly $y \in \partial\Omega_{\text{sing}}$, otherwise we should come to a contradiction with the definition of τ_+. This observation shows that

$$Z_{\text{fin}} = \bigcup_{t \in \mathbb{Q}_+} (\Omega_t \cap T^{-t}(A))\,.$$

Taking into account 1.1.3 we obtain the result. □

In addition to the assertion of the lemma the preceding proof shows that Z_{fin} is in a certain sense of codimension 1. The next and the main step is to show that

$$m(Z_\infty) = 0\,.$$

Let $x \in Z_\infty$ be given. Define

$$I_\infty(x) = \{i : \operatorname{Card}\{t \in D_c(x) \cap \mathbb{R}_+ : T^t(x) \in \Gamma_i\} = \infty\}\,.$$

Since $x \in Z_\infty$, this set is not empty. Let $D_c(x) \cap \mathbb{R}_+ = \{\tau_1 < \ldots < \tau_k < \ldots\}$. We have

$$\lim_{k \to +\infty} \tau_k = \tau_+\,. \tag{1.1.10}$$

This follows from the fact that $D_c(x)$ is a discrete subset of $D(x)$ and $x \in Z_\infty$.

Lemma 1.1.11. *Let $x \in Z_\infty$ be as above. There exist:*

a) $q_+(x) = \lim_{t \uparrow \tau_+(x)} q(T^t(x))$;

b) $v_+(x) = \lim_{t \uparrow \tau_+(x)} v(T^t(x))$.

Proof. Due to assumption (1.1.4) we see that

$$\sup\{||v(T^t(x)|| : t \in [0,\ \tau_+(x))\} < C(x) < \infty\,. \tag{1.1.12}$$

Thus there exists

$$q_+(x) = \lim_{t\uparrow\tau_+(x)} q(T^t(x)) = q(x) + \int_0^{\tau_+-} v(T^s(x))\,ds\,. \tag{1.1.13}$$

Moreover,

$$\vartheta_j(q_+(x)) = 0 \tag{1.1.14}$$

for any $j \in I_\infty(x)$. Indeed:

$$0 = \liminf_{s\uparrow\tau_+(x)} |\vartheta_j(q(T^s(x)))| = |\vartheta_j(q_+(x))|$$

To prove b) we notice that

$$v(T^t(x)) = v(x) + \int_0^t F(q(T^s(x))\,ds + \sum_0^t \Delta v(T^s(x))\,, \tag{1.1.15}$$

where $\Delta v(T^s(x)) = v(T^s(x)) - v(T^{s-}(x))$ differs from 0 only for

$$s \in [0\ ,t] \cap D_c(x)\,.$$

Thus the sum $\sum_0^t \Delta v(T^s(x))$ contains a finite number of terms and is correctly defined for any $t < \tau_+(x)$. The second term in (1.1.15) converges as $t \uparrow \tau_+(x)$ since

$$\sup\{||F(q(T^s(x))|| : s < \tau_+(x)\} < \mathbf{F} < \infty\,.$$

To estimate the last term in (1.1.15) choose $c > 0$ and $\zeta \in \mathbb{R}^N$ such that

$$\vartheta_i(\zeta_0) > c \quad \text{for all} \quad i = 1, \ldots, K\,.$$

Set $\zeta = \zeta_0 - q_+(x)$. Taking into account (1.1.14) we conclude that

$$\langle\theta_j, \zeta\rangle > c\,,$$

as $j \in I_\infty(x)$.

By (1.1.12) and the previous observation we see that

$$\sup\Big\{\Big\|\sum_0^t \Delta v(T^s(x))\Big\| : t \in [0,\ \tau_+(x))\Big\} < \infty\,.$$

Thus

$$\sup\Big\{\Big|\Big\langle \zeta\ ,\ \sum_0^t \Delta v(T^s(x))\Big\rangle\Big| : t \in [0,\ \tau_+(x))\Big\} < \infty\,.$$

On the other hand the boundary reflection rule gives, provided that $T^s(x) \in \Gamma_j$ and $j \in I_\infty(x)$,

$$\langle \zeta\ ,\ \Delta v(T^s(x))\rangle = 2\langle \zeta\ ,\ \theta_j\rangle ||\Delta v(T^s(x))||\,.$$

Thus

$$\langle \zeta\ ,\ \Delta v(T^s(x))\rangle > 2c||\Delta v(T^s(x))||\,.$$

Note that for all $s \in D_c(x)$ except a finite number holds:

$$T^s(x) \in \Gamma_j \text{ and } j \in I_\infty(x)$$

Hence

$$\sum_0^{\tau(x)-} ||\Delta v(T^s(x))|| < C(x) + \frac{1}{2c}\sup\Big\{\Big|\Big\langle \zeta, \sum_0^t \Delta v(T^s(x))\Big\rangle\Big| : t \in [0, \tau_+(x))\Big\} < \infty\,.$$

Summarizing:

$$\sum_0^{\tau(x)-} ||\Delta v(T^s(x))|| < \infty\,. \tag{1.1.16}$$

This implies the assertion b). ∎

Lemma 1.1.17. *Under the previous assumptions*

$$\vartheta_j(q_+(x)) = \langle\ v_+(x)\ ,\ \theta_j\rangle = 0$$

for any $j \in I_\infty(x)$.

Proof. The equality $\vartheta_j(q(x)) = 0$, as $\theta \in I_\infty(x)$ is already proved. (See above.) If $T^s(x) \in \Gamma_j$ then

$$\Delta v(T^s(x)) = 2\langle\, v(T^s(x))\ ,\ \theta_j\rangle\theta_j\,.$$

(1.1.16) implies

$$0 = \liminf_{s\uparrow\tau_+(x)} |\langle\, v(T^s(x))\ ,\ \theta_j\rangle| = |\langle\, v_+(x)\ ,\ \theta_j\rangle|\,. \qquad \square$$

Define

$$Z_{R,t} = \{x \in Z_\infty : ||x|| < R,\ \tau_+(x) < t\}\,.$$

We complete the proof of Theorem 1.1.5 by the following

Lemma 1.1.18. *For any $R > 0$ and $t > 0$*

$$m(Z_{R,t}) = 0\,.$$

Proof. Let $x \in Z_{R,t}$. We introduce

$$\beta(x) = \sup\{|\langle v(T^s(x))\ ,\ \theta_j\rangle| : t \in [0,\ \tau_+(x)), j \in I_\infty(x)\}\,.$$

Note that $\beta(T^s(x))$ is a non-increasing function of s. Moreover, by (1.1.11), (1.1.17)

$$\lim_{s\uparrow\tau_+(x)} \beta(T^s(x)) = 0\,. \tag{1.1.19}$$

Furthermore, choose $\overline{R}(R,t)$ as in assumption 1.1.4. This choice guarantees that $T^s(x) \in B_{\overline{R}}$ for all

$$x \in Z_{R,t},\ s < \tau_+(x)\,.$$

We define then

$$\gamma(x) = \min\{|\vartheta_j(q(x))| : j \in I_\infty(x)\}\,.$$

Clearly,

$$m\{x \in \overline{R} : \gamma(x) \le \delta\} \le C \cdot \delta \tag{1.1.20}$$

for some constant $C(\overline{R}) > 0$. Note that

$$\gamma(x) \le \beta(x) \cdot \tau_+(x), \quad \text{if } x \in Z_\infty\,. \tag{1.1.21}$$

To see this, note that there exist $s \in [0,\ \tau_+(x))$ and $j \in I_\infty(x)$ such that

$$\vartheta_j(q(T^s(x))) = 0\,.$$

Thus:

$$\gamma(x) \le\ |\vartheta_j(q(x))|\ \le\ \beta(x)s\ \le\ \beta(x)\tau_+(x)\,.$$

Set

$$\tau^{(k)}(x) = \max\Big\{\frac{i}{2^k} : \frac{i}{2^k} < \tau_+(x),\ i = 0, 1, \dots\Big\};$$

$$G_i^{(k)} = \Big\{x \in Z_{R,t} : \tau^{(k)}(x) = \frac{i}{2^k}\Big\},\ i = 0, 1, \dots, [2^k t]\,.$$

Let $\beta^{(k)}(x) = \beta(T^{\tau^{(k)}(x)}(x))$. Clearly $\tau(x) - \tau^{(k)}(x) \le \frac{1}{2^k}$. Thus

$$\lim_{k\to+\infty} \beta^{(k)}(x) = 0\,.$$

To conclude the proof, choose $\varepsilon > 0$ and arbitrary integer $k > 0$. Consider $A_\varepsilon^{(k)} = \{x \in Z_{R,t} : \beta^{(k)}(x) < \varepsilon\}$. Denoting for simplicity $s_j = \frac{j}{2^k}$ we may write

$$m(A_\varepsilon^{(k)}) = \sum_{i=0}^{[2^k t]} m(A_\varepsilon^{(k)} \cap G_i^{(k)}) = \sum_{i=0}^{[2^k t]} m(T^{s_i}(A_\varepsilon^{(k)} \cap G_i^{(k)}))\,. \tag{1.1.22}$$

Let

$$y \in T^{s_i}(A_\varepsilon^{(k)} \cap G_i^{(k)}),\ y = T^{s_i}(x), \quad \text{where}\ \ x \in A_\varepsilon^{(k)} \cap G_i^{(k)}\,.$$

We have

$$\beta(y) = \beta^{(k)}(x) < \varepsilon\,, \tag{1.1.23}$$

since $x \in A_\varepsilon^{(k)}$. Moreover,

$$\tau_+(y) = \tau_+(x) - \tau^{(k)}(x) \le \frac{1}{2^k}\,. \tag{1.1.24}$$

In view of (1.1.21) this implies that

$$\gamma(y) \le \frac{\varepsilon}{2^k}\,.$$

Hence

$$T^{s_i}(A_\varepsilon^{(k)} \cap G_i^{(k)}) \subset \left\{ y : \gamma(y) \le \frac{\varepsilon}{2^k} \right\} \cap B_{\overline{R}}\,.$$

Taking into account (1.1.20), we see that

$$m(T^{s_i}(A_\varepsilon^{(k)} \cap G_i^{(k)})) \le \frac{C\varepsilon}{2^k}\,.$$

Hence

$$m(\{x \in Z_{R,t} : \beta^{(k)}(x) < \varepsilon\}) \le Ct\varepsilon\,.$$

Since $\lim_{k\to+\infty} \beta^{(k)}(x) = 0$, the previous inequality implies that $m(Z_{R,t}) \le Ct\varepsilon$. Since ε is arbitrary, $m(Z_{R,t}) = 0$. ∎

1.1.4 – Dimension of Z_∞: some examples

In this section, we develop sufficient conditions for the set Z_∞ to be of codimension 1 or even empty. The results obtained here are of independent interest and much stronger than those provided by Theorem 1.1.5. Throughout, a set $B \subset X = \mathbb{R}^{2N}$ is said to be of (at least) codimension L if there exists a countable family $\Lambda_i,\ i = 1, \dots$ of smooth submanifolds such that $\dim(\Lambda_i) \le N - L$ and satisfying $B \subseteq \cup_{i=1}^\infty \Lambda_i$. We will write $\operatorname{codim}(B) \ge L$.

For a subset $I \subseteq \{1, \dots, K\}$ let us denote by Θ_I the linear space generated by the vectors $\theta_i,\ i \in I$ and by P_I the orthogonal projection onto Θ_I. In the case of *polyhedral angle* a self-adjoint linear operator $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ is said to be compatible (with the polyhedral angle) if

$$P_I \Psi = \Psi P_I$$

for any $I \subseteq \{1, \ldots, K\}$. By definition

$$\Psi_I = P_I \Psi = \Psi P_I = P_I \Psi P_I \, .$$

From now on we make the following assumptions on the force field F:

There exist an open neighborhood O containing the boundary of the polyhedron Ξ, a constant vector F_* and, in case, where Ξ is a polyhedral angle, a compatible self-adjoint operator Ψ such that

$$F(q) = F_* - \Psi(q) \tag{1.1.25}$$

for any $q \in O \cap \Xi$.

Proposition 1.1.26. *Under the assumptions above* $\operatorname{codim}(Z_\infty) \geq 1$.

Proof. We expose the proof for the case of a polyhedral angle. In the general case we assume just that the force field is constant near the boundary and the proof becomes simpler. A point $x \in Z_\infty$ is said to be *regular* if for any $t \in [0, \ \tau_+(x))\, q(T^t(x)) \in O$ and $T^t(x) \in \Gamma_i \ \Rightarrow i \in I_\infty(x)$. By W_∞ we denote the set of all regular points. Clearly, for any $x \in Z_\infty$ there exists a rational number $r \in [0, \ \tau_+(x))$ satisfying $T^r(x) \in W_\infty$. Thus it is reasonable to estimate first $\operatorname{codim}(W_\infty)$. For $I \subseteq \{1, \ldots, K\}$ we define

$$H_I(x) = \frac{1}{2} ||P_I[v(x)]||^2 + \frac{1}{2} \langle \Psi_I[q(x)] \ , \ q(x) \rangle - \langle F_*, P_I[q(x)] \rangle \, .$$

Set

$$H_*(x) = H_{I_\infty(x)}(x) \, .$$

Let $x \in W_\infty$ be given. Consider the trajectory

$$q_*(t) = P_{I_\infty(x)}[q(T^t(x))], v_*(t) = P_{I_\infty(x)} \, [v(T^t(x))] \, .$$

It is governed by the equations $\dot{q}_* = v_*$, $\dot{v}_* = P_{I_\infty(x)}[F_*] - \Psi_{I_\infty(x)}(q_*)$ and by elastic reflections at the boundary of $\Theta_{I_\infty(x)} \cap \{q : \vartheta_i(q) \geq 0, \ i \in I_\infty(x)\}$. Hence we have the following conservation low:

$$H_*(T^t(x)) = H_*(x), t \in [0, \ \tau_+(x)) \, . \tag{1.1.27}$$

On the other hand Lemma 1.1.18 implies that

$$P_{I_\infty(x)}(v_+(x)) = P_{I_\infty(x)}(q_+(x)) = 0\,.$$

That is, $\lim_{t\uparrow\tau_+(x)} H_*(T^t(x)) = 0$. Thus

$$H_*(x) = 0\,. \tag{1.1.28}$$

We conclude that

$$W_\infty \subset \bigcup_{I\subseteq\{1,\dots,K\}} \{x : H_I(x) = 0\}\,. \tag{1.1.29}$$

Thus

$$Z_\infty \subset \bigcup_{I\subseteq\{1,\dots,K\}} \bigcup_{r\in\mathbb{Q}_+} T^{-r}(\{x : H_I(x) = 0\} \cap \Omega_{-r})\,.$$

This representation implies the result. ∎

Proposition 1.1.30. *Suppose that in addition to the assumptions of proposition* (1.1.26)*:*

$$\langle F_*, P_I[q]\rangle = \langle P_I[F_*], q\rangle \le 0 \tag{1.1.31}$$

for any

$$I \subseteq \{1,\dots,K\},\ q \in O \cap \Xi$$

and, in the case of a polyhedral angle, Ψ_I is non-negative defined as $I \subseteq \{1,\dots,K\}$:

$$\Psi_I \ge 0 \tag{1.1.32}$$

Then $Z_\infty = \emptyset$.

Proof. We present the proof for the case of a polyhedral angle. Let $x \in W_\infty$ be given. We have

$$-\langle P_I[F_*], q\rangle + \frac{1}{2}\langle \Psi_I[q(x)]\ ,\ q(x)\rangle \geq 0\,,$$

for any $I \subseteq \{1, \ldots, K\}$. Since $H_*(x) = 0$,

$$-\langle P_{I_\infty(x)}[F_*], q(x)\rangle + \frac{1}{2}\langle \Psi_{I_\infty(x)}[q(x)]\ ,\ q(x)\rangle + \frac{1}{2}||P_{I_\infty(x)}[v(x)]||^2 = 0\,.$$

Hence

$$-\langle P_{I_\infty(x)}[F_*], q(x)\rangle + \frac{1}{2}\langle \Psi_{I_\infty(x)}[q(x)]\ ,\ q(x)\rangle = \frac{1}{2}||P_{I_\infty(x)}[v(x)]||^2 = 0\,.$$

Thus

$$P_{I_\infty(x)}[v(T^t(x))] = 0,\ t \in [0,\ \tau_+(x))\,,$$

and $\vartheta_i(T^t(x)) = \text{const}$ for any $i \in I_\infty(x)$. That is, $\vartheta_i(T^t(x)) = 0,\ i \in I_\infty(x), t \in [0,\ \tau_+(x))$. It is impossible, since $D_c(x)$ is discrete. □

Remark to Proposition 3.2. In the particular case, where $\Psi = 0$, it is sufficient to require that (1.1.31) holds for all $I : \text{Card}(I) \geq 2$. It follows from the observation that $\text{Card}(I_\infty(x)) \geq 2$ for any $x \in Z_\infty$. To see this suppose that $I_\infty(x) = \{i\}$. Without any loss of generality we may assume that $x \in W_\infty$. Due to the previous argument $\langle F_*, \theta_i\rangle > 0$. But it implies that at most one collision with Γ_i is possible.

Example 1. Particle system in a constant force field.

Let us return to the particle system. Assume that all the external forces are constant : $F^{(i)}(Q^1, \ldots, Q^N) = F^{(i)}$. The scalar product we have to consider corresponds to the quadratic form

$$\langle Q,\ Q\rangle = \sum_{i=1}^{N} m_i (Q^i)^2\,.$$

The polyhedral angle is given by $N-1$ linear inequalities (in coordinates $Q^1, \ldots, Q^N$)

$$Q^{i+1} - Q^i \geq 0,\ i = 1, \ldots N-1\,.$$

Note that in this case $|i-j|>1$ implies that $\langle \theta_i,\ \theta_j\rangle = 0$. That means that we may verify condition (1.1.31) only for each integer *subinterval* I of $\{1,\dots,N-1\}$ containing at least 2 points. This observation has a clear mechanical interpretation. Let$x \in W_\infty$. Thus $I_\infty(x)$ may be decomposed into subintervals:

$$I_\infty(x) = \cup I^{(n)}$$

such that $I^{(n)} = \{i^{(n)}, i^{(n)}+1,\dots,i^{(n)}+|I^{(n)}|-1\}$ and $|i-j|>1$ provided $i \in I^{(n_1)},\ j \in I^{(n_2)}, n_1 \neq n_2$. It is easy to see that each $I^{(n)}$ corresponds to a group of particles (cluster) moving separately, i.e. without collisions with other particles. Hence $|I^{(n)}| \geq 2$ which means that there are at least three particles in each cluster. Condition (1.1.31) applied to a given cluster of particles assumes the following form.

Consider a cluster labeled by $J \subset \{1,\dots,N\}, J = \{j,\ j+1,\ j+2,\dots,j+L-1\}$, where L denotes *the number of particles* in the cluster. Set

$$M_J = \sum_{i\in J} m_i\text{- the mass of the cluster}\,,$$

$$A_J = \frac{1}{M_I}\sum_{i\in J} F^{(i)}\text{- the barycenter acceleration}\,.$$

Then

$$\sum_{i\in J}(F^{(i)} - m_i A_J)Q^i \leq 0, \text{for any } Q^j \leq Q^{j+1} \leq \cdots \leq Q^{j+L-1}\,. \tag{1.1.33}$$

This is equivalent to the following *monotonicity condition*:

For any subcluster with the same leftmost particle, that is, labeled by $J^{(r)} \subset J, J^{(r)} = j,\dots,j+r$ holds:

$$A_{J^{(r)}} \geq A_J\,, \tag{1.1.34}$$

or, equivalently

$$A_{J^{(r)}} \geq A_{J\setminus J^{(r)}}\,. \tag{1.1.35}$$

Summarizing we obtain the following assertion.

Proposition 1.1.36. *If the above monotonicity condition holds for any cluster J containing at least three particles, then $Z_\infty = \emptyset$.*

In particular, this is the case when the particles move with nonincreasing accelerations:

$$A_{\{1\}} \geq A_{\{2\}} \geq \dots A_{\{N\}}\,.$$

If all the inequalities above (1.1.35) are strict then we have a particle system which provides infinitely many collisions between any two neighboring particles but satisfying $Z_\infty = \emptyset$. If all the accelerations are equal:

$$A_{\{1\}} = A_{\{2\}} = \dots A_{\{N\}}$$

then the system is equivalent to that of Sinai and Galperin, since we may place the origin at the barycenter.

EXAMPLE 2. A system of one - dimensional balls with gravity (due to M.P. Wojtkowski).

This is a particle system satisfying

$$\frac{F^{(i)}}{m_i} = -g,\, g > 0$$

and with the following additional restriction: $Q^1 \geq 0$. In other words, the particles are restricted by an elastic wall from the left. From M. P. WOJTKOVSKI [30] this system, under certain conditions on the particle masses, has nonvanishing Lyapunov exponents almost everywhere.

PROPOSITION 1.1.37. *$Z_\infty = \emptyset$ for any masses and any number of particles.*

PROOF. The conditions of Proposition 1.1.30 have the following form. First of all the monotonicity condition introduced in the previous example has to be valid - that is, no accumulation of collisions is possible when the particles move separately and do not interact with the wall. This condition holds automatically, since the accelerations are equal (See above). In addition to that, the following inequalities have to hold:

$$\sum_{i=1}^{L} F^{(i)} Q^i \leq 0 \text{ for any } L > 1,\ 0 \leq Q^1 \leq \dots \leq Q^L\,.$$

The l.h.s. equals $-g \sum_{i=1}^{L} m_i Q^i \leq 0$. $\square$

Example 3. Harmonic Oscillator in a Polyhedral angle.
Set

$$\Psi = \psi E,\ \psi > 0\,;$$
$$F_* = 0\,.$$

Obviously, $Z_\infty = \emptyset$ independently of the angle geometry.

1.2 – Infinite systems

In this section we study the dynamics of one - dimensional Rayleigh gas and its modified version introduced above. Our aim is to construct the dynamics, corresponding to the interaction we have assumed, and correctly defined on a "large" set of phase points. We present a detailed proof for Rayleigh gas, the argument for modified Rayleigh gas requires just an unessential change of notations. Recall that we work in the *extended phase space* of the system

$$\widehat{\Omega} = \mathbb{R}^2 \times \mathbf{Y}\,,$$

where

$$\mathbf{Y} = \{Y \subset \mathbb{R}^2 : \mathrm{card}(Y \cap K) < \infty \text{ for any compact } K \subset \mathbb{R}^2\}\,.$$

Any point of Y is called a particle of the ideal gas and denoted by $(q\ ,\ v) \in Y \in \mathbf{Y}$. Here q and v are the position and the velocity of the particle respectively. A point $\hat{\omega} \in \hat{\Omega}$ is given by $\hat{\omega} = ((Q, V), Y)$, where Q, V are the position and the velocity of the massive particle (M.P.). The basic reference measure we are dealing with is (infinite) Gibbs measure

$$\hat{\mu}(d\hat{\omega}) = \sqrt{\frac{\beta M}{2\pi}}\exp\Big(-\beta M\frac{V^2}{2} + \beta f Q\Big) dV dQ\ \mathcal{P}(dY)\,,$$

where $\mathcal{P}$ denotes the Poisson field on the one-particle phase space $\mathcal{M} = \mathbb{R}^2$ with the intensity measure

$$n(dq\ dv) = \rho\sqrt{\frac{\beta m}{2\pi}}\exp\Big(\frac{-\beta m v^2}{2}\Big) dv\ dq\,.$$

Introducing the energy of M.P. by

$$H(Q,V) = M\frac{V^2}{2} - fQ\,,$$

we see that

$$\hat{\mu}(d\hat{\omega}) = \sqrt{\frac{\beta M}{2\pi}}\exp(-\beta H(Q,V))dV\,dQ\;\mathcal{P}(dY)\,.$$

We recall some elementary properties of the ideal gas $(\mathbf{Y}, \mathcal{P}(dY))$. Let $B \subset \mathcal{M}$ be a measurable subset of the one-particle phase space and $t \in \mathbb{R}$. Define

$$\mathsf{C}_t(B) = \{(q,v) \in \mathcal{M} : q + sv \in B \text{ for some } s \in [0,t]\}\,.$$

That is, $\mathsf{C}_t(B)$ contains all the phase points whose (free dynamics) trajectory up to time t intersects the set B. Furthermore, set

$$v^+(B) = \sup\{|v| : (q,v) \in B\}\,.$$

For $L > 0$

$$\Lambda_L = \{(q,v) : |q| \le L\}\,.$$

Elementary calculations show that

$$\mathcal{P}\{Y : v^+(Y \cap \mathsf{C}_t(\Lambda_L)) > a\}$$

is estimated from above by

$$2\rho L\sqrt{\frac{m\beta}{2\pi}}\int_{|v|>a}\exp\Big(-\frac{m\beta v^2}{2}\Big)dv + 2\rho|t|\sqrt{\frac{2}{m\beta\pi}}\exp\Big(-\frac{m\beta a^2}{2}\Big)\,.$$

Therefore, for $C > 1$ and arbitrary t holds:

$$\sum_{L\in\mathbb{N}}\mathcal{P}\Big\{Y : v^+(Y \cap \mathsf{C}_t(\Lambda_L)) > 2C\sqrt{\frac{\log L}{m\beta}}\Big\} < \infty\,. \tag{1.2.1}$$

This implies

Proposition 1.2.2. *Let $C > 1$ be given. There exists a measurable subset $\mathbf{Y}_* \subset \mathbf{Y}$ such that*

$$\mathcal{P}(\mathbf{Y} \setminus \mathbf{Y}_*) = 0$$

and for $Y \in \mathbf{Y}_*$ holds:

$$\limsup_{L \to \infty} \frac{v^+(Y \cap \mathsf{C}_t(\Lambda_L))}{\sqrt{\log L}} < \frac{2C}{\sqrt{m\beta}}$$

for any t. Moreover, the set $\mathbf{Y}_*$ is invariant with respect to the free dynamics $\{T_0^t\}$.

Proof. Set

$$\mathbf{Y}_* = \bigcap_{t \in \mathbb{Z}} \bigcup_{L_0 \in \mathbb{N}} \bigcap_{\substack{L \geq L_0, \\ L \in \mathbb{N}}} \Big\{ Y : \frac{v^+(Y \bigcap \mathsf{C}_t(\Lambda_L))}{\sqrt{\log L}} < \frac{2C}{\sqrt{m\beta}} \Big\} .$$

Note that

$$v^+(T_0^s(Y) \cap \mathsf{C}_t(\Lambda_L)) \leq v^+(Y \cap \mathsf{C}_\tau(\Lambda_L)) ,$$

as $[s, t+s] \subseteq [0, \tau]$. ∎

From now on we identify $\mathbf{Y}$ and $\mathbf{Y}_*$. To construct the infinite-particle dynamics we approximate the system by a finite-particle one.

Definition. We say that the dynamics is well defined up to time t for $\hat{\omega} = ((Q, V), Y)$ if there exists

$$L_0 \in \mathbb{N}, L_0 > |Q|$$

such that for any $L \in \mathbb{N}, L \geq L_0$ the dynamics is well defined for the finite system

$$\hat{\omega}_L = ((Q, V), Y) \cap \mathsf{C}_t(\Lambda_L) .$$

The evolution of $\hat{\omega}_L$ will be denoted by

$$\widehat{T}^s(\hat{\omega}_L),\ s \in [0, t] .$$

For L large enough velocities of all the particles contained in $\hat{\omega}_L$, including M.P., satisfy

$$|v| \le \text{Const}\,\sqrt{\log L}\,.$$

Without loss of generality we may also assume that

$$|Q(\hat{\omega})| \le \sqrt{\log L}\,.$$

By the evolution $\{\widehat{T}^s\}$, $|s| \le t$ the following estimate holds:

$$v^+(T^s(\hat{\omega}_L)) \le \text{Const}\,\sqrt{\log L} + \frac{|f|}{M}|s|\,.$$

As L is large enough, this guarantees that M.P. is up to time t inside the segment $[-\sqrt{L}, \sqrt{L}]$. Therefore, no particles of $Y \cap \mathsf{C}_t(\Lambda_L)^c$ moving according to the free dynamics interact with M.P. Thus for $s \in [0, t]$ it makes sense to write:

$$T^s(\hat{\omega}) = T^s(\hat{\omega}_L) \cup T_0^s(Y \cap \mathsf{C}_t(\Lambda_L)^c)\,.$$

Clearly the expression above is independent of L, as L is large enough. Denote by $\widehat{\Omega}_t$ the set of all configurations for that the dynamics is well defined up to t. Set

$$\widehat{\Omega}_\infty = \bigcap_{t\in\mathbb{Z}} \widehat{\Omega}_t\,.$$

It is easily seen that $\widehat{\Omega}_\infty$ is a measurable $\{T^t\}$ invariant set. We want to show that this set is of full $\hat{\mu}$ measure and that

$$\hat{\mu}\widehat{T}^t = \hat{\mu}\,.$$

To this end for $N \in \mathbb{N}$ and $\hat{\omega} \in \widehat{\Omega}$ such that $|Q(\hat{\omega})| \le N$ we define $\widehat{T}_N^t(\hat{\omega})$ as follows. First,

$$\widehat{T}_N^t(\hat{\omega}) \cap \Lambda_N^c = \hat{\omega} \cap \Lambda_N^c\,,$$

i.e. the configuration outside the interval $[-N, N]$ remains constant. Inside the interval $[-N, N]$ all the particles, including M.P., move according to the usual dynamics and *reflecting elastically at the boundary* $\{-N, N\}$. For $|Q(\hat{\omega})| > N$ we set

$$\widehat{T}_N^t(\hat{\omega}) = \hat{\omega}\,.$$

Our previous results concerning dynamics of finite particle systems show that $\widehat{T}^t_N(\hat\omega)$ is correctly defined for all t and $\hat\omega$ contained in a full $\hat\mu$ - measure invariant set $\widehat\Omega_N$. Moreover, each $\widehat{T}^t_N$ preserves the measure $\hat\mu$. Set

$$\widehat\Omega_* = \bigcap \widehat\Omega_N\,.$$

For $\hat\omega \in \hat\Omega_*$ holds:

$$\widehat{T}^t(\hat\omega) = \lim_{N\to\infty} \widehat{T}^t_N(\hat\omega)\,. \tag{1.2.3}$$

Moreover, for fixed t and any given $L > 0$ there exists $N_0(L,t,\hat\omega)$ such that

$$\widehat{T}^t_N(\hat\omega)\bigcap \Lambda_L = T^t_{N_0}(\hat\omega)\bigcap \Lambda_L\,,$$

as $N \geq N_0$. To see this, note again that for N large enough velocities of all the particles inside $[-N,N]$, including M.P., satisfy

$$|v| \leq \text{Const}\,\sqrt{\log N}\,.$$

Without loss of generality we may also assume that

$$|Q(\hat\omega)| \leq \sqrt{\log N}\,,$$

and

$$v^+(T^s_N(\hat\omega)\cap\Lambda_N) \leq \text{Const}\,\sqrt{\log N} + \frac{|f|}{M}|s|\,.$$

As N is large enough this guarantees that M.P. is up to time t inside the segment $[-\sqrt{N},\sqrt{N}]$ and

$$\hat\omega \cap C_t(\Lambda_{\sqrt{N}}) \subset \Lambda_{\frac{N}{2}}\,.$$

Therefore, no particle of $\hat\omega\cap \mathsf{C}_t(\Lambda_{\sqrt{N}})$ may collide with the boundary, since any particle, which does, will never achieve the segment $[-\frac{N}{2},\frac{N}{2}]$. Choosing N_0 so large that the previous assertions hold true and $L \leq \sqrt{N_0}$ we see that particles of $\hat\omega$ contained in

$$\mathsf{C}_t(\Lambda_{\sqrt{N_0}})$$

evolute separately and their evolution is independent of $N \geq N_0$:

$$\widehat{T}^s_N(\hat{\omega}) \cap \Lambda_{\sqrt{N}0} = \widehat{T}^s(\hat{\omega} \cap \mathsf{C}_t(\Lambda_{\sqrt{N_0}})) \cap \Lambda_{\sqrt{N}0}\,.$$

This implies

$$\widehat{\Omega}_* \subseteq \widehat{\Omega}_\infty\,.$$

As $\widehat{\Omega}_*$ is of full $\hat{\mu}$ measure so is $\widehat{\Omega}_\infty$.

Proposition 1.2.4. *The measure $\hat{\mu}$ is invariant with respect to the shift $\{T^t\}$.*

Proof. Suppose, a function ϕ satisfies the following conditions:

$$||\phi||_\infty = \sup\{|\phi(\hat{\omega})|\} < \infty\,; \tag{1.2.5}$$

$$\phi(\hat{\omega}_2) = \phi(\hat{\omega}_1)\,, \tag{1.2.6}$$

as $\hat{\omega}_1 \cap \Lambda_L = \hat{\omega}_2 \cap \Lambda_L$;

$$\phi(\hat{\omega}) = 0\,, \tag{1.2.7}$$

as $|Q(\hat{\omega})| \geq L$.

