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On the mathematical contributions of Giorgio Valli

J. EELLS

Pubblicando questo articolo, il comitato di redazione intende onorare la memoria di Giorgio Valli, scomparso prematuramente nel 1999.

Presentazione: *L'equazione delle applicazioni armoniche costituisce nella teoria dei sistemi integrabili un utile mezzo di interpretazione di fenomeni legati alle varietà minime ed alla equazione di campo di Toda. In quest'ultimo ambito, l'equazione delle applicazioni armoniche compare come coppia di Lax con parametro, facendo così entrare in gioco il gruppo dei cappi* (*loop group*)*. In un celebre articolo pubblicato nel* 1989*, K. Uhlenbeck ha riscoperto ed unificato tale teoria per mezzo di applicazioni armoniche tra superficie di Riemann e gruppi di Lie compatti o spazi simmetrici. In tal caso l'applicazione armonica si solleva ad un'applicazione dalla superficie di Riemann al gruppo dei cappi del gruppo di Lie compatto. Nel caso in cui la superficie di Riemann sia* S^2 *e il gruppo sia* $U(n)$*, Atiyah ha dimostrato che il gruppo dei cappi possiede una struttura complessa naturale, rispetto alla quale il sollevamento a tale gruppo è un'applicazione olomorfa.*

Tali costruzioni sono presentate in rassegna in questo articolo con particolare riguardo al lavoro del giovane matematico italiano Giorgio Valli, che ha portato decisivi contributi di chiarificazione e riformulazione alle costruzioni ed ai risultati sopra menzionati. In particolare vengono esposti i risultati di Valli riguardo:

- (*i*) *una versione con "unitoni" del teorema di fattorizzazione di Uhlenbeck che è stato fonte di importanti raffinamenti da parte di Eells-Lemaire;*
- (*ii*) *l'interpretazione del sollevamento canonico come geodetica nel gruppo di "gauge";*
- (*iii*) *l'utilizzo dell'estensione del sollevamento olomorfo di Atiyah nel caso di un qualsiasi gruppo compatto per la descrizione della fattorizzazione di applicazioni olomorfe in* $U(n)$ *mediante un'ingegnosa applicazione del teorema di Gohberg di interpolazione di funzioni a valori matriciali con elementi razionali;*
- (*iv*) *le grassmanniane bi-invarianti come applicazione della teoria delle funzioni a valori matriciali con elementi razionali;*

(*v*) *le applicazioni* 1*-armoniche tra superficie di Riemann e le immersioni lagangiane.*

Abstract: *Giorgio left us much too soon. But not before making several decisive and influential contributions to global analysis and geometry. Here is a brief account of some of these:*

Firstly, Giorgio's reformulation and clarification of Karen Uhlenbeck's factorization theorem for certain harmonic maps — with implications on energy spectra. Then his interpretation in terms of geodesics in gauge groups. Next, his applications of Gohberg's interpolation theory to maps into loop groups.

In quite a new and different direction — in collaboration with Stefano Trapani: An existence theorem for 1*-harmonic diffeomorphisms between compact Riemann surfaces. That has broken new ground-first steps along a path with much promise.*

1 – On the energy spectrum of harmonic maps $S^2 \to U(n)$ [V 1988]

Loosely speaking, various geometric and physical theories have useful interpretations as integrable systems. Explicit instances in the physical literature include [Pohlmeyer 1976], [Zakharov-Mikhailov 1978, [Zakharov-Shabat 1979]. The harmonic map equation appears there, as Lax pairs with parameter (especially in relation to the Toda field equation). That parameter brings loop groups into play — providing basic motivation throughout.

Quite independently, K. Uhlenbeck [JDG 1989] rediscovered much of that theory, through harmonic maps of Riemann surfaces into compact Lie groups and symmetric spaces. For basic constructions and recent applications, the text [Guest 1997] is highly recommended, following the earlier description of [Eells-Lemaire 1988, §§ 8,9].

In particular, Uhlenbeck gives a factorization of harmonic maps $S^2 \to U(n)$ (= the unitary group on C^n) into the product of essentially holomorphic maps into complex Grassmannians $G_r(C^n)$. That latter is identified with $\{B \in U(n) : B^2 = I$ and its $(+1)$-eigenspace is r-dimensional$\}$. Given an Hermitian projection P onto an r-dimensional subspace of C^n, $P - P^\perp = B \in U(n)$. That is a totally geodesic embedding $\psi : G_r(C^n) \to U(n)$.

