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The inverse eighth degree anharmonic oscillator

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RIASSUNTO: *Si applica la trasformata di Laplace e la relativa inversa per un sistema risolvibile di equazioni differenziali lineari e il risultato è supportato dall'uso di MAPLE V (computer algebra), per ottenere una quadrupla ipergeometrica rappresentazione convergente di una soluzione dell'equazione di Schroedinger di un oscillatore inverso monodimensionale di ottavo grado.*

ABSTRACT: *The Laplace transform and its inverse are applied to a soluble system of linear differential equations and the result combined with the use of the computer algebra package MAPLE V to obtain a quadruple convergent hypergeometric representation of a solution of a Schroedinger equation of an inverse eighth degree one dimensional oscillator.*

1 – Introduction

The purpose of this study is to deduce a quadruple hypergeometric series representation of an explicit solution of the Schroedinger equation governing a one dimensional anharmonic with the inverse eighth degree interaction. The method employed is an extension of that used in the treatment of the doubly-confluent Heun equation by Exton [4].

The equation concerned is

$$U'' + [-A^2/4 + B/2x^{-1} + C/4x^{-2} - D/2x^{-3} - E/4x^{-4} + \\ - F/2x^{-5} - G/2x^{-6} + H/2x^{-7} - K^2/4x^{-8}]U = 0\,, \tag{1.1}$$

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where the quantities A to K depend upon the physical characteristics of the system. Any equation of higher inverse degree does not lend itself to solution by the technique employed here for algebraic reasons. The equation has two singularities, both irregular. That at the origin is of the sixth type and the other at infinity is of the second type.

A soluble system of two linear differential equations is considered such that the inverse Laplace transform is of the second order which has a normal form the same as (1.1). This gives rise to a non-linear system of algebraic equations which is shown to be consistent by the computer algebra package MAPLE V. The working out is completed by evaluating the appropriate Laplace transform.

All the indices of summation are taken to run over all of the non-negative integers and any values of parameters leading to results which do not make sense are tacitly excluded as are any inconsequential constant multipliers. The interchanging of the operations of integration and summation is justified in each case by the convergence of the series concerned and the Pochhammer symbol $(a, n) = \Gamma(a + n)/\Gamma(a)$ is used below.

2 – A soluble differential system

Consider the differential system

$$(kt + a)u' + bu = v \tag{2.1}$$

and

$$(t/k + c)v'''' + fv''' + gv'' + hv' + pv = 0 \tag{2.2}$$

which are equivalent to the fifth-order equation

$$\begin{aligned}&[t^2 + (ck + a/k)t + ac]u^v + [(fk + b/k + 4)t + 4ck + af + bc]u''''+\\ &\quad + (gkt + ag + bf + 3fk)u''' + (hkt + ah + bg + 2gk)u''+\\ &\quad + (kpt + ap + bh + hk)u' + bpu = 0\,.\end{aligned} \tag{2.3}$$

The inverse Laplace transform is written as

$$u(t) = \int \exp(-xt)y(x)dx\,, \tag{2.4}$$

where the contour of integration consists of a simple path closed on the Riemann surface of the integrand such that this integrand remains unchanged after the completion of one circuit.

It is found that the function $y(x)$ is determined by the differential equation

$$\begin{aligned} x^5y''+&[(ck+a/k)x^5+(6-fk-b/k)x^4+gkx^3-hkx^2+pkx]y'+\\ &+[acx^5+(ck-af-bc+5a/k)x^4+(ag+bf+4-4b/k-fk)x^3+\\ &+(gk-ah-bg)x^2+(ap+bh-hk)x-bp+pk]y=0\,. \end{aligned} \tag{2.5}$$

the normal form of (2.5) is the same as (1.1), in which

$$\begin{aligned} &A=a/k-ck\,,\quad B=4k^3c-4ka\,,\\ &C=(2ck^4g+b^2+6kb+f^2k^4-2k^2ag-6k^3f+8k^2-2k^2bf)/k^2\,,\\ &D=2gk-fk^2g-ck^2h+ah+bg\,,\\ &E=g^2k^2-10hk+2ck^2p+6bh+2fk^2h+6ap\,,\\ &F=pb-fk^2p-gk^2h\,,\ G=h^2k^2+2gk^2p\,,\ H=hk^2p \text{ and } K=pk\,. \end{aligned} \tag{2.6}$$

Introduce a scaling parameter q, such that

$$x=qX\,, \tag{2.7}$$

when A, B, D, E, F, G, H and K respectively by

$$Aq\,,\quad Bq\,,\quad D/q\,,\quad E/q^2\,,\quad F/q^3\,,\quad G/q^4\,,\quad H/q^5 \text{ and } K/q^3\,. \tag{2.8}$$

The equations (2.6) and (2.8) can be shown to be consistent in the variables a, b, c, f, g, h, k, p and q by means of MAPLE V and

$$\begin{aligned} U=X^{3-fk/2-b/(2k)}\exp[&(ck+a/k)qX/2-gk/(2qX)+\\ &+hk/(4q^2X^2)-pk/(6q^3X^3)]y(X)\,. \end{aligned} \tag{2.9}$$

The equations (2.1) and (2.2) must now be examined.