Consider $\phi(\widehat{T}^t_N(.))$. Since ϕ depends on the configuration of particles inside the segment $[-L, L]$, it is easily seen that for any t

$$\lim_{N\to\infty} \phi(\widehat{T}^t_N(\hat{\omega})) = \phi(T^t(\hat{\omega}))\,, \tag{1.2.8}$$

as $\hat{\omega} \in \Omega_*$. Let us study the convergence above. Evidently,

$$\phi(\widehat{T}^t_N) \in \mathcal{L}_1(\hat{\mu})$$

and

$$||\phi(\widehat{T}^t_N)||_1 = ||\phi||_1\,.$$

We state that for a given t there exists $c > 0$ such that

$$\int_{|Q(\hat{\omega})|>R} \phi(\widehat{T}^t_N(\hat{\omega}))\hat{\mu}(d\hat{\omega}) \leq \frac{1}{c}\exp(-cR^2)\,. \tag{1.2.9}$$

Indeed,

$$\int_{\{|Q(\hat{\omega})|>R\}} \phi(\widehat{T}_n^t(\hat{\omega}))\hat{\mu}(d\hat{\omega}) \leq$$
$$\leq C \int_{|Q|>R}^{\infty} e^{\beta f Q} dQ \int \exp\Big(-\frac{\beta M V^2}{2}\Big) dV \mathcal{P}(dY) I_{D_{N,Q}}(V,Y)\,,$$

$I_{D_{N,x}}(V,Y)$ denotes the characteristic function of the set

$$D_{N,x} = \{(V,Y) : |Q(\widehat{T}_N^t((V,Q=x),Y))| \leq L\}\,.$$

On the other hand

$$D_{N,Q} \subseteq \Big\{(V,Y) : \max\{|V|, v^+(\mathsf{C}_t(\Lambda_Q))\} > \frac{|Q|-L}{t} - \frac{|f|}{M}|t|\Big\}\,,$$

as $|Q| > L$. Thus

$$\int_{D_{N,Q}} \exp\Big(-\frac{\beta M V^2}{2}\Big) dV \mathcal{P}(dY) \leq C \exp(-cQ^2)\,.$$

and therefore (1.2.9) holds. It is easily seen that the estimate (1.2.9) provides uniform $\hat{\mu}$ -integrability of $\phi(\widehat{T}_N^t)$, $N = 1, \ldots,$. (t is fixed.) Hence

$$\int \phi(\widehat{T}^t(\hat{\omega}))\hat{\mu}(d\hat{\omega}) = \lim_{N\to\infty} \int \phi(\widehat{T}_N^t(\hat{\omega}))\hat{\mu}(d\hat{\omega}) = \int \phi(\hat{\omega})\hat{\mu}(d\hat{\omega})\,.$$

Noting that the functions ϕ under consideration are dense in $\mathcal{L}_1(\hat{\mu})$ we complete the argument. $\square$

2 – One - dimensional Rayleigh gas in equilibrium

The goal of this section is to present several rigorous results concerning the asymptotic behavior of a massive particle (M.P.) of mass M moving in one direction under the action of elastic collisions with particles of equal masses m. We do not have a goal to represent all the known results, instead we derive the *balance equations* - at the moment, the main tool of obtaining rigorous results on the one-dimensional Rayleigh gas.

The external force field acting on the M.P. assumed to be trivial: $f = 0$. Rigorous definitions and notations are given in the introduction. For a mathematically correct construction of the dynamics we refer the reader to the previous section. We work in the phase space Ω of configurations related to the position of the M.P. By convention, the position of M.P. is identified with the origin. The main reference measure μ is invariant under the dynamics $\{T^t\}$.

2.1 – The case of equal masses

We start by studying the simplified case, $m = M$.

THEOREM 2.1.1 [26]. *Assume that $M = m$. Then the random process*

$$\frac{1}{\sqrt{N}} Q(tN, .)$$

converges (as $N \to \infty$) in the sense of week convergence of path measures to the Wiener process $W_{\sigma_+^2 t}$. The limit variance

$$\sigma_+^2 = \sqrt{\frac{2}{\pi}} \frac{1}{\rho\sqrt{m\beta}} \, .$$

PROOF. We present briefly the argument which is due to Dürr, Goldstein, and Lebowitz. Let us label the positions of *all the particles*, i.e. gas particles and the M.P., at time t in their natural order:

$$\cdots < q_{-2}(t) < q_{-1}(t) < q_0(t) \leq 0 < q_1(t) < \dots$$

For instance $q_0(0) = Q = 0$. Whenever a particle crosses the origin the above labeling shifts by one unit. Set

$$\mathcal{N}^+(t) = \{(q, v) \in Y : q < 0, q + tv > 0\} \, ,$$
$$\mathcal{N}^-(t) = \{(q, v) \in Y : q > 0, q + tv < 0\} \, .$$

Define

$$N^+(t) = \mathrm{card}\{\mathcal{N}^+(t)\} \, ,$$
$$N^-(t) = \mathrm{card}\{\mathcal{N}^-(t)\} \, .$$

Evidently $N^+(t)$ and $N^-(t)$ are independent identically distributed Poisson variables with parameter

$$\frac{\rho t}{\sqrt{2\pi m\beta}}\,.$$

Set

$$\psi(\omega) = I_{\{V>0\}}(\omega)$$

and let

$$j(t) = N^+(t) - N^-(t) + \psi\,.$$

The key observation is

$$Q(t) = q_{j(t)}\,.$$

By stationarity the ideal gas particles including the M.P. are Poisson distributed at any time t. Thus estimating "large deviations" we see that

$$\mu\Big\{\sup_{|n|\le N}\Big\{\Big|q_n - \frac{n}{\rho}\Big|\Big\} \ge N^{\frac{1}{2}+\alpha}\Big\} \le \exp(-c_1 N^{2\alpha})$$

for some constant c_1. A similar estimate holds for $j(t)$:

$$\mu\{\sup_{|t|\le T}\{|j(t)|\} \ge T^{\frac{1}{2}+\alpha}\} \le \exp(-c_2 T^{2\alpha})\,.$$

The estimates above imply that

$$\lim_{N\to\infty}\mu\Big\{\sup_{t\le T}\frac{1}{\sqrt{N}}\Big|Q(Nt) - \frac{j(Nt)}{\rho}\Big| > N^{-\frac{1}{8}}\Big\} = 0\,.$$

Further,

$$\frac{1}{\rho\sqrt{N}}j(Nt)$$

converges as $N\to\infty$, to the Wiener process $W_{\sigma_+^2 t}$, where

$$\sigma_+^2 = \sqrt{\frac{2}{\pi}}\frac{1}{\rho\sqrt{m\beta}}\,. \qquad \square$$

2.2 – The case of unequal masses $M \neq m$

The case where $M \neq m$ does not provide a precise expression for $Q(t)$. The dynamics is too complicated for to be analyzed directly. The arguments working relatively effective in this situation may be called "balance arguments." They make use of various conservation laws and stationarity of the reference measure.

2.2.1 – An estimate of the displacement of the M.P

LEMMA 2.2.1. *For a given $\epsilon > 0$ there exists a constant $c_1 > 0$ such that*

$$\mu\{\omega : \max_{0 \le s \le t} |V(T^s(\omega))| > t^\epsilon\} \le \exp(-c_1 t^{2\epsilon}) .$$

PROOF. Take $L(t) = (1 + t^2)$. Set

$$B_t = \{\omega : |V(\omega)| \le t^\epsilon, v_+(\mathsf{C}_t(\Lambda_L)) \le t^\epsilon\} .$$

Evidently $\mu(B_t) > 1 - \exp(-ct^{2\epsilon})$. For $\omega \in B_t$

$$\max_{0 \le s \le t} |V(T^s(\omega))| \le t^\epsilon \qquad \square$$

THEOREM 2.2.2. *For any $\epsilon > 0$ and μ - almost every ω there exists $\mathbf{t}_0(\omega, \epsilon) > 0$, such that for all $\mathbf{t} > \mathbf{t}_0(\omega, \epsilon)$*

$$\sup\{|Q(s, \omega)| : 0 \le s \le \mathbf{t}\} \le \mathbf{t}^{1/2+\epsilon} .$$

There exists a constant $c > 0$ such that

$$\mu\{\omega : \sup\{|Q(s, \omega)| : 0 \le s \le \mathbf{t}\} > \mathbf{t}^{1/2+\epsilon}\} \le \exp(-c\mathbf{t}^{2\epsilon}) .$$

Proof. Let us start by proving the second assertion. Suppose,

$$Q(t,\omega) > R(\mathbf{t}),$$

where $t \in \mathbb{Z}$, $0 \le t \le \mathbf{t}$. Set $R(\mathbf{t}) = \mathbf{t}^{1/2+\epsilon}$, $d(t,\omega) = Q(t,\omega) - \frac{1}{2}R(\mathbf{t})$. We will consider the configuration of particles at the moment t inside the segment

$$\Delta(t,\omega) = [d(t,\omega), Q(t,\omega)].$$

The M.P. at the time t coincides with the right vertex of the segment Δ. Since μ is an invariant measure, the distribution of particles inside $\Delta(t)$ is equal to that of particles inside $[-\frac{1}{2}R(\mathbf{t}), 0]$ at the moment $t = 0$. Set

$$C^{(t)} = \{\omega : Q(t,\omega) > R(\mathbf{t})\}.$$

For $\omega \in C^{(t)}$ we define

$$\tau(\omega) = \max\{s : Q(s,\omega) = d(t,\omega);\ 0 \le s \le t\}.$$

Then

$$C^{(t)} = \bigcup_{k=0}^{t-1} C_k^{(t)},$$

where

$$C_k^{(t)} = \{\omega : k < \tau(\omega) \le k+1\}.$$

Our next goal is to estimate the μ measure of each $C_k^{(t)}$. Let us define the following set

$$\mathcal{N}_-(t,\omega) = \{(q,v) \in Y : q(t) \in \Delta(t,\omega), v(t) < 0\},$$

where $q(t)$ and $v(t)$ denote the position and velocity of the particle at time t. Set

$$N_-(t,\omega) = \mathrm{card}\{\mathcal{N}_-(t,\omega)\}.$$

The random value $N_-(t,\omega)$ is easy to handle since

$$N_-(t,\omega) = \mathrm{card}\Big\{(q,v) \in Y(T^t(\omega)) : q \in \Big[-\frac{1}{2}R(\mathbf{t}), 0\Big], v < 0\Big\}.$$

Taking into account that the measure μ is stationary under the dynamics we conclude that $N_-(t,\omega)$ is a Poisson variable with the parameter $\frac{1}{4}\rho R(\mathbf{t})$. Further, define

$$\mathcal{K}(t,Y)=\{(q,v)\in Y(\omega): q(t)<d(t), v(t)<0, q(s)=Q(s) \text{ for some } S\in[\tau,t]\}.$$

$\mathcal{K}(t,Y)$ contains the particles which had a collision with the M.P. in the time interval $[\tau,t]$ and then left the segment $\Delta(t)$. Set

$$K(t,\omega)=\operatorname{card}\{\mathcal{K}_-(t,Y)\}$$

By a similar manner we define

$$\mathcal{M}(t,\omega)=\{(q,v)\in Y : q<0, q(s)=Q(s) \text{ for some } s:\tau\le s\le t\},$$

and

$$M(t,\omega)=\operatorname{card}\{\mathcal{M}(t,\omega)\}.$$

The key observation is contained in the obvious inequality:

$$N_-(t,\omega)\le M(t,\omega)-K(t,\omega) \tag{2.2.3}$$

As we mentioned above, the left hand side of this inequality is easy to handle: it is a Poisson variable. The values on the right hand side are of a more complicated structure. We will estimate them by poisson variables as well. First, if some particle at the moment t is on the left of $d(t)$, has negative velocity and under the inverse free dynamics will cross the point $d(t)$ up to time $k+1$, then this particle belongs to $\mathcal{K}(t,\omega)$, as $\omega\in C_k^{(t)}$. This long statement may be expressed formally as follows. For $\omega\in C_k^{(t)}$.

$$\mathcal{K}^{(1)}(t,\omega)\subseteq\mathcal{K}(t,\omega),$$

where

$$\mathcal{K}^{(1)}(t,\omega)=\{(q,v)\in Y: q(t)<d(t), v(t)<0, q(t)-(t-(k+1))v(t)\ge d(t)\}.$$

Thus

$$K(t,\omega)\ge K^{(1)}(t,\omega),$$

where

$$K^{(1)}(t,\omega) = \mathrm{card}\{\mathcal{K}^{(1)}(t,\omega)\}\,.$$

The random variable $K^{(1)}(t,\omega)$ has Poisson distribution with parameter

$$\gamma = \frac{\rho}{\sqrt{2\pi m\beta}}(t-(k+1))\,.$$

It remains to estimate the value $M(t,\omega)$ from above. To this end set

$$\mathcal{D}_t = \{\omega : \max_{0\le s\le t} V(T^s(\omega)) \le \mathbf{t}^\epsilon\}\,.$$

We know that there exists a constant $c_1 > 0$ such that

$$\mu(\mathcal{D}_t) \ge 1 - \exp(-c_1\mathbf{t}^{2\epsilon})\,.$$

Put

$$\begin{aligned}\mathcal{M}^{(1)}(t,\omega) = \{(q,v)\in Y :\, & q(k)<Q(k),\, v(k)>0\,,\\ & q(k)+(t-k)v(k) > Q(k)-\mathbf{t}^\epsilon\}\,,\end{aligned}$$

and

$$M^{(1)}(t,\omega) = \mathrm{card}\{\mathcal{M}^{(1)}\}\,.$$

The random variable $M^{(1)}$ has Poisson distribution with parameter

$$\delta = \frac{\rho\sqrt{m\beta}}{\sqrt{2\pi}}\int_{-\infty}^{0} dq \int_{\max\{0,-\frac{q+\mathbf{t}^\epsilon}{t-k}\}}^{\infty} \exp\Big(-\frac{m\beta v^2}{2}\Big)dv\,.$$

Evidently

$$\begin{aligned}\delta &\le \frac{\rho\sqrt{m\beta}}{\sqrt{2\pi}}\int_{-\infty}^{0} dq \int_{-\frac{q+\mathbf{t}^\epsilon}{t-k}} \exp\Big(-\frac{m\beta v^2}{2}\Big)dv = \\ &= \frac{\rho(t-k)}{\sqrt{2\pi m\beta}}\exp\Big(-\frac{m\beta\mathbf{t}^{2\epsilon}}{2(t-k)^2}\Big) + \frac{\rho\mathbf{t}^\epsilon\sqrt{m\beta}}{\sqrt{2\pi}}\int_{-\frac{\mathbf{t}^\epsilon}{t-k}}^{\infty}\exp\Big(-\frac{m\beta v^2}{2}\Big)dv\,.\end{aligned}$$

Thus

$$\delta \le \frac{\rho(t-k)}{\sqrt{2\pi m\beta}} + \rho\mathbf{t}^\epsilon\,. \tag{2.2.4}$$

The inequality

$$M^{(1)}(t,\omega) > M(t,\omega)$$

holds under the condition

$$Q(k,\omega) \le d(t) + \mathbf{t}^{\epsilon}\,.$$

On the other hand

$$Q(\tau,\omega) = d(t)$$

for $k < \tau \le k+1$.

Thus if $Q(k) > d(t) + \mathbf{t}^{\epsilon}$ then

$$\max_{k<s\le k+1} V(T^s(\omega)) > \mathbf{t}^{\epsilon}$$

Thus

$$M^{(1)}(t,\omega) > M(t,\omega), \text{ as } \omega \in \mathcal{D}_t\,. \tag{2.2.5}$$

This means that

$$\mu(C_k^{(t)} \cap \mathcal{D}_t) \le \mu\{\omega : N_-(t,\omega) \le M^{(1)}(t,\omega) - K^{(1)}(t,\omega)\} \tag{2.2.6}$$

The right hand side of (2.2.6) contains Poisson variables only. Thus it may be easily estimated by using "Large deviations" of a Poisson distribution. For a given constant $c_2 > 0$ we set

$$A_+ = \left\{\omega : M^{(1)}(t,\omega) \ge \frac{\rho(\mathbf{t}-k)}{\sqrt{2\pi m\beta}} + \rho\mathbf{t}^{\epsilon} + c_2 R(\mathbf{t})\right\},$$
$$A_- = \left\{\omega : K^{(1)}(t,\omega) \le \frac{\rho(\mathbf{t}-(k+1))}{\sqrt{2\pi m\beta}} - c_2 R(\mathbf{t})\right\}.$$

From general properties of Poisson distribution it follows that for $c_3 > 0$

$$\mu(A_\pm) \le \exp(-c_3\mathbf{t}^{2\epsilon})\,.$$

Further, for

$$\omega \in C_k^{(t)} \cap \mathcal{D}_t \setminus (A_+ \cup A_-)$$

we have:

$$N_-(t,\omega) \le \rho \mathbf{t}^\epsilon + 2c_2 R(\mathbf{t}).$$

But $N_-(t,\omega)$ has Poisson distribution with parameter $\frac{\rho}{4}R(\mathbf{t})$. If c_2 is chosen sufficiently small then for a constant $c_4 > 0$ we have:

$$\begin{aligned}&\mu(C_k^{(t)} \cap \mathcal{D}_t \setminus (A_+ \cup A_-)) \le \\ &\le \mu\Big\{\omega : N_-(t,\omega) \le \frac{\rho}{4}R(t) + \rho t^\epsilon + \Big(c_2 - \frac{\rho}{4}\Big)R(\mathbf{t})\Big\} \le \exp(-c_4 \mathbf{t}^{2\epsilon}).\end{aligned}$$

Thus, taking into account the estimations of $\mu(A_\pm)$ and $\mu(\mathcal{D}_t)$, we get

$$\mu(C_k^{(t)}) \le \exp(-c_5 \mathbf{t}^{2\epsilon}).$$

Hence

$$\mu(C^{(t)}) \le \sum_{k=0}^{t-1} \mu(C_k^{(t)}) \le \exp(-c_6 \mathbf{t}^{2\epsilon}),$$

where the constant $c_6 > 0$. Thus

$$\mu\{\omega : \sup\{|Q(s,\omega)| : s \in \mathbb{Z},\ 0 \le s \le \mathbf{t}\} > \mathbf{t}^{1/2+\epsilon}\} \le \exp(-c_7 \mathbf{t}^{2\epsilon}),$$

for some constant $c_7 > 0$. Taking into account that

$$\mu\{\omega : \sup_{0 \le s \le 1} |V(T^{t+s})| > \mathbf{t}^\epsilon\} \ \le \exp(-c_8 \mathbf{t}^{2\epsilon}), \tag{2.2.7}$$

we obtain the required assertion.

Further, Borell-Cantelly argument shows that

$$\sup\{|Q(t,\omega)|,\ 0 \le t \le \mathbf{t}\} \le \mathbf{t}^{1/2+\epsilon/2},$$

as

$$\mathbf{t} \in \mathbb{Z}_+, \mathbf{t} \ge \mathbf{t}_0(\omega).$$

This implies the result. □

2.2.2 – Restriction on the number of collisions

The following theorem may be considered as an infinite - particle analog of that of Sinai and Galperin (see the previous section).

THEOREM 2.2.8. *For μ - almost every ω each particle of the ideal gas has finitely many collisions with M.P.*

Proof. Let us denote by $\Gamma^\infty(\omega)$ the set of all particles of $Y(\omega)$ that are at the left of M.P. and undergo infinitely many collisions as $t > 0$. It follows from the previous theorem that the velocity of any particle contained in $\Gamma^\infty(\omega)$ has to be positive for each $t > 0$, because if it becomes negative, then the particle escapes to infinity. Define

$$(q^+(\omega), v^+(\omega)) \in \Gamma^\infty(\omega)$$

by

$$q^+(\omega) = \max\{q : (q,v) \in \Gamma^\infty(\omega)\}\,.$$

Set $w(\omega) = v^+(\omega)$ if $\Gamma^\infty(\omega) \neq \emptyset$ and $w(\omega) = 0$ otherwise. Evidently $w(\omega) \geq 0$. The key observation is:

$$\int_0^t w(T^s(\omega))ds \leq Q(t,\omega) - q^+(\omega)\,,$$

as $\Gamma^\infty(\omega) \neq \emptyset$. It follows from the previous theorem that

$$\lim_{t\to\infty} \frac{1}{t} Q(t,\omega) = 0 \ \text{ a.e.}$$

Thus

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t w(T^s(\omega))ds = 0 \ \text{ a.e.}$$

and by the ergodic theorem $w = 0$ a.e. This is equivalent to $\Gamma^\infty = \emptyset$ a.e. The same argument works for particles at the right of M.P. ∎

2.2.3 – An expression for the displacement of the Massive Particle

We start with several notations:

$$\begin{aligned} \mathcal{A}^+(t,\omega) &= \{(q,v) \in Y(\omega) : q < 0,\ q + tv > 0\}\,,\\ \mathcal{A}^-(t,\omega) &= \{(q,v) \in Y(\omega) : q < Q(t),\ q - tv > Q(t)\}\,. \end{aligned}$$

The values $A^\pm(t,\omega) = \operatorname{card}\{\mathcal{A}^\pm(t,\omega)\}$ are identically distributed Poisson variables. Note that these variables are, however, strongly dependent and we have no information on their joint distribution. Our purpose is to present the displacement of the M.P. as a function of $A^\pm$. Set

$$\mathcal{B}(t,\omega) = \{(q,v) \in Y(\omega) : q < 0,\ q(s) = Q(s) \ \text{ for some } \ s : 0 \leq s \leq t\}\,,$$

and

$$B(t,\omega) = \mathrm{card}\{\mathcal{B}(t,\omega)\}\,.$$

Lemma 2.2.9.

$$A^+(t,\omega) = \frac{\rho}{2}Q(t,\omega) + B(t,\omega) + \alpha(t,\omega)\,,$$

where

$$\frac{\alpha(t,.)}{\sqrt{t}} \to 0$$

in probability as $t \to \infty$.

Proof. In view of Theorem 2.2.2 we may consider only those ω and t for which

$$\sup\{|Q(s,\omega)|,\ 0 \le s \le t\} \le t^{1/2+\epsilon}$$

for sufficiently small ϵ. Assume that $Q(t,\omega) > 0$. The set

$$\mathcal{L}(t,\omega) = \mathcal{A}^+(t,\omega) \setminus \mathcal{B}(t,\omega)$$

consists of particles $(q,v) \in Y(\omega)$ such that:

1. $q(t) \in [0, Q(t,\omega)]$,
2. $v(t) > 0$,
3. $q(s) < Q(s)$, for all $s \in [0,t]$.

Denote

$$\begin{aligned}\mathcal{E}(t,\omega) &= \{(q,v) \in Y : v(t) > 0, q(t) \in [0, Q(t)]\}\,,\\ E(t,\omega) &= \mathrm{card}\{\mathcal{E}(t,\omega)\}\,.\end{aligned}$$

Evidently

$$\mathcal{L}(t,\omega) \subseteq \mathcal{E}(t,\omega)\,.$$

Considering the difference

$$\mathcal{E}_1(t,\omega) = \mathcal{E}(t,\omega) \setminus \mathcal{L}(t,\omega)\,,$$

we see that it consists of particles with positive velocities which are at the moment t in the segment $[0, Q(t)]$ and interacted with the M.P. at the moment $s \in [0, t]$. We state that

$$\frac{E_1(t, .)}{\sqrt{t}} \to 0$$

in probability, where

$$E_1(t, \omega) = \mathrm{card}\{\mathcal{E}_1(t, \omega)\} .$$

To see this, for $(q, v) \in \mathcal{E}_1(t, \omega)$ we introduce the moment $\tau((q, v), \omega)$ equal to the last moment of collision with M.P. in the time interval $[0, t]$. Consider separately two cases.

CASE 1. $\tau < 1 - t^{3/4}$. Denote by $p_1(t, \omega)$ the number of particles with this property. Each of these particles moves freely in the time interval (τ, t) with a positive velocity and therefore, for the displacement $\delta q(\tau, t)$ of this particle during the interval (τ, t) we have:

$$v(t) \leq \frac{\delta q(\tau, t)}{t^{3/4}} \leq \frac{Q(t, \omega) - Q(\tau, \omega)}{t^{3/4}} \leq t^{-1/4+\epsilon} .$$

Also for these particles holds the inclusion:

$$q(t) \in [0, Q(t)] \subset [Q(t) - t^{1/2+\epsilon}, Q(t)] .$$

Thus the $p_1(t, \omega)$ is no more than the number of particles $(q, v) \in Y(T^t(\omega))$ such that $q \in [-t^{1/2+\epsilon}, 0]$ and $0 \leq v \leq t^{-1/4+\epsilon}$. Thus $p_1(t, \omega)$ is estimated from above by the Poisson variable with parameter

$$\gamma < \mathrm{const}\, t^{1/4+\epsilon} .$$

For sufficiently small ϵ we have

$$\frac{p_1(t, .)}{\sqrt{t}} \to 0$$

in probability as $t \to \infty$.

Case 2. $\tau \geq t - t^{3/4}$. The number of particles with this property is denoted by $p_2(t,\omega)$. Let us put

$$\theta(t,\omega) = \max_{s\in[t-t^{3/4},t]} \{|Q(t,\omega) - Q(s,\omega)|\}$$

It follows from Theorem 2.2.2 that a.e

$$\theta(t,\omega) \leq t^{3/8+2\epsilon}$$

as t is sufficiently large. Assume that we deal with so large t. Thus

$$q(t) \in [Q(t) - \theta(t), Q(t)]\,.$$

The number of particles satisfying $q(t) \in [Q(t) - \theta(t), Q(t)]$ has Poisson distribution with parameter

$$\gamma \leq \mathrm{const}\, t^{3/8+\epsilon}\,.$$

Assuming ϵ to be sufficiently small, we conclude that

$$\frac{p_2(t,.)}{\sqrt{t}} \to 0$$

in probability as $t \to \infty$.

Finally we get

$$\frac{1}{\sqrt{t}} E_1(t,\omega) = \frac{1}{\sqrt{t}} (E(t,\omega) - \mathrm{card}\{\mathcal{A}^+(t,\omega) \setminus \mathcal{B}(t,\omega)) \to 0\,.$$

Taking into account the equality

$$A^+(t,\omega) = B(t,\omega) + E_1(t,\omega) - \mathrm{card}\{\mathcal{B}(t,\omega) \setminus \mathcal{A}^+(t,\omega)\}\,,$$

we see that

$$A^+(t,\omega) = B(t,\omega) + E(t,\omega) - \mathrm{card}\{\mathcal{B}(t,\omega) \setminus \mathcal{A}^+(t,\omega)\} + \alpha_1(t,\omega)\,,$$

where

$$\frac{\alpha_1(t,.)}{\sqrt{t}} \to 0$$

in probability. From the strong low of large numbers

$$E(t,\omega) = \frac{\rho}{2} Q(t,\omega) + \alpha_2(t,\omega)\,,$$

where

$$\frac{\alpha_2(t,.)}{\sqrt{t}} \to 0$$

in probability. To complete our argument (in the case $Q(t) > 0$) we have to show that

$$\mathrm{card}\{\mathcal{B}(t,\omega) \setminus \mathcal{A}^+(t,\omega)\}$$

is $o(\sqrt{t})$ in probability. The set $\mathcal{B}(t,\omega) \setminus \mathcal{A}^+(t,\omega)$ consists of particles $(q,v) \in Y$ with $q < 0, v < 0$ and of particles $(q,v) \in X$ with $q < 0, v > 0, q + tv < 0$ and interacting with the M.P. during the interval $(0,t)$. The set of particles of the first (second) group is denoted by $\mathcal{P}_3(t,\omega)(\mathcal{P}_4(t,\omega))$. Set

$$\mathrm{card}\{\mathcal{P}_i\} = p_i, i = 3,4\,.$$

For the particles of the first group we have

$$q \in [-t^{1/2+\epsilon}, 0]\,.$$

Again we introduce the moment τ of the last collision with the M.P. If $\tau < t^{3/4}$, then as above $q \in [-t^{3/4(1/2+\epsilon)}, 0]$. and the number of such particles is $o(\sqrt{t})$ in probability. If $\tau > t^{3/4}$ then the velocity of the particle must be small:

$$0 < v < t^{1/4}\,,$$

which together with the inclusion

$$q \in [-t^{1/2+\epsilon}, 0]\,.$$

shows that

$$p_3(t,.) = o(\sqrt{t})$$

in probability. Now we shall estimate $p_4(t,\omega)$. Define

$$\tau_1(q,v,\omega) = \inf\{s : q(s) = Q(s)\},\ (q,v) \in \mathcal{P}_4\,,$$

and put

$$n_1(t,\omega) = \mathrm{card}\{(q,v) \in \mathcal{P}_4 : Q(\tau_1) > -t^{1/2-\epsilon}\}\,,$$
$$n_1(t,\omega) = \mathrm{card}\{(q,v) \in \mathcal{P}_4 : Q(\tau_1) \le -t^{1/2-\epsilon}\}\,.$$

We have

$$p_4(t,\omega) = n_1(t,\omega) + n_2(t,\omega)\,.$$

To estimate $n_1(t,\omega)$ we observe that if $Q(\tau_1) > -t^{1/2-\epsilon}$, then

$$q + tv \ge q + \tau_1 v = Q(\tau_1) > -t^{1/2-\epsilon}\,,$$

that is,

$$q + tv \in [-t^{1/2-\epsilon}, 0]\,.$$

The number of particles satisfying the last inclusion has the Poisson distribution with the parameter $\gamma \le \mathrm{const}\, t^{1/2-\epsilon}$. This gives

$$n_1(t,.) = o(\sqrt{t})$$

in probability. In order to estimate $n_2(t,\omega)$, we remark that from inequalities

$$Q(\tau_1) \le -t^{1/2-\epsilon}$$

and

$$Q(t) \ge 0$$

we have $Q(t) - Q(\tau_1) > t^{1/2-\epsilon}$. Thus we may restrict our consideration to the case

$$(t-\tau_1)^{1/2+\epsilon} > t^{1/2-\epsilon}\,.$$

Hence

$$t - \tau_1 > t^{1/2-5\epsilon}\,,$$

as ϵ is sufficiently small. From another side

$$q + v\tau_1 + v(t-\tau_1) = Q(\tau_1) + v(t-\tau_1) < 0\,.$$

Thus

$$v(t-\tau_1) \le -Q(\tau_1) \le t^{1/2+\epsilon}$$

and

$$v \le t^{-1/2+6\epsilon}\,.$$

This yields

$$q + tv \in [-t^{1/2+\epsilon}, 0],\ 0 \le v \le t^{1/2+6\epsilon}\,.$$

This immediately gives $n_2(t,.) = o(\sqrt{t})$ in probability. Thus the lemma is proven in case $Q(t) > 0$.