Key Words and Phrases: *Lax pairs – Toda fields – Lagrangian immersions – harmonic gauges – Loop groups – Pluriharmonic map – Bi-invariant Grassmannian – 1-harmonic map.*
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The Euler-Lagrange equations for harmonicity of φ take the form [Eells-Lemaire, (8.7)]

$$\overline{\partial} A_z + [A_{\bar{z}}, A_z] = 0, \text{ or equivalently, } \partial A_{\bar{z}} + [A_z, A_{\bar{z}}] = 0\,,$$

where $2A = \varphi^{-1} d\varphi$ is the pull-back of the Maurer-Cartan form of $U(n)$.

The main results of Uhlenbeck, with a different, clarifying proof and refinement by Valli [V 1988] are these:

Let M be a compact Riemann surface and

$$\varphi : M \to U(n) \textit{ a harmonic map. Then for}$$
$$P : M \to \text{Grass } (C^n) = \bigcup_{r=0}^{\infty} G_r(C^n)$$

satisfying

$$P^{\perp} A_z P = 0 \textit{ and } P^{\perp}(\overline{\partial} P + A_{\bar{z}} P) = 0\,,$$

the map $\tilde{\varphi} = \varphi.\ (P - P^{\perp}) : M \to U(n)$ is harmonic. Furthermore,

$$E(\tilde{\varphi} - E(\varphi) = \text{ Area } (M) c_1(\underline{P}))$$

where $\underline{P}$ is the vector subbundle in $M \times C^r$ whose fibre $\underline{P}_x$ is the point P_x.

In case $M = S^2$ and φ is nonconstant, P can be chosen so that $E(\tilde{\varphi}) - E(\varphi) < 0$.

That requires use of the Birkhoff-Grothendieck theorem on the structure of holomorphic vector bundles over S^2.

An immediate consequence of Valli's version of Uhlenbeck's factorization theorem (adding a uniton):

Associated to each harmonic map $\varphi : S^2 \to U(n)$ is a sequence $\varphi_0, \dots, \varphi_k$ of harmonic maps $S^2 \to U(n)$ with φ_0 constant

$$\begin{aligned} &\varphi_k = \varphi\,, \\ &\varphi_j = \varphi_{j-1}(P_j - P_j^{\perp}) \end{aligned} \qquad (1 \le j \le k \le E(\varphi)/4\pi)$$

and $E(\varphi_j) - E(\varphi_{j-1}) \ge 4\pi$.

Thus we have the canonical factorization

$$\varphi = \varphi_0(P_1 - P_1^\perp)\dots(P_k - P_k^\perp)\,.$$

Each factor is holomorphic with respect to a specific connection.

We observe that

$$E(\varphi) = -4\pi\sum_{j=1}^{k} c_1(\underline{P})\,.$$

Thus *the energy of every harmonic map* $\varphi : S^2 \to U(n)$ *is an integral multiple of* 4π.
For further refinements and extensions, see [Eells-Lemaire, (8.19)-(8.35)].

2 – Harmonic gauges on Riemann surface [V 1987], [V 1989]

Let G be a compact Lie group, with Maurer-Cartan form μ (= $L(G)$-valued 1-form on G given by $\mu(v) = v$ for all $v \in L(G)$). If $\varphi : M \to G$ is a map of a Riemann surface M into G, let $\alpha = \varphi^*\mu$. Then $\alpha = \varphi^{-1}d\varphi$. Its complexification has type decomposition $\alpha = \alpha' + \alpha''$ with $\alpha'' = \overline{\alpha'}$. Then φ is harmonic iff $d^*(\varphi^*\mu) = 0$ iff $\overline{\partial}\alpha' = \partial\alpha''$.

For $\lambda \in C^*$ form the $L^C(G)$-valued 1-form on M:

$$\alpha_\lambda = \frac{1}{2}(1-\lambda^{-1})\alpha' + \frac{1}{2}(1-\lambda)\alpha''\,.$$

If $\varphi : M \to G$ *is a harmonic map, then* α_λ *satisfies the structural equation*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \text{ for all } \lambda \in C^*\,.$$

I.e., α_λ is a *flat* G-connection form on M.

