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An extension of the Hausdorff-Young theorem to the Besicovitch-Orlicz space of almost periodic functions

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RIASSUNTO: *In un recente lavoro* [1] *è stato esteso il teorema di Hausdorff-Young ad una classe di funzioni di Besicovitch quasi periodiche* $B^p a.p.$ *Qui si considera tale estensione nel contesto degli spazi di Orlicz, ovvero allo spazio Besicovitch-Orlicz delle funzioni quasi periodiche.*

ABSTRACT: *In a recent papers* (*cf.* [1]), *the Hausdorff-Young Theorem was extended to the class of Besicovitch almost periodic functions* $B^p a.p.$ *We consider here such an extension in the context of Orlicz spaces, namely the Besicovitch-Orlicz space of almost periodic functions* $B^\phi a.p.$

1 – Introduction

The classical Hausdorff-Young theorem for L^p spaces has been subject to various generalizations. In [9], this result was extended to the context of Orlicz spaces and recently considered in the class of Besicovitch spaces of almost periodic functions $B^p a.p.$ (cf. [1]). More precisely:

Let $\mathcal{P}$ denotes the linear set of generalized trigonometric polynomials.

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Define for $p \in [1,+\infty[$ the norm

$$\|f\|_{B^p} = \lim_{T\to\infty}\Big(\frac{1}{2T}\int_{-T}^{+T}|f(t)|^p dt\Big)^{\frac{1}{p}},\ f \in \mathcal{P}\,.$$

The space $B^p a.p.$ is the completion of $\mathcal{P}$ with respect to the norm $\|.\|_{Bp}$.

The Hausdorff-Young Theorem in the class $B^p a.p.$ states that if $f \in B^p a.p.$ and $\hat{f} = (c_k)_{k\in\mathbb{Z}}$ is the corresponding sequence of it's Fourier coefficients, then

i) $\|\hat{f}\,\|_{l^q} \le \|f\|_{B^p}$ if $p \in]1,2]$.
ii) $\|f\|_{B^p} \le \|\hat{f}\|_{l^q}$ if $p \in [2,+\infty[$.

Here we extend these properties to the context of Orlicz space.

2 – Preliminaries

In the sequel, the notation ϕ will stand for an Orlicz function, i.e. a function $\phi : \mathbb{R} \to \mathbb{R}^+$ which satisfies the conditions: ϕ is even, convex, $\phi(0) = 0$, $\phi(u) > 0$ if $u \ne 0$ and $\lim_{u\to 0}\frac{\phi(u)}{u} = 0$, $\lim_{u\to\infty}\frac{\phi(u)}{u} = \infty$.

This function is called of Δ_2 type (δ_2 type respectively) when there exist constants $K > 2$ and $u_0 \ge 0$ for which $\phi(2u) \le K\,\phi(u)$, $\forall u \ge u_0$ (respectively for $0 \le u \le u_0$).

An Orlicz function admits a derivative ϕ' unless on a denumerable set of points. It satisfies $\phi'(0) = 0$, $\phi'(|u|) > 0$ if $u \ne 0$ and $\lim_{|u|\to\infty}\phi'(|u|) = +\infty$ so that ϕ is strictly increasing to infinity.

The derivative ϕ' satisfies the inequality (cf. [4], [7], [8])

$$u\,\phi'(u) \le \phi(2u) \le 2\,u\;\phi'(2\,u) \qquad \forall u \ge 0\,. \tag{2.1}$$

From [8], we know that if ϕ is an Orlicz function then, for every $\varepsilon > 0$ there exists an Orlicz function ϕ_ε with a continuous derivative and satisfying

$$\phi_\varepsilon(x) \le \phi(x) \le (1+\varepsilon)\phi_\varepsilon(x) \qquad \forall x \in \mathbb{R}\,.$$

In view of this we may assume ϕ' to be continuous in all what follows.