3 – The solution of the equations (2.1) and (2.2)

In (2.2), put

$$t = T - ck \tag{3.1}$$

and obtain

$$Tv'''' + fkv''' + gkv'' + hkv' + pkv = 0\,. \tag{3.2}$$

Put

$$v(T) = \int \exp(sT)\eta(s)ds\,, \tag{3.3}$$

where the contour of integration is a simple loop beginning and ending at $-\infty$ and encircling the origin once in the positive direction. The function $\eta(s)$ is given by

$$\eta'/\eta = (C_0 - C_1')/C_1\,, \tag{3.4}$$

where

$$C_0 = fks^3 + gks^2 + hks + pk \tag{3.5}$$

and

$$C_1 = s^4\,. \tag{3.6}$$

Hence,

$$\eta = s^{fk-4}\exp(-gks^{-1} - hk/2s^{-2} - pk/3s^{-3})\,. \tag{3.7}$$

Thus,

$$v(T) = \int \exp(sT - gks^{-1} - hk/2s^{-2} - pk/3s^{-3})s^{fk-4}ds\,, \tag{3.8}$$

which is proportional to

$$\sum \frac{(-gk)^m(-hk)^n(pk)^M T^{3-fk+m+2n+3M}}{m!n!M!(4 - fk, m + 2n + 3M)}. \tag{3.9}$$

From (2.1),

$$u' = -b/k(t+a/k)^{-1} + v/k(t+a/k)^{-1}\,, \tag{3.10}$$

so that, apart from a constant multiple,

$$u(t) = (t+a/k) - b/k \int (t+a/k)b/k - 1v(t)dt\,. \tag{3.11}$$

Recalling (3.1) and (3.9) we see that

$$\begin{aligned} u(T) = (T - ck + a/k)^{-b/k} \sum [&(-gk)^m(-hk)^n(-pk)^M \times \\ &\times (ck-a/k)^{-N}(1-b/k,N)\times \\ &\times (4-fk, m+2n+3M+N)T^{4-fk+m+2n+3M+N}] \Big/ \\ &\quad [m!n!M!N!(4-fk, m+2n+3M)\times \\ &\times (5-fk, m+2n+3M+N)]. \end{aligned} \tag{3.12}$$

From (2.4), by inversion

$$y(x) = \int \exp(xt)u(t)dx \tag{3.13}$$

and from (2.7) and (3.1),

$$y(X) = \exp(-ckqX)\int \exp(qXT)u(T)dT\,. \tag{3.14}$$

Let the contour of integration be a Pochhammer double loop slung around the origin and the point $1/(ck - a/k)$ in the T-plane and we see, after some reduction, that $y(X)$ is proportional to

$$\begin{aligned} &\exp(-ckqX)\sum[(-gk)^m(-hk)^n(-pk)^M(qX)^P\times \\ &\times (ck-a/k)^{m+2n+3N+P}(1-b/k,N)\times(4-fk,m+2n+3M+N)\times \\ &\times (5-fk,m+2n+3M+N+P)]\Big/[(4-fk,m+2n+3M+N)\times \\ &\times (6-b/k-fk,m+2n+3M+N+P)m!n!M!N!P!]\,. \end{aligned} \tag{3.15}$$

This expression can be expanded as a convergent quadruple series of confluent hypergeometric functions. After the application of Kummer's first theorem ([5], 6) and re-arranging, the following quadruple series of classical hypergeometric functions of unit argument is obtained:

$$
\begin{aligned}
&y(X) = \exp(-aqX/k) \sum [(-gk)^m (-hk)^n (-pk)^M \times \\
(3.16) \quad &\times (ck - a/k)^{m+2n+3m+P} (-qX)^P (1 - b/k, P)] \Big/ \\
&[m!n!M!P!(6 - b/k - fk, m + 2n + 3M + P)] \times \\
&\times {}_2F_1[4-fk+m+2n+3M, 1-b/k; 6-b/k-fk+m+2n+3M+P; 1] \, ,
\end{aligned}
$$

Hence, by means of Gauss's summation theorem ([1], 104), apart from a constant multiple

$$
\begin{aligned}
&y(X) = \exp(-aqX/k) \sum [(-gk)^m (-hk)^n (-pk)^M \times \\
(3.17) \quad &\times (ck - a/k)^{m+2n+3M+P} (-qX)^P (1 - b/k, P)] \Big/ \\
&[(m!n!M!P!(2 - b/k, P)(5 - fk, m + 2n + 3M + P)] \, .
\end{aligned}
$$

A convergent series representation of (1.1) can be deduced from (2.9) and (3.17) and the family of equations (2.6) and (2.7).

If the parameters are not too large, the series (3.17) can be implemented numerically by direct summation if $|X|$ is not greater than 200, by means of MAPLE V.

The singularity of (1.1) at infinity is not considered in this paper and it is recalled that convergent series representation of solutions of linear differential equations relative to irregular singularities are rather unusual. For other similar examples, the reader is referred to [2] and [3].

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