If M is a Riemann surface with base point a and $\mathrm{Hom}(\pi_1 M, G) = 0$, then associated to a harmonic map $\varphi : (M, a) \to (G, e)$, there is a unique map

$$\Phi : C^* \times M \to G^C \text{ satisfying } d\Phi_\lambda = \Phi_\lambda \alpha_\lambda \text{ with } \Phi_1 = e\,,$$
$$\Phi_{-1} = \varphi, \Phi_\lambda(a) = e \text{ for all } \lambda \in C^*\,.$$

Furthermore, Φ_k is holomorphic in λ.

Restrict λ to $S^1 \subset C$, and write

$$\Phi_\lambda^{-1} d\Phi_\lambda = \frac{1}{2}(1-\lambda^{-1})\Phi_\lambda^{-1}\partial'\Phi_\lambda + \frac{1}{2}(1-\lambda)\Phi_\lambda^{-1}\partial''\Phi_\lambda\,.$$

Then $\Phi_\lambda^{-1}\partial\Phi_\lambda$ takes the form $-\frac{1}{2}\chi_1\alpha'_\lambda$ so the restriction $\Phi|S^1 \times M \to G$ defines called the *canonical lift of* φ:

$$\begin{array}{ccc} & & (\Omega G, e) \\ & \overset{\Phi}{\nearrow} & \big\downarrow {\scriptstyle \rho-1} \\ (M, a) & \longrightarrow & (G, e) \end{array}$$

Valli has interpreted the canonical lift as a geodesic in the gauge group $C(M, G)$, in case $G = U(n)$ [V 1987]. And in greater generality, in [V 1989].

If G is a compact simple group, then $H^2(\Omega G, Z) = Z$. If $\varphi : S^2 \to G$ is harmonic, then canonical lift Φ has degree given by

$$E(\varphi) = \frac{16\pi}{|\vartheta^2|} \deg \Phi\,,$$

where $|\vartheta^2|$ is the length^2 of the highest root of G [V 1988].

3 – Maps to loop groups [V 1993]

Motivation for the next steps comes from two sources:

1) Uhlenbeck's factorization theorem, via completely integrable systems; and

2) [Atiyah] and Donaldson: For any classical group (such as $G = SU(n)$) the space of based holomorphic maps $S^2 \to \Omega G$ is diffeomorphic to the space of Yang-Mills structures on S^4, modulo based gauge transformations.

To a harmonic map $\varphi : S^2 \to U(n)$ is associated a finite set of holomorphic maps of S^2 into complex Grassmannians. And then, via methods of integrable systems, the canonical lift Φ of φ into a loop group $\Omega U(n)$.

Now that loop group has a natural complex structure (based on Fourier series expansions [Atiyah 1984]) with respect to which Φ is holomorphic. Specifically and briefly, for any compact Lie group G take as ΩG the Sobolev space

$$L^2_1(S^1, 1; G, e)$$

with left invariant 2-form S on its Lie algebra given by

$$S(\xi, \eta) = f < \xi', \quad \eta > dt\,.$$

Then S determines an integrable complex structure on $\Omega\ G$ (and a sort of Kaehler structure on $L^2_{1/2}(S^1, 1; G, e)$).

[Eells-Lemaire 1988, § 9] and references there.

[Segal 1989] has shown that any holomorphic map into $\Omega U(n)$ has image in the rational loops. (I.e., those γ having finite Laurent decomposition ($= \gamma : S^1 \to U(n)$ with $\gamma(1) = 1$ and

$$\gamma(z) = \sum_{k=-N}^{N} A_k z^k$$

for some N, where A_k are $n \times n$ matrices). That has permitted Valli — the main thrust in [V 1994] and [V 1991] — to apply the interpolation (or, realization) theory of rational matrix — valued functions on S^2 of [Gohberg 1988], in particular, a method to reconstruct such functions from their divisors (= zero and pole data). Factorizations become Blaschke products. Valli also constructed a commutative meromorphic product on $QU(n)$, generalizing the ordinary product to rational functions.

Valli [V 1993] described a factorization of holomorphic maps into $U(n)$: Start with the totally geodesic embedding $\psi : G_r(C^n) \to U(n)$

of § 1. That induces an isomorphism $H_2(G_r(C^n)) \to H_2(U(n))$ in homology.

If $V \in G_r(C^n)$ let $p : C^n \to V$ denote the Hermitian projection operator; thus $p^* = p$, $p^2 = p$. Define $\psi(V) = p - p^\perp$ (Cartan embedding). For any α in the open disc $D \subset C$ let $V \to p + \xi_\alpha p^\perp$, where $\xi_\alpha \in \Omega U(n)$ is given by $\xi_\alpha(\lambda) = (\lambda - \alpha)(\overline{\alpha - 1})/\bar{\alpha}\lambda - 1)(1 - \lambda)$.