Now we assume that $Q(t) \le 0$. Introduce

$$\mathcal{F}(t,\omega) = \{(q,v) \in Y : v > 0, q < 0, q + tv > Q(t,\omega)\}\,.$$

If $Q(t,\omega) < 0$, then

$$\mathcal{A}^+(t,\omega) \subseteq \mathcal{F}(t,\omega)\,.$$

Our next goal is to show that

$$\operatorname{card}\{\mathcal{B}(t,\omega) \setminus \mathcal{F}(t,\omega)\} = o(\sqrt{t})$$

in probability. We have

$$\mathcal{B}(t,\omega) \setminus \mathcal{F}(t,\omega) = \mathcal{L}_+(t,\omega) \cup \mathcal{L}_-(t,\omega)\,,$$

where

$$\mathcal{L}_-(t,\omega) = \{(q,v) \in \mathcal{B}(t,\omega) : v < 0\}\,,$$
$$\mathcal{L}_+(t,\omega) = \{(q,v) \in \mathcal{B}(t,\omega) : v > 0, q + tv < Q(t)\}\,.$$

For a particle $(q,v) \in \mathcal{L}_\pm(t,\omega)$ we define

$$\tau((q,v),\omega) = \inf\{s : s \in [0,t], q(s) = Q(s)\}\,.$$

Let us estimate the number of particles contained in $\mathcal{L}_-$. As above, we consider separately two groups of particles:

$$\mathcal{L}_-^{(1)} = \{(q,v) : \tau((q,v)) < t^{3/4}\}\,,$$
$$\mathcal{L}_-^{(2)} = \{(q,v) : \tau((q,v)) \ge t^{3/4}\}\,.$$

For $(q,v) \in \mathcal{L}_-^{(1)}$ we have:

$$q \ge Q(\tau) \ge -\tau^{1/2+\epsilon} \ge -t^{3/4(1/2+\epsilon)}\,,$$

which implies

$$q \in [-t^{3/8+3/4\epsilon}, 0]\,.$$

and therefore
$$\mathrm{card}\{\mathcal{L}_-^{(1)}\} = o(\sqrt{t})$$
in probability. If $(q,v) \in \mathcal{L}_-^{(2)}$, then
$$q \geq q + t^{3/4}v \geq \min_{0\leq s\leq t} Q(s) \geq -t^{1/2+\epsilon}\,.$$
This implies
$$|v| \leq t^{-1/4+\epsilon}\,.$$
Taking into account that
$$q \in [-t^{1/2+\epsilon}, 0]$$
we conclude:
$$\mathrm{card}\{\mathcal{L}_-^{(2)}\} = o(\sqrt{t})$$
in probability. Estimating
$$\mathrm{card}\{\mathcal{L}_+\}$$
we remark that
$$q + \tau v = Q(\tau) \in [-t^{1/2+\epsilon}, 0]\,.$$
Again we decompose
$$\mathcal{L}_+ = \mathcal{L}_+^{(1)} \cup \mathcal{L}_+^{(2)}\,,$$
where
$$\begin{aligned}\mathcal{L}_+^{(1)} &= \{(q,v) : (q,v) \in \mathcal{L}_+, q + \tau v < -t^{1/2-\epsilon}\}\,,\\ \mathcal{L}_+^{(2)} &= \{(q,v) : (q,v) \in \mathcal{L}_+, q + \tau v \geq -t^{1/2-\epsilon}\}\,.\end{aligned}$$
In the first case as above
$$Q(t) - Q(\tau) > t^{1/2-\epsilon}\,.$$
Considering convergence in probability we may restrict to the set where
$$(t-\tau)^{1/2+\epsilon} > t^{1/2-\epsilon}$$
and thus
$$t - \tau > t^{1-4\epsilon}\,.$$

Notice that

$$Q(t) > q + tv > q + \tau v + t^{1-4\epsilon} v = Q(\tau) + vt^{1-4\epsilon}\,,$$

which gives

$$vt^{1-4\epsilon} \le Q(t) - Q(\tau)\,.$$

Thus

$$0 < v \le 2t^{1/2+\epsilon} t^{4\epsilon-1} = 2t^{-1/2+5\epsilon}\,.$$

Taking into account the inclusion

$$q + tv \in [-t^{1/2+\epsilon}, 0]\,,$$

we easily get that

$$\mathrm{card}\{\mathcal{L}_+^{(1)}(t,.)\}/\sqrt{t} \to 0$$

in probability. If $(q, v) \in \mathcal{L}_+^{(2)}$, then

$$q + tv \in [Q(\tau), Q(t)] \subseteq [Q(t) - t^{1/2-\epsilon}, Q(t)] \subseteq [-2t^{1/2+\epsilon}, 0]\,. \tag{2.2.10}$$

This implies that

$$\mathrm{card}\{\mathcal{L}_+^{(2)}(t,.)\}/\sqrt{t} \to 0$$

in probability. The last step is to show that in probability

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}}\Big(\mathrm{card}\{\mathcal{F}(t,\omega) \setminus \mathcal{A}^+(t,\omega)\} - \frac{\rho}{2}|Q(t,\omega)|\Big) = 0\,. \tag{2.2.11}$$

The set $\mathcal{F}(t,\omega) \setminus \mathcal{A}^+(t,\omega)$ consists of particles for which

$$q + tv \in [Q(t), 0], v > 0\,.$$

Setting

$$\zeta_{\epsilon,t}(\omega) = \sup_L \frac{|\,\mathrm{card}\{(q,v) : v > 0, q + tv \in [-L, 0]\} - \frac{\rho}{2} L|}{L^{1/2+\epsilon}}$$

and noting that the average value of $\zeta_{\epsilon,t}$ is uniformly (in t) bounded we get (2.2.11). Summarizing we have

$$\begin{aligned}
&\mathrm{card}\{\mathcal{F}(t,\omega)\} = \mathrm{card}\{\mathcal{B}(t,\omega)\} + o(\sqrt{t}) = \\
&= \mathrm{card}\{\mathcal{A}^+(t,\omega)\} + \mathrm{card}\{\mathcal{F}(t,\omega) \setminus \mathcal{B}(t,\omega)\} + o(\sqrt{t}) = \\
&= A^+(t,\omega) + \frac{\rho}{2}|Q(t,\omega)| + o(\sqrt{t})\,.
\end{aligned}$$

Thus

$$A^+(t,\omega) = B(t,\omega) + \frac{\rho}{2}Q(t,\omega) + o(\sqrt{t})\,. \qquad \square$$

THEOREM 2.2.12.

$$Q(t,\omega) = \frac{1}{\rho}(A^+(t,\omega) - A^-(t,\omega)) + \varepsilon(t,\omega)\,,$$

where

$$\frac{1}{\sqrt{t}}\varepsilon(t,.) \to 0$$

in probability with respect to μ *as* $t \to \infty$.

PROOF. Consider the involution

$$\phi : \Omega \to \Omega$$

changing the signs of velocities. It is clear that

$$\phi \circ T^t = T^{-t} \circ \phi\,.$$

We remark that

$$A^+(t, \phi \circ T^t(\omega)) = A^-(t,\omega)\,.$$

The number of particles colliding with the M.P is the same for the "backward" dynamics:

$$B(t, \phi \circ T^t(\omega)) = B(t,\omega)\,.$$

At the same time

$$Q(t,\omega) = -Q(t, \phi \circ T^t(\omega))\,.$$

We know that

$$A^+(t,\omega) = B(t,\omega) + \frac{\rho}{2}Q(t,\omega) + \alpha(t,\omega)\,, \tag{2.2.13}$$

where

$$\lim_{t\to\infty} \frac{\alpha(t,.)}{\sqrt{t}} = 0$$

in μ - probability. Substituting $\phi \circ T^t$ instead of ω we get

$$\begin{aligned} A^-(t,\omega) = B(t,\omega) - \frac{\rho}{2}Q(t,\omega) + \alpha(t,\phi\circ T^t(\omega))\,, \\ \alpha(t,\phi\circ T^t(\omega)) = o(\sqrt{t}) \end{aligned} \tag{2.2.14}$$

since $\phi \circ T^t$ is measure preserving. Comparing (2.2.13) and (2.2.14) we conclude that

$$Q(t,\omega) = \frac{1}{\rho}(A^+(t,\omega) - A^-(t,\omega)) + o(\sqrt{t})\,. \qquad \square$$

Let us write the assertion of the previous theorem in the following form. Set

$$a^\pm(t,\omega) = A^\pm(t,\omega) - \int A^\pm(t,\omega)\mu(d\omega)\,.$$

Since

$$\int A^+(t,\omega)\mu(d\omega) = \int A^-(t,\omega)\mu(d\omega)$$

we get:

$$Q(t,\omega) = \frac{1}{\rho}(a^+(t,\omega) - a^-(t,\omega)) + \varepsilon(t,\omega)\,. \tag{2.2.15}$$

$a^\pm(t,\omega)$ are identically distributed centered Poisson processes and the distribution of

$$\frac{a^\pm(t,.)}{\sqrt{t}}$$

converges as $t\to\infty$ to the Gaussian one with the variance

$$\frac{\rho}{\sqrt{2\pi m\beta}}\,.$$

This implies immediately that the family of distributions corresponding to the variables

$$\frac{Q(t,\omega)}{\sqrt{t}},\ t \in \mathbb{R}_+$$

is tight. Hence we may introduce the set of limit points $\mathfrak{D}$, which contains all the distributions representable as

$$\lim_{t_n\to\infty} \frac{Q(t_n,\omega)}{\sqrt{t_n}}\,.$$

Given a random variable ξ we shall write $\xi \in R(\mathfrak{D})$ if the distribution of ξ is contained in $\mathfrak{D}$. We know that $\mathfrak{D}$ is not empty and weakly compact. Our next goal is to study the distributions contained in $\mathfrak{D}$. Unfortunately we are not able to prove the Central Limit Theorem, which says that $\mathfrak{D}$ consists of the unique Gaussian distribution. We may, however, get some precise results in this direction.

THEOREM 2.2.16. *Any random variable $\xi \in R(\mathfrak{D})$ may be represented in the form*

$$\xi = \xi^+ + \xi^-\,,$$

where $\xi^\pm$ are identically distributed Gaussian variables with the variance

$$\frac{1}{\rho\sqrt{2\pi m\beta}}\,.$$

The variance of $\xi \in R(\mathfrak{D})$ is estimated from above by

$$\sqrt{\frac{8}{\pi}}\frac{1}{\rho\sqrt{m\beta}}\,.$$

PROOF. Assume that

$$\frac{Q(t_n,.)}{\sqrt{t_n}} \to \xi$$

in distribution. Due to the Prochorov theorem the joint distributions

$$\Big(\frac{a^+(t_n,.)}{\rho\sqrt{t_n}},\ -\frac{a^-(t_n,.)}{\rho\sqrt{t_n}}\Big)$$

are tight. Thus, considering if necessarily a suitable subsequence of t_n, we may and will assume that

$$\Big(\frac{a^+(t_n,.)}{\rho\sqrt{t_n}}, -\frac{a^-(t_n,.)}{\rho\sqrt{t_n}}\Big) \to (\xi^+,\xi^-)\,.$$

Thus

$$\xi = \xi^+ + \xi^-\,,$$

where

$$\xi^\pm = \pm\lim \frac{a^\pm(t_n,.)}{\rho\sqrt{t_n}}$$

are the required identically distributed Gaussian variables. The estimate for the variance of ξ follows from the Schwartz inequality. ∎

Let us emphasize the surprising fact that the upper bound for the variance does not depend on M, i.e. on the mass of the M.P. On the other hand the results we have do not exclude that the scaling rate $\sqrt{t}$ is trivial and $\mathfrak{D}$ consists of the distribution concentrated in 0. We know that it is not true as $M = m$. The general case needs additional arguments.

2.2.4 – Lower bounds for the limit variance

In this section we briefly represent an elegant argument, invented by D. Szasz and B. Toth [28], to show that that the limit variance of the normalized displacement of the M.P. is bounded from below:

$$\liminf \mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) \geq \sigma_-^2\,,$$

where

$$\sigma_-^2 = \sqrt{\frac{\pi}{8}}\frac{1}{\rho\sqrt{m\beta}}\,.$$

1. Return first to the Lemma 2.2.9. The assertion of the lemma deals with a *number of particles* contained in certain regions of the phase space. It is easily seen that replacing the counting function

$$\mathrm{card}(A) = \sum_{(q,v)\in A} 1$$

by an additive translation invariant function

$$\mathrm{card}_f(A) = \sum_{(q,v)\in A} f(v)$$

we do not violate neither the assertion nor the proof of the lemma. Let us formulate the refinement, we obtain this way.

Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ satisfy

$$\int f^2(v) \exp\Big(-\frac{m\beta v^2}{2}\Big) dv < \infty\,.$$

Set as before

$$\mathcal{A}^+(t,\omega) = \{(q,v) \in Y(\omega) : q < 0,\, q + tv > 0\}\,,$$

and define $A_f^+(t,\omega) = \mathrm{card}_f\{\mathcal{A}^\pm(t,\omega)\}$. Analogously for

$$\mathcal{B}(t,\omega) = \{(q,v) \in Y(\omega) : q < 0,\, q(s) = Q(s) \text{ for some } s : 0 \le s \le t\}\,,$$

set

$$B_f(t,\omega) = \mathrm{card}_f\{\mathcal{B}(t,\omega)\}\,.$$

Lemma 2.2.17.

$$A_f^+(t,\omega) = a^+(f)Q(t,\omega) + B_f(t,\omega) + \alpha(t,\omega)\,,$$

where

$$a^+(f) = \rho\sqrt{\frac{m\beta}{2\pi}} \int_0^\infty f(v) \exp\Big(-\frac{m\beta v^2}{2}\Big) dv\,,$$

and

$$\frac{\alpha(t,.)}{\sqrt{t}} \to 0$$

in probability as $t \to \infty$.

PROOF. A straightforward inspection of the arguments, applied to Lemma 2.2.9, but replacing card( ) by $\mathrm{card}_f(\,)$ everywhere. See also [28]. ∎

LEMMA 2.2.18. *Under the previous assumptions the correction*

$$\frac{\alpha(t,.)}{\sqrt{t}}$$

converges to zero in $\mathcal{L}_2(\Omega,\mu)$.

PROOF. See [28]. The way of reasoning is close to that of (2.2.9) but makes use of a more precise information concerning the fluctuations of a Poisson point process. ∎

2. Which choice of the function f is reasonable? To make arguments close to those of the previous section work, the function f should correspond to a certain conservation law. For instance, $f = 1$ corresponds to the number of particles. Consider in addition:

1. $f(v) = mv$. Conservation of momentum.
2. $$f(v) = \frac{mv^2}{2}\,.$$

   Conservation of energy.

The second case contains no new information in addition to the case where $f = 1$, since in the both cases the function f is even. The first case is expected to be more interesting. To make the notations more transparent, set

$$P^+(t,\omega) = A_v^+(t,\omega)\,,$$
$$R^+(t,\omega) = B_v^+(t,\omega)\,.$$

We have

$$P^+(t) = R^+(t) + \frac{\rho}{\sqrt{2\pi m\beta}}Q(t) + o(\sqrt{t})\,. \tag{2.2.19}$$

In addition we have, as before,

$$A^+(t) = \frac{\rho}{2}Q(t,\omega) + B^+(t) + 0(\sqrt{t})\,, \tag{2.2.20}$$

where

$$A^+(.) = A_1^+(.), B^+(.) = B_1^+(.)\,.$$

3. There exist two natural measure preserving involutions of Ω. The involution $\phi : \Omega \to \Omega$ changing the signs of velocities was used in the preceding section. The involution $\psi : \Omega \to \Omega$ reflects the configuration with respect to the origin:

$$\psi((V,Y)) = (-V,-Y)\,,$$

where $-Y = \{(q,v) : (-q,-v) \in Y\}$. Clearly

$$\psi \circ T^t = T^t \circ \psi\,,$$

and

$$Q(t,\psi(\omega)) = -Q(t,\omega)\,.$$

Set:

$$\begin{aligned} A(t,\omega) &= A^+(t,\omega) - A^+(t,\psi(\omega)),\ B(t,\omega) = B^+(t,\omega) - B^+(t,\psi(\omega))\,,\\ P(t,\omega) &= P^+(t,\omega) - P^+(t,\psi(\omega)),\ R(t,\omega) = R^+(t,\omega) - R^+(t,\psi(\omega))\,. \end{aligned}$$

From what is said above we conclude that in $\mathcal{L}_2(\Omega,\mu)$:

$$P(t) = R(t) + \rho\sqrt{\frac{2}{\pi m\beta}}Q(t) + o(\sqrt{t})\,, \tag{2.2.21}$$

$$A(t) = B(t) + \rho Q(t) + o(\sqrt{t})\,. \tag{2.2.22}$$

Denote by Z_t^* the process

$$Z^*(t,\omega) = Z(t,\phi\circ T^t(\omega))\,.$$

The values involved into equations (2.2.21), (2.2.22) have the following properties with respect to the transformation *:

$$B^*(t) = B(t), Q^*(t) = -Q(t), R(t) + R^*(t) = (V(t) - V(0))M\,.$$

Since $\phi \circ T^t$ is a measure preserving transformation, we conclude from the preceding equations that

$$\mathbb{E}(Q(t)B(t)) = 0\,.$$

By (2.2.22)

$$\mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) = \frac{1}{\rho^2}\Big(\mathbb{E}\Big(\frac{A^2(t)}{t}\Big) - \mathbb{E}\Big(\frac{B^2(t)}{t}\Big)\Big) + o(1)\,.$$

Thus

$$\limsup_{t\to\infty} \mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) \le \frac{1}{\rho^2} \lim_{t\to\infty} \mathbb{E}\Big(\frac{A^2(t)}{t}\Big) = \frac{1}{\rho}\sqrt{\frac{2}{\pi m\beta}}\,.$$

Further,

$$\mathbb{E}(B(t)R(t)) = \mathbb{E}(B^*(t)R^*(t)) = -\mathbb{E}(B(t)R(t)) + M\mathbb{E}(B(t)(V(t)-V(0)))\,.$$

Since $B(t)/\sqrt{t}$ and $V(t)$ are both $\mathcal{L}_2$ bounded,

$$\mathbb{E}(B(t)R(t)) = O(\sqrt{t})\,.$$

Let us multiply the both sides of (2.2.22) by $P(t)/t$. Taking then the expectation we have

$$\mathbb{E}\Big(\frac{A(t)P(t)}{t}\Big) = \mathbb{E}\Big(\frac{B(t)P(t)}{t}\Big) + \rho\mathbb{E}\Big(\frac{Q(t)P(t)}{t}\Big) + o(1)\,.$$

Replace in the first term of the right hand side above $P(t)$ by its representation in (2.2.21):

$$\mathbb{E}\Big(\frac{A(t)P(t)}{t}\Big) = \mathbb{E}\Big(\frac{B(t)R(t)}{t}\Big) + \rho\mathbb{E}\Big(\frac{Q(t)P(t)}{t}\Big) + o(1)\,.$$

We have used the equality

$$\mathbb{E}\Big(\frac{Q(t)B(t)}{t}\Big) = 0\,.$$

Recalling then that

$$\mathbb{E}(B(t)R(t)) = O(\sqrt{t})\,,$$

we obtain

$$\mathbb{E}\Big(\frac{A(t)P(t)}{t}\Big) = \rho\mathbb{E}\Big(\frac{Q(t)P(t)}{t}\Big) + o(1)\,.$$

By Schwartz inequality

$$\rho^2 \mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) \geq \Big(\mathbb{E}\Big(\frac{A(t)P(t)}{t}\Big)\Big)^2 \Big(\mathbb{E}\Big(\frac{P^2(t)}{t}\Big)\Big)^{-1} + o(1)\,.$$

Note then that the values

$$\mathbb{E}\Big(\frac{A(t)P(t)}{t}\Big),\ \mathbb{E}\Big(\frac{P^2(t)}{t}\Big)$$

do not depend on t and may be easily calculated directly. Omitting this elementary step we obtain

$$\mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) \geq \frac{1}{\rho}\sqrt{\frac{\pi}{8m\beta}} + o(1)\,.$$

Summarizing we have

Proposition 2.2.23.

$$\frac{1}{\rho}\sqrt{\frac{\pi}{8m\beta}} \leq \liminf_{t\to\infty} \mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) \leq \limsup_{t\to\infty} \mathbb{E}\Big(\frac{Q^2(t)}{t}\Big) \leq \frac{1}{\rho}\sqrt{\frac{2}{\pi m\beta}}\,.$$

The upper estimate is sharp and coincides with the limit variance in the case where $M = m$.

Remark. A completely different method of proving the lower estimate is contained in [22]. The arguments there are more complicated, essentially make use of the Gaussian distribution of velocities, but give more information concerning the structure of possible limit distributions.

2.2.5 – Numerical evidence

It is a very nontrivial numerical and statistical problem to simulate the process of displacement of the M.P. for large times and to test whether it is asymptotically Wiener. From our point of view the highest level of computational precision was achieved in [1]. The results obtained there support the following assertions:

1. The process

$$\xi^\epsilon(t) = \sqrt{\epsilon}\,Q(t/\epsilon)$$

is, for small ϵ, asymptotically Wiener for all values of the mass M.

2. The limit variance $\sigma^2_\infty(M)$ of the normalized displacement depends on the mass M. The theoretical lower bound obtained above is sharp and appears as $M \to \infty$.
3. The Einstein relation holds for any value of M.

3 – Complete diffusion theory for the modified Rayleigh gas

This section is the center of the text. We consider here a mechanical system in the plane, consisting of a vertical rod of length ℓ, with its center moving on the horizontal axis, subject to elastic collisions with the particles of a free gas, and to a constant horizontal force f acting on the rod only. The rigorous construction of the system and the basic notations the reader may find in the introduction. Recall that the initial measure μ is chosen so that the vertical velocity distribution of a gas particle has a "hole" in 0, i.e. we assume that the vertical velocity v_2 satisfies $|v_2| > u_0 > 0$ for all particles. This ensures that all particles which are at a given time available for the rod will get out of it after a renewal time $\tau = \ell/u_0$. The distribution "in the horizontal direction" is assumed to be a Gibbs measure for the system for $f = 0$, corresponding to some particle density ρ and some inverse temperature β. That is, particle positions are Poisson distributed in the plane $\mathbb{R}^2$, and the horizontal velocities of the rod and of the particles have maxwelian distribution. The verticle velocity distribution is denoted by $h(dv_2)$ and is assumed to have at least finite first moment. The special choice of the vertical velocity distribution and the geometrical restrictions on the motion of the test object remove, at least partially, the problem connected with possible recollisions. The dynamics of the system is still very nontrivial: an arbitrary number of recollisions of a particle with the rod are possible, though we control their duration in time. The evolution of the rod is not markovian and, what is very special for this system, the presence of the external force destroys the equilibrium: there is no natural invariant measure given in advance, unless $f = 0$. We consider the displacement of the rod $Q(t)$ in the hydrodynamic limit

$$\sqrt{\epsilon}Q(t\epsilon^{-1})$$

and establish existence of the two basic macroscopic values: drift $d(f)$

and diffusion constant $\sigma^2(f)$. From the probabilistic point of view we prove the *central limit theorem* for the displacement of the rod. As the values $d(f)$ and $\sigma^2(f)$ are correctly determined (which is alone a very nontrivial problem) we establish the famous *Einstein relation* between them:

$$\lim_{f\to 0}\frac{d(f)}{f}=\frac{\beta}{2}\sigma^2(0)\,.$$

Let us emphasize that we follow the physically correct order of the limit transitions: first we consider the limit as $t\to\infty$ and then the limit $f\to 0$. Abusing a bit the standard terminology, we say that the Einstein relation is valid in the strong form. (There exists a weak form of the Einstein relation where the limit transitions are coupled so that $f=f_0/\sqrt{t}$.) It is worth mentioning that, so far, it is the unique example of a deterministic mechanical system, describing Brownian motion, for which all axioms of the classical diffusion theory are proved completely and in the strong form. Another example of the diffusion theory constructed for a deterministic (but not mechanical) system is Lorentz gas with Gaussian dynamics.

3.1 – Basic properties of the dynamics

3.1.1 – Definitions and notations

In this section we deal with a family of dynamical systems $\{T_f^t\}$ depending on the parameter $f\in\mathbb{R}$ (external force acting on the rod.) The phase space of $\{T_f^t\}$ denoted by $\Omega_f\subset\Omega$ consists of configurations for which the dynamics is well defined. We know that for any f

$$\mu(\Omega_f)=1\,.$$

Without loss of generality we may and will assume that the external force $f\geq 0$. Moreover, it suffices for our purposes to consider f contained in a finite interval $[0,f_{\max}]$.

A particular role will be played by the region of the plane accessible for the rod:

$$\mathcal{S}=\{q\in\mathbb{R}^2:|q_2|<\ell/2\}\,.$$

The corresponding subset of the one - particle phase space $\mathcal{M}$ is denoted by

$$\mathcal{M}^S=\mathcal{S}\times\mathbb{R}^2$$

and, more generally, for $t \geq 0$ we set

$$\mathcal{M}^S{}_t = \bigcup_{0 \leq s \leq t} \overset{\circ}{T}{}^{-s}(\mathcal{M}^S)\,.$$

Here $\overset{\circ}{T}{}^t$ denotes the free dynamics on the phase space $\mathcal{M}$. Thus $\mathcal{M}^S{}_t$ consists of the points that pass through $\mathcal{M}^S$ by the free dynamics in the time interval $[0, t]$. Recall that the distribution of vertical velocities $h(dv_2)$ is chosen so that for some $u_0 > 0$ $h([-u_0, u_0]) = 0$. Set

$$\tau = \frac{\ell}{u_0}\,.$$

The notation $\mathfrak{S}$ will be used for the subset $\mathcal{M}^S{}_\tau \subset \mathcal{M}$. Let us also introduce some special notations related to $\mathfrak{S}$:

$$\mathcal{M}^{\pm} = \{(q, v) \in \mathcal{M} \setminus \mathfrak{S} : \pm q_2 v_2 > 0\}\,,$$

and

$$Y^{\mathfrak{S}} = Y \cap \mathfrak{S},\ Y^{\text{in}} = Y \cap \mathcal{M}^+,\ Y^{\text{out}} = Y \cap \mathcal{M}^-\,.$$

$\mathcal{M}^+(\mathcal{M}^-)$ is the portion of the phase space where the particles that visit $\mathfrak{S}$ in the future (in the past) are located. Y^{in} and Y^{out} are called the "ingoing" and "outgoing" configurations respectively. The corresponding spaces are

$$\mathbf{Y}^{\text{in}} = \{Y^{\text{in}} : Y \in \{\mathbf{Y}\}$$

and

$$\mathbf{Y}^{\text{out}} = \{Y^{\text{out}} : Y \in \{\mathbf{Y}\}\,.$$

The marginal distributions of $Y^{\text{in}}, Y^{\text{out}}$ induced by $\mathcal{P}$ will be denoted by the same symbol $\mathcal{P}$.

An important object of interest is the subsystem in $\mathfrak{S}$. That subsystem is described by

$$X = X(\omega) = (V, Y^{\mathfrak{S}})\,.$$

The corresponding phase space and σ - algebras are denoted by $\mathfrak{X}$ and $\mathcal{B}^{\mathfrak{S}}$ respectively. Let π be the measure induced by μ on $\mathfrak{S}$:

$$\pi(A) = \mu\{\omega : X(\omega) \in A\},\ A \in \mathcal{B}^{\mathfrak{S}}\,.$$

The evolution of the configuration inside $\mathfrak{S}$ in terms of the entire configuration is given by

$$X_f(t,\omega) = X(T_f^t(\omega))\,,$$

where the process $X_f(t)$ is *markovian*. The transition probabilities are given in terms of the Poisson measure:

$$P_f^t(X,A) = \mathcal{P}\{Y : T_f^t(X \cup Y^{\mathrm{in}}) \in A\}\,. \tag{3.1.1}$$

To be precise, we define the phase space of the Markov process $X_f(t)$:

$$\mathfrak{X}_f = \{X : \mathcal{P}\{Y : X \cup Y^{\mathrm{in}} \in \Omega_f\} = 1\}\,.$$

Clearly $\pi(\mathfrak{X}_f) = 1$. We leave to the reader a slightly tedious inspection of the fact that $\mathfrak{X}_f$ is an absorbing set with respect to the transition probabilities indicated above.

3.1.2 – Transformation of the main measure

As just asserted the initial measure μ is not invariant under the dynamics as the external force $f \neq 0$. We start considering evolution of μ under the dynamics $\{T_f^t\}$.

LEMMA 3.1.2. *Let $\mu_t = \mu(T_f^{-t}(.))$ be the family of the measures generated by the dynamics. Then for each $t \in \mathbb{R}$ the measure μ_t is equivalent to the measure μ and the Radon Nikodym derivative is equal to*

$$\frac{d\mu_t}{d\mu} = \exp\Big(\beta f \int_0^t V(T_f^{-t+s}(\omega))\,ds\Big)\,.$$

PROOF. Let us consider the dynamics in the extended phase space. In coordinates $\hat\omega = (Q,\omega)$ we have:

$$\widehat{T}_f^t(\hat\omega) = \Big(Q + \int_0^t V(T_f^s(\omega))\,ds, T_f^t(\omega)\Big)\,.$$

The Gibbs measure $\hat\mu_f = \exp(\beta f Q) dQ \mu(d\omega)$ is invariant for $\{\widehat{T}_f^t\}$. Define $\psi(\hat\omega) = \mathbb{I}_J(Q)\phi(\omega)$, where $\phi(.)$ is a bounded measurable function and $\mathbb{I}_J$

denotes the indicator function of the finite interval (a,b). ψ is summable with respect to $\hat{\mu}_f$. The invariance condition for the Gibbs measure implies that $\hat{\mu}_f(\psi) = \hat{\mu}_f(\psi(\widehat{T}^t_f(.))$ After elementary manipulations we get

$$\int_\Omega \mu(d\omega)\phi(\omega)\cdot \int_a^b \exp(\beta f Q)dQ = \\ = \int_\Omega \mu(d\omega)\phi(T^t_f(\omega))\cdot \int_a^b \exp\Big(\beta f\Big(Q - \int_0^t V(T^s_f(\omega))ds\Big)\Big)\, dQ\,.$$

Thus

$$\int \mu(d\omega)\phi(\omega) = \int \mu(d\omega)\phi(T^t_f(\omega))e^{-\beta f\int_0^t V(T^s\omega)ds}$$

and the result follows if we replace ω by $T_f^{-t}(\omega)$. ∎

Note that the same argument works for the standard Rayleigh gas as well. The previous proposition implies the following *summation rule*:

$$\int_\Omega \mu(d\omega)\exp\Big(-\beta f\int_0^t V(T^s_f(\omega))ds\Big) = 1\,. \tag{3.1.3}$$

LEMMA 3.1.4. *For any $t\ge 0$ the measures πP^t_f and π are equivalent on $\mathcal{B}^{\mathfrak{S}}$ and the Radon Nikodym derivative $\varrho_f(t,X)$ of πP^t_f with respect to π is given by the formula*

$$\varrho_f(t,X) = \frac{d\pi P^t_f}{d\pi}(X) = \int \mathcal{P}(dY)\exp\Big(\beta f\int_0^t V(T_f^{-s}(X\cup Y^{out}))ds\Big)\,.$$

PROOF. For $A\in\mathcal{B}^{\mathfrak{S}}$ we have:

$$\pi P^t_f(A) = \mu_t\{\omega : X(\omega)\in A\}\,.$$

Thus, in view of the previous lemma

$$\pi P^t_f(A) = \int_A \pi(dX)\int_{\mathbf{Y}} \mathcal{P}(dY)\exp\Big(\beta f\Big(\int_0^t V(T_f^{-s}(X\cup Y^{out})ds\Big)\Big)\,. \quad ∎$$

3.1.3 – Construction of a Lyapunov function

In this section we construct a function W defined on $\mathfrak{X}$. This function, roughly speaking, controls the deviations of the subsystem in $\mathfrak{S}$ from the "normal" or in certain sense typical state, where the energy of the rod and the neighboring particles is relatively small. The term "Lyapunov function" will be justified somewhat later.

For $B \subset \mathcal{M}$ and $X \in \mathfrak{X}$ introduce the quantities

$$W(B) = \sup\left\{|v_1| : (q, v) \in B : \tau|v_1| > \frac{1}{4}|q_1|\right\},$$

$$W(X) = \max\{|V(X)|, W(Y^{\mathfrak{S}}(X))\}\,. \tag{3.1.5}$$

Let $X = (V, Y^{\mathfrak{S}})$ The condition $W(X) < U$ implies that $|V| < U$ and

$$Y^{\mathfrak{S}} \cap \{(q, v) : |q_1| \leq 4\tau|v_1|, |v_1| > U\} = \emptyset\,.$$

Assume that $W(X) < U$ and consider the dynamics $\{T_f^s(X), s \in \mathbb{R}\}$, i.e. the dynamics of the point $\omega = (V, Y^{\mathfrak{S}})$. If $(q, v) \in Y^{\mathfrak{S}}$ is a particle with $|v_1| > U$ then

$$|v_1| \leq \frac{|q_1|}{4\tau}\,.$$

If no collisions occurs then the horizontal position at time 2τ

$$q_1(2\tau) = q_1 + 2\tau v_1$$

and

$$|q_1(\tau)| \geq 4\tau|v_1| - 2\tau|v_1| \geq 2\tau U\,.$$

So its distance from the origin (I.e. from the position of the rod at tome 0 is larger then $2U\tau$ for all $s \in [0, 2\tau]$. If the rod does not collide with particle having velocity larger than U in absolute value it cannot travel more than a distance

$$U\tau + \frac{f}{2M}(2\tau)^2\,.$$

If U is so large that

$$\frac{2f\tau}{UM} \leq \frac{1}{4},$$

then $U\tau + \frac{f}{2M}(2\tau)^2 \leq \frac{3}{2}U\tau$. Hence the condition $W(X) < U$ implies that in the dynamics

$$\{T_f^s(X)\}$$

the rod does not collide in the time interval $[0,\tau]$ with particles having velocity larger in absolute value than U. The preceding assertion holds true for all $U > U_0$, where U_0 may be chosen uniformly in all values of f under consideration. (Recall that $0 \leq f \leq f_{\max}$.)