The function $\psi(y) = \sup\{x\,|y| - \phi(x),\ x \ge 0\}$ is called conjugate to ϕ. It is an Orlicz function when ϕ is.

The pair (ϕ, ψ) satisfies the Young's inequality:

$$x\, y \leq \phi(x) + \psi(y) \quad x \in \mathbb{R},\, y \in \mathbb{R}\,.$$

Let us note that equality holds in the Young's inequality iff $x = \psi^{'}(y)$ or $y = \phi'(x)$.

In the following we shall consider normalized pairs of conjugate functions, i.e. such that $\phi(1) + \psi(1) = 1$. There is no restriction since for every pair we may define an equivalent normalized one in the sense that they define the same respective spaces and equivalent norms (cf. [9]).

In the class of Orlicz functions a partial order may be defined by setting,

$$\phi_1 \leq \phi_2, \text{ when } \begin{cases} \phi_1(ax) \leq b\, \phi_2(x) & \text{for } |x| \geq x_0 > 0 \\ \text{and} & \\ \phi_2(cx) \leq d\, \phi_1(x) & \text{for } |x| \leq x_1 \end{cases} \tag{2.2}$$

where a, b, c, d, x_0 and x_1 are constants depending on ϕ_1 and ϕ_2.

This order is natural in the L^p spaces for power functions with $a = b = c = d = x_0 = x_1 = 1$.

Let ϕ be an Orlicz function. The Orlicz sequence space l^ϕ is the set of sequences of scalars,

$$\begin{aligned} l^\phi &= \Big\{a = (a_1, a_2, \dots), \sum_{n\geq 1} \phi\Big(\frac{|a_n|}{k}\Big) < +\infty, \text{ for some } k > 0\Big\} = \\ &= \Big\{a = (a_1, a_2, \dots), \lim_{\lambda\to 0} \sum_{n\geq 1} \phi(\lambda |\, a_n|) = 0\Big\} \end{aligned}$$

We will use the notation $\rho_{l^\phi}(a) = \sum_{n\geq 1} \phi(|\, a_n|)$.

The space l^ϕ equiped with the Luxemburg norm,

$$\|x\,\|_{l^\phi} = \inf \Big\{k > 0, \rho_{l^\phi}\Big(\frac{x}{k}\Big) \leq \phi(1)\Big\}$$

is a Banach space (cf. [7], [8]).

A second norm (called Orlicz norm) is defined by means of the formula:

$$|||a|||_{l^\phi} = \sup \Big\{\Big|\sum_{n\geq 1} a_n\, b_n\Big|; \rho_{l^\psi}(b) \leq 1\Big\}$$

(cf. [7], [8]), where $a = (a_1, a_2, \dots)$.

These norms are equivalent and satisfy,

$$\phi(1).\|a\|_{l^\phi} \le |||a|||_{l^\phi} \le \frac{1}{\psi(1)}\|a\|_{l^\phi} \qquad a \in l^\phi\,. \tag{2.3}$$

3 – The Besicovitch-Orlicz space of almost periodic functions

Let $M(\mathbb{R})$ be the set of all real Lebesgue mesurable functions. The functional,

$$\rho_{B^\phi} : M(\mathbb{R}) \to [0,\infty]\,, \quad \rho_{B^\phi}(f) = \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{-T}^{+T} \phi|f(t)|)dt$$

is a pseudomodular (cf. [3], [5], [6]).

The associated modular space,

$$\begin{aligned} B^\phi(\mathbb{R}) &= \{f \in M(\mathbb{R}),\ \lim_{\alpha\to 0} \rho_{B^\phi}(\alpha f) = 0\} = \\ &= \{f \in M(\mathbb{R}),\ \rho_{B^\phi}(\lambda f) < +\infty,\ \text{for some } \lambda > 0\} \end{aligned}$$

is called the Besicovitch-Orlicz space.