Then *any rational loop* γ *admits a factorization*

$$\gamma = \xi_\alpha \dots \xi_{\alpha_s}(p_1 + \xi_{\beta_1} p_1^\perp) \dots (p_r + \xi_{\beta_r} p_r^\perp)\,.$$

Thus any rational map $f : M \to \Omega U(n)$ *of a Riemann surface has a Blaschke product decomposition*

$$f = Q(p_1 + \xi_{\beta_1} p_1^\perp) \dots (p_r + \xi_{\beta_r} p_r^\perp)$$

where Q *denotes a constant loop.*

There is a form of uniqueness, as well.

Much in the above sections has been generalized to pluriharmonic maps $\varphi : M \to U(n)$, where M is a compact simply connected complex manifold; here φ is *pluriharmonic* if its restriction to every holomorphic curve in M is harmonic. That is developed in detail in [OV 1990].

4 – Bi-invariant Grassmannians [V 2000]

Valli continued with the theory of rational matrix-valued functions mentioned in § 3 — and applied it extensively to the following situation:

Denote by $M_{d,n}$ the moduli space of holomorphic vector bundles on CP^r, of rank n and second Chern class $c_2 = d$ together with a fixed trivialization at the line at infinity. $M_{d,n}$ is a noncompact, connected, complex manifold of $\dim_C = 2dn$. There are maps (Taubes) $M_{d,n} \to M_{d,n+1}$ corresponding to adding a uniton in a neighbourhood of infinity.

A suitable completion $\overline{M}_{d,n}$ (through Barth-Donaldson monad matrices) embeds algebraically as a closed submanifold of some product $G_d(C^{m_1}) \times C^{m_2}$.

Under natural conditions (vanishing of the algebraic invariants of the monad matrices) the theory produces a deformation retract $G_{d,n}$ of $\overline{M}_{d,n}$ called the *bi-invariant Grassmannian* (a Lagrangian subvariety of

$\overline{M}_{d,n}/C^2$). It is a compact singular variety, representable as the space of d-dimensional subspaces of a certain vector space, fixed by two commuting nilpotent transformations.

Valli computed the homology of $\overline{M}_{d,n}$ (thus that of $G_{d,n}$, as well), and the map $\overline{M}_{d,n} \to \overline{M}_{d,n+1}$ (which induce isomorphisms in j-homology for $j < 2d+2$. In fact, a theorem of Bialynicki-Birula provides dual cellular decompositions of $G_{d,n}$.

There is a natural map $\overline{M}_{d,n} \to \Omega^3 SU(n)$, the triple loop group of $SU(n)$. Through it Valli related his results above to Atiyah-Jones homological stability.

5 – One-harmonic maps [TV 1995]

The 1-*energy* of a map $\varphi : (M,g) \to (N,h)$ between smooth Riemannian manifolds is given by the functional E_1 on the Sobolev space $L_1^1(M,N)$, where

$$E_1(\varphi) = \int_M |d\varphi| \,.$$

Formally speaking, its Euler-Lagrange operator has the form

$$\tau_1(\varphi) = \operatorname{div}\Big(\frac{d\varphi}{|d\varphi|}\Big) \,.$$

Suitably differentiable maps satisfying $\tau_1(\varphi) = 0$ are called 1-*harmonic maps.* In case $N = R$ these solutions are sometimes said to be of *mean curvature type* [Gilbarg-Trudinger (1977). Second edition (1998)]. The equation $\tau_1(\varphi) = 0$ certainly has serious degeneracy. For its analytic study (in particular, in matters of partial regularity) there is an important extension of the space $L_1^1(M,N)$ to $BV(M,N)$, the space maps of bounded variation. For that, see [Ambrosio 1990] and [Aviles-Giga (1991)].

In this general context, Trapani and Valli [TV 1995] have recognized the geometric importance of 1-harmonic maps between Riemann surfaces. Only first steps have been taken so far-but the future looks bright!