LEMMA 3.1.6. *There is a constant $c > 0$ such that for all U large enough*

$$\pi\{X : W(X) \geq U\} < e^{-cU^2}.$$

PROOF. Set $d_k = \max\{1, |k| - 1\}$ and

$$A_k = \{\max\{|v_1| : (q,v) \in Y^{\mathfrak{S}}, q_1 \in [4k\tau, 4(k+1)\tau U)\} > d_k U\}.$$

We have

$$\{W(X) > U\} \subset \bigcup_{k\in\mathbb{Z}} A_k \bigcup \{|V| > U\}.$$

Since $\pi(A_k) \leq c_1 e^{-c_2 d_k^2 U^2}$ for some $c_1, c_2 > 0$ we get the result summing over k. □

As a consequence we have $W(X) < \infty$ π, a.e. We may and will assume that the condition

$$W(X) < \infty$$

is included into the definition of $\mathfrak{X}$. For $U > 0$ we set

$$A_U = \{X \in \mathfrak{X} : W(X) < U\}. \tag{3.1.7}$$

3.1.4 – Properties of the transition kernels $P_f^t(X_0, dX)$

In this section we prove some important properties of the dynamics $\{T_f^t\}$. We formulate these properties as assertions concerning the transition probabilities $P_f^t(X_0, dX)$. First, introduce some notations. For $(q, v) \in \mathcal{M}$ we consider the parameters

$$t^e(q,v) = \inf\{t : q + tv \in \mathcal{M}^S\},\ t^o(q,v) = \sup\{t : q + tv \in \mathcal{M}^S\}\,,$$

which are the entrance and exit times in $\mathcal{S}$. Introduce also the horizontal coordinates of the particles at those times:

$$q_1^e(q,v) = q_1 + t^e(q,v)v_1,\ q_1^o(q,v) = q_1 + t^o(q,v)v_1\,.$$

For $t_1 < t_2$ and $B \subset \mathcal{M}$ we denote by

$$\mathcal{N}(t_1, t_2, B) = \bigcup_{t \in (t_1, t_2)} \mathring{T}^{-t}(B)\,.$$

the set of the points that pass through B by the free dynamics in the time interval (t_1, t_2). For $L > 0$ set

$$\mathcal{S}_L = \{q \in \mathcal{S} : |q_1| < L\}\,,$$
$$\mathcal{M}^S{}_L = \{(q,v) \in \mathcal{M}^S : |q_1| < L\}$$

and

$$\mathfrak{S}_L = \{(q,v) \in \mathfrak{S} : |q_1| < L\}\,.$$

Lemma 3.1.8. *For any $X_0 \in \mathfrak{X}_f$ and $t > 4\tau$ the measure π is absolutely continuous with respect to $P_f^t(X_0, dX)$.*

Proof. We begin with brief intuitive arguments which explain our strategy. The main idea of the proof relies on a simple mechanism that brings the initial point X_0 into any other fixed but arbitrary point X at time t. Consider the image $T_f^{-s_1}(X)$ of X under the backward dynamics in the absence of particles outside $\mathfrak{S}$. Here $s_1 > \tau$.

$$T_f^{-s_1}(X) = (V(T_f^{-s_1}(X)), Y(T_f^{-s_1}(X)))\,,$$

and the particles of $Y(T_f^{-s_1}(X))$ are outside $\mathcal{S}$. Let now Y^{in} be such that in the dynamics $T_f^s(X_0 \cup Y^{in})$ the following happens:

(i) no particles of Y^{in} enter the strip $\mathcal{S}_L$ up to time $s_0 > 2\tau$, except one particle (q^*, v^*). Suppose that $L > 4s_0U$, where $U > W(X_0)$ is large enough. The particle (q^*, v^*) is such that it collides with the rod at time s_0 and the velocity of the rod jumps from $V(T_f^{s_0}(X_0))$ to $V(T_f^{-s_1}(X))$.

(ii) the configuration of the particles of Y^{in} that at time $t = s_0 + s_1$ are in $\mathfrak{S}$, shifted forward by s_0 in time and seen from the rod position at that time (i.e. the configuration

$$\overset{\circ}{T}{}^{s_0}(Y^{in}) \cap \overset{\circ}{T}{}^{-s_1}(\mathfrak{S}) - Q(s_0, X_0))$$

coincides with $Y(T_f^{-s_1})(X)$. If no recollisions with the particle (q^*, v^*) occur it is easy to see that $X(T_f^t(X_0 \cup Y^{in}))) = X$, since the particle (q^*, v^*) is out of $\mathfrak{S}$ by time t and the dynamics $T_f^{s_1}$ applied to

$$(V(T_f^{-s_1}(X)), Y(T_f^{-s_1}(X)))$$

will restore X.

The technical part of the proof consists in showing that the configurations Y^{in} constructed in this way such that $X(T_f^t(X_0 \cup Y^{\mathrm{in}})) \in A$ with $\pi(A) > 0$ have positive measure $\mathcal{P}$. Condition (i) may be changed into a positive measure condition for each X_0 by assuming that the colliding particle is in a small neighborhood $\mathcal{U}$ of (q^*, v^*). To accomplish the proof we then use the fact that the velocity of the rod after collision is a C^1 map of full rank on $\mathcal{U}$ and the absolute continuity of the time shift of μ with respect to μ.

Remark. Some of the formulas we use are valid only for $f \neq 0$. The case $f = 0$ is simpler but requires to rewrite certain expressions. We omit this evident modification leaving details to the reader.

Let $X_0 \in \mathfrak{X}$ be fixed. Choose a positive number $U > W(X_0)$. Fix some positive numbers τ', τ'' and L so that

$$2\tau + \tau'' < t - \tau' < t - \tau,\ L > 4\Big(Ut + \frac{f}{M}t^2\Big)\,. \tag{3.1.9}$$

We split the incoming configuration Y^{in} as $Y = \widehat{Y}_1 \cup \widehat{Y}_2 \cup \widehat{Y}_3$. The particles of $\widehat{Y}_1$ cross $\mathfrak{S}$ and are out of $\mathfrak{S}$ at time $t - \tau'$. That is, $t^o(q,v) < t - \tau'$ for any particle $(q,v) \in \widehat{Y}_1$. The particles of $\widehat{Y}_3$ are defined by the condition $t^e(q,v) > t + \tau$ Those particles enter $\mathcal{M}_S$ only after time $t + \tau$ and $\mathfrak{S}$ after t. Set

$$\widehat{Y}_2 = Y^{\text{in}} \setminus (\widehat{Y}_1 \cup \widehat{Y}_3)\,.$$

For the particles of $\widehat{Y}_1$ holds:

$$t^e(q,v) \le t + \tau, t^0(q,v) \ge t - \tau'\,.$$

We further set

$$\begin{aligned} Y_1 &= \widehat{Y}_1 \cap \mathcal{N}(0, t - \tau', M_{SL})\,, \\ \widehat{Y}_2 &= Y_2 \cup \overline{Y_2}\,, \end{aligned}$$

where

$$Y_2 = \widehat{Y}_2 \cap \mathring{T}^{-t}(\mathfrak{S}),\ \overline{Y_2} = \widehat{Y}_2 \setminus Y_2\,.$$

We shall construct a subset $\mathbf{Y}_U \subset \mathbf{Y}^{\text{in}}$ by giving Y_1, Y_2, and $\overline{Y_2}$ a special form so that at some time $s \in (2\tau, t-\tau)$ the velocity of the rod is brought close to the value $V(T_f^{-t+s}(X))$ for $X \in A_U$. The difference

$$\Delta(X) = V(T_f^{-t+s}(X)) - V(T_f^s(X_0))$$

is constant for $s \in (2\tau, t - \tau)$ as no collision take place in the dynamics $T_f^u(X)$ for $u < -\tau$ or $u > 2\tau$ (all particles are out of $\mathcal{S}$).

We fix $v_2^* \in \text{supp}\ h(dv_2)$ and denote by

$$\tau^* = \ell / v_2^* < \tau$$

the corresponding "crossing time". Let $X \in A_U$ be fixed and such that

$$|\Delta(X)| > (1 - \alpha)\tau \frac{f}{M} = \Delta_0\,, \tag{3.1.10}$$

where the collision parameter

$$\alpha = \frac{M - m}{M + m}\,.$$

We choose

$$Y_1 = \{(q^*, v^*)\}$$

with the value of v_2^* fixed above and q_2^* such that $t_*^e = t^e(q^*, v^*) \in (2\tau, 2\tau + \tau'')$. This ensures, by the first inequality in (3.1.9), that $t_*^o = t^o(q^*, v^*) < t_*^e + \tau^* < t - \tau'$. That is, the particle (q^*, v^*) is out of $\mathfrak{S}$ at time $t - \tau'$. The values q_1^*, v_1^* are chosen as functions of

$$\overline{V}_0 = V(T_f^{t_*^e}(X_0)),\ V^* = V(T_f^{t-t_*^e}(X)),\ \overline{Q}_0 = \int_0^{t_*^e} V(T_f^s(X_0))ds\,.$$

Suppose that $V^* - \overline{V}_0 = \Delta(X) > 0$. We should have

$$\hat{q}_1^* = q_1^* + v_1^* t_*^e - \overline{Q}_0 < 0\,,$$

since the velocity of the rod has to increase, and the collision is on the left. We fix the collision time $t_* \in (t_*^e, t_*^o)$ and determine q_1^* and v_1^* solving the equations

$$(v_1^* - \overline{V}_0)^2 + 2\hat{q}_1^* \frac{f}{M} = \frac{\Delta^2}{(1-\alpha^2)}\,; \tag{3.1.11}$$

$$t_* - t_*^e = \frac{M}{f}\Big(v_1^* - \overline{V}_0 - \frac{\Delta}{1-\alpha}\Big)\,. \tag{3.1.12}$$

The dynamics $T_f^s(X_0 \cup Y_1)$ is then such that a single collision with (q^*, v^*) takes place at the time t_*, and the velocity of the rod jumps by Δ. For $\Delta < 0$ the collision is on the right, i.e. $q_1^* > 0$. We can again determine (q_1^*, v_1^*) by the conditions (3.1.11), (3.1.12) but we have to discuss possible recollisions. Inequality (3.1.10) shows that no recollision occurs for the configuration $X_0 \cup (q^*, v^*)$, since recollision time is

$$t_* + 2\frac{M}{f}(V(T_f^{t_*}(X_0)) - v_1^*) > t_*^e + 2\frac{M}{f}(1-\alpha)|\Delta| > t_*^e + \tau^*\,.$$

Only for $|\Delta| \le \Delta_0/2$ recollisions may occur for negative Δ. We then construct Y_1 as a configuration of two particles . The first one (q^+, v^+) is chosen exactly as above, for a velocity jump $\Delta' = -2\Delta_0$. Denote by t_+^e and t_+ the entrance and the collision time of (q^+, v^+). The second particle

$(q^*, v^*) \in Y_1$ is now fixed in such a way that the entrance time t_*^e and the collision time t_* satisfy the inequalities

$$t_*^e \in (2\tau, 2\tau + \tau'')$$

and

$$t_* \in (t_+, t_+ + \tau^*)\,.$$

q_1^* and v_1^* are then determined by equations analogue to (3.1.11), (3.1.12) in such a way as to provide further jump of the velocity of the rod by $2\Delta_0 + \Delta > \Delta_0$. The collision with (q^*, v^*) is on the left, since the velocity of the rod has to increase. Possible recollisions with (q^+, v^+) are avoided by choosing t_* so that $t_* - t_*^e \in (\delta^*, \tau^*)$ where δ is some constant close to τ^*, depending on Δ_0 and on the parameters of the model. Summarizing, we have shown that for all $X \in A_U$ we can construct a configuration $Y_1, \mathrm{card}(Y_1) \leq 2$ depending on X, such that in the dynamics of $X_0 \cup Y_1$ the rod collides with a particle (q^*, v^*) at some time $t_* \in (2\tau, 2\tau + \tau'')$, and the outgoing velocity of the rod for $s > t_*$ is given by $V(T_f^{-t+s}(X))$. From now on q_2^*, v_2^* and t_* are assumed to be fixed and independent of $X \in A_U$. Let $\mathcal{O}_\delta(x, y)$, $(x, y) \in \mathbb{R}^2$ denote the open disk with radius δ and

$$\mathcal{U}_\delta(q, v) = \mathcal{O}_\delta(q_1, v_1) \times \mathcal{O}_\delta(q_2, v_2)\,.$$

For $X \in A_U$, and denoting symmetrization by $[.]_\Sigma$, we set

$$\mathbf{Y}_X^{(1)} = \mathcal{U}_\delta(q^*(X), v^*(X)) \ \text{ if } \ |\Delta| > \Delta_0\,, \tag{3.1.13}$$

$$\mathbf{Y}_X^{(1)} = [\mathcal{U}_\delta(q^+(X), v^+(X)) \times \mathcal{U}_\delta(q^*(X), v^*(X))]_\Sigma \ \text{ else}\,. \tag{3.1.14}$$

Let $Z(X) = (q^*(X), v^*(X))$ if $|\Delta(X)| > \Delta_0$ and $Z(X) = (q+(X), v^+(X)) \cup (q^*(X), v^*(X))$ if $|\Delta(X)| \leq \Delta_0$. For any $X \in A_U$ and $Y_1 \in \mathbf{Y}_X^{(1)}$ the dynamics $\{T_f^s(X_0 \cup Z(X)\}$ and $\{T_f^s(X_0 \cup Y_1\}$ are close for $s \in [0, t - \tau')$ if δ is small. (Moreover, the mentioned dynamics are close uniformly in $f \in [0, f_{\max}]$) In particular the particles of Y_1 collide only once, before $t - \tau'$ and produce a jump in velocity which is continuous function of $Y_1 \in \mathbf{Y}_X^{(1)}$ and close to $\Delta(X)$. Set

$$\mathbf{Y}^{(1)} = \bigcup_{X \in A_U} \mathbf{Y}_X^{(1)}\,. \tag{3.1.15}$$

For $Y_1 \in \mathbf{Y}^{(1)}$ we set:

$$(3.1.16)\quad Q_0(Y_1) = \int_0^{t-\tau'} V(T_f^s(X_0 \cup Y_1))ds,\; V_0(Y_1) = V(T_f^{t-\tau'}(X_0 \cup Y_1))\,.$$

Introduce the set

$$\mathbf{Y}_U = \{Y \in \mathbf{Y}^{\text{in}} : Y_1 \in \mathbf{Y}^{(1)}, (\overline{Y}_2 + Q_0(Y_1)) \cap \mathcal{N}(t-\tau', t, \mathfrak{S}_L) = \emptyset\,.$$

Remind that $\overline{Y}_2$ consists of particles which are in $\mathfrak{S} \cup \mathcal{M}^+$ at time $t-\tau'$ and by time t have left $\mathfrak{S}$. Our next goal is to find a proper expression for the $\mathcal{P}$ measure of $\mathbf{Y}_U$. For the sack of simplicity we use the same notation $\mathcal{P}$ to denote restriction of the measure $\mathcal{P}$ on some subset of $\mathbf{Y}$. We have

$$\begin{aligned}\mathcal{P}(\mathbf{Y}_U) &= \int_{\mathbf{Y}^{(1)}} \mathcal{P}(dY_1)\mathcal{P}(\{Y : (\overline{Y}_2 - Q_0(Y_1)) \cap \mathcal{N}(t-\tau', t, \mathfrak{S}_L) = \emptyset\}) = \\ &= \mathcal{P}(\mathbf{Y}^{(1)}) \cdot p\,,\end{aligned}$$

where

$$\begin{aligned}p &= \mathcal{P}(\{Y : (\overline{Y}_2 - Q_0(Y_1)) \cap \mathcal{N}(t-\tau', t, \mathfrak{S}_L) = \emptyset\}) = \\ &= \mathcal{P}(\{Y : Y \cap (\mathcal{M} \setminus \mathcal{M}^-) \cap \mathring{T}^{-\tau'}(\mathcal{M}^-) \cap \mathcal{N}(0, \tau', \mathfrak{S}_L) = \emptyset\}) > 0\,.\end{aligned}$$

Here we have used the independence of Y_1 and $\widehat{Y}_2$, as configurations in nonintersecting regions of $\mathcal{M}^+$, and the invariance of $\mathcal{P}$ with respect to horizontal translations and to the free dynamics. Let now A be fixed and such that $\pi(A) > 0$. We can assume that $A \subset A_U$ for some $U < \infty$. If $Y \in \mathbf{Y}_U$, then

$$(3.1.17)\quad X_t = X(T_f^t(X_0 \cup Y)) = X(T_f^{\tau'}(V_0(Y_1), \mathring{T}^{t-\tau'}(Y_2) - Q_0(Y_1)))\,.$$

In fact, in the time interval $(t-\tau', t]$ the rod can only collide with the particles of $\mathring{T}^{t-\tau'}(Y_2)$, since, by the last inequality in (3.1.9) and by the definition of A_U, it cannot get out of $\mathcal{S}_L$ and thus cannot collide with the particles of $\mathring{T}^{t-\tau'}(\overline{Y}_2)$. Hence X_t is a function of Y_1 and $\widetilde{Y} = \mathring{T}^{t-\tau'}(Y_2) - Q_0(Y_1)$. Taking into account that Y_1 and Y_2 are independent and that the distribution $\mathcal{P}(dY_2)$ is invariant with respect to horizontal translations and the free dynamics we see that Y_1 and $\widetilde{Y}$ are independent and the distribution of $\widetilde{Y}$ coincides with the restriction of $\mathcal{P}$ on $\mathring{T}^{-\tau'}(\mathfrak{S})$. Set

$$\mathcal{G} = \mathring{T}^{-\tau'}(\mathfrak{S}) ,$$
$$Y^{\mathcal{G}} = Y \cap \mathcal{G} .$$

Suppose that $|\Delta(X)| > \Delta_0$. Then

$$\mathbf{Y}_A^{(1)} = \bigcup_{X\in A} \mathbf{Y}_X^{(1)} \subset \mathbf{Y}_U^{(1)} .$$

is an open set of one particle configurations. Set

$$\mathcal{U} = \{(q_1, v_1) : (q, v) \in \mathbf{Y}_A^{(1)}\} .$$

We have

$$\begin{aligned} P^t(X_0, A) &\geq \int_{\mathbf{Y}_U} \mathcal{P}(dY) \mathrm{I\!I}_A(X(T_f^t(X_0 \cup Y))) \geq \\ &\geq \int_{\mathbf{Y}_A^{(1)}} \mathcal{P}(dY_1)\mathcal{P}(\{Y^{\mathcal{G}} : (V_0(Y_1), Y^{\mathcal{G}}) \in T_f^{-\tau'}(A)\}) = \\ &= \int_{\mathcal{U}} \mathfrak{m}(dq_1, dv_1)\mathcal{P}(\{Y^{\mathcal{G}} : (V_0(q_1, v_1), Y^{\mathcal{G}}) \in T_f^{-\tau'}(A)\}) , \end{aligned}$$

where $\mathfrak{m}$ is the measure induced on $\mathcal{U}$ by the restriction $\mathcal{P}(dY_1)$ on $\mathbf{Y}_A^{(1)}$. $\mathfrak{m}$ is equivalent to the Lebesgue measure $dq_1 dv_1$ on the open set $\mathcal{U}$, since

$$\int_{\mathcal{O}_\delta(q^*, v^*)} dq_2 h(dv_2) > 0 .$$

The function $V_0(Y_1)$ depends, as $Y_1 \in \mathbf{Y}_A^{(1)}$ only on the horizontal coordinates (q_1, v_1). That justifies the notation $V_0(q_1, v_1)$ in the above formula. The same holds for $Q_0(Y_2) = Q_0(q_1, v_1)$. Set $C_V = \{\widetilde{Y} : (V; \widetilde{Y}) \in T_f^{-\tau'}(A)\}$ and note that we have to prove that the set

$$\mathcal{U}' = \{(q_1, v_1) \in \mathcal{U} : \mathcal{P}(C_{V_0(q_1,v_1)}) > 0\}$$

is of positive Lebesgue measure. First we show that the set

$$\mathcal{V} = \{V : \mathcal{P}(C_V) > 0\}$$

is of positive Lebesgue measure. To do this let us add to all configurations $X \in A$ configurations in $\mathcal{M} \setminus \mathfrak{S}$ constructed in a special way. Namely, set

$$\mathcal{D} = \mathcal{M}^- \cap \overset{\circ}{T}{}^{\tau'}(\mathcal{M} \setminus \mathcal{M}^-)\,.$$

In words $(q, v) \in \mathcal{D}$ means that (q, v) is an outgoing particle, which by the backward free dynamics

$$\overset{\circ}{T}{}^{-s}, s \in [0, \tau')\,,$$

cross $\mathfrak{S}$. By $Y^{\mathcal{D}} = Y \cap \mathcal{D}$ we denote the particles of Y contained in $\mathcal{D}$. Set

$$\mathcal{A} = \Big\{\omega : X(\omega) \in A, \Big(Y^{\mathcal{D}}(\omega) - \int_0^{\tau'} V(T_f^{-s}(X))ds\Big) \cap \mathcal{N}(-\tau', 0, \mathfrak{S}_{4L}) = \emptyset\Big\}. \tag{3.1.18}$$

By translation invariance of $\mathcal{P}(dY)$ and independence of X and Y, we have

$$\mu(\mathcal{A}) = \pi(A)\mathcal{P}(\{Y : Y^{\mathcal{D}} \cap \mathcal{N}(-\tau', 0, \mathfrak{S}_{4L}) = \emptyset\}) > 0\,.$$

By Lemma 3.1.2

$$\mu(T_f^{-\tau'}(\mathcal{A})) > 0\,.$$

Let us consider what the set $T_f^{-\tau'}(\mathcal{A})$ is. As $X \in A_U$,

$$\Big|\int_0^r V(T_f^{-s}(X))ds\Big| < L, r \in [0, \tau')\,.$$

This means that the particles of $Y^{\mathcal{D}}(\omega)$ do not intersect $\mathcal{S}_L$ and do not interact with the rod. Recall that

$$\mathcal{G} = \overset{\circ}{T}{}^{-\tau'}(\mathfrak{S})\,,$$
$$Y^{\mathcal{G}} = Y \cap \mathcal{G}\,,$$

and set

$$Y^*(\omega) = Y \cap (\mathcal{M} \setminus \mathcal{M}^-) \cap \mathring{T}^{-\tau'}(\mathcal{M}^-)\,.$$

We have

$$T_f^{-\tau'}(\mathcal{A}) = \{\omega : (V(\omega), Y^{\mathcal{G}}(\omega)) \in T_f^{-\tau'}(A);\ Y^*(\omega) \cap \mathcal{N}(0, \tau', \mathfrak{S}_{4L}) = \emptyset\}\,.$$

Hence, using the independence of $Y^{\mathcal{G}}$ and Y^* we find:

$$\mu(T_f^{-\tau'}(\mathcal{A})) = p_1 \int g(dV)\mathcal{P}(dY^{\mathcal{G}})\mathbb{I}_{C_V}(Y^{\mathcal{G}}) > 0\,. \tag{3.1.19}$$

Here p_1 is a positive constant and g denotes the Gaussian distribution of Y induced by μ. Thus $\mathcal{V}$ is of positive Lebesgue measure. Return to the mapping

$$V_0 : \mathcal{U} \to \mathbb{R}^1, V_0 = V_0(q_1, v_1)\,.$$

By our construction $V_0^{(-1)}(\mathcal{V}) \subset \mathcal{U}'$. The function V_0 is differentiable with derivatives

$$\begin{aligned}
\frac{\partial V_0}{\partial q_1} &= -(1-\alpha)\frac{f}{M(V(T_f^{t_c}(X_0)) - v_1)}\,,\\
\frac{\partial V_0}{\partial v_1} &= (1-\alpha)\Big(1 - \frac{f}{M(V(T_f^{t_c}(X_0)) - v_1)}\Big)\,,
\end{aligned}$$

where t_c is the collision time which is close to t_* as δ is small. The expression above shows that $dV_0 \neq 0$ on $\mathcal{U}$ and thus V_0 is a C^1 mapping of full rank. Hence the Lebesgue measure of $V_0^{-1}(\mathcal{V}) \subset \mathcal{U}'$ is positive and the Lebesgue measure of $\mathcal{U}'$ is also positive.

If A is such that $|\Delta(X)| > \Delta_0$ on A the proof is somewhat more complicated but goes the same lines. We omit the details. ∎

We prove next a property of nondegeneracy of the distribution of the displacement which will be used for to establish nondegeneracy of the limiting brownian motion. In what follows we denote by λP_f the distribution of the markov process $X_t(\omega) = X(T_f^t(\omega))$ with initial measure λ. By $\mathbb{E}_f(.|X_0, X_t)$ we denote the conditional expectation with respect to the σ-algebra generated by X_0 and X_t. Formally the previous value involves the initial distribution of the process X_s. In fact it is of no importance, since it may be represented as $\mathbb{E}_f(.|X_t)$, assuming that the expectation corresponds to the distribution $\delta_{X_0} P_f^t$. Recall that for a given σ-algebra $\mathfrak{L}$ the conditional dispersion of a random variable ξ

$$\mathbb{D}(\xi|\mathfrak{L}) = \mathbb{E}((\xi - \mathbb{E}(\xi|\mathfrak{L}))^2|\mathfrak{L})$$

where $\mathbb{E}(.|\mathfrak{L})$ denotes the conditional expectation with respect to the σ-algebra $\mathfrak{L}$. Recall some basic properties of the conditional dispersion we shall use. If σ-algebras $\mathfrak{L}$ and $\mathfrak{K}$ are such that $\mathfrak{L} \subset \mathfrak{K}$, then

$$\mathbb{D}(\xi|\mathfrak{L}) \geq \mathbb{E}(\mathbb{D}(\xi|\mathfrak{K})|\mathfrak{L}) .$$

For a finite σ-algebra $\mathfrak{L}$:

$$\mathfrak{L} = \{C_1, \ldots, C_n\},\ C_i \cap C_j = \emptyset,\ i \neq j\ ,$$

we have:

$$\mathbb{D}(\xi|\mathfrak{L}) = \sum_{j=1}^{n} \mathbb{I}_{C_j} \mathbb{D}_{C_j}(\xi) ,$$

where $\mathbb{D}_{C_j}$ denotes the dispersion with respect to the probability restricted to C_j and properly normalized.

By the sigma algebra generated by X_0, X_t we use the obvious notation

$$\mathbb{D}_f(.|X_0, X_t) .$$

Introduce

$$D(X_0, X_t) = \mathbb{D}_f(Q(t)|X_0, X_t) .$$

Corollary 3.1.20. *Let t be as in the previous lemma and $A \in \mathcal{B}^{\mathfrak{S}}$ be of positive π-measure. Then the function*

$$J_A(X_0) = \int_A P_f^t(X_0, dX_t) D(X_0, X_t)$$

is π-almost everywhere positive. In particular

$$J_{\mathfrak{X}}(X_0) = \int P_f^t(X_0, dX_t) D(X_0, X_t)$$

is π-almost everywhere positive.

Proof. The result is intuitively obvious, since $D(X_0, X_t) = 0$ on a set of positive measure would imply that on that set $Q(t)$ is a function of X_0, X_t only, which is impossible since X_0 and X_t are relative to the rod position and cannot determine the absolute shift $Q(t)$.

Let $X_0 \in \mathfrak{X}_f$ be fixed. It is not restrictive to assume that for some U large enough we have both $W(X_0) < U$ and $A \subset A_U$, where as above

$$A_U = \{X : W(X) < U\}\,.$$

Going back to the mechanical system we express $X_s, 0 \le s \le t$ as a function of $Y^{in} \in \mathbf{Y}^{in}$:

$$X_s = X(T_f^s(X_0 \cup Y^{in})\,.$$

We denote by $\mathfrak{M}_0^t$ the σ-algebra of subsets of $\mathbf{Y}^{in}$ generated by

$$X_s,\, 0 \le s \le t\,.$$

By $\mathfrak{M}_t$ we denote the σ-algebra generated by X_t. Obviously $\mathfrak{M}_t \subset \mathfrak{M}_0^t$. $\mathbb{E}_{f,X_0}$ denotes the expectation with respect to the measure induced by $\mathcal{P}$ on $\mathfrak{M}_0^t$. Clearly,

$$J_A(X_0) = \int_{X_t \in A} \mathbb{E}_{f,X_0}((Q(t) - \mathbb{E}_{f,X_0}(Q(t)|\mathfrak{M}_t))^2|\mathfrak{M}_t)\mathcal{P}(dY^{\mathrm{in}})\,.$$

Assume that A is such that $|\Delta(X)| > \Delta_0$ for all $X \in A$. Let $\mathbf{Y}_U$ and $\mathbf{Y}_A^{(1)}$ be the same as in the course of the previous proof. Set

$$\mathcal{Y} = \{Y^{in} \in \mathbf{Y}_U : Y^{(1)} \in \mathbf{Y}_A^{(1)}\}\,.$$

We again consider the variables $Y_2, \widetilde{Y}$. By (3.1.17) the restriction of $\mathfrak{M}_t$ on $\mathcal{Y}$ coincides with the σ-algebra generated by

$$V_0(Y_1) = V_0(q_1, v_1),\, (q_1, v_1) \in \mathcal{U}$$

and by $\widetilde{Y} = \overset{\circ}{T}{}^{t-\tau'}(Y_2) - Q_0(Y_1)$. We denote this restriction by $\mathfrak{M}_t^*$. Observe that on $\mathcal{Y}$ holds

$$Q(t) = Q_0(Y_1) + Q_t^*, Q_t^* = \int_{t-\tau'}^{t} V(T_f^s(V_0(Y_1), \widetilde{Y}))ds\,.$$

The value Q_t^* is $\mathfrak{M}_t^*$-measurable, in other words it is a function of X_t only. Thus, with respect to the distribution $\delta_{X_0} P_f^t$, restricted to $\mathcal{Y}$ and properly normalized, we have:

$$\mathbb{D}(Q(t)|\mathfrak{M}_t^*) = \mathbb{D}(Q_0|\mathfrak{M}_t^*) = \mathbb{D}(Q_0|\mathfrak{V}),$$

where $\mathfrak{V}$ is the σ-algebra of subsets of $\mathcal{U}$ generated by V_0. The equality

$$\mathbb{D}(Q_0|\mathfrak{M}_t^*) = \mathbb{D}(Q_0|\mathfrak{B}),$$

holds, since Y_1 and $\widetilde{Y}$ are independent. By the properties of the conditional distribution we have:

$$\begin{aligned} J_A(X_0) &\geq \int_{\mathcal{Y}\cap\{X_t\in A\}} \mathbb{D}(Q(t)|\mathfrak{M}_t)\mathcal{P}(dY) = \\ &= \mathcal{P}(\mathcal{Y}) \int_{\{X_t\in A\}} \mathbb{E}((Q(t) - \mathbb{E}(Q(t)|\mathfrak{M}_t))^2|\mathfrak{M}_t^*)\mathcal{P}_{\mathcal{Y}}(dY) \geq \\ &\geq \mathcal{P}(\mathcal{Y}) \int_{\{X_t\in A\}} \mathbb{D}(Q_0|\mathfrak{V})\mathcal{P}_{\mathcal{Y}}(dY), \end{aligned}$$

where $\mathcal{P}_{\mathcal{Y}}(dY)$ denotes the normalized restriction of $\mathcal{P}$ on $\mathcal{Y}$. The value

$$\mathbb{D}(Q_0|\mathfrak{V}) = \mathbb{D}(Q_0|V_0)$$

is almost everywhere positive. Indeed, V_0 and Q_0 are functions of Y_1. Recall that Y_1 may be identified with $(q_1, v_1) \in \mathcal{U}$ and the distribution $\mathcal{P}(dY_1)$ coincides with some measure $\mathfrak{m}(dq_1, dv_1)$ on $\mathcal{U}$ equivalent to the Lebesgue measure. On the other hand the jacobian of the transformation

$$(q_1, v_1) \to (V_0(q_1, v_1), V_0(q_1, v_1))$$

is positive everywhere. (In fact the jacobian is equal to the constant $(1-\alpha)^2$.) Hence $J_A(X_0)$ is positive if $\mathcal{P}(\mathcal{Y} \cap \{X_t \in A\}) > 0$. But this property is, as shown in the course of the previous proof, a consequence of the condition $\pi(A) > 0$. The proof when A is such that $|\Delta(X)| > \Delta_0$ on A follows the same line. ∎

COROLLARY 3.1.21. *For given $U > 0$ and $t > 4\tau$ there exists a positive nontrivial measure λ_U on $\mathfrak{X}$ such that for all $f \in [0, f_{\max}]$*

$$P_f^t(X_0, dX) \geq \mathbb{I}_{A_U}(X_0)\lambda(dX). \tag{3.1.22}$$

Proof. The proof again makes use of the construction of the previous lemma. Let $X_0 \in A_U$ be fixed. Set $\Delta(X_0) = -V(T^t(X_0))$. Suppose that $|\Delta| > \Delta_0$. We determine (q^*, v^*) as a function of Δ and $t^* < t - \tau'$ by equation (3.1.11), (3.1.12):

$$(v_1^* - \overline{V}_0)^2 + 2\hat{q}_1^* \frac{f}{M} = \frac{\Delta^2}{(1-\alpha^2)}\,;$$
$$t_* - t_*^e = \frac{M}{f}\Big(v_1^* - \overline{V}_0 - \frac{\Delta}{1-\alpha}\Big)\,.$$

Here, as above

$$\overline{V}_0 = V(T_f^{t_*^e}(X_0)),\ V^* = V(T_f^{t-t_*^e}(X)),\ \overline{Q}_0 = \int_0^{t_*^e} V(T_f^s(X_0))ds\,,$$
$$\hat{q}_1^* = q_1^* + v_1^* t_*^e - \overline{Q}_0\,.$$

In words, we choose the particle (q^*, v^*) so, that after a single collision the velocity of the rod becomes equal to 0 at time t.