This space is endowed with the pseudonorm (cf. [3], [5], [6])

$$\|f\|_{B^\phi} = \inf\Big\{k > 0, \rho_{B^\phi}\Big(\frac{f}{k}\Big) \le \phi(1)\Big\}$$

called the Luxemburg norm.

As usual, one may define an Orlicz pseudonorm in the $B^\phi a.p.$ space by setting,

$$|||f|||_{B^\phi} = \sup\{M(|fg|), g \in B^\psi a.p., \rho_{B^\psi}(g) \le 1\}$$

Let now $\mathcal{P}$ be the set of generalized trigonometric polynomials, i.e.;

$$\mathcal{P} = \Big\{P(t) = \sum_{j=1}^{n} a_j\, e^{i\,\lambda_j\, t},\ \lambda_j \in \mathbb{R},\ a_j \in \mathbb{C}, n \in \mathbb{N}.\Big\}$$

The Besicovitch-Orlicz space of almost periodic functions, denoted by$B^\phi a.p.$ is the closure of the linear set $\mathcal{P}$ in $B^\phi(\mathbb{R})$, with respect to the pseudonorm $\|.\|_{B^\phi}$:

$$B^\phi a.p. = \{f \in B^\phi(\mathbb{R}), \exists P_n \in \mathcal{P},\ n = 1,2,\dots\,;\ \text{s.t.}\ \lim_{n\to\infty} \|f - P_n\|_{B^\phi} = 0\}$$

Some structural and topological properties of this space are considered in [3], [5], [6].

From [3], [5], we know that $\phi(|f|) \in B^1 a.p$ if $f \in B^\phi a.p$, then by a classical result (cf. [2]) the limit exists in the expression of $\rho_{B^\phi}(f)$, i.e.:

$$\rho_{B^\phi}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \phi(|f(t)|dt), \quad f \in B^\phi a.p.$$

This fact will be very useful in our computations.

Let us denote by $C^o a.p.$the classical algebra of Bohr's almost periodic functions, or what is the same the uniform closure of the linear set $\mathcal{P}$. It is known that

$$\phi(|f|) \in C^o a.p. \text{ when } f \in C^o a.p. \text{ (cf. [2])} \tag{3.1}$$

Also, from [2] we have,

$$\mathbf{M}(|\,f|) > 0 \text{ when } f \in C^o a.p., \ \ f \neq 0 \tag{3.2}$$

where the notation $\mathbf{M}(f)$ is used for $\mathbf{M}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)\,dt$.

To every $f \in B^\phi a.p.$, because of the inclusion $B^\phi a.p. \hookrightarrow B^1 a.p.$ (see Proposition 3.1 below), we can associate a formal Fourier series, more precisely: define the Bohr transform of $f \in B^\phi a.p.$, $a(\lambda, f) = \mathbf{M}(f\, e^{-i\,\lambda\, t})$, $\lambda \in \mathbb{R}$.

There is at most a denumerable set $\{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ of scalars for which $a(\lambda, f) \neq 0$, (these are called the Fourier-Bohr's exponents). The associated coefficients $\{a(\lambda_i, f)\}_{i\geq 1}$ are the Fourier-Bohr's coefficients. The sequence of Fourier-Bohr's coefficients of the function f will be denoted by $(\hat{f})$.

Questions concerning the convergence of the formal Fourier series $S(f)(x) = \sum_{n\geq 1} a(\lambda_n, f)\, e^{-i\lambda_n x}$ are not trivial and only partial results are available.

The Bochner approximation result will be of importance here:

Let $f \in B^\phi a.p.$ and $S_n(f)(x) = \sum_{K=1}^{n} a(\lambda_K, f)\, e^{i\,\lambda_K\, x}$, be the partial sums of it's formal Fourier series. There exists a sequence $\sigma_m(f)$, $m \geq 1$ of trigonometric polynomials (the Bochner-Fejér's approximation polynomials) of the form,

$$\sigma_m(f)(x) = \sum_{K=1}^{r_m} \mu_{m\,K}\, a(\lambda_K, f) e^{i\lambda_K x}$$

where the convergence factors $\{\mu_{m\,K}\}$ depend only on the sequence of characteristic exponents $\{\lambda_K\}$ of the function f and satisfy $0 < \mu_{m\,K} \le 1$.

