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Chains of lines and intersections of quadrics in $PG(n,q)$

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Riassunto: *Sia q una potenza di un primo e $X \subseteq PG(n,q)$ l'intersezione di s ipersufici quadriche. Si dimostra qui che se $n \geq 6s$ per tutti i $P,Q \in X$ ci sono rette $D_1, D_2 \subseteq X$ con $P \in D_1$, $Q \in D_2$ e $D_1 \cap D_2 \neq \emptyset$*

Abstract: *Let q be a power of a prime. Let $X \subseteq PG(n,q)$ be the intersection of s quadric hypersurfaces. Assume $n \geq 6s$. Here we prove that for all $P,Q \in X$ there are lines $D_1, D_2 \subseteq X$ with $P \in D_1$, $Q \in D_2$ and $D_1 \cap D_2 \neq \emptyset$*

1 – Introduction

Let p be a prime integer and q a power of p. Fix a subset X of a finite n-dimensional projective space $PG(n,q)$ with $X \neq \emptyset$. X is a linear space if and only if for all $P,Q \in X$ there is a line $D \subseteq X$ with $\{P,Q\} \subseteq D$. We will say that X is *connected by lines in at most t steps* if for all $P,Q \in X$ there are lines D_i, $1 \leq i \leq t$, with $D_i \subseteq X$ for every i, $P \in D_1$, $Q \in D_t$ and $D_j \cap D_{j+1} \neq \emptyset$ for every j with $1 \leq j \leq t-1$. We will say that X is *connected by lines* if there is an integer $t \geq 1$ such that X is connected by lines in at most t steps. The aim of this paper is to show that quite often intersection of quadrics are connected by lines and even connected by lines in at most two steps. Here is our main result.

Key Words and Phrases: *Finite projective space, Quadric hypersurface, Line*
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THEOREM 1.1. *Fix integers n, s with $n \geq 6s > 0$. Let $X \subseteq PG(n,q)$ be the intersection of s quadrics. Then X is connected by lines in two steps.*

Several nice varieties (e.g. Grassmannians or linearly normal curves with high degree with respect to their genus, say degree d and genus $g := d-n$ with $d \geq 3g$) are intersection of quadric hypersurfaces and indeed of hyperbolic quadric hypersurfaces of rank at most 4. The existence of lines (and even more of many lines) is however a very strong restriction on X. Hence it is natural to make some strong assumption and here we assume that X is the intersection of very few quadrics with respect to n. We do not know what should be the best possible lower bound for n in term of s. For $s = 1$ everything is clear: $n \geq 4$ is enough and it is sharp for the following reason. If X is a smooth quadric in $PG(n,q)$, $n \geq 4$, see the classification in [2], Ch. 22. If $X \subset PG(3,q)$ is a smooth quadric, it contains lines and any two points of it are connected by lines in two steps if and only if X is hyperbolic ([2], p. 4). If $X \subset PG(n,q)$ is a singular quadric, then it is a cone and it is easy to check that it is connected by lines in two steps. If $s = 2$ we proved a weaker result (every point of X is contained in a line) for every $n \geq 7$ ([1], Th. 1.1) and applied it to the so-called Finite Field Nullstellensatz. It seems to be interesting to consider the same problem for an arbitary intersection of s hypersurfaces of degree $d_1, \dots, d_s$ in $PG(n,q)$ when $n \gg d_1 + \ldots + d_s$. Instead of lines one could consider conics or, more generally, fix an integer $t \leq n$ and consider connectedness with respect to some rational normal curves in $PG(h,q)$, $h \leq t$. If q is a square we think it is very interesting to consider intersection of Hermitian hypersurfaces ([2], Ch. 23). In this case, however, instead of lines one should take elliptic quadric surfaces with respect to the p-power $q^{1/2}$. We also consider the case of affine spaces $A(n,q)$ and affine lines. As a very easy application of Theorem 1.1 we obtain the following result.

COROLLARY 1.2. *Fix integers n, s with $n \geq 6s > 0$. Let $X \subseteq A(n,q)$ be the intersection of s quadrics. Then for all $P, Q \in X$ there are affine lines D_1, $D_2 \subseteq X$ with $P \in D_1$, $P \in D_2$ and $D_1 \cup D_2$ spanning an affine plane.*

The same problem (connectedness in finitely many steps by lines) may be considered over an arbitrary field. It is easy to obtain from Theorem 1.1 the corresponding result for an algebraically closed base field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = p$; then from this result it easy (although cumbersome) to obtain the corresponding result for an algebraically closed base field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. However these problems are much easier over an algebraically closed base field and we knew the result corresponding to Theorem 1.1 and several other similar results before proving Theorem 1.1: we used the algebraically closed case as a guide for the case of a finite field.

2 – Proof of 1.1 and 1.2

Proof of 1.1. Set $X := Q_1 \cap \ldots \cap Q_s = 0$ with $Q_i = \{q_i = 0\}$ and q_i homogeneous quadratic form. We do not assume that the quadratic forms q_i $1 \leq i \leq s$, are linearly independent (we even allow $q_i \equiv 0$), but if the forms q_i, $1 \leq i \leq s$, are not linearly independent, one could obviously reduce to a case with $s' < s$. Fix $P, Q \in X$. If either $q_i \equiv 0$ (i.e. $Q_i = PG(n, q)$) or $q_i \neq 0$, but Q_i is singular at P, set $V(i, P) := PG(n, q)$. If $q_i \neq 0$ and Q_i is smooth at P, let $V(i, P) \subset PG(n, q)$ be any hyperplane of the tangent space of Q_i at P not containing P. Hence in all cases every point of $V(i, P) \cap Q_i$ is contained in a line contained in Q_i and passing through P. Do the same for the point Q and define the linear spaces $V(i, Q)$, $1 \leq i \leq s$. Set $V := V(1, P) \cap V(1, Q) \cap \ldots \cap V(s, Q)$ and $Y := X \cap V$. Hence V is a linear subspace of $PG(n, q)$ with $\dim(V) \geq n - 4s$ and Y is defined in V by s quadratic equations. Since $n - 4s \geq 2s$, by [3], Th. 6.11, or [4], Th. 3.1, we have $Y \neq \emptyset$. Fix $A \in Y$. First assume $A \notin \{P, Q\}$. By the choice of V the line $\langle P, A\rangle$ spanned by P and A is contained in every quadric Q_i, i.e. it is contained in X. For the same reason the line $\langle Q, A\rangle$ is contained in X. Hence P and Q may be connected inside X just using two lines. Similarly if $A \in \{P, Q\}$, then P and Q are contained in a line contained in X.

Proof of 1.2. See $A(n, q)$ as the complement of a hyperplane in $PG(n, q)$. Set $X := Q_1 \cap \ldots \cap Q_s$ with $Q_i = \{q_i = 0\}$ and q_i quadratic equation. Let q'_i be the homogeneous form associated to q_i and $X' := Q'_1 \cap \ldots \cap Q'_s$ with $Q'_i := \{q'_i = 0\}$. Hence $X \subseteq X'$. Define the linear subspaces

$V(i, P)$, $V(i, Q)$, $1 \le i \le s$ and V of $PG(n, q)$ with respect to X' and take $A \in V \cap X'$. Set $D_1 := A(n, q) \cap \langle P, A\rangle$ and $D_2 := A(n, q) \cap \langle Q, A\rangle$. Since $A \in \langle P, A\rangle \cap \langle Q, A\rangle$. D_1 and D_2 are coplanar, as wanted.

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