Consider the dynamics $\{T^s(X_0 \cup Y_1)\}$ for $Y_1 \in \mathcal{U}_\delta(q^*, v^*)$. δ may be chosen here uniformly in $X_0 \in A_U,\ f \in [0, f_{\max}]$. Let $V_t = V(T_f^t(X_0 \cup Y_1))$ and

$$Q_t = \int_0^t V(T_f^s(X_0 \cup Y_1))ds$$

denote the velocity and the displacement at time t. Define the mapping $\Psi_{X_0,f} : \mathcal{O}_\delta(q_1^*, v_1^*) \to \mathbb{R}^2$ by

$$\Psi_{X_o,f}(q_1, v_1) = (Q(t), V(t))\,.$$

Note that

$$\det\,(d\Psi_{X_o,f}) = (1-\alpha)^2$$

and there exists a constant $C_1 > 0$ such that

$$||d\Psi_{X_o,f}|| < C_1 < \infty$$

uniformly in $X_0 \in A_U,\ f \in [0, f_{\max}]$. These two properties imply that there exists a constant C such that

$$\max(||d\Psi_{X_o,f}||, ||(d\Psi_{X_o,f})^{(-1)}||) < C\,.$$

Since the dynamics brings the velocity of the rod to 0, we have:

$$\Psi_{X_o,f}(q_1^*, v_1^*) = (Q^*, 0)\,.$$

The image of $\mathcal{O}_\delta(q_1^*, v_1^*)$ is an open neighborhood of $(Q^*, 0)$ which, due to the properties of the mapping $\Psi_{X_o,f}$ indicated above, contains a square

$$|Q - Q^*| < \theta, |V| < \theta\,,$$

where $\theta > 0$ may be chosen uniformly in $X_0 \in A_U,\ f \in [0, f_{\max}]$. Taking into account that

$$\det\,(d\Psi_{X_o,f}) = (1-\alpha)^2\,,$$

we conclude that for all X_0 and f under consideration the measure induced on

$$\mathcal{J} = \{V : |V| < \theta\}$$

by the Lebesgue measure on $\mathcal{O}_\delta(q_1^*, v_1^*)$, via the mapping V_t is absolutely continuous with respect to dV, with some density $G_{X_0,f}$: $0 < a < G_{X_0,f}(V)$, where the constant a does not depend on X_0 and f. For a given $X_0 \in A_U$ let

$$\mathcal{Y}^{(X_0)} = \{Y : Y_1 \in \mathcal{U}_\delta(q^*, v^*), (\hat{Y}_2 - Q(T_f^t(X_0 \cup Y_1)) \cap \mathcal{N}(t-\tau', t, \mathfrak{S}_L) = \emptyset\}\,.$$

Then, by repeating the arguments used in the proof of the lemma , we see that:

$$P^t(X_0, A) \geq \int \mathcal{P}(dY) \mathrm{I\!I}_A(X_t) \mathrm{I\!I}_{\mathcal{Y}^{(X_0)}}(Y) \geq \mathrm{const}\hat{\lambda}(A)\,,$$
$$\hat{\lambda}(dV, dY^{\mathfrak{S}}) = dV \mathcal{P}(dY^{\mathfrak{S}}) \mathrm{I\!I}_J(V) \mathrm{I\!I}_{\mathcal{N}_L}(Y^{\mathfrak{S}})\,,$$

where

$$\mathcal{N}_L = \{Y : Y \cap \mathcal{N}(-\tau', 0, \mathfrak{S}_L) = \emptyset\}\,.$$

Hence the result follows. On the set $X_0 \in A_U : |\Delta| \leq \Delta_0$ we can repeat the same arguments, by taking Y_1 in a set of two - particles configurations. The procedure is a bit more complicated but straightforward. We omit the details. $\square$

In the following lemma we turn to the Lyapunov function $W(X)$. We use this function for to describe the phenomenon of energy relaxation in the system. Roughly speaking, we show that large velocities in a neighborhood of the rod slow down with large probability. Recall the definition of W. $W(B)$ is defined for any $B \subset \mathcal{M}$ by

$$W(B) = \sup\Big\{|v_1| : (q,v) \in B : \tau|v_1| > \frac{1}{4}|q_1|\Big\}.$$

$W(X)$ is also defined for $X \in \mathfrak{X}$ by

$$W(X) = \max\{|V(X)|, W(Y^{\mathfrak{S}}(X))\}. \tag{3.1.23}$$

Lemma 3.1.24. *Let $4\tau < t \leq 10\tau$. There exists a positive number U_0 such that the following inequalities hold for $U > U_0$ and some positive constants κ, c_1, c_2:*

$$\begin{aligned} &P_f^t(X, \{W \geq U\}) < \exp(-c_1 U^2), \ \textit{if}\ W(X) < U - \frac{f}{M}t\,,\\ &P_f^t(X, \{W \geq U\}) < \exp(-c_2 U), \ \textit{if}\ W(X) < U\exp(\kappa U)\,. \end{aligned}$$

The constants $c_{1,}c_2, U_0, \kappa$ do not depend on f.

Proof. We begin with a short description of the main idea. Suppose that the rod keeps a high velocity $\overline{U}$ for some time $\bar{t}$. Then typically it will collide with particles with horizontal velocity not exceeding $\log(\overline{U}\bar{t})$ in absolute value, except for a set with measure of the order $e^{-\operatorname{const}\log \overline{U}}$. This follows from the Gaussian distribution of incoming horizontal velocities. If no particles with velocities larger than $\log(\overline{U}\bar{t})$ collide with the rod, then the rod can keep a velocity larger than $O(\log(\overline{U}))$ only if it collides with a small number of slow particles. Indeed, each time that it collides with a slow particle its velocity drops by a factor α. The probability to collide with a small number of slow particles turns to be of the order $e^{-c\log(\overline{U})}$ Passing to the proof of the first assertion set

$$\widehat{U} = \Big(U - \frac{f}{M}t\Big)$$

and

$$L = 100tU\,.$$

Consider the event B_U defined by the following condition on the incoming particles: all incoming particles passing through $\mathfrak{S}_L$ until the time t have horizontal velocities in absolute value not larger than $\gamma\hat{U}$, where $\gamma \in (0,1)$ is a fixed constant. Formally this event is defined as follows. Set

$$\overline{Y} = Y^{\mathrm{in}} \cap \mathring{T}^{-t}(\mathcal{M} \setminus \mathcal{M}^{+})\,.$$

$\overline{Y}$ consists of particles crossing $\mathfrak{S}$ up to time t. Divide $\overline{Y}$ into two groups:

$$\overline{Y}_L = \overline{Y} \cap \mathcal{N}(0, t, \mathfrak{S}_L),\ Z_L = \overline{Y} \setminus \overline{Y}_L\,.$$

Then

$$B_U = \{Y : \sup\{|v_1| : (q_1, v_1) \in \overline{Y}_L\} < \gamma\widehat{U}\}\,.$$

Elementary estimates show that for some constant $\theta_1 > 0$ and U large enough

$$\mathcal{P}(B_U) > 1 - \exp(-\theta_1 U^2)\,. \tag{3.1.25}$$

Consider the dynamics $T_f^s(X \cup Y),\ 0 \le s \le t$ for $X : W(X) < U - \frac{f}{M}t$ and $Y \in B_U$. Evidently the conditions $W(X) < U - \frac{f}{M}t$ and $Y \in B_L$ imply that as long as the rod collides with particles of $\overline{Y}_L \cup Y^{\mathfrak{S}}$ its velocity is bounded in absolute value by U. Thus the displacement of the rod is bounded by $tU < L/100$. That shows that the rod does not collide with particles of Z_L at all:

$$T_f^s(X \cup \overline{Y}) = T_f^s(X \cup \overline{Y}_L) \cup (\mathring{T}^s(Z_L) - Q_s(Y_L))\,.$$

for $0 \le s \le t$. Here

$$Q_s(\overline{Y}_L) = \int_0^s V(T_f^r(X \cup \overline{Y}_L))dr\,.$$

All the particles included into $X \cup \overline{Y}_L$ keep the horizontal velocities not larger in absolute value than U and thus may not violate the condition $W(X_t) < U$. Hence the condition $W(X_t) \ge U$ implies that

$$W\Big(\mathring{T}^t(Z_L) \cap \mathfrak{S} - \int_0^t V(T_f^s(X \cup \overline{Y}_L))ds\Big) > U\,. \tag{3.1.26}$$

The configuration $Z_t^* = \overset{\circ}{T}^t(Z_L) \cap \mathfrak{S}$ is $\mathcal{P}$ - independent of $\overline{Y}_L$. Since $\mathcal{P}$ is translation invariant, we have:

$$\mathcal{P}(\{W(Z^* - Q_t) \geq U\} | \overline{Y}_L) = \mathcal{P}(\{W(Z_t^* - Q_t) \geq U\}) \leq \mathcal{P}(\{W(Y^{\mathfrak{S}}) \geq U\}).$$

We have then

$$\mathcal{P}(\{Y : W(X(T_f^t(X \cup Y)) \geq U\}) \leq \mathcal{P}(B_U^c) + \mathcal{P}(B_U)\mathcal{P}(\{W(Y^{\mathfrak{S}}) \geq U\}).$$

Taking into account lemma (3.1.6) and (3.1.25) we obtain the required assertion.

Consider the second assertion of the lemma. Choose some constant $\kappa > 0$. The restrictions on κ will be formulated somewhat later. We set

$$L = 100\Big(tUe^{\kappa U} + \frac{f}{M}t^2\Big)$$

and define $\overline{Y}_L, Z_L, B_U$ as above. We may and will assume that U is large enough to imply (3.1.25). Repeating our analysis of $T_f^s(X \cup Y)$, for $Y \in B_U$ we see that the rod cannot travel by the time t more than a distance $L/4$ and no collisions with particles of Z_L occur. The resulting configuration

$$X_s = T_f^s(X \cup Y)$$

may be again represented as follows:

$$X_s = \widehat{X}_s \cup (Z_s^* - Q_s),$$

where

$$\widehat{X}_s = X(T_f^s(X \cup \overline{Y}_L)),$$
$$Z_s^* = \overset{\circ}{T}^s(Z_L) \cap \mathfrak{S},$$

and Q_s is defined as above. Let

$$D_U = \{Y \in B_U : \inf_{s \in (2\tau, t-\tau)} |V(X_s)| < \widehat{U}\}.$$

Suppose that $Y \in D_U$, then $W(\widehat{X}_t) < U$. In fact all particle that collide in the time interval $(2\tau, t)$ come from $\overline{Y}_L$ and have velocity less than $\gamma\widehat{U}$

in absolute value. Some of those particles may be accelerated by the rod but then become "outgoing" i.e. directed away from the rod and cannot accelerate it. Once the absolute value of the velocity of the rod has fallen under $\widehat{U}$ it can only increase by the acceleration up to the value $\widehat{U}+tf<U$. The particles of $\overline{Y}_L$ that have been accelerated by the rod to velocities larger than U will leave $\mathfrak{G}$ during the time interval $(t-\tau,t]$. Hence all the particles involved into $\widehat{X}_t$ (and the rod) have velocities not exceeded U. This implies

$$W(\widehat{X}_t)<U\,.$$

Then the conditional probability

$$\mathcal{P}(\{W(X_t)\geq U\}\cap D_U|\overline{Y}_L)\leq\mathcal{P}(W(Z_t^*-Q_t(\overline{Y}_L))\geq U)$$

for $\overline{Y}_L\in B_U$. As above,

$$\mathcal{P}(W(Z_t^*-Q_t(\overline{Y}_L))\geq U)<\mathcal{P}(W(Y^{\mathfrak{G}})\geq U)<\exp(-\mathrm{const}U^2)\,.$$

Thus

$$\mathcal{P}(\{W(X_t)\geq U\}\cap D_U)<\exp(-\mathrm{const}U^2)\,.$$

It remains to estimate the probability of

$$\Gamma_U=B_U\setminus D_U\,.$$

In other words, we estimate the probability of configurations in B_U satisfying

$$\inf_{s\in(2\tau,t-\tau)}|V(X_s)|\geq\widehat{U}\,.$$

The collision rule

$$V'=\alpha V+(1-\alpha)v_1$$

shows that if V and V' have opposite signs, then

$$|V'|<(1-\alpha)|v_1|\,.$$

Hence in the dynamics

$$T_f^s(X\cap Y),\ Y\in B_U\,,$$

the velocity of the rod cannot change sign in the time interval $(2\tau, t-\tau)$ without dropping below $\gamma\widehat{U} < \widehat{U}$. Thus Γ_U splits into two nonintersecting subsets $\Gamma_U^{\pm}$, according to the sign of the rod velocity. The arguments we use in the sequel are analogous for Γ_+ and Γ^-, so we may and will consider Γ_+ only.

Another consequence of the collision rules is that for $Y \in B_U$, a particle that collides at some time $s \in (2\tau, t-\tau)$, when the incoming velocity of the rod

$$V^-(X_s) = \lim_{\epsilon \nearrow 0} V(X_{s-\epsilon}) > \widehat{U}\,,$$

cannot recollide. To see this, note that for such a particle $|v_1| < \gamma\widehat{U}$, and the outgoing velocity

$$v_1' = V^- + \alpha(V^- - v_1) \geq V^- + \alpha(1-\gamma)\widehat{U}\,.$$

The outgoing velocity of the rod

$$V(X_s) = V^- + (1-\alpha)(v_1 - V^-) \leq V^-\,.$$

The next incoming particles can not accelerate the rod up to $V^- + \alpha(1-\gamma)\widehat{U}$ since their horizontal velocities are smaller than $\gamma\widehat{U}$. If U is so large that

$$\alpha(1-\gamma)\widehat{U} > (t-\tau)\frac{f}{M}\,,$$

then any recollision can occur only after the moment t when the particle will leave the strip $\mathcal{S}$.

From now on we take

$$B_U, Pr(.) = \mathcal{P}(.|B_U)$$

as the main probability space. $V_s = V(X_s)$ is a random process on $B_U, Pr(.)$. Define a stopping time

$$t^* = \min\{s \geq 0 : |V_s| \leq \widehat{U}\}$$

and set

$$\widetilde{V}_s = V_{s \wedge t^*}\,.$$

We consider then the process $\widetilde{V}_s$ at the time interval $s \in [3\tau, t-\tau]$. The basic observation is that $\widetilde{V}$ restricted to the mentioned time interval is a Markov process with jumps and absorbing boundary conditions. In fact, by the observation above, the condition $|V_s| > \widehat{U}$ implies that all particles that collide in the interval $[3\tau, t-\tau]$ cannot recollide, while those that collided in the time interval $(0, 2\tau]$ are out of the game. Assuming positive velocities $\widetilde{V}_s > \widehat{U}$ we conclude that collisions may occur only with "fresh" particles on the right. That is, for given $\widetilde{V}_r > \widehat{U}, r \in [3\tau, t-\tau)$ the distribution of

$$\widetilde{V}_s, s \geq r$$

depends on V_r and the configuration of particles

$$Y_r = \overset{\circ}{T}{}^r(Y^{\mathrm{in}}) \cap (\{(q,v) \in \mathcal{M} \setminus \mathcal{M}^- : q_1 \in (0, L/4)\} + Q_r)\,.$$

This observation relies on the fact that at time r no particles with $q_{1>Q_r}$ and $|v_1| < \gamma\widehat{U}$ could collide with the rod in the past. The distribution of Y_r induced by Pr is Poisson with intensity measure

$$n(dq, dv)\mathbb{I}_{\mathcal{A}_r}\,,$$
$$\mathcal{A}_r = (\{(q,v) \in \mathcal{M} \setminus \mathcal{M}^- : q_1 \in (0, L/4),\, |v_1| < \gamma\widehat{U}\} + Q_r)\,.$$

Using invariance of the Poisson distribution with respect to translations and free dynamics we conclude that the process $\widetilde{V}_s,\ 3\tau \leq s \leq t-\tau$ corresponds to a collision process with Poisson distributed particles, no recollisions, instantaneous collision rate

$$R(V_s) = \rho\ell\sqrt{\frac{\beta m}{2\pi}} \int_{|v_1|<\gamma\widehat{U}} |V_s - v_1| e^{(-\beta m v_1^2/2)} dv_1$$

and absorbing boundary conditions in the region $|V| < \widehat{U}$. To estimate

$$Pr(D_U^+ | \widetilde{V}_{3\tau}) = Pr(\inf_{3\tau \leq s \leq t-\tau} \widetilde{V}_s \geq \widehat{U} \mid \widetilde{V}_{3\tau})$$

we may and will assume that

$$\widetilde{V}_{3\tau} = \overline{V} \in \left[\widehat{U},\ Ue^{\kappa U} + \frac{f}{M} 2\tau\right].$$

Summarizing and simplifying notations we may reduce the problem the following one. Let

$$\mathsf{V}_t,\ t \geq 0$$

be a right continuous Markov process with jumps governed by the infinitesimal operator A:

$$\mathsf{A}F(V) = \rho\ell\sqrt{\frac{\beta m}{2\pi}}\int_{|v_1|<\gamma\widehat{U}} |V-v|(F(\alpha V+(1-\alpha)v)-F(V))e^{(-\beta m v^2/2)}dv + \\ + \frac{f}{M}\frac{\partial F}{\partial V}\,.$$

The collision rate of this process is

$$R(V) = \rho\ell\sqrt{\frac{\beta m}{2\pi}}\int_{|v_1|<\gamma\hat{U}} |V-v|e^{(-\beta m v^2/2)}dv\,.$$

Suppose that the starting point

$$\mathsf{V}_0 \in \left[\widehat{U},\ Ue^{\kappa U} + \frac{f}{M}2\tau\right].$$

We are interested in estimating

$$P_{\mathsf{V}_0}(\inf_{s\in I} V_s \geq \widehat{U})\,,$$

where $I = [0, t-4\tau]$ is a fixed interval of time.

Let $0 = t_o < t_1 < t_2 < \dots$ be the moments of jumps and

$$\mathsf{V}_{t_1}, \mathsf{V}_{t_2}, \dots$$

the corresponding velocities. Introduce

$$\tau_k = t_k - t_{k-1}\,.$$

The conditional distribution of τ_k, as $\mathsf{V}_{t_{k-1}}$ is fixed, is given by

$$\mathcal{F}_{\mathsf{V}_{t_{k-1}}}(t) = P_{\mathsf{V}_{t_{k-1}}}(\tau_k \leq t)\,,$$

where

$$\mathcal{F}_V(t) = 1 - \exp\Big(- \int_o^t R\Big(V + \frac{f}{M}s\Big)ds\Big) .$$

If $V \geq \widehat{U}$ then $R(V) \geq CU$ for some suitable constant $C > 0$. Set $\nu(U) = CU$ and introduce

$$\mathcal{G}(t) = 1 - \exp(-\nu(U)t) .$$

Evidently $\mathcal{G}(t) \leq \mathcal{F}_V(t)$ provided

$$V + \frac{f}{M}s > \widehat{U},\ 0 \leq s \leq t .$$

Define new random variables $\sigma_k,\ k = 1, \dots$ by

$$\sigma_k = \mathcal{G}^{(-1)} \circ \mathcal{F}_{\mathsf{V}_{t_{k-1}}}(\tau_k) .$$

That is,

$$\sigma_k = \frac{1}{\nu(U)} \int_o^{\tau_k} R(\mathsf{V}_s)ds .$$

By construction, the distribution of σ_k is exponential and independent of $\mathsf{V}_{t_{k-1}}$:

$$P_{\mathsf{V}_{t_{k-1}}}(\tau_k \leq t) = \mathcal{G}(t) .$$

Thus $\sigma_k,\ k = 1, \dots$ is a sequence of independent identically distributed exponential variables. On the set where

$$\inf_{t_{k-1} \leq s \leq t_k} \mathsf{V}_s \geq \widehat{U}$$

we have $\sigma_k \geq \tau_k$. Let j be the number of jumps inside the time interval $I = [0, t - 4\tau]$. For any integer n the inequality $j > n$ is equivalent to

$$\sum_{k=1}^{n} \tau_k < |I| = t - 4\tau .$$

Thus, setting as above

$$D_U^+ = \{\inf_{s \in I} \mathsf{V}_s \geq \widehat{U}\} ,$$

we have:

$$\Big\{\sum_{k=1}^{n}\sigma_k < |I|\Big\} \cap D_U^+ \subset \{j > n\} \cap D_U^+ .$$

The probability of

$$\Big\{\sum_{k=1}^{n}\sigma_k < |I|\Big\}$$

is easy to estimate since it coincides with the probability that a Poisson variable with parameter $\nu(U)|I|$ is larger than n. Take for definiteness

$$n(U) = \Big[\frac{1}{2}\nu(U)|I|\Big] .$$

We have, using the standard large deviations estimate for a Poisson variable:

$$P\Big(\Big\{\sum_{k=1}^{n}\sigma_k < |I|\Big\}\Big) > 1 - \exp(-aU) ,$$

where $a > 0$ is a suitable constant. On the other hand, on D_U^+ by each jump inside the interval I value of V decreases by a constant factor:

$$\mathsf{V}_{t_k} < \alpha \mathsf{V}_{t_k-} + (1-\alpha)\gamma\widehat{U} \leq \theta \mathsf{V}_{t_k-} ,$$

where $\theta = \alpha + (1-\alpha)\gamma < 1$. After $n(U)$ jumps we have:

$$\mathsf{V} \leq \theta^n(U)\Big(Ue^{\kappa U} + 2\tau\frac{f}{M}\Big) .$$

This value is $o(1)$ for U large enough if we assume κ to be sufficiently small:

$$\kappa < -\frac{C}{2}\log(\theta)|I| .$$

Hence, for U large enough

$$\{j > n(U)\} \cap D_U^+ = \emptyset$$

and thus

$$\Big\{\sum_{k=1}^{n}\sigma_k < |I|\Big\} \cap D_U^+ = \emptyset .$$

This implies

$$P(D_U^+) < 1 - P\Big(\Big\{\sum_{k=1}^{n} \sigma_k < |I|\Big\}\Big) \leq \exp(-aU)\,.$$

The proof is complete. ∎

3.1.5 – Transition to a Markov chain

We now discretize time and consider the Markov chain $(\mathfrak{X}_f, \mathbb{P}_f)$, where $\mathbb{P}_f = P_f^\tau$. The results above imply some important properties of the chain $(\mathfrak{X}_f, \mathbb{P}_f)$ that we formulate in the following

Proposition 3.1.27. *The chain P_f is defined on the absorbing set $\mathfrak{X}_f$ of full π - measure. Moreover:*

(i) *The chain $(\mathfrak{X}_f, \mathbb{P}_f)$ is π - irreducible and aperiodic.*
(ii) *$\pi\mathbb{P}_f$ are equivalent to π and π is a maximal irreducibility measure.*
(iii) *For all $X \in \mathfrak{X}_f$ π is absolutely continuous with respect to $\mathbb{P}_f^5(X, .)$.*
(iv) *For a π - almost everywhere positive function W, defined on $\mathfrak{X}$, and $U > U_0$ the following conditions hold:*

$$\int \exp(bW^2(X))\pi(dX) < C < \infty$$

$$\mathbb{P}_f(X, \{W \geq U\}) < \exp(-c_1U^2), \quad \textit{if} \;\; W(X) < U - \frac{f}{M}\tau\,,$$

$$\mathbb{P}_f^n(X, \{W \geq U\}) < \exp(-c_2U), \quad \textit{if} \;\; W(X) < U\exp(\kappa U)\,,$$

for any $n : 5 \leq n \leq 10$. Here the constants $U_o, c_1, c_2, \kappa, b, C$ do not depend on $f \in [0, f_{\max}]$.
(v) *The set $A_U = \{X : W(X) < U\}, U > U_0$ is a uniformly "small" set for the family of chains $\mathbb{P}_f, f \in [0, f_{\max}]$. That is, there exists a nontrivial positive measure λ_U such that*

$$\mathbb{P}_f^5(X_0, dX) \geq \mathbb{I}_{A_U}(X_0)\lambda_U(dX)\,.$$

Our main object, the displacement of the rod, may be represented in terms of the Markov chain introduced above. Set

$$\mathcal{V}_f(X) = \int_0^\tau V(T_f^s(X))ds\,.$$

Then

$$Q_f(n\tau) = \sum_{i=0}^{n-1} \mathcal{V}_f(X_i)\,,$$

where X_i denotes a trajectory of the chain. It easily seen that the function $\mathcal{V}_f$ satisfies the following inequality

$$|\mathcal{V}_f(X)| \leq \mathrm{Const}(1 + W(X))\,, \tag{3.1.28}$$

where Const does not depend on $f \in [0, f_{\max}]$.

Suppose we forget the concrete dynamical system under consideration and just deal with an abstract family of Markov chains $\mathbb{P}_f$ and functions $\mathcal{V}_f$ satisfying the above conditions. Which results may be obtained in this general situation? We find that this problem (in a bit more general setting) is of independent interest and devote a separate section to it. The corresponding considerations appeal to relatively advanced probabilistic techniques. Now we just formulate the main results.

THEOREM 3.1.29. *Under the conditions on $\mathbb{P}_f$ and $\mathcal{V}_f$ formulated above the following statements hold true:*

1. *For each f there exists a unique probability measure ν_f, which is invariant with respect to the chain $\mathbb{P}_f$:*

$$\nu_f \mathbb{P}_f = \nu_f\,.$$

The measure ν_f is equivalent to π.

2. $$\int \nu_f(dX) \exp(\mathsf{b}W(X)) \leq \mathrm{Const}\,,$$

where b *and* Const *do not depend on f.*

There exist $\kappa > 0, \mathrm{Const} > 0$ *and* $\alpha_{\max} > 0$ *such that*

$$\mathbb{E}_\pi\Big(\exp\Big(\alpha \sum_{j=0}^{N-1} W(X_j)\Big)\Big) < \mathrm{Const}\exp(\kappa N\alpha)$$

for $0 < \alpha \le \alpha_{\max}$*, arbitrary* $N = 1, 2, \dots$ *and* $f : |f| \le f_{\max}$.

3. *There exists an absorbing subset*

$$\mathfrak{H}_f \subseteq \mathfrak{X}_f : \pi(\mathfrak{X}_f \setminus \mathfrak{H}_f) = 0$$

such that the chain $(\mathbb{P}_f, \mathfrak{H}_f)$ *is Harris recurrent. Moreover there exist exist positive constants* A, B *and* γ_0 *independent of* f *such that for any* $X \in \mathfrak{H}_f$

$$||\mathbb{P}_f^n(X, dY) - \nu_f(dY)|| \le \mathsf{A}\exp(-\gamma_0 n)\log(2 + W(X)) \tag{3.1.30}$$

and

$$||\pi\mathbb{P}_f^n(dY) - \nu_f(dY)|| \le \mathsf{B}\exp(-\gamma_0 n) \tag{3.1.31}$$

for all $n = 1, 2 \dots$ *and* $f : |f| \le f_{\max}$.

4. *There exist positive constants* γ_1, D *independent of* f *such that*

$$\begin{aligned} &\Big| \int \mathbb{P}_f^n(X_0, dX)\,(\mathcal{V}_f(X) - \nu_f(\mathcal{V}_f)) \Big| \le \\ &\le \mathsf{D}\,(1 + \log W(X_0))\exp(-\gamma_1 n)\,, \end{aligned} \tag{3.1.32}$$

for $n \ge m_0 = 5$.

5. *For any initial distribution of* X_0 *the sum*

$$S_{f,n} = \frac{Q_f(\tau n) - d(f)\tau n}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(\mathcal{V}_f(X_j) - \nu_f(\mathcal{V}_f))$$

converges in distribution to the centered Gaussian variable with the variance

$$\begin{aligned} \tau\sigma^2(f) = &\int \nu_f(dX)(\mathcal{V}_f(X) - \nu_f(\mathcal{V}_f))^2 + \\ &+2\sum_{n=1}^{\infty}\int \nu_f(dX)(\mathcal{V}_f(X) - \nu_f(\mathcal{V}_f))\int \mathbb{P}^n(X, dY)(\mathcal{V}_f(Y) - \nu_f(\mathcal{V}_f))\,. \end{aligned}$$

6. *The sum*

$$\sum_{n=1}^{\infty}\int \nu_f(dX)(\mathcal{V}_f(X)-\nu_f(\mathcal{V}_f))\int \mathbb{P}_f^n(X,dY)(\mathcal{V}_f(Y)-\nu_f(\mathcal{V}_f))$$

converges absolutely and uniformly in f.

7. *There exist positive constants ϱ_0, L such that*

$$\mathbb{E}_\pi \exp(\varrho S_{f,n}) \le \mathsf{L} \tag{3.1.33}$$

for any $\varrho : |\varrho| < \varrho_0,\ n \ge 1$ and $f : |f| \le f_{\max}$.Each function

$$\Phi_{f,n}(w) = \mathbb{E}_\pi \exp(wS_{f,n})$$

is analytic inside the strip Γ defined by the inequality $|\Re(w)| < \varrho$. The family of analytic functions

$$\{\Phi_{f,n}(w) :\ n \ge 1,\ |f| \le f_{\max}\}$$

is tight with respect to the topology of uniform convergence on each compact subset of Γ. A sequence $\Phi_{f_k,n_k} : n_k \to \infty$ converges in the mentioned topology to a limit analytic function Φ_∞ if and only if $\sigma^2(f_k)$ converges to some σ^2_∞ and

$$\Phi_\infty(w) = \exp\Big(\frac{w^2\tau\sigma^2_\infty}{2}\Big).$$

3.2 – Drift and diffusion

In this section we introduce the drift and diffusion constant. In view of Theorem 3.1.29 the existence of the mentioned values is pretty obvious. The nontrivial part of the section contains the proof of their nondegeneracy.

We first proof that the measure ν_f is invariant with respect to P_f^t and has strong mixing properties.

LEMMA 3.2.1. *For any $t \geq 0$*

$$\nu_f P_f^t = \nu_f \,.$$

Moreover there exists constants $\chi > 0, C > 0$ such that for $X \in \mathfrak{H}$

$$||P_f^t(X,.) - \nu_f|| < C \log(2 + W(X)) \exp(-\chi t) \,.$$

PROOF. By Theorem 3.1.29 $\nu_f P_f^\tau = \nu_f$ and by 3.1.30 we have

$$\begin{aligned} ||P_f^{\tau n}(X, dY) - \nu_f(dY)|| = ||\mathbb{P}_f^n(X, dY) - \nu_f(dY)|| \leq \\ \leq \mathsf{A} \exp(-\gamma_0 n) \log(2 + W(X))\,, \end{aligned}$$

provided $X \in \mathfrak{H}_f$. Note first that $\nu_f P^t$ is equivalent to π. (We shall denote it by $\nu_f P^t \sim \pi$.) Indeed, $\nu_f \sim \pi$ implies that

$$\nu_f P^t \sim \pi P^t \sim \pi \,.$$

Thus

$$\nu_f P^t(\mathfrak{X}_f \setminus \mathfrak{H}_f) = 0 \,.$$

Hence

$$\nu_f P_f^t = \nu_f P_f^{n\tau} P_f^t = \nu_f P_f^t P_f^{n\tau} \to \nu_f$$

as $n \to \infty$. Taking

$$m(t) = \Big[\frac{t}{\tau}\Big]\,,$$

we have, for any $X \in \mathfrak{H}$:

$$\begin{aligned} ||P_f^t(X,\cdot) - \nu_f|| = ||(\mathbb{P}_f(X,\cdot) - \nu_f) P^{t-\tau m(t)}|| \leq ||\mathbb{P}_f(X,\cdot) - \nu_f|| \leq \\ \leq C \log(2 + W(X)) \exp(-\gamma m(t)) \leq \\ \leq C \log(2 + W(X)) \exp\Big(-\frac{\gamma}{\tau} t\Big)\,. \end{aligned}$$

Hence the result. ∎

PROPOSITION 3.2.2. *There are finite constants $d(f)$ and $\sigma^2(f)$ such that such that for any initial distribution Λ of the Markov process $\{X_t, t \geq 0\}$ for which*

$$\lim_{t\to\infty} ||\lambda P_f^t - \nu_f|| = 0$$

the following assertions hold:

(i) (*Existence of the drift*)

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t V(X_s)ds = \nu_f(V) = d(f)$$

πP_f *or* $\nu_f P_f$ *a.e. In particular*

$$\lim_{t\to\infty}\frac{Q(T_f^t(\omega))}{t} = d(f)$$

for μ *- typical* ω.

(ii) (*Diffusion*) *The distribution of the process*

$$\xi_t^\epsilon = \sqrt{\epsilon}\int_0^{\frac{t}{\epsilon}}(V(X_s)-d_f)ds$$

induced by λP_f *converges as* $\epsilon \to 0$ *to that of Wiener process* $\mathcal{W}_{\sigma^2(f)t}$.