Suppose that M is a compact oriented surface of genus $M > 1$. Let g and h denote Riemannian metrics on M, and for each give M the holomorphic structure determined by its conformal equivalence. In terms of these we have the local holomorphic representations

$$g = \rho^2 dz\, d\bar{z}, \qquad h = \sigma^2 dw\, d\overline{w}\,.$$

A map $\varphi : (M, g) \to (N, h)$ has a complex differential which decomposes to $\partial'\varphi + \partial''\varphi$ in terms of its types [Eells-Lemaire 1988, § 4]. Define the functionals

$$E_1(\varphi) = E_1(\varphi; g, h) = \int_M |d\varphi|$$
$$E_1'(\varphi) = \int_M |\partial'\varphi|, \qquad E_1''(\varphi) = \int_M |\partial''\varphi|\,.$$

Their Euler-Lagrange operators are $\tau_1'(\varphi) = \operatorname{div} \frac{\partial'\varphi}{|\partial'\varphi|}$, $\tau_1''(\varphi) = \operatorname{div} \frac{\partial''\varphi}{|\partial''\varphi|}$.

Henceforth we concentrate on E_1'-beginning with its first properties:

a) *Let $\varphi : (M, g) \to (N, h)$ be an orientation-preserving diffeomorphism. Then*

$$E_1'(\varphi; g, h) = E_1'(\varphi^{-1}; h, g)\,.$$

Similarly for E_1'' and E_1. Furthermore,

$$|\partial'\varphi| \geq \{|\partial'\varphi|^2 - |\partial''\varphi|^2\}^{1/2} > 0 \text{ on } M\,.$$

Consequently, E_1' is well defined on all M.

b) *Assume that g and h both have strictly negative curvatures K_g and K_h.* Then φ preserves curvature forms:

$$K_g dx_g = \varphi^* K_h dy_h$$

for any smooth E_1'-critical point.

In particular, φ is an orientation-preserving diffeomorphism.

The main theorem of [TV 1995] is the following:

Let g and h be Riemannian metrics on M, both with strictly negative curvature; and let H denote a homotopy class of self-maps of M. Then

(i) *if H contains an orientation-preserving diffeomorphism, then there exists a unique E_1'-critical point in H. It preserves curvature forms.*

(ii) *If H contains the constants, then they are the only E_1'-critical points in H.*

(iii) *If H contains a nonconstant antiholomorphic map, then it is the unique $E;_1$-critical point in H.*

No other homotopy class contains an E_1'-critical point.

The proof of that theorem involves a delicious blend of geometry and hard-core analysis:

Basic existence and uniqueness require application of the continuity method to various compactness results [Mumford (1971)], [Bethuel-Ghidaglia (1993)].

A Lorentz distance is defined on the space $\underline{M}$ of Hermitian metrics on M; that can be expressed in terms of E_1'. Furthermore, its restriction to $\underline{M_{-1}} = \{g \in \underline{M} : K_g \equiv -1\}$ is positive definite; it induces the Weil-Petersson metric on the Teichmüller space $T = \underline{M_{-1}}/D_0$, where D_0 denotes the identity component of the group of orientation-preserving diffeomorphisms on M. [Eells-Lemaire 1988, §§ 5.46-5.57]. The theorem of Trapani-Valli can be formulated by saying that any two points of T can be joined by a unique Lorentz geodesic.

6 – Lagrangian immersions

Let (M, g) be a compact oriented Riemannian $2n$-manifold, and (N, h) a Kaehler-Einstein manifold of complex dimension $2n$, and negative Ricci curvature (i.e., $\mathrm{Ricci}^N = \mathrm{Scal}^N g$ with $\mathrm{Scal}^N < 0$).

Various restrictions on an isometric immersion $f : M \to N$ have been studied in [SV 1998, 1999, 2000], to insure that it is *Lagrangian* (i.e., $f^*\omega^N = 0$, ω^N being the Kaehler form of (N, h)). Apparently, the following consequence of these efforts has recently been derived by Salavessa:

Say that *complex direction of f* at a point p in M is a real 2-plane P in $T_p(M)$ such that $df(p)P$ is a complex line of $T_{f(p)}(M)$. The *Kaehler angles* of f at p have cosines in a symplectic diagonalization of the 2-form $f^*\omega^N$ at p. Then

If f is minimal with equal Kaehler angles and no complex directions, then f is Lagrangian.

A corresponding result in case $n = 2$ is due to [Wolfson 1989].

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Indirizzo dell'Autore:

James Eells – Inst. Maths. – Warwick Univ. – Coventry CV4 7AL – Cambridge (England)
E-mail: je208@cus.cam.ac.uk