Proof. The first statement follows from Theorem 3.1.29 and the fact that $|V(X)| < W(X)$ and the function V is exponentially integrable with respect to ν_f. As for assertion (ii), let $\mathfrak{M}^t$ and $\mathfrak{M}_t$ be the σ - algebras generated by the variables $\{X_s, s \geq t\}$ and $\{X_s, s \leq t\}$ respectively. Consider the operator

$$\Pi_{p,r}^t:\ L_p(\Omega,\mathfrak{M}^t,\nu_f P_f) \to L_r(\Omega,\mathfrak{M}_0,\nu_f P_f),\ p,r\geq 1$$

defined by setting

$$\Pi_{p,r}^t(\xi) = \mathbb{E}(\xi|\mathfrak{M}_0) - \mathbb{E}(\xi)\,,$$

where the expectation $\mathbb{E}$ refers to the measure $\nu_f P_f$. By Lemma 3.2.1

$$||\Pi_{\infty,1}^t|| \leq 2C\exp(-\chi t)\int \nu_f(dX)\log(2+W(X))\,,$$

and obviously

$$||\Pi_{p,p}^t|| \leq 2$$

for any $p>1$. Taking $p=4$ and noting that

$$\frac{1}{3}+\frac{1}{4}\Big(1-\frac{1}{3}\Big) = \frac{1}{2}\,,$$

we get using the Riesz-Thorin Interpolation theorem (see [20]) that

$$||\Pi^t_{6,2}|| \le \mathrm{Const}\exp\Big(-\frac{\chi}{3}t\Big).$$

Since $V(X_t) \in L_p(\Omega, \mathfrak{M}^t, \nu_f P_f)$ for any $p \ge 1$, we have:

$$||\mathbb{E}(V(X_t) - d(f))|\mathfrak{M}_0||_{L_2} \le ||V||_{L_6}||\Pi^t_{6,2}|| \le \mathrm{Const}\exp\Big(-\frac{\chi}{3}t\Big).$$

We can apply, for example, Theorem 3.79 of [14] and obtain the result. □

The constants $d(f)$ and $\sigma^2(f)$ defined above coincide, of course, with those mentioned in Theorem 3.1.29. The following proposition concludes the proof of the (nontrivial) diffusive asymptotic behavior of the rod displacement.

Proposition 3.2.3. *The following assertions hold.*

1. $\sigma^2(f) > 0$ *for all* $f \in \mathbb{R}$
2. $fd(f) > 0$ *if* $f \neq 0$.

Proof. We denote by

$$\Delta Q_n = \int_{5\tau n}^{5\tau n+1} V(X(T^s_f\omega))ds$$

the displacement between $5\tau n$ and $5\tau(n+1)$, and by

$$S_m = \sum_{k=0}^{m-1} \Delta Q_k$$

the total displacement up to time $5m\tau$. Let $\mathfrak{T}$ denote the σ - algebra generated by the variables $X_0, X_{5\tau}, \dots, X_{5\tau k}, \dots$ The displacements ΔQ_n, $n = 1, 2, \dots$ are conditionally independent with respect to $\mathfrak{T}$. Denoting by $\mathbb{D}(\cdot\,|\mathfrak{T})$ the conditional variance, we get

$$\mathbb{D}(S_m|\mathfrak{T}) = \sum_{k=0}^{m-1} \mathbb{D}(\Delta Q_k|\mathfrak{T}) = \sum_{k=0}^{m-1} \mathbb{D}(\Delta Q_k|X_{5\tau k}, X_{5\tau(k+1)})$$

By Corollary 3.1.20, we have

$$\mathbb{E}_{\nu_f}(\mathbb{D}(\Delta Q_k|X_{5\tau k}, X_{5\tau(k+1)})) = a > 0\,.$$

Hence

$$\begin{aligned}\frac{1}{m}\mathbb{D}(S_m) &\geq \frac{1}{m}\mathbb{E}_{\nu_f}(\mathbb{D}(S_m)\,|\mathfrak{T}) = \\ &= \frac{1}{m}\sum_{k=0}^{m-1}\mathbb{E}_{\nu_f}(\mathbb{D}(\Delta Q_k|X_{5\tau k}, X_{5\tau(k+1)})) = a > 0\,.\end{aligned}$$

The first assertion is proved.

As for the next assertion, assume $f > 0$ and suppose that $d(f) \leq 0$ Recall the *summation rule* 3.1.3:

$$\int_\Omega \mu(d\omega)\exp\Big(-\beta f\int_0^t V(T_f^s(\omega))ds\Big) = 1\,.$$

By the Chebyshev inequality

$$\mu(\{\omega : Q(t,\omega) \leq -\varepsilon\sqrt{t}\}) \leq \exp(-\beta f\varepsilon\sqrt{t})\,. \tag{3.2.4}$$

On the other hand, by the central limit theorem and the first assertion

$$\begin{aligned}&\mu(\{\omega : Q(t,\omega) - td(f) \leq \\ &\leq -\varepsilon\sqrt{t}\}) \to \frac{1}{\sqrt{2\pi}\sigma(f)}\int_\varepsilon^{+\infty}\exp\Big(-\frac{s^2}{2\sigma^2(f)}\Big)ds\,.\end{aligned} \tag{3.2.5}$$

The right-hand side of (3.2.5) is positive, and the right-hand side of (3.2.4) tends to 0 as $t \to \infty$, so that they are incompatible if $f > 0$ and $d(f) \leq 0$. ∎

Let us summarize the results we have proved concerning the displacement $Q_f(t,\omega)$ and the initial distribution $\mu(d\omega)$.

I. Existence of the drift. The limit

$$d(f) \equiv \lim_{t\to\infty}\frac{Q_f(t,\omega)}{t} = \nu_f(V)$$

exists, is finite, and does not depend on ω for μ-a.a. $\omega \in \Omega$. Moreover $fd(f) > 0$ for $f \neq 0$.

II. Diffusion. The process

$$\xi_t^\epsilon = \sqrt{\epsilon}\Big(Q_f\Big(\frac{t}{\epsilon},\, \omega\Big) - d(f)\frac{t}{\epsilon} \Big)$$

converges weakly, as $\epsilon \to 0$ in the space of continuous functions of t to the Wiener process $\mathcal{W}_{\sigma^2(f)t}$ with nondegenerate diffusion constant $\sigma^2(f) > 0$.

3.3 – Einstein relation

In this section we, assuming that the assertions of Theorem 3.1.29 hold true, prove our main result:

III. Einstein relation. The drift $d(f)$ and the diffusivity $\sigma^2(f)$ are continuous functions of f. Moreover, the Einstein relation holds, i.e.,

$$\lim_{f\to 0} \frac{d(f)}{f} = \frac{\beta}{2}\sigma^2(o)\,.$$

As above we consider $f : 0 \leq f \leq f_{\max}$.

Lemma 3.3.1. *The family of stationary measures $\nu_f\,:\, 0 \leq f \leq f_{\max}$ is continuous with respect to f in the variation norm.*

Proof. By inequality 3.1.31 of Theorem 3.1.28

$$||\pi \mathbb{P}_f^n(dY) - \nu_f(dY)|| \leq \mathsf{B}\exp(-\gamma_0 n)\,,$$

where the constants B, γ_0 do not depend on f and $||\;||$ denotes the variation norm. Thus

$$||\nu_f - \pi P_f^n||$$

converges to zero uniformly in f. It remains to show that the single terms

$$\pi P_f^n$$

are continuous with respect to f. As asserted in Lemma 3.1.4, the exact expression for the density

$$\varrho_{f,n} = \frac{\partial \pi P_f^n}{\partial \pi}$$

is given by the formula

$$\varrho_{f,n}(X) = \frac{d\pi P_f^{\tau n}}{d\pi}(X) = \int \mathcal{P}(dY) \exp\Big(\beta f \int_0^{\tau n} V(T_f^{-s}(X \cup Y^{\text{out}}))ds\Big).$$

We have to prove that $\varrho_{f,n}$ is continuous in $L_1(\pi)$. Further, ϱ_f is nothing else than the conditional expectation of the density

$$R_{f,n}(\omega) = \frac{\partial \mu(T_f^{-n\tau}(.))}{\partial \mu}(\omega),$$

given by Lemma 3.1.2, with respect to the sigma-algebra generated by $X(\omega)$. Thus it is sufficient to show that $R_{f,n}$ is continuous with respect to f in $L_1(\mu)$. We first show that it is continuous in probability. We have:

$$R_{f,n}(\omega) = \exp\Big(\beta f \int_0^{\tau n} V(T_f^{-s}(\omega))ds\Big).$$

In the previous expression we assume that ω belongs to the full μ-measure subset $\Omega_f^{(n)} \subset \Omega$ of those $\omega \in \Omega$, where the dynamics T_f^t is well defined up to time $n\tau$ and $-\tau n$ and 0 are not collision times. Set by convention

$$R_{f,n}(\omega) = 0,$$

as $\omega \notin \Omega_f^{(n)}$.

Suppose $\omega \in \Omega_{f_*}^{(n)}$. Note that if we take f in some small interval containing f_*, then the dynamics

$$T_f^t(\omega),\ t \in (-\tau n, 0]$$

is well defined and the rod will collide with the same particles as for f_* and the collision times and incoming velocities vary continuously with f. Hence the function

$$f \int_0^{\tau n} V(T_f^{-s}(\omega))ds$$

is continuous at f_* with respect to f. Thus $R_{f,n}(\omega)$ is continuous with respect to f at the point f_*, provided $\omega \in \Omega_{f_*}^{(n)}$. Consequently, $R_{f,n}$ is continuous with respect to f in μ - probability. At the same time the family of functions $R_{f,n}$ is uniformly μ - integrable:

$$\int (R_{f,n}(\omega))^{(1+\delta)} \mu(d\omega) \leq \int \exp(K_n v_n(\omega))\mu(d\omega)\,, \tag{3.3.2}$$

where $K_n = \tau n f \beta(1+\delta)$, and

$$\mathsf{v}_n(\omega) = \sup_{s\in[0,\tau n]} |V(T^s_{-f}(\omega))|\,.$$

It is easily seen (Section 1) that

$$\mu\{\omega : \mathsf{v}_n > a\} \leq c_1 n \exp(-c_2 a^2)\,,$$

where c_1, c_2 are independent of f for f in any bounded interval.

Hence the integral in (3.3.2) is uniformly bounded and $R_{f,n}$ is continuous with respect to f in the space $L_1(\mu)$. This achieves the proof that ν_f is continuous in f. ∎

Lemma 3.3.3. *Let $H_f(X)$ be a family of tempered functions, that is satisfying*

$$|H_f(X)| \leq \mathrm{Const}(1 + W(X))^p$$

and π - stochastically (i.e. in π -probability) continuous with respect to f. Then

$$\nu_f(H_f) = \int H_f(X)\, \nu_f(dX)$$

is continuous in f.

Proof. Observe that

$$\int \nu_f(dX) W^p \mathbb{I}_{\{W^p \geq N\}} \to 0$$

as $N \to \infty$ uniformly in f. This follows from the inequality

$$\int \nu_f(dX) \exp(\mathsf{b} W(X)) \leq \mathrm{Const}\,,$$

where b and Const do not depend on f. Thus it suffices to show that for each N

$$\int \nu_f(dX)H_f(X)I_{\{W^p<N\}}(X)$$

is continuous in f. Note: $H_f^{(N)}(X) = H_f(X)\mathbb{I}_{\{W^p<N\}}(X)$ is uniformly bounded and π - stochastically continuous in f. At the same time ν_f is continuous in the variation norm. Suppose that $f_j \to f_*$.

$$|\nu_{f_*}(H_{f_*}^{(N)}) - \nu_{f_j}(H_{f_j}^{(N)})| \leq \text{Const}\,||\nu_{f_*} - \nu_{f_j}|| + |\nu_{f_*}(H_{f_*}^{(N)} - H_{f_j}^{(N)})|\,.$$

The first term here converges to zero, as for the second one, we observe that since $\pi \sim \nu_{f_*}$,

$$H_{f_j}^{(N)} \to H_{f_*}^{(N)}$$

in ν_{f_*} probability as well. As $|H_{f_j}^{(N)}| \leq \text{Const}$

$$|\nu_{f_*}(H_{f_*}^{(N)} - H_{f_j}^{(N)})| \to 0\,.$$

Hence the result. $\square$

Since the function

$$\mathcal{V}_f(X) = \int_0^\tau V(T_f^s(X))ds$$

satisfies the conditions of the previous lemma, we have the following

Corollaryt 3.3.4. *The drift*

$$d(f) = \frac{1}{\tau}\nu_f(\mathcal{V}_f)$$

is continuous in f.

Lemma 3.3.5. *The limit variance*

$$\sigma^2(f)=\frac{1}{\tau}\Big(\nu(\mathcal{V}_f-\tau d(f))^2+2\sum_{n=1}^{\infty}\int\nu(dX)(\mathcal{V}_f-\tau d(f))(X)P_f^n(X,\mathcal{V}_f-\tau d(f))\Big)$$

is a continuous function of f.

Proof. We know that the sum above converges uniformly in f. It remains to show that each term is continuous. In view of the previous lemma we have just to show that $P^n_f(X, \mathcal{V}_f)$ is π stochastically continuous with respect to f. (It is evidently a tempered function of X). But

$$P^n_f(X, \mathcal{V}_f) = \mathcal{P}(dY)\Big(\int_{n\tau}^{(n+1)\tau} V(T^s_f(X \cup Y))ds\Big) .$$

This representation implies the continuity with respect to f. To be precise, fix f_*. The family of functions

$$\int_0^{n\tau} V(T^s_f(X \cup Y))ds$$

is $\mathcal{P}$ stochastically continuous at f_* with respect to f for π a.e. X. On the other hand this family is $\mathcal{P}$ uniformly integrable:

$$\mathcal{P}(dY)\Big(\int_{n\tau}^{(n+1)\tau} V(T^s_f(X \cup Y))ds\Big)^2 = P^n_f(X, \mathcal{V}^2_f) \le \mathrm{Const}(1+\log(W(X))^2 e^{-n}).$$

Hence $P^n_f(X, \mathcal{V}_f)$ is continuous at f_* for π a.e. X (may be depending on f_*). This completes the proof. ∎

Consider the characteristic functions:

$$\Phi_{n,f}(w) = \mu\Big(\exp\Big(\frac{w}{\sqrt{n}}\int_0^{\tau n}(V(T^s_f) - d(f)\Big)ds\Big) .$$

We know that

$$|\Phi_{n,f}(w)| \le \mathsf{L}$$

as $|\Re(w)| \le \varrho_0$. This family of analytic functions is tight. Combining results of Theorem 3.1.29 and the continuity of $\sigma^2(f)$ we conclude that for any sequence $f_n : f_n \to 0$

$$\Phi_{f_n,n}(w)$$

converges as $n \to \infty$ uniformly on compact sets to

$$\exp\Big(\frac{\sigma^2(0)\tau w^2}{2}\Big) .$$

Choose a sequence $\{f_n\}$ satisfying:

$$f_n = F_n/\sqrt{n}; \quad 0 < a < F_n < b\,,$$

where a, b are (small) fixed constants. As $\Phi_{f_n,n}(w)$ converges uniformly in w from a compact subset of Γ we may take $w_n = -F_n\beta$ and get:

$$\Big|\mu\Big(\exp\Big(\frac{w_n}{\sqrt{n}}\int_0^{\tau n}(V(T^s_{f_n}) - d(f_n))ds\Big)\Big) - \exp\Big(\frac{\sigma^2(0)\tau w_n^2}{2}\Big)\Big| \to 0\,.$$

That is,

$$\Big|\mu\Big(\exp\Big(-\beta f_n\int_0^{\tau n}(V(T^s_{f_n}) - d(f_n))ds\Big)\Big) - \exp\Big(\frac{\sigma^2(0)\tau(\beta F_n)^2}{2}\Big)\Big| \to 0\,.$$

At the same time summation rule (3.1.3) says that:

$$\mu\Big(\exp\Big(-\beta f_n\int_0^{\tau n}V(T^s_{f_n})ds - d(f_n)\Big)\Big) = \exp(\beta\tau f_n n d(f_n)) = \\ = \exp\Big(\beta F_n^2\tau\frac{d(f_n)}{f_n}\Big)\,.$$

Hence

$$\Big|\exp\Big(\beta F_n^2\tau\frac{d(f_n)}{f_n}\Big) - \exp\Big(\frac{\sigma^2(0)\tau(\beta F_n)^2}{2}\Big)\Big| \to 0\,.$$

This implies:

$$F_n^2\Big(\frac{d(f_n)}{f_n} - \beta\frac{\sigma^2(0)}{2}\Big) \to 0\,.$$

Since $F_n > a > 0$, we may conclude that

$$\frac{d(f_n)}{f_n} \to \beta\frac{\sigma^2(0)}{2}\,,$$

provided

$$a < f_n\sqrt{n} < b\,. \tag{3.3.6}$$

To avoid the restriction (3.3.6) notice that for *any* sequence $\{f_n\} : f_n > 0$, $f_n \to 0$ one may construct a sequence $\{\hat{f}_n\}$ satisfying (3.3.6) and such that

$$\mathrm{card}(\{\hat{f}_n\} \cap \{f_n\}) = \infty\,.$$

This is because the intervals

$$\Big(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}}\Big)$$

cover without leaving gaps an interval $(0,r)$ for some $r>0$. Thus

$$\lim_{f\to 0}\frac{d(f)}{f}=\frac{\beta}{2}\sigma^2(0)$$

as claimed. The Einstein relation is proved.

3.4 – Markov chains satisfying relaxation conditions

In this section we prove Theorem 3.1.29 which is a corollary of more general assertions we consider. We think that the techniques developed here is of independent interest.

Let $(\mathcal{X},\ \mathcal{B})$ be a separable measurable space. That is, the σ- algebra $\mathcal{B}$ is generated by a countable family of subsets. For a Markov chain on $(\mathcal{X},\ \mathcal{B})$ we denote by $P(x,dy)$ the corresponding transition probability. $X=(X_j,\ j=0,1,)\dots$ denotes a trajectory (path) of the chain. Let F be a measurable function defined on the space of trajectories:

$$F:\,(\mathcal{X}^\infty,\mathcal{B}^\infty)\,\to\,\mathbb{R}^1\,.$$

By $\mathbb{E}^n_x(F)$ we denote the n-shifted expectation:

$$\mathbb{E}^n_x(F)=\mathbb{E}_x(\mathsf{T}^n\circ F);\ \mathsf{T}^n\circ F(X)=F(X_n,X_{n+1},\dots)\,.$$

In the same sense we understand the notation P^n_x. $\mathcal{M}_1(\mathcal{X})$ denotes the space of probability measures on $\mathcal{X}$.

Suppose that the following objects are given:

- A positive valued, $\mathcal{B}$ measurable function $W:\mathcal{X}\to\mathbb{R}_+$;
- A probability measure π satisfying

$$\int_{\mathcal{X}}\exp(\mathsf{b}W^2(x))\,\pi(dx)<\infty$$

for an appropriate positive constant b;

- A mapping $\mathbb{R}_+ \to \mathcal{M}_1(\mathcal{X})$ which corresponds to $U \in \mathbb{R}_+$ a probability measure λ_U absolutely continuous with respect to π;
- A real valued function θ defined on $\mathbf{R}_+$ such that $0 < \theta(U) < 1$ for any $U \in \mathbb{R}_+$;
- An (at least) exponentially increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. (The reader may have in mind $\phi(t) = \exp(\mathsf{k}t), \mathsf{k} > 0$);
- Positive constants: U_0, c, a, u_0;
- An integer number $m_0 \geq 1$.

Introduce the class of Markov chains on $(\mathcal{X},\ \mathcal{B})$ with transition probabilities P satisfying the following *relaxation conditions*:

(0) P is well defined and π - irreducible on a measurable absorbing set $\mathcal{X}_P \subset \mathcal{X}$ such that $\pi(\mathcal{X}_P)$.

(i) $\pi \sim \pi P$;

(ii) $\pi \preceq P^n(x, .)$ for any $x \in \mathcal{X}_P$ and $n \geq m_0$;

(iii) Set $\mathbf{W}(X) = \max\{W(X_j) : 0 \leq j < m_0\}$. If $U \geq U_0$ and $W(x) \leq \phi(U)$ then:

$$P_x^{m_0}(\{\mathbf{W} > U\}) < \exp(-\mathsf{a}U)\,;$$

(iv)

$$P_x(\{\mathbf{W} > U\}) \leq \exp(-\mathsf{b}U^2)\,,$$

provided $U > W(x) + \mathsf{u}_0$;

(v) Set $A_U = \{x : W(x) \leq U\}$. Then the following inequality holds:

$$P^{m_0}(x, dy) \geq \theta(U)\mathbb{I}_{A_U}(x)\lambda_U(dy)\,.$$

The class of Markov chains introduced above will be denoted by

$$\wp = \wp(\pi, W, m_0, \dots)\,.$$

We shall write $P \in \wp$ indicating that the Markov chain with the transition probability P is contained in the mentioned class. The objects $\pi, W, \dots$ defining the class $\wp$ will be called *parameters of* $\wp$.

Let us briefly comment conditions (i)-(v).

If no ambiguity arises, we will identify $\mathcal{X}_P$ and $\mathcal{X}$ as long as we deal with a given Markov chain. Since only sufficiently large values of W are essential, we may and will assume that $W(x) > W_0 > 0$, where W_0 may be chosen large enough. More precisely, the conditions above hold true if W is replaced by $\max\{W, U_0\}$.

Without any loss of generality we may assume that $\lambda_U(A_{\overline{U}}) = 1$ for $\overline{U}(U)$ large enough. Otherwise we could replace λ_U by

$$\lambda_U I_{A_{\overline{U}}} / \lambda_U(A_{\overline{U}})$$

and $\theta(U)$ by $\theta(U)\lambda_U(A_{\overline{U}})$.

Given a Markov chain X, we introduce the *coupled chain* $\overline{X}$ with the state space $\overline{\mathcal{X}} = \mathcal{X} \times \mathcal{X}$ assuming that the projections are independent Markov chains, coinciding (in the sense of distribution) with X. The transition probability $\overline{P} = P \times P$. If $P \in \wp$ then, obviously,

$$\overline{P} \in \overline{\wp} = \wp(\overline{\pi}, \overline{W}, m_0, \dots)\,,$$

where, for instance,

$$\overline{W}(x^{(1)}, x^{(2)}) = \max\{W(x^{(1)}), W(x^{(2)})\},\ \overline{\pi} = \pi \times \pi\,.$$

Note that the m_0 - iterated chain $\widehat{X} = (X_{m_0 j}; j = 0, 1, \dots)$ with the transition probability $\widehat{P} = P^{m_0}$ belongs to $\wp$ with $m_0 = 1$.

Our main goal is to estimate the rate of convergence of $P^n; n \to \infty$ to an equilibrium measure ν_P *uniformly in* $P \in \wp$. It will be often needed to emphasize that certain constants $\mathsf{C}, \alpha, \gamma \dots$ depend on the parameters of $\wp$ rather than on a concrete choice of $P \in \wp$. We shall indicate this writing $\mathsf{C}(\wp)$, $\mathsf{C} = \mathsf{C}(\wp), \alpha(\wp), \dots$

3.4.1 – Preliminary estimates

Let ξ be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and taking values in $\mathbb{R}_+$. Suppose that the following inequality holds:

$$\mathsf{P}\{\omega \in \Omega : \xi(\omega) \geq t\} \leq f(t) \leq 1, \text{ provided } t \geq a \geq 0\,. \tag{3.4.1}$$

The function f is assumed to be decreasing and vanishing in infinity. Notice that (3.4.1) is valid only for $t \geq a$, concerning $t < a$ no assumptions are made. Furthermore, let g be a continuous, positive valued, strictly increasing function on $\mathbb{R}_+$ satisfying

$$\int_0^\infty f(g^{(-1)}(s))\,ds < \infty\,, \tag{3.4.2}$$

where $g^{(-1)}$ denotes the inverse function. We are interested in estimating

$$\mathsf{E}(g(\xi)) = \int g(\xi(\omega))\mathsf{P}(d\omega)\,.$$

Clearly

$$\begin{aligned}\mathsf{E}(g(\xi)) &= \int_0^\infty \mathsf{P}\{g(\xi) \geq s\}ds = \\ &= \int_0^\infty \mathsf{P}\{\xi \geq g^{(-1)}(s)\}ds = \\ &= \int_0^{g(a)} \mathsf{P}\{\xi \geq g^{(-1)}(s)\}ds + \int_{g(a)}^\infty \mathsf{P}\{\xi \geq g^{(-1)}s\}ds \leq \\ &\leq g(a) + \int_{g(a)}^\infty f(g^{(-1)}(s))\,ds\,.\end{aligned}$$

Summarizing,

$$\mathsf{E}(g(\xi)) \leq g(a) + \int_{g(a)}^\infty f(g^{(-1)}(s))\,ds\,. \tag{3.4.3}$$

This implies

$$\mathsf{E}(g(\xi)) \leq g(a) + C(f,g)\,, \tag{3.4.4}$$

where

$$C(f,g) = \int_0^\infty f(g^{(-1)}(s))\,ds\,.$$

In the particular case of (3.4.3), where

$$g(s) = f^{-\gamma}(s) :\ 0 < \gamma < 1\,,$$

(3.4.3) becomes:

$$\mathsf{E}(g(\xi)) \leq g(a) + \int_{g(a)}^\infty \frac{ds}{s^{1/\gamma}} = g(a)\Big(1 + \frac{\gamma}{1-\gamma} f(a)\Big)\,.$$

That is,

$$\mathsf{E}(g(\xi)) \leq g(a)\Big(1 + \frac{\gamma}{1-\gamma} f(a)\Big) \leq g(a)\Big(1 + \frac{\gamma}{1-\gamma}\Big)\,. \tag{3.4.5}$$

It is sufficient to consider $\gamma : \frac{1}{2} > \gamma > 0$ and thus to replace in (3.4.5)

$$\Big(1+\frac{\gamma}{1-\gamma}\Big)$$

by $\exp(2\gamma)$. Finally:

$$\mathsf{E}(g(\xi)) \le g(a)e^{2\gamma} = \Big(\frac{e^2}{f(a)}\Big)^{\gamma}; \quad \frac{1}{2} > \gamma > 0\,. \tag{3.4.6}$$

Let us apply the above assertions to the relaxation conditions introduced above. Consider first condition (iii). The role of ξ is played by $\mathbf{W}$.

$$a = \max\{U_0, \phi^{(-1)}(W(x))\} = \chi(W(x))\,,$$

where we set for brevity $\chi(t) = \max\{U_0, \phi^{(-1)}(t)\}$. Evidently $f(s) = \exp(-\mathsf{a}s)$. We have:

$$\begin{gathered} E_x^{m_0}(g(\mathbf{W})) \le g(\chi(W(x))) + \mathsf{C}(g)\,. \\ \mathsf{C}(g) = \int_0^\infty \exp(-\mathsf{a}(g^{(-1)}(s)))\,ds\,. \end{gathered} \tag{3.4.7}$$

We may choose g. Take first

$$g(s) = f^{-\gamma}(s); 0 < \gamma < \frac{1}{2}\,.$$

That is,

$$g(s) = \exp(\gamma \mathsf{a} s)\,.$$

(3.4.6) gives:

$$\mathbb{E}_x^{m_0}(\exp(\gamma\mathsf{a}\mathbf{W})) \le \exp(\gamma\mathsf{a}\chi(W(x)))e^{2\gamma} \le \tag{3.4.8}$$

$$\le (W(x))^{\frac{\gamma\mathsf{a}}{\mathsf{k}}}\exp(\gamma(2+\mathsf{a}U_0))\,. \tag{3.4.9}$$

In the sequel we will make use of the following estimate which easily follows from the previous one:

$$\mathbb{E}_x^{m_0}(\exp(\alpha\mathbf{W})) \le (\exp(r\alpha W(x)))e^{\alpha\kappa}\,,$$

where $\kappa(\wp) > 0$ and $r(\wp) < 1$ are some suitable constants. α is assumed here to be sufficiently small, namely:

$$\alpha < \frac{\mathsf{a}}{2}\,.$$

Iterating we obtain:

$$\text{(3.4.10)}\quad \mathbb{E}_x^{m_0 n}(\exp(\alpha\mathbf{W})) \le \exp(r^n \alpha W(x)) \exp\Big(\frac{\kappa\alpha(1-r^n)}{1-r}\Big) \le$$

$$\text{(3.4.11)}\quad \le \exp(r^n \alpha W(x)) \exp\Big(\frac{\kappa\alpha}{1-r}\Big)\,.$$

By the same manner we may estimate the moments of $\mathbf{W}$. For this purpose set $g(s) = s^p$, $p \ge 1$. We get immediately:

$$\mathbb{E}_x^{m_0}(\mathbf{W}^p) \le (\chi(W(x))^p + \int_0^\infty e^{-\mathsf{a}\sqrt{[p]s}} ds\,.$$

This obviously implies:

$$\text{(3.4.12)}\quad \mathbb{E}_x^{m_0}(\mathbf{W}^p) \le \psi_p(W(x))\,,$$

where $\psi_p(t) = \log(\mathsf{c}_p + t)^p$. The constant c_p is chosen so that the function ψ_p is concave and has a unique stationary point. Iterating we have:

$$\text{(3.4.13)}\quad \mathbb{E}_x^{m_0 n}(\mathbf{W}^p) \le \psi_p^{(n)}(W(x)),\ n \ge 1$$

where $\psi_p^{(n)}$ denotes the n-th iteration of ψ_p. We may and will assume that these iterations converge superexponentially to the unique stable stationary point. Combining (3.4.13) and (3.4.9) we may obtain that for $n \ge 2$

$$\text{(3.4.14)}\quad \mathbb{E}_x^{m_0 n}(\exp(\alpha\mathbf{W})) \le (\psi_1^{(n-1)}(W(x)))^{\frac{\alpha}{\mathsf{k}}} \exp\Big(\alpha\Big(\frac{2}{\mathsf{a}} + U_0\Big)\Big)$$

Finally consider condition (iv). The same method gives:

$$\mathbb{E}_x(g(\mathbf{W})) \le g(W(x) + \mathsf{u}_0) + \mathrm{Const}(g)\,.$$

In particular,

$$\mathbb{E}_x(\exp(\alpha \mathbf{W}^2)) \leq \mathrm{Const}^\alpha \exp(\alpha W(x)^2)\,,$$

and analogously:

$$\mathbb{E}_x(\exp(\alpha \mathbf{W})) \leq \mathrm{Const}^\alpha \exp(\alpha W(x))\,.$$

Simplifying notations we summarize:

LEMMA 3.4.15. *There exist constants* $\alpha_{\max}(\wp) > 0, \kappa(\wp) > 0, \delta(\wp) > 0$ *and* $r(\wp) : 1 > r > 0$ *such that for any* $\alpha : \alpha_{\max} > \alpha > 0$ *and for any Markov chain in* $\wp$*:*

(1):

$$\mathbb{E}_x^{nm_0}(\mathbf{W}^p)) \leq \psi_p^{(n)}(W(x)),\ n \geq 1\,,$$

where $\psi_p(s) = \log(\mathsf{c}_p + t)^p$ *is a concave function having a unique stable stationary point.* c_p *depends on the parameters of* $\wp$ *only;*

(2):

$$\mathbb{E}_x^{m_0 n}(\exp(\alpha \mathbf{W})) \leq (\,\psi_1^{(n-1)}(W(x))\,)^{\delta\alpha} \exp(\kappa\alpha),\ \ n \geq 2\,;$$

(3):

$$\mathbb{E}_x^{m_0 n}(\exp(\alpha \mathbf{W})) \leq \exp(r^n \alpha W(x)) \exp(\kappa\alpha),\ \ n \geq 1\,;$$

(4):

$$\mathbb{E}_x(\exp(\alpha \mathbf{W}^2)) \leq e^{\kappa\alpha} \exp(\alpha W(x)^2)\,,$$
$$\mathbb{E}_x(\exp(\alpha \mathbf{W})) \leq e^{\kappa\alpha} \exp(\alpha W(x))\,.$$

LEMMA 3.4.16. *There exist* $\kappa_5 = \kappa_5(\wp) > 0$ *and* $\alpha_5(\wp) > 0$ *such that*

$$\mathbb{E}_x\Big(\exp\Big(\alpha \sum_{j=0}^{N-1} W(X_j)\Big)\Big) < \exp(\kappa_5 N\alpha) \exp\Big(\frac{\alpha m_0}{1-r} W(x)\Big)$$

for $0 < \alpha \leq \alpha_5$*, arbitrary* $N = 1, 2, \ldots$ *and uniformly in* $P \in \wp$*. Constant* $0 < r < 1$ *is here the same as in Lemma* 3.4.15

Proof. Set

$$J_k(N) = \{j :\ 0 \le j < N, j = k \bmod m_0\}, \quad k = 0, 1 \ldots, m_0 - 1\,;$$
$$\mathrm{card}(J_k(N)) = N_k\,;$$
$$\Sigma_k^N(X) = \sum_{j \in J_k(N)} W(X_j)\,.$$

Since

$$\mathbb{E}_x\Big(\exp\Big(\alpha \sum_{j=0}^{N} W(X_j)\Big)\Big) \le \prod_{k=0}^{m_0-1} (\mathbb{E}_x(\exp(\alpha m_0 \Sigma_k^N(X))))^{\frac{1}{m_0}}\,,$$

it is sufficient to estimate each term

$$\mathcal{E}_k(x, N) = \mathbb{E}_x(\exp(\alpha m_0 \Sigma_k^n(X)))$$

separately. Let $\alpha_{\max}$ be the same as in Lemma 3.4.15. Choose then α satisfying

$$\frac{m_0 \alpha}{1 - r} < \alpha_{\max}\,.$$

Fix k and set for simplicity of notations

$$Y_n = X_{k+(n-1)m_0};\ n = 1, \ldots, N_k$$

We may apply iterated conditioning and note that by Lemma 3.4.15

$$\mathbb{E}_{Y_{N_k-1}}(\exp(\alpha m_0 W(Y_{N_k}))) \le e^{\alpha m_0 \kappa} \exp(r \alpha m_0 W(Y_{N_k-1}))\,.$$

Thus

$$\mathcal{E}_k(x, N) \le e^{\alpha m_0 \kappa} \mathbb{E}_x\Big(\exp\Big(\alpha m_0\Big(\sum_{j=1}^{N_k-2} W(Y_j)\Big) + \\ + \alpha m_0 (1 + r) W(Y_{N_k-1})\Big)\Big)\,.$$

As $\alpha m_0(1 + r) < \alpha_{\max}$, we may apply Lemma 3.4.15 to

$$\mathbb{E}_{Y_{N_k-2}}(\exp(\alpha m_0 (1 + r) W(Y_{N_k-1})))$$

and obtain the following estimate from above:

$$\exp(\alpha m_0 \kappa(1+r)) \exp(m_0(r+r^2)\alpha W(Y_{N_k-2}))\,.$$

Hence

$$\begin{aligned}\mathcal{E}_k(x,N) &\le \exp(\alpha m_0 \kappa(2+r))\times \\ &\times \mathbb{E}_x\Big[\exp\Big(m_0\alpha\Big(\sum_{j=1}^{N_k-3} W(Y_j)\Big) + \alpha m_0(1+r+r^2)W(Y_{N_k-2})\Big)\Big]\,.\end{aligned}$$

We may continue iteratively and obtain:

$$\mathcal{E}_k(x,N) \le \exp(\alpha c_k(N))\mathbb{E}_x\Big(\exp\Big(\frac{\alpha m_0}{1-r}(W(X_k))\Big)\Big)\,,$$

where

$$c_k(N) = m_0\kappa \sum_{j=0}^{N_k-1} r^j(N_k - j) \le \mathrm{Const}N\,.$$

In view of Lemma 3.4.15 the result follows. ∎

3.4.2 – Stationary measures ν_P

LEMMA 3.4.17.

1. *For any chain $P \in \wp$ there exists an absorbing set $\mathcal{H}_P \subset \mathcal{X}$ such that the chain restricted to $\mathcal{H}_P$ is Harris recurrent. There exists a unique invariant probability measure ν_P. ν_P is equivalent to the measure π and $\nu_P(\mathcal{H}_P) = 1$.*
2. *There exists a positive constant $\mathsf{b}_1(\wp)$ such that for sufficiently small $\alpha < \alpha_{\max}(\wp)$ and any Markov chain in $\wp$ holds:*

$$\mathbb{E}_{\nu_P}(\exp(\alpha\mathbf{W})) \le \exp(\mathsf{b}_1\alpha)\,.$$

3. $$\lim_{n\to\infty} ||P^n(x,dy) - \nu_P(dy)|| = 0$$

for any $x \in \mathcal{H}_P$.

Proof. Obviously the chain P is π-irreducible. Consider the potential kernel

$$G(x,\cdot)=\sum_{j=0}^{\infty}P^{j}(x,\cdot)\,.$$

Inequality (3.4.11) implies that

$$\liminf_{n\to\infty}P_x^n(A_U)\geq 1-\mathsf{C}(\wp)e^{-\alpha U}>0$$

for U sufficiently large. Thus

$$G(x,A_U)=\infty \tag{3.4.18}$$

for any x. As A_U is a small set, (3.4.18) implies (see [19]) that the chain is recurrent. Hence one can find an absorbing set $\mathcal{H}_P$ and a P-invariant σ-finite measure ν_P, supported on $\mathcal{H}_P$, such that π is absolutely continuous with respect to ν_P, and $\nu_P(B)>0$ implies that for $x\in\mathcal{H}_P$

$$P_x\Big(\sum_{j=0}^{\infty}\mathrm{I\!I}_B(X_j)=\infty\Big)=1\,.$$

Since $\pi P\sim\pi$, the measure π is a maximal irreducibility measure. The above equality shows that the chain P restricted to the absorbing set $\mathcal{H}_P$ is ν_P irreducible. Hence $\nu_P\sim\pi$. According to standard terminology P is a Harris recurrent chain. Relaxation condition (ii) implies that P is aperiodic. We next show that ν_P is a finite measure. Since π is a maximal irreducibility measure, $\lambda_U(dx)$ has to be absolutely continuous with respect to π:

$$\lambda_U(dx)=\varrho(x)\pi(dx)\,.$$

We can find a positive γ and a set

$$D\subset\{x:\varrho(x)>\gamma\}$$

such that $\pi(D)>0$ and $\nu_P(D)<\infty$. For any choice of x and $n>n_0(x)$ large enough we have:

$$\begin{aligned}P^{nm_0}(x,D)\geq\int_{A_U}&P^{(n-1)m_0}(x,dy)P^{m_0}(y,D)\geq\\ &\geq\gamma\pi(D)P^{(n-1)m_0}(x,D)>\overline{\gamma}>0\,.\end{aligned}$$

Thus

$$\nu_P(D) = \int_{\mathcal{X}} \nu_P(dx) P^{nm_0}(x, D) \geq \int_{\mathcal{X}} \nu_P(dx) \liminf_n P^{nm_0}(x, D) \geq \overline{\gamma} \nu_P(\mathcal{X}) .$$

Since $\nu_p(D) < \infty$, we conclude that

$$\nu_P(\mathcal{X}) < \infty .$$

From now on we assume that ν_P is normalized to a probability measure. As the chain is Harris recurrent and aperiodic, we have by Orey's theorem that for $x \in \mathcal{H}_P$

$$\lim_{n\to\infty} ||P(x, \cdot) - \nu_P(\cdot)|| = 0 .$$

Uniqueness of the invariant measure ν_P is a standard fact. The inequality

$$\mathbb{E}_{\nu_P}(\exp(\alpha \mathbf{W})) \leq \exp(\mathsf{b}_1 \alpha)$$

follows from (3.4.11) since

$$\mathbb{E}_x^{nm_0}(\exp(\alpha \mathbf{W})) \leq \exp(\mathsf{b}_1 \alpha) ,$$

provided $n > n_0(x)$. More precisely, for any $L > 0$

$$\nu_P(\exp(\alpha(\mathbf{W} \wedge L))) = \lim_{n\to\infty} \mathbb{E}_x^{nm_0}(\exp(\alpha(\mathbf{W} \wedge L))) \leq \exp(\mathsf{b}_1 \alpha) .$$

As L is arbitrary, the result follows. ∎

3.4.3 – Recurrence time τ_{A_U}

For a measurable set $B \subset \mathcal{X}$ we define

$$\tau_B(X) = \min\{j \geq 1 : X_{jm_0} \in B\} .$$

Set $\tau_B^{(1)}(X) = \tau_B(X)$ and iteratively:

$$\tau_B^{(i)}(X) = \min\{j > \tau_B^{(i-1)}(X) : X_{jm_0} \in B\} .$$

Set

$$\mathcal{K}_U(x) = \min\{n : W(x) \leq \phi^{(n)}(U)\} ,$$

where $\phi^{(n)} = \phi^{(n-1)} \circ \phi$, $n = 1, \dots$. For U fixed $\mathcal{K}_U(x)$ is extremely slowly increasing with respect to $W(x)$, an appropriate upper bound is given, for instance, by $\mathrm{Const}(U) \log\log(W(x))$.

LEMMA 3.4.19. *There exist positive constants* C_3 *and* b_3 *such that for* U *large enough and for all chains in* $\wp$

$$P_x(\tau_{A_U} > n + k) < \mathsf{C}_3 \exp(-\mathsf{b}_3 k U)\,,$$

provided $\mathcal{K}_U(x) < n$.

PROOF. It is sufficient to prove the assertion for $m_0 = 1$, the general case follows immediately by considering the chain P^{m_0}.

Recall that for all U large enough $\phi(U) > W(x)$ implies that

$$P(x, A_U^c) < \exp(-\mathsf{a}U)\,. \tag{3.4.20}$$

Let n and k be given. Suppose that x satisfies the condition

$$W(x) < \phi^{(n)}(U)\,.$$

Consider $P^n(x, A_U^c)$.

$$\begin{aligned} P^n(x, A_U^c) &= \int P^{n-1}(x, dy) P(y, A_U^c) = \\ &= \int_{A^c_{\phi(U)}} P^{n-1}(x, dy) P(y, A_U^c) + \int_{A_{\phi(U)}} P^{n-1}(x, dy) P(y, A_U^c)\,. \end{aligned}$$

According to (3.4.20) the second integral is not larger than

$$\exp(-\mathsf{a}U)$$

is. As for the first one, it may be estimated from above by

$$P^{n-1}(x, A^c_{\phi(U)})\,.$$

Thus:

$$P^n(x, U) \le P^{n-1}(x, A^c_{\phi(U)}) + \exp(-\mathsf{a}U)\,.$$

We may continue iteratively and obtain that

$$P^n(x, A_U^c) \le \sum_{j=0}^{n-2} \exp(-\mathsf{a}\phi^{(j)}(U)) + P(x, A^c_{\phi^{(n-1)}(U)})\,.$$

As $W(x) < \phi^{(n)}(U)$, we may apply (3.4.20) to the last term and get

$$P^n(x, U) \le \sum_{j=0}^{n-1} \exp(-\mathsf{a}\phi^{(j)}(U))\,.$$

From here on we may and will assume that the values of U under consideration are large enough for to imply

$$\phi^{(j)}(U) > (j+1)U,\ j = 0, 1, \dots.$$

Hence we may conclude that

$$P^n(x, A_U^c) \le \delta^{(n)}(U) = \frac{e^{-\mathsf{a}U}}{1 - e^{-\mathsf{a}(n-1)U}}\,.$$

Let us define

$$G_k^{(n)}(x) = \int_{A_U^c} P^n(x, dy_0) \int_{A_U^c} P(y_0, dy_1) \cdots \int_{A_U^c} P(y_{k-2}, dy_{k-1}) \int_{A_U^c} P(y_{k-1}, dy_k).$$

Clearly

$$P_x(\tau_{A_U} > n + k) \le G_k^{(n)}\,.$$

Splitting the integration in y_{k-1} over the regions $A_{\phi(U)}$ and $A^c_{\phi(U)}$ we obtain that

$$\int_{A_U^c} P(y_{k-2}, dy_{k-1}) P(y_{k-1}, A_U^c) \le \exp(-\mathsf{a}U) + P(y_{k-2}, A^c_{\phi(U)})\,.$$

Thus

$$G_k^{(n)}(x) \le \exp(-\mathsf{a}U) G_{k-1}^{(n)}(x) + \\ + \int_{A_U^c} P^n(x, dy_0) \int_{A_U^c} P(y_0, dy_1) \dots \int_{A_U^c} P(y_{k-3}, dy_{k-2}) P(y_{k-2}, A^c_{\phi(U)}).$$

Iterating we get the following recursive inequality

$$G_k^{(n)}(x) \le \sum_{s=1}^{k} \exp(-\mathsf{a}\phi^{(s-1)}(U)) G_{k-s}^{(n)}(x) + P^n(x, A_{\phi^{(k)}(U)}) , \tag{3.4.21}$$

where

$$G_o^{(n)}(x) = P^n(x, A_U^c) .$$

Since $\phi^{(k)}(U) \ge (k+1)U$, we have, setting $\epsilon(U) = \exp(-\mathsf{a}U)$:

$$G_k^{(n)}(x) \le \sum_{s=1}^{k} \epsilon^s G_{k-s}^{(n)}(x) + \delta^{(n)}(\phi^{(k)}(U)) .$$

Let us emphasize that the above inequality is similar to that arising in the renewal theory. Suppose that we have

$$G_r^{(n)}(x) < C\theta^r, \ \theta < 1 .$$

for $r = 0, 1, \ldots k-1$ and consider $G_K^{(n)}(x)$:

$$G_k^{(n)}(x) \le C \sum_{s=1}^{k} \epsilon^s \theta^{k-s} + \delta^{(n)}(\phi^{(k)}(U)) = C\theta^k \Delta_k ,$$

where

$$\Delta_k = \sum_{s=1}^{k} \left(\frac{\epsilon}{\theta}\right)^s + \frac{\delta^{(n)}(\phi^{(k)}(U))}{C\theta^k} .$$

If $\Delta_k < 1$ then we may conclude that

$$G_k^{(n)}(x) \le C\theta^k .$$

Thus the condition sufficient to apply the inductive argument for all k is

$$\sum_{s=1}^{\infty} \left(\frac{\epsilon}{\theta}\right)^s + \frac{\delta^{(n)}(\phi^{(k)}(U))}{C\theta^k} < 1, \quad k = 0, 1, \ldots$$

Suppose, we consider U large enough to imply

$$\epsilon(U) = \exp(-\mathsf{a}U) < \frac{1}{4} ,$$

and take

$$\theta(U) = 4\epsilon\,.$$

For $k = 0$ we have

$$G_o^{(n)}(x) = P^n(x, A_U) < \delta^{(n)}(U) < C = \frac{1}{1 - e^{-\mathsf{a}U_o}}\,.$$

It remains to prove that for U large enough

$$\frac{\delta^{(n)}(\phi^{(k)}(U))}{C\theta^k} < \frac{1}{2}$$

for all k, n. Taking into account the precise expressions for $\delta^{(n)}(U)$ and $\theta(U)$ we have

$$\frac{\delta^{(n)}(\phi^{(k)}(U))}{\theta^k} < \frac{\exp(-\mathsf{a}(\phi^{(k)}(U) - kU))}{4^k(1 - e^{-\mathsf{a}U_0})} \le \mathrm{Const}\exp(-\mathsf{a}U)\,. \qquad \square$$

The next is an obvious consequence of the preceding lemma.

Lemma 3.4.22. *There exists a positive constant $\kappa_4(\wp, U)$ such that for all chains in $\wp$ the following assertion holds:*

$$\mathbb{E}_x(\exp(\alpha\tau_{A_U})) < e^{\kappa_4\alpha}\exp(\alpha\mathcal{K}_U(x))\,,$$

provided $\alpha < \alpha_{\max}(\wp)$.

Corollary 3.4.23.

$$\mathbb{E}_x(\exp(\alpha\tau_{A_U}^{(n)})) < q^{\alpha n}\exp(\alpha\mathcal{K}_U(x))\,,$$

where $q = e^{\kappa_4}\sup\{\exp(\alpha\mathcal{K}_U(x)) : x \in A_U\}$.

3.4.4 – The split chain. Uniform convergence to equilibrium

Here and in the sequel we fix $U > U_0$ large enough. As U is fixed, we will often, when no ambiguity arises, omit the symbol U in the notations. Suppose that a chain $(X_n; n \geq 0)$ with a transition probability P is contained in the class $\wp$. Until otherwise stated we assume that $m_0 = 1$ (or, equivalently, consider the iterated chain $\widehat{X} = (X_{m_0 j}; j \geq 0)$). We correspond to P the *split chain* defined on the phase space $\mathcal{Z} = \mathcal{X} \times \{0, 1\}$. The points of $\mathcal{Z}$ will be denoted as $z = (x, \varepsilon)$, with $\varepsilon \in \{0, 1\}$. The construction is as follows (see [19]). Set

$$
\begin{aligned}
s(x) &= \theta(U) \mathrm{I\!I}_{A_U}(x)\,, \\
Q(x, B) &= (1 - s(x))^{-1} (P(x, B) - s(x) \lambda_U(B))\,,
\end{aligned}
$$

where $x \in \mathcal{X}, B \in \mathcal{B}$. Clearly Q is a transition probability on $\mathcal{X}$. Let a point $z_0 \in \mathcal{Z}; z_0 = (x_0, \varepsilon_0)$ be given. Define a rule of transition from z_0 to a new state z. First, we choose $x = x(z)$ according to the following distribution:

$$
\mathrm{Prob}(x \in B) = \varepsilon_0 \lambda(B) + (1 - \varepsilon_0) Q(x_0, B) = \text{ def } = G_{\varepsilon_0}(x_0, B)\,.
$$

As $x = x(z)$ is chosen, we obtain $\varepsilon = \varepsilon(z)$ by a random experiment, setting

$$
\mathrm{Prob}\ (\{\varepsilon = 1\}) = s(x)\,.
$$

Note, that $\varepsilon = 0$, whenever $x \notin A_U$. The rule described above defines the transition probability $\mathbf{P}(z_0, dz)$, which is easy to write down.

Let $Z = \{Z_0, Z_1, \ldots Z_n, \ldots\}$ denote the Markov chain, corresponding to the transition probability $\mathbf{P}(z_0, dz)$. Here $Z_n = (X_n, \varepsilon_n)$. For the conditional expectation and probability in the path space we will use the symbols $\mathbf{P}_z$ and $\mathbf{E}_z$ respectively. The pleasing property of the *split chain* defined above is that, roughly speaking, the natural projection $\mathsf{p} : \mathcal{Z} \to \mathcal{X}$ transforms Z to the original chain X. To be precise,

$$
\begin{aligned}
&\mathrm{Prob}\ (\mathsf{p}(Z_j) \in B \,|\, \mathsf{p}(Z_{j-1}) = z_0) = \mathrm{Prob}\ (\{\varepsilon(Z_{j-1}) = 1\}) G_1(\mathsf{p}(z_0), B) + \\
&\quad + \mathrm{Prob}\ (\{\varepsilon(Z_{j-1}) = 0\}) G_0(\mathsf{p}(z_0), B) = s(\mathsf{p}(z_0)) \lambda(B) + \\
&\quad + (1 - s(\mathsf{p}(z_0))) Q(\mathsf{p}(z_0), B) = P(\mathsf{p}(z_0), B)\,.
\end{aligned}
$$

In other words, with respect to the increasing family of sigma-algebras

$$\{\mathcal{F}_n\} : \mathcal{F}_n = \Sigma\{X_0, \dots, X_n\,; \varepsilon_0, \dots, \varepsilon_{n-1}\}$$

$X_j = \mathsf{p}(Z_j)$ is a Markov chain with transition probability P. The next important property of the split chain Z is the existence of a *proper atom.* Namely, consider

$$\Delta = \mathcal{X} \times \{1\}\,.$$

Clearly

$$\mathbf{P}(z_1, dz) = \mathbf{P}(z_2, dz) = \text{ def } = \mathbf{P}(\Delta, dz)$$

for any $z_1, z_2 \in \Delta$. The stopping times

$$\tau_\Delta(Z) = \tau_\Delta^{(1)}(Z),\, \tau_\Delta^{(2)}(Z), \dots\,,$$

called renewal moments for the chain Z, will be denoted by $\mathbf{t}^{(j)}; j = 1, \dots$. Instead of $\mathbf{t}^{(1)}$ we shall write $\mathbf{t}$. Let $\mathfrak{m}$ be a probability measure on $\mathcal{X}$. We correspond to m the measure $\hat{\mathfrak{m}} \in \mathcal{M}_1(\mathcal{Z})$ as follows.

$$\hat{\mathfrak{m}}(dx, d\varepsilon) = \mathfrak{m}(dx)(s(x)\delta_{\{1\}}(d\varepsilon) + (1 - s(x))\delta_{\{0\}}(d\varepsilon)\ ,.$$

If $F = F(X_0, \dots, X_n, \dots)$ is a measurable function on the path space of the chain X, then

$$\mathbf{E}_{\hat{\mathfrak{m}}}(F(\mathsf{p}(.))) = \mathbb{E}_{\mathfrak{m}}(F)\,.$$

Note that

$$\mathbf{E}_z^n(F(\mathsf{p}(.))) = G_{\varepsilon(z)}(\mathsf{p}(z), dx)\mathbb{E}_x^{n-1}(F)\,; n = 1, \dots$$

LEMMA 3.4.24. *There exist positive constants $\mathsf{C}_6(\wp)$ and $\gamma_6(\wp)$ such that the following inequality holds for all $\alpha \in [0, \alpha_6)$, $P \in \wp$ and $z = (x, \varepsilon) \in \mathcal{Z}$:*

$$\mathbf{E}_z(e^{\alpha \mathbf{t}}) \le \mathsf{C}_6^\alpha \exp(\alpha \mathcal{K}_U(x))\,.$$

Proof. Set, as above, $X = \mathsf{p}(Z)$ and consider the sequence

$$\eta_j = \varepsilon(\tau^{(j)}_{A_U}(X)); \qquad j = 1, \dots .$$

Note that η_j are independent identically distributed variables taking values in $\{0,1\}$ and $\mathbf{P}_z\{\eta_j = 1\} = \theta$. Moreover, η_n is independent of

$$\tau^{(1)}_{A_U}(X), \dots, \tau^{(n)}_{A_U}(X); \eta_1, \dots \eta_{n-1} .$$

Let

$$J = \min\{j : \eta_j = 1\} .$$

We have

$$\mathbf{t} = \tau^{(J)}_{A_U}(X) .$$

Thus

$$\mathbf{E}_z(\exp \alpha \mathbf{t}) = \sum_{j=1}^{\infty} \mathbf{E}_z(\mathbb{I}_{\{J=j\}} \exp(\alpha \tau^{(j)}_{A_U}(X))) .$$

This implies

$$\begin{aligned}\mathbf{E}_z(\exp \alpha \mathbf{t}) &\le \sum_{j=1}^{\infty} \sqrt{\mathbf{P}_z\{J = j\} \mathbf{E}_z(\exp(2\alpha \tau^{(j)}_{A_U}(X)))} = \\ &= \sum_{j=1}^{\infty} \sqrt{\theta(1-\theta)^{j-1}} \sqrt{\mathbf{E}_z(\exp(2\alpha \tau^{(j)}_{A_U}(X)))} .\end{aligned}$$

Consider

$$\mathbf{E}_z(\exp(2\alpha \tau^{(j)}_{A_U}(X))) ,$$

where $z = (x, \varepsilon)$. Suppose first that $x \in A_U$. In this case $s(x) = \theta$ and, since the averaging with respect to ε reproduces the original transition probability, we have

$$[\theta \mathbf{E}_{(x,1)} + (1-\theta) \mathbf{E}_{(x,1)}](\exp(2\alpha \tau^{(j)}_{A_U})) = \mathbb{E}_x(\exp(2\alpha \tau^{(j)}_{A_U})) .$$

Combining this with the assertion of Corollary 3.4.23 we obtain that:

$$[\theta \mathbb{E}_{(x,1)} + (1-\theta) \mathbb{E}_{(x,0)}](\exp(2\alpha \tau^{(j)}_{A_U})) < q^{2\alpha j} ,$$

provided $\alpha < \alpha_{\max}/2$ and q is a sufficiently large constant. Hence, applying the Jensen inequality, we conclude that for $\alpha < \alpha_1$ and q_1 large enough

$$\mathbf{E}_{(x,\varepsilon)}(\exp(2\alpha\tau_{A_U}^{(j)})) < q_1^{2\alpha j}\,.$$

Let now $x \in A_U^c$ and $\varepsilon = 0$. In this case

$$\mathrm{Prob}\ (X_1 \in B) = P(x, B)\,.$$

That is,

$$\mathbf{E}_{(x,\varepsilon)}(\exp(2\alpha\tau_{A_U}^{(j)})) = \mathbb{E}_x(\exp(2\alpha\tau_{A_U}^{(j)})) < q^{2\alpha j}\exp(2\alpha\mathcal{K}_U(x))\,.$$

The estimate for $z = (x, 1), x \in A_U^c$ is the same as for $x \in A_U$, since $\mathbf{P}_{(x,1)}$ does not depend on X. Finally,

$$\mathbf{E}_z(\exp(2\alpha\tau_{A_U}^{(j)})) < q^{2\alpha j}\exp(2\alpha\mathcal{K}_U(x))\,.$$

Thus

$$\mathbf{E}_z(\exp(\alpha\mathbf{t})) \le \exp(\alpha\mathcal{K}_U(x))\sum_{j=1}^{\infty}\sqrt{\theta(1-\theta)^{j-1}}q^{\alpha j}\,.$$

For $\alpha \le \alpha_6(\wp)$ small enough the sum in the r.h.s. of the preceding inequality converges. The result follows then from the Jensen inequality. $\square$

For each $P \in \wp$ we consider the product chain $\overline{Z} = (Z^{(1)}, Z^{(2)})$ consisting of two independent copies of the split chain Z.

$$\overline{\mathcal{Z}} = \mathcal{Z} \times \mathcal{Z}$$

is the phase space of the product chain and

$$\overline{\mathbf{P}} = \mathbf{P} \times \mathbf{P}$$

is the transition probability. Expectations are denoted by $\overline{\mathbf{E}}$. We also use the obvious notations $\overline{z} = (\overline{x}, \overline{\varepsilon})$ with

$$\begin{aligned}
\overline{x} &= (x^{(1)}, x^{(2)}), \overline{\varepsilon} = (\varepsilon^{(1)}, \varepsilon^{(2)})\,,\\
\overline{W}(\overline{x}) &= \max\{W(x^{(1)}), W(x^{(2)})\}\,,\\
\mathcal{K}_U(\overline{x}) &= \max\{\mathcal{K}_U(x^{(1)}), \mathcal{K}_U(x^{(2)})\}\\
\overline{\Delta} &= \Delta \times \Delta\,.
\end{aligned}$$

Consider the returning time to $\overline{\Delta}$

$$\mathfrak{t} = \min\{j \geq 1 : Z_j^{(1)} \in \Delta \text{ and } Z_j^{(2)} \in \Delta\}\,.$$

Lemma 3.4.25. *There exist positive constants* $\mathsf{C}_8(\wp), \alpha_8(\wp)$ *such that for any* $\alpha \in [0, \alpha_8)$ *and all* $P \in \wp$

$$\overline{\mathbf{E}}_{(\overline{z})}(\exp(\alpha \mathfrak{t})) \leq \mathsf{C}_8^\alpha \exp(\alpha \mathcal{K}_U(\overline{x}))\,.$$

Proof. The way of reasoning is similar to that of the preceding lemma. Consider

$$\mathbf{s}_j = \tau^{(j)}_{A_U \times A_U}(\overline{X}) = \tau^{(j)}_{A_U \times A_U}(X^{(1)}, X^{(2)})\,,$$

where as above $X^{(i)} = \mathsf{p}Z^{(i)}$. The transition probability $\overline{P} = P \times P$ of $\overline{X}$ is contained in the class $\overline{\wp}$ depending on $\wp$. In particular the Lyapunov function

$$\overline{W}(X^{(1)}, X^{(2)}) = \max\{W(X^{(1)}), W(X^{(2)})\}\,.$$

Thus by (3.4.23):

$$\mathbb{E}_{(x^{(1)}, x^{(2)})}(\exp(\alpha \mathbf{s}_j)) \leq \mathrm{Const}(\wp)^{\alpha j} \exp(\alpha \mathcal{K}_U(\overline{x}))\,.$$

Set

$$\eta_j = \varepsilon(Z^{(1)}_{\mathbf{s}_j}) \cdot \varepsilon(Z^{(2)}_{\mathbf{s}_j}); \qquad j = 1, \dots\,.$$

Note that η_j are independent identically distributed variables taking values in $\{0, 1\}$ and

$$\overline{\mathbf{P}}_{(z^{(1)}, z^{(2)})}\{\eta_j = 1\} = \theta^2\,.$$

We proceed in very much the same way as we did in the proof of (3.4.24). Set

$$J(\eta) = \min\{j : \eta_j = 1\}\,.$$

Next, note that

$$\mathfrak{t} = \mathbf{s}_J\,.$$

Thus

$$\overline{\mathbf{E}}_{\overline{z}}(\exp\gamma\mathfrak{t}) \le \sum_{j=1}^{\infty} \sqrt{\overline{\mathbf{P}}_{\overline{z}}\{J=j\}\overline{\mathbf{E}}_{\overline{z}}(\exp 2\gamma\mathbf{s}_j)}\,.$$

Repeating then arguments of Lemma 3.4.24 word to word we obtain the result. ∎

Observation. Let f be an arbitrary measurable function on $\mathcal{Z}$ satisfying

$$||f||_\infty = \sup\{|f(z)|;\ z\in Z\} \le 1\,.$$

Let $\overline{z} = (z^{(1)}, z^{(2)}) \in \overline{Z}$. We want to estimate

$$|\mathbf{E}^n_{z^{(1)}}(f) - \mathbf{E}^n_{z^{(1)}}(f)|\,.$$

For this purpose consider the product chain $\overline{Z}$. Set

$$F(\overline{z}) = f(z^{(1)}) - f(z^{(2)})\,.$$

We have:

$$\mathbf{E}^n_{z^{(1)}}(f) - \mathbf{E}^n_{z^{(2)}}(f) = \overline{\mathbf{E}}^n_{\hat{z}}(F)\,.$$

Furthermore:

$$\overline{\mathbf{E}}^n_{\overline{z}}(F) = \overline{\mathbf{E}}_{\overline{z}}(\mathbb{I}_{\{\mathfrak{t}\ge n\}}F(\overline{Z}_n)) + \overline{\mathbf{E}}_{\overline{z}}(\mathbb{I}_{\{\mathfrak{t}<n\}}F(\overline{Z}_n))\,.$$

The first term here is estimated from above by

$$\overline{\mathbf{P}}_{\overline{z}}(\{\mathfrak{t}\ge n\})||F||_\infty \le 2\,\overline{\mathbf{P}}_{\overline{z}}(\{\mathfrak{t}\ge n\})$$

The second one is equal to zero:

$$\begin{aligned}\overline{\mathbf{E}}_{\overline{z}}(\mathbb{I}_{\{n>\mathfrak{t}\}}F(\overline{Z}_n)) &= \overline{\mathbf{E}}_{\overline{z}}(\mathbb{I}_{\{n>\mathfrak{t}\}}\overline{\mathbf{E}}^{n-\mathfrak{t}}_{\overline{Z}_\mathfrak{t}}(F))) = \\ &= \overline{\mathbf{E}}_{\overline{z}}(\mathbb{I}_{\{n>\mathfrak{t}\}}\overline{\mathbf{E}}^{n-\mathfrak{t}}_{\Delta\times\Delta}(F))) = 0\,,\end{aligned}$$

since Δ is a proper atom and $\overline{\mathbf{E}}^k_{\Delta\times\Delta}(F) = 0$ for any $k\ge 1$. Hence

$$|\mathbf{E}^n_{z^{(1)}}(f) - \mathbf{E}^n_{z^{(1)}}(f)| \le 2\overline{\mathbf{P}}^n_{\overline{z}}(\{n\le\mathfrak{t}\})\,.$$

Taking supremum over f we obtain the *coupling inequality*:

$$(3.4.26)\quad ||\mathbf{P}^n(z^{(1)}, dz) - \mathbf{P}^n(z^{(2)}, dz)|| \leq 2\,\overline{\mathbf{P}}_{\overline{z}}(\{n \leq \mathfrak{t}\});\ \ \overline{z} = (z^{(1)}, z^{(2)})\,.$$

Here $||\ \ ||$ denotes the variation norm. Making use of the estimate given by the preceding lemma we obtain that for a fixed $\alpha < \alpha_8$

$$\begin{aligned}&||\mathbf{P}^n(z^{(1)}, dz) - \mathbf{P}^n(z^{(2)}, dz)|| \leq \\ &\leq 2\,\mathsf{C}_8^\alpha \exp(-\alpha n)) \exp(\alpha \mathcal{K}_U(\overline{x}))\,.\end{aligned}$$

This inequality implies that for two given initial distributions ν_1 , ν_2 on $\mathcal{Z}$ holds true:

$$||\nu_1 \mathbf{P}^n - \nu_2 \mathbf{P}^n|| \leq 2\,\mathsf{C}_8^\alpha \exp(-\alpha n)(\nu_1(H) + \nu_2(H))\,,$$

where $H(z) = \exp(\alpha \mathcal{K}_U(x))$.

We may choose $\nu_i = \hat{\delta}_{\{x_i\}}$ where $x_i \in \mathcal{X}$ and obtain that

$$\begin{aligned}&||P^n(x_1, dy) - P^n(x_2, dy)|| \leq \\ &\leq \mathrm{Const} \exp(-\alpha n)(\log(2 + W(x^{(1)})) + \log(2 + W(x^{(2)})))\,.\end{aligned}$$

Thus

$$(3.4.27)\quad \begin{aligned}&||P^n(x, \cdot) - \nu_P(\cdot)|| \leq \\ &\leq \mathrm{Const} \exp(-\alpha n)(\log(2 + W(x)) + \int \nu_P(dy) \log(2 + W(y)))\,,\end{aligned}$$

where ν_P, as above, denotes the stationary distribution of the chain with the transition probability P. Although ν_P depends on P, the integral

$$\int \nu_P(dy) \log(2 + W(y)) \leq \mathrm{Const}(\wp)$$

for some suitable $\mathrm{Const}(\wp)$ *independent* of $P \in \wp$. We have obtained the estimate (3.4.27) under the assumption $m_0 = 1$, and thus for the chain X_{jm_0}. This restriction is of no importance. Indeed, setting $L(n) = [n/m_0]$ we have in the general case:

$$||P^n(x, \cdot) - \nu_P(\cdot)|| \leq ||P^{m_0 L(n)}(x, \cdot) - \nu_P(\cdot)|| \leq \mathrm{Const} \exp(-\alpha L(n)) \ldots$$

Thus (3.4.27) holds for arbitrary m_0 (with $\alpha = \alpha/m_0$ and, may be, another suitable Const). Simplifying notations we formulate the main result of the present section towards which our efforts have been directed:

PROPOSITION 3.4.28. *There exist positive constants* $\mathsf{A}(\wp)$ *and* $\gamma_0(\wp)$ *such that the following inequality holds for any* $n = 1, 2, \ldots$ *and for any choice of* $P \in \wp$*:*

$$||P^n(x,dy) - \nu_P(dy)|| \leq \mathsf{A} \exp(-\gamma_0 n) \log(2 + W(x)) \,. \tag{3.4.29}$$

This implies, in particular, that

$$||\pi P^n - \nu_P|| \leq \mathsf{A} \exp(-\gamma_0 n) \int \pi(dx)\, \log(2 + W(x)) \,,$$

and

$$||\pi P^n - \nu_P|| \leq \mathsf{B}(\wp) \exp(-\gamma_0 n) \tag{3.4.30}$$

uniformly in $P \in \wp$*.*

3.4.5 – The Central Limit Theorem for tempered functions

A real valued function $\mathcal{V} : \mathcal{X} \to \mathbf{R}$ is said to be *tempered* if for some $p \geq 0$

$$||\mathcal{V}||_*^{(p)} = \quad \text{def} \quad = \sup\Big\{\frac{|\mathcal{V}(x)|}{(1 + W(x))^p} : x \in \mathcal{X}\Big\} < \infty \,.$$

Let $\mathsf{B}_R^{(p)}$ denote the set

$$\{\mathcal{V} : ||\mathcal{V}||_*^{(p)} \leq R\} \,.$$

Fix p and $\mathcal{V}$ such that $||\mathcal{V}||_*^{(p)} < \infty$. Let $P \in \wp$ and ν_P be the corresponding stationary measure. Clearly,

$$||(\mathcal{V} - \nu(\mathcal{V}))||_*^{(p)} \leq ||\mathcal{V}||_*^{(p)}(1 + \nu((1 + W)^p)) \leq ||\mathcal{V}||_*^{(p)} \text{Const}(\wp, p) \,.$$

Note that the r.h.s. here does not depend on $P \in \wp$. Let $\mathcal{V}$ be a tempered function, satisfying $\nu_P(\mathcal{V}) = 0$. Consider

$$\int P^n(x, dy) \mathcal{V}(y) \,,$$

where $n \geq m_0$. By linearity we may assume that $||\mathcal{V}||_*^{(p)} = 1$. Write

$$\int P^n(x,dy)\mathcal{V}(y)$$

in the following form:

$$\int_{\{|\mathcal{V}|\leq \exp(\delta n)\}} P^n(x,dy)\mathcal{V}(y) + \int_{\{|\mathcal{V}|> \exp(\delta n)\}} P^n(x,dy)\mathcal{V}(y)\,, \tag{3.4.31}$$

where

$$\frac{\gamma_0}{4} > \delta > 0\,,$$

γ_0 is the same as in Proposition 3.4.28. By Proposition 3.4.28 the first term equals

$$\int \mathrm{I\!I}_{\{|\mathcal{V}|\leq \exp(\delta n)\}}(y)\mathcal{V}(y)\nu_P(dy) + \kappa_n(x)\,,$$

where

$$|\kappa_n(x)| \leq \mathsf{A}\exp(-(\gamma_0-\delta)n)\log(2+W(x))\,.$$

At the same time

$$\begin{aligned}\int \mathrm{I\!I}_{\{|\mathcal{V}|\leq \exp(\delta n)\}}(y)\mathcal{V}(y)\nu_P(dy) = \int \mathrm{I\!I}_{\{|\mathcal{V}|> \exp(\delta n)\}}(y)\mathcal{V}(y)\nu_P(dy) \leq \\ \leq \int \mathrm{I\!I}_{\{(1+W)^p> \exp(\delta n)\}}(y)(1+W(y))^p\nu_P(dy) \leq \\ \leq \sqrt{\nu_P((1+W)^{2p})\nu_P\{y:(1+W)^p>\exp(\delta n)\}}.\end{aligned}$$

Recall that

$$\int \nu_P(dx)\exp(\mathsf{b}_* W(x)) < \mathsf{C}_*\,.$$

Thus

$$\int \nu_P(dx)\,(1+W(x))^{2p} < \mathrm{Const}(p)\,.$$

Hence $\nu_P((1+W)^{2p})\nu_P\{y:(1+W(y))^p > \exp(\delta n)\}$ may be estimated from above by

$$\mathrm{Const}(\wp,p)\exp(-2\delta n)\,.$$

Thus the first term in (3.4.31) is estimated from above by

$$\mathrm{Const}(\wp,p)\exp(-\delta n)\,.$$

Estimating the second term in (3.4.31), note that

$$\Big|\int_{\{|\mathcal{V}|>\exp(\delta n)\}} P^n(x,dy)\mathcal{V}(y)\Big| \le \int_{\{(1+W)^p>\exp(\delta n)\}} P^n(x,dy)(1+W(y))^p \le$$
$$\le \sqrt{P^n(x,\{(1+W)^p > \exp(\delta n)\})\int P^n(x,dy)(1+W(y))^{2p}}\,.$$

By Lemma 3.4.15

$$\int P^n(x,dy)(1+W(y))^{2p} \le \mathsf{C}\psi_{2p}^{(L(n))}(W(x)) \le \mathrm{Const}(p)(1+\log_+ W(x))^{2p}\,,$$

for $n \ge m_0$. This implies

$$P^n(x,\{(1+W)^p > \exp(\delta n)\} \le \exp(-2\delta n)\mathrm{Const}(p)(1+\log_+ W(x))^{2p}\,.$$

Summarizing,

$$\Big|\int_{\{|\mathcal{V}|>\exp(\delta n)\}} P^n(x,dy)\mathcal{V}(y)\Big| \le \mathrm{Const}(p)(1+\log_+ W(x))^{2p}\exp(-\delta n)\,.$$

Setting $\gamma_1 = \min(\delta, \gamma_0 - \delta)$, we may finally write

$$\Big|\int P^n(x,dy)\,\mathcal{V}(y)\Big| \le \mathrm{Const}||\mathcal{V}||_*^{(p)}(1+\log_+ W(x))^{2p}\exp(-\gamma_1 n);\ n \ge m_0$$

If $n < m_0$, then, in view of Lemma 3.4.15 (4):

$$\Big|\int P^n(x,dy)\,\mathcal{V}(y)\Big| \le \mathrm{Const}(\wp)(1+W(x))^p\,.$$

We have proved the following

LEMMA 3.4.32. *There exist positive constants $\gamma_1(\wp), \mathsf{D}(\wp, p)$ such that*

$$\Big|\int P^n(x,dy)(\mathcal{V}-\nu(\mathcal{V}))(y)\Big| \le \mathsf{D}||\mathcal{V}||_*^{(p)}(1+\log_+ W(x))^{2p}\exp(-\gamma_1 n), \tag{3.4.33}$$

provided $n \ge m_0$. If $n < m_0$, then

$$\Big|\int P^n(x,dy)\,(\mathcal{V}-\nu(\mathcal{V}))(y)\Big| \le \mathsf{D}||\mathcal{V}||_*^{(p)}(1+W(x))^p$$

We denote the r.h.s. of (3.4.33) by

$$D(x,p)||\mathcal{V}||_*^{(p)}\exp(-\gamma_1 n)\,.$$

Lemma 3.4.33 states nothing else but the exponential rate of mixing for the process $\mathcal{V}(X_n)$. The important point is that the estimate is uniform in $P\in\wp$ and $\mathcal{V}\in\mathsf{B}_R^{(p)}$.

LEMMA 3.4.34.

(i) *For a tempered function $\mathcal{V}$ and any initial distribution of X_0*

$$S_n=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(\mathcal{V}-\nu(\mathcal{V}))(X_j)$$

converges weakly to the centered Gaussian variable with the variance

$$\sigma^2(P,\mathcal{V})=\int\nu_P(dx)((\mathcal{V}-\nu(\mathcal{V}))(x))^2+ \\ +2\sum_{n=1}^{\infty}\int\nu_P(dx)(\mathcal{V}-\nu(\mathcal{V}))(x)\int P^n(x,dy)(\mathcal{V}-\nu(\mathcal{V}))(y).$$

(ii) *The sum*

$$\sum_{n=1}^{\infty}\int\nu_P(dx)(\mathcal{V}-\nu(\mathcal{V}))(x)\int P^n(x,dy)(\mathcal{V}-\nu(\mathcal{V}))(y)$$

converges absolutely and uniformly in

$$P\in\wp,\ \mathcal{V}\in\mathsf{B}_R^{(p)}\,.$$

(iii) *The sum*

$$\sum_{n=0}^{\infty}\int P^n(x,dy)(\mathcal{V}-\nu(\mathcal{V}))(y)$$

converges absolutely and uniformly in $P\in\wp$, $\mathcal{V}\in\mathsf{B}_R^{(p)}$. The function

$$\mathcal{U}(x)=\sum_{n=0}^{\infty}\int P^n(x,dy)(\mathcal{V}-\nu(\mathcal{V}))(y)$$

is correctly defined and belongs to the class of tempered functions as well:

$$|\mathcal{U}(x)| \leq \mathsf{C}_{11}\, ||\mathcal{V}||_*^{(p)}\, (1 + W(x))^p\,.$$

(iv)

$$\mathbb{E}_\pi(((\mathcal{V} - \nu(\mathcal{V}))(X_j))((\mathcal{V} - \nu(\mathcal{V}))(X_k))) \leq$$
$$\leq (||\mathcal{V}||_*^{(p)})^2 \mathrm{Const}(\wp, p) \exp(-\gamma_1 |k - j|)$$

for a suitable constant $\mathrm{Const}(\wp, p)$*.*

Proof. (i) The assertion is standard, provided the mixing is sufficiently fast. (See [16]) or any suitable textbook).

(ii) Consider

$$\int P^n(x, dy)(\mathcal{V} - \nu(\mathcal{V}))(y)\,.$$

If $n < m_0$, then

$$\Big|\int P^n(x, dy)(\mathcal{V} - \nu(\mathcal{V}))(y)\Big| \leq \mathrm{Const}||\mathcal{V}||_*^{(p)} W^p(x)\,.$$

According to 3.4.33, as $n \geq m_0$,

$$\Big|\int P^n(x, dy)(\mathcal{V} - \nu(\mathcal{V}))(y)\Big| \leq ||\mathcal{V}||_*^{(p)} D(x) \exp(-\gamma_1 n)\,.$$

Thus each term of the sum labeled by $n \geq m_0$ is bounded in absolute value by

$$\mathrm{Const}(||\mathcal{V}||_*^{(p)})^2 \exp(-\gamma_1 n) \int \nu_P(dx) W(x) D(x, p)\,.$$

As $n < m_0$ the estimate becomes

$$\mathrm{Const}(||\mathcal{V}||_*^{(p)})^2 \int \nu_P(dx)(1 + W(x))^{p+1}\,.$$

Note that the inequality

$$\int \nu_P(dx) \exp(\mathsf{b}_* W(x)) \leq \mathsf{C}_*(\wp)\,.$$

implies that each term of the sum in (ii) is estimated by

$$\mathrm{Const}(\wp, p)(||\mathcal{V}||_*^{(p)})^2 \exp(-\gamma_1 n)\,.$$

Hence the result. (iii), (iv) In view of the preceding the assertions are straightforward. ∎

The function $\mathcal{U}(x)$ satisfies

$$(\mathcal{V} - \nu(\mathcal{V}))(x) = \mathcal{U}(x) - \int P(x, dy)\mathcal{U}(y) .$$

Thus

$$\sum_{j=0}^{n-1}(\mathcal{V} - \nu(\mathcal{V}))(X_j) = \sum_{j=0}^{n-1}(\mathcal{U}(X_j) - \mathbb{E}_{X_j}(\mathcal{U})) =$$

$$= \mathcal{U}(X_0) - \mathcal{U}(X_n) + \sum_{j=1}^{n}(\mathcal{U}(X_j) - \mathbb{E}_{X_{j-1}}(\mathcal{U})) .$$

Set

$$\eta_j(X) = \eta(X_j, X_{j-1}) = \mathcal{U}(X_j) - \mathbb{E}_{X_{j-1}}(\mathcal{U}) \quad j = 1, 2, \ldots$$

and

$$\eta_j(X) = 0, \text{ for } j \leq 0 .$$

Rewrite the previous expression in the following form

$$\sum_{j=0}^{n-1}(\mathcal{V} - \nu(\mathcal{V}))(X_j) = \mathcal{U}(X_0) - \mathcal{U}(X_n) + \sum_{j=1}^{n} \eta_j(X) . \tag{3.4.35}$$

The advantage this representation has is that

$$\mathbb{E}_{X_{j-1}}(\eta_j(X)) = 0 .$$

Proposition 3.4.36. *Consider the tempered functions $\mathcal{V} \in \mathsf{B}_R^{(1)}$. There exist positive constants $\varrho_o(\wp, R), \mathsf{L}(\wp, R)$ such that*

$$\mathbb{E}_\pi \exp(\varrho S_n) \leq \mathsf{L} \tag{3.4.37}$$

for any $n \geq 0$, $P \in \wp$, $\varrho : |\varrho| \leq \varrho_0$ and $\mathcal{V} \in \mathsf{B}_R^{(1)}$. Here

$$S_n = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1}(\mathcal{V} - \nu(\mathcal{V}))(X_j) .$$

Proof.

$$\mathbb{E}_\pi \exp(\varrho S_n) = E_\pi \exp\Big(\varrho\Big(\mathcal{U}(X_0) - \mathcal{U}(X_n) + \sum_{j=1}^n \eta_j(X)\Big)\Big) \le$$
$$\le \frac{1}{3}\Big(\mathbb{E}_\pi \exp\Big(\frac{3\varrho}{\sqrt{n}} \sum_{j=1}^n \eta_j(X)\Big) + \mathbb{E}_\pi \exp\Big(\frac{3\varrho}{\sqrt{n}} \mathcal{U}(X_0)\Big) +$$
$$+ \mathbb{E}_\pi \exp\Big(-\frac{3\varrho}{\sqrt{n}} \mathcal{U}(X_n)\Big)\Big).$$

The last two terms are easy to estimate, since according to (3.1.17)

$$|\mathcal{U}(x)| \le \mathrm{Const}(\wp, R)(1 + W(x))\,.$$

We have for instance,

$$\mathbb{E}_\pi \exp\Big(\frac{3\varrho}{\sqrt{n}} \mathcal{U}(X_0)\Big) \le \mathrm{Const}(\wp)\pi\Big(\exp\Big(\frac{\mathsf{c}_{12}(\wp)|\varrho|}{\sqrt{n}} W\Big)\Big) = \mathsf{L}_1(\wp) < \infty,$$

as $|\varrho|$ is small enough.

$$\mathbb{E}_\pi \exp\Big(-\frac{3\varrho}{\sqrt{n}} \mathcal{U}(X_n)\Big) \le \mathrm{Const}(\wp)\mathbb{E}_\pi^n\Big(\exp(\frac{\mathsf{c}_{12}(\wp)|\varrho|}{\sqrt{n}} W\Big) \le$$
$$\le \mathbb{E}_\pi^n\Big(\exp\Big(\frac{\mathsf{c}_{12}(\wp)\varrho}{\sqrt{n}} W\Big)\Big) \le \mathsf{L}_2(\wp) < \infty\,.$$

The main point is to estimate

$$\mathbb{E}_\pi\Big(\exp\Big(\frac{3\varrho}{\sqrt{n}} \sum_{j=1}^n \eta_j(X)\Big)\Big)\,.$$

Set

$$J^{(k)}(n) = \{j : 1 \le j \le n,\, j = k \mod 4m_0\};\ k = 0, \dots 4m_0 - 1\,,$$

and

$$\Sigma_n^k = \frac{1}{\sqrt{n}} \sum_{j \in J^{(k)}(n)} \eta_j(X)\,.$$

Obviously

$$\mathbb{E}_\pi\Big(\exp\Big(\frac{\alpha}{\sqrt{n}}\sum_{j=1}^{n}\eta_j(X)\Big)\Big)$$

is estimated from above by

$$\prod_{k=0}^{4m_0-1}[\mathbb{E}_\pi(\exp(4m_0\alpha\Sigma_n^k))]^{\frac{1}{4m_0}}\,.$$

It is sufficient to estimate

$$\mathbb{E}_\pi(\exp(\gamma\Sigma_n^k))$$

for each k. First consider the value $\mathsf{e}(x,\alpha)$ defined by

$$\mathsf{e}(X_0,\alpha)=\mathbb{E}_{X_0}(\exp(\alpha\eta(X_{4m_0},X_{4m_0-1})))$$

($|\alpha|$ is supposed to be small). Since

$$\mathbb{E}_{X_0}(\eta(X_{4m_0},X_{4m_0-1}))=0\,,$$

and in view of the elementary inequality $e^t\le(1+t+\frac{t^2}{2}e^{|t|})$ we may conclude that

$$\begin{aligned}&\mathsf{e}(X_0,\alpha)\le\\&\le\mathbb{E}_{X_0}\Big[1+\frac{\alpha^2\eta(X_{4m_0},X_{4m_0-1})^2}{2}\exp(\alpha|\,\eta(X_{4m_0},X_{4m_0-1})\,|)\Big]\,.\end{aligned}$$

Making use of Lemma 3.4.15 and of the inequality $|\mathcal{U}(x)|\le R\mathrm{Const}(\wp)W(x)$ we may choose positive constants $\kappa_*(\wp)$ and $r_*(\wp)<1$ such that the right hand side of the above inequality is estimated from above by

$$\exp(\alpha^2\kappa_*)\exp(r_*\alpha^2W(X_0))\tag{3.4.38}$$

for any $\alpha:|\alpha|<\alpha_{\max}(\wp)$. To see this note that

$$|\eta(X_{4m_0},X_{4m_0-1})|\le\mathrm{Const}(1+W(X_{4m_0})+W(X_{4m_0-1}))$$

and taking in account (3.4.15) it is easy to obtain that

$$\begin{aligned}&\mathbb{E}_{X_0}\Big[1+\frac{\alpha^2\eta(X_{4m_0},X_{4m_0-1})^2}{2}\exp(\alpha|\,\eta(X_{4m_0},X_{4m_0-1})\,|)\Big]\le\\&\le 1+\alpha^2\mathsf{C}(1+\sqrt{W(X_0)})\le\\&\le\exp(\alpha^2\mathsf{C}(1+\sqrt{W(X_0)}))\le\exp(\alpha^2\mathsf{C}(r_*^{-1}+r_*W(X_0)))\,,\end{aligned}$$

where $r_*\in(0,1)$. The last estimate implies the required one. We have

$$\mathsf{e}(X_0,\alpha)\le\exp(\alpha^2\kappa_*)\exp(r_*\alpha^2W(X_0))\,. \tag{3.4.39}$$

Denote by r the value $\max(r_*,r)<1$ and by κ - $\max(\kappa,\kappa_*)$, where r and κ are introduced in Lemma 3.4.15. Consider then

$$\Psi(X_0,\gamma,\delta)=\mathbb{E}_{X_0}\Big[\exp\Big(\frac{\gamma}{\sqrt{n}}\eta(X_{4m_0},X_{4m_0-1})+\frac{\delta\gamma^2}{n}W(X_{4m_0})\Big)\Big]\,.$$

Set $s_n=\frac{n}{n-1}$. Without loss of generality we may assume that $s_n\le 2$. Using then Hölder inequality we observe that:

$$\begin{aligned}&\Psi(X_0,\gamma,\delta)\le\\&\le\Big(\mathbb{E}_{X_0}\Big(\exp\Big(\frac{s_n\gamma}{\sqrt{n}}\eta(X_{4m_0},X_{4m_0-1})\Big)\Big)\Big)^{\frac{1}{s_n}}(\mathbb{E}_{X_0}\exp(\gamma^2\delta W(X_{4m_0})))^{\frac{1}{n}}\,.\end{aligned}$$

In view of (3.4.39) and Lemma 3.4.15 we have:

$$\Psi(X_0,\gamma,\delta)\le\exp\Big(\gamma^2\frac{rs_n+\delta r^4}{n}W(X_0)\Big)\exp\Big(\gamma^2\frac{\kappa\delta+\kappa s_n}{n}\Big)\,.$$

This implies:

$$\Psi(X_0,\gamma,\delta)\le\exp\Big(\frac{\gamma^2\hat\delta}{n}W(X_0)\Big)\exp\Big(\frac{\kappa\gamma^2(2+\delta)}{n}\Big)\,, \tag{3.4.40}$$

where

$$\hat\delta=h(\delta)=2r+\delta r^4\,.$$

Return to the proof of the proposition.

$$\mathbb{E}_\pi(\exp(\gamma\Sigma_n^k))=\mathbb{E}_\pi\Big(\exp\Big(\frac{\gamma}{\sqrt{n}}\eta_k(X)\Big)\mathbb{E}_{X_k}\Big[\exp\Big(\frac{\gamma}{\sqrt{n}}\sum_{j=1}^{L_k(n)}\eta_{k+4m_0j}\Big)\Big]\Big)\,.$$

Consider

$$\mathcal{E}_{n,k}(X_k) = \mathbb{E}_{X_k}\Big(\exp\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{L_k(n)}\eta_{k+4m_0 j}\Big)\Big)$$

Applying the iterative conditioning and (3.4.39) we obtain:

$$\mathcal{E}_{n,k}(X_k) \leq E_{X_k}\Big(\exp\Big(\frac{\gamma}{\sqrt{n}}\sum_{j=1}^{L_k(n)-2}\eta_{k+4m_0 j}\Big)\Psi(X_{k+4m_0(L_k(n)-2)},\gamma,\delta_0)\Big)\times$$
$$\times\exp(\kappa\gamma^2/n)\,,$$

where $\delta_0 = r$. Applying (3.4.40) and conditional expectation iteratively we obtain:

$$\mathcal{E}_{n,k}(X_k) \leq \exp\Big(\frac{\gamma^2 h^{(L_k(n)-2)}(\delta_0)}{n}W(X_k)\Big)\exp\Big(\gamma^2\kappa\Big(2+\frac{1}{n}\sum_{j=1}^{L_k(n)-2}h^{(j)}(\delta_0)\Big)\Big).$$

Elementary properties of the function $h(t) = 2r + tr^4$ imply:

$$\mathcal{E}_{n,k}(X_k) \leq \exp(\overline{\delta}(\wp)\gamma^2 W(X_k))\exp(\gamma^2\overline{\kappa}(\wp))\,,$$

where

$$\overline{\delta}(\wp)) = \sup\{h^{(n)}(r), n = 0,1,\dots\} < \infty$$
$$\overline{\kappa}(\wp) = \kappa\Big(1+\sup_n\Big\{\frac{1}{n}\sum_{j=1}^{n}h^{(j)}(r)\Big\}\Big) < \infty\,.$$

Hence

$$\mathbb{E}_\pi(\exp(\gamma\Sigma_n^k)) = \mathbb{E}_\pi(\mathcal{E}_{n,k}(X_k)) < \mathrm{Const}(\wp)$$

for any $k \leq 4m_0 - 1$, $\quad n \geq 0$. Hence the result. ∎

For $\mathcal{V} \in \mathsf{B}_R^{(1)}$ and $P \in \wp$ set

$$\Phi_{P,\mathcal{V},n}(w) = \mathbb{E}_\pi(\exp(wS_n))\,,$$

where $w \in \mathbb{C}$ is a complex number satisfying $|\Re(w)| < \varrho_0(\wp)$. Proposition 3.4.36 implies

Corollary 3.4.41. *Each function $\Phi_{P,\mathcal{V},n}$ is analytic inside the strip $\Gamma = \{|\Re(w)| < \varrho(\wp, R)\}$ and satisfies*

$$|\Phi_{P,\mathcal{V},n}(w)| < \mathsf{L}(\wp, R)\,.$$

The family of analytic functions

$$\mathcal{A} = \{\Phi_{P,\mathcal{V},n}; \quad P \in \wp, \mathcal{V} \in \mathsf{B}_R^{(1)},\, n \geq 0\}$$

is tight in Γ. *This means that each sequence* $F_j \in \mathcal{A}$; $j = 1, 2, \ldots$ *contains a subsequence* F_{j_k} *which converges with all derivatives uniformly on compact subsets of* Γ *to some analytic function* F.

Proof. Straightforward in view of Proposition 3.1.21 and of the fact that any bounded family of analytic functions is tight. ∎

Introduce

$$\Phi_{P,\mathcal{V},\infty}(w) = \lim_{n\to\infty} \Phi_{P,\mathcal{V},n}(w) = \exp\Big(\frac{w^2\sigma^2(P,\mathcal{V})}{2}\Big)\,.$$

Proposition 3.4.42. *For any* $\epsilon > 0$ *and* $\alpha_0 > 0$ *there exists an integer number*

$$N_0(\epsilon, \wp, R) > 0$$

such that for any $n \geq N_0$ *and any* $P \in \wp$, $\mathcal{V} \in \mathsf{B}_R^{(1)}$:

$$|\Phi_{P,\mathcal{V},n}(\imath\alpha) - \Phi_{P,\mathcal{V},\infty}(\imath\alpha)| \leq \epsilon\,, \tag{3.4.43}$$

provided $|\alpha| < \alpha_0$.

Proof. We have to study

$$\mathbb{E}_\pi(\imath\alpha S_n)\,.$$

Recall that

$$S_n = \frac{1}{\sqrt{n}}\Big(\mathcal{U}(X_0) - \mathcal{U}(X_n) + \sum_{j=1}^n \eta_j\Big)\,,$$

where

$$\eta_j(X) = \mathcal{U}(X_j) - \mathbb{E}_{X_{j-1}}(\mathcal{U}(X_j))\,.$$

According to the preceding results

$$\mathbb{E}_\pi \exp\Big(\imath\alpha\frac{1}{\sqrt{n}}(\mathcal{U}(X_0) - \mathcal{U}(X_n))\Big)$$

converges to 1 uniformly in $P \in \wp, \mathcal{V} \in \mathsf{B}_R^{(1)}$ and α. Consider the martingale

$$\mathsf{M}_n = \frac{1}{\sqrt{n}}\Big(\sum_{j=1}^{n} \eta_j\Big).$$

The quadratic characteristic of this martingale is given by

$$D_n = \frac{1}{n}\Big(\sum_{j=1}^{n} f(X_{j-1})\Big),$$

where

$$f(x) = \mathbb{E}_x(\mathcal{U}^2) - (\mathbb{E}_x(\mathcal{U}))^2.$$

Note that $0 \le f(x) \le \mathrm{Const}(\wp) W(x)^2$ and $\sigma^2(P,\mathcal{V}) = \nu(f)$. We shall make use of the following general statement on the rate of convergence in the the central limit theorem for martingales:

Statement [Lipzer, Shiraev, Mori]. *Let*

$$\mathsf{M}_n = \sum_{j=1}^{n} \zeta_j : \quad \mathbb{E}(\zeta_j | \mathcal{F}_{j-1}) = 0$$

be an L_2 summable martingale with quadratic characteristic D_n and σ^2 be a positive number. Set $d_n = \mathbb{E}|\, D_n - \sigma^2\,|$. There exists another L_2 summable martingale $\widehat{\mathsf{M}}_n$ defined on the same probability space and satisfying:

$$\mathbb{E}|\widehat{\mathsf{M}}_n - \mathsf{M}_n|^2 \le d_n;$$

$$\Big|\mathbb{E}(\exp(\imath\alpha\widehat{\mathsf{M}}_n)) - \exp\Big(-\frac{\alpha^2\sigma^2}{2}\Big)\Big| \le c|\alpha|^3\mathbb{E}\Big(\sum_{j=1}^{n} |\zeta_j|^3\Big) + \frac{\alpha^2}{2} d_n, \tag{3.4.44}$$

where

$$c = \frac{1}{2\sqrt{2}} + \frac{1}{6}.$$

The proof is contained in [16] (Ch. 7 sec. 5), see also [17]. We shall make use of the following obvious corollary of the above statement:

Under the same assumptions

$$(3.4.45)\quad \Big|\mathbb{E}(\exp(\imath\alpha\mathsf{M}_n)) - \exp\Big(-\frac{\alpha^2\sigma^2}{2}\Big)\Big| \le c|\alpha|^3\mathbb{E}\sum_{j=1}^{n}|\zeta_j|^3 + \frac{\alpha^2}{2}d_n + |\alpha|\sqrt{d_n}.$$

In our case $\zeta_j = \frac{\eta_j(X)}{\sqrt{n}}$ and

$$d_n = \mathbb{E}_\pi\Big(\Big|\frac{1}{n}\sum_{j=0}^{n-1}(f(X_j) - \nu_P(f))\Big|\Big),$$

where,

$$f(x) = \int P(x,dy)\mathcal{U}^2(y) - \Big(\int P(x,dy)\mathcal{U}(y)\Big)^2.$$

Take

$$\sigma^2 = \sigma^2(P,\mathcal{V}) = \nu_P(f).$$

Consider

$$H_n = \frac{1}{n}\sum_{j=0}^{n-1}(f(X_j) - \nu_P(f)).$$

Set $F(x) = f(x) - \nu_P(f)$. Clearly,

$$\mathbb{E}_\pi(H_n^2) = \frac{1}{n^2}\sum_{j=0}^{n-1}\mathbb{E}_\pi(F^2(X_j)) + \frac{2}{n^2}\sum_{j<k\le n-1}\mathbb{E}_\pi(F(X_j)F(X_k)).$$

The function F satisfies:

$$\nu_P(F) = 0;$$
$$F(x) \le \mathrm{Const}(\wp, R)(1+W(x))^2$$

In other words, it is a centered tempered function. Our previous results on tempered functions guarantee that:

$$\sup_n \mathbb{E}_\pi(F^2(X_n)) \le \mathsf{C}_1(\wp, R);$$
$$\mathbb{E}_\pi(F(X_j)F(X_k)) \le \mathsf{C}_2(\wp, R)\exp(-\gamma_1|k-j|).$$

Thus

$$\mathbb{E}_\pi(H_n^2) \le \mathsf{C}_3(\wp, R)/n.$$

This implies

$$d_n = \mathbb{E}_\pi(|H_n|) \le \frac{\mathrm{Const}(\wp, R)}{\sqrt{n}}\,.$$

It remains to estimate the value

$$\mathbb{E}_\pi\Big(\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^{n} |\eta_j(X)|^3\Big)\,.$$

Since

$$|\eta_j(X)| \le \mathrm{Const}(\wp)(W(X_j) + W(X_{j-1}))$$

and in view of previous estimates

$$\mathbb{E}_\pi\Big(\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^{n} |\eta_j(X)|^3\Big) \le \mathrm{Const}(\wp)/\sqrt{n}\,.$$

The assertion of Proposition 3.4.42 is now straightforward. ∎

Lemma 3.4.46. *If the sequence $\Phi_{P_n,\mathcal{V}_n,n}$ converges (in the sense of analytic functions in Γ) to Φ then:*

$$\Phi = \exp\Big(\frac{w^2\sigma^2}{2}\Big)\,.$$

Moreover, the sequence $\sigma^2(P_n,\mathcal{V}_n)$ converges, and

$$\sigma^2 = \lim_{n\to\infty} \sigma^2(P_n,\mathcal{V}_n)\,.$$

Proof. Since Φ is an analytic function, it is uniquely determined by the restriction to any segment $|\alpha| < \alpha_0$ of the imaginary axis. At the same time Proposition 3.4.42 implies that

$$|\Phi_{P_n,\mathcal{V}_n,n}(\imath\alpha) - \Phi_{P_n,\mathcal{V}_n,\infty}(\imath\alpha)| < \mathrm{Const}(\wp,\alpha_0)\epsilon_n\,,$$

where $\epsilon_n \to 0$, as $n \to \infty$ (in fact $\epsilon_n \approx \frac{1}{\sqrt{[4]n}}$). Thus

$$\Phi_{P_n,\mathcal{V}_n,\infty}(\imath\alpha) = \exp\Big(-\frac{\alpha^2\sigma^2(P_n,\mathcal{V}_n)}{2}\Big) \to \Phi(\alpha)$$

uniformly on each segment $|\alpha| < \alpha_0$. It is possible only if the sequence $\sigma^2(P_n, \mathcal{V}_n)$ converges to some σ^2. Hence

$$\Phi(\imath\alpha) = \exp\Big(-\frac{\alpha^2\sigma^2}{2}\Big).$$

Since Φ is analytic in Γ, we obtain that

$$\Phi(w) = \exp\Big(\frac{\sigma^2 w^2}{2}\Big).$$

$\square$

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