The sequence $\{\sigma_m(f)\}$ has the following approximation properties (cf. [2], [3]):

$$\begin{aligned} &\text{i)} \quad \|\sigma_m(f)\|_{B^\phi} \le \|f\|_{B^\phi}, \ \ m = 1, 2, \dots \\ &\qquad \rho_{B^\phi}(\sigma_m(f)) \le \rho_{B^\phi}(f), \ m = 1, 2, \dots \\ &\text{ii)} \quad \|\sigma_m(f) - f\|_{B_\phi} \to 0, \text{ when } \ m \to \infty\,. \end{aligned} \tag{3.3}$$

Notice that from (3.1), (3.2) and the properties i)-ii) of the Bochner-Fejér's approximation polynomials we deduce easily that $\|.\|_{B^\phi}$ is in fact a norm on $C^o a.p.$.

To end this section, we summarize in the following the inclusion relations in the class of Besicovitch-Orlicz spaces and between Orlicz sequence spaces (cf. [3], [9]).

Proposition 3.1. *Let ϕ_1 and ϕ_2 be two Orlicz functions such that $\phi_1 \le \phi_2$ (see (2.2) for the definition) then, there hold:*

i) $B^{\phi_2}(\mathbb{R}) \subseteq B^{\phi_1}(\mathbb{R})$; $B^{\phi_2}a.p. \subseteq B^{\phi_1}a.p.$

ii) $\|.\|_{B^{\phi_1}} \le \alpha\ \|.\|_{B^{\phi_2}}$ *for some $\alpha > 0$ depending on ϕ_1 and ϕ_2.*

iii) $l^{\phi_1} \subseteq l^{\phi_2}$.

iv) $\|.\|_{l^{\phi_2}} \le \beta \|.\|_{l^{\phi_1}}$ *for some $\beta > 0$ depending on ϕ_1 and ϕ_2.*

v) *if $(\phi_i, \psi_i), i = 1, 2$ are two pairs of complementary Orlicz functions, we have,*

$$\phi_1 \le \phi_2 \Rightarrow \psi_2 \le \psi_1\,.$$

4 – Convergence results in $B^\phi a.p.$

A sequence $\{f_k\}_{k\ge1}$ from $B^\phi(\mathbb{R})$ is called modular convergent to some $f \in B^\phi(\mathbb{R})$ when $\lim_{k\to\infty} \rho_{B^\phi}(f_k - f) = 0$.

Let $P(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ and$\Sigma(\mathbb{R})$ the Σ- algebra of it's Lebesgue mesurable sets. For $A \in \Sigma$, we define the set function,

$$\begin{aligned} \overline{\mu}(A) &= \overline{\lim_{T\to\infty}} \frac{1}{2\,T} \int_{-T}^{+T} \chi_A(t) d\mu = \\ &= \overline{\lim_{T\to\infty}} \frac{1}{2\,T}\, \mu\Big(A \bigcap [-T, T]\Big) \end{aligned}$$

Clearly, $\overline{\mu}$ is null on sets with μ-finite measure. Moreover, $\overline{\mu}$ is not σ-additive.

As usual, a sequence $\{f_k\}_{k\geq 1}$ will be called $\overline{\mu}$-convergent to f when, for all $\varepsilon > 0$,

$$\lim_{k\to+\infty} \overline{\mu}\{x \in \mathbb{R}, |f_k(x) - f(x)| > \varepsilon\} = 0 .$$

We now state some fundamental convergence results that will be used below (cf. [5], [6]):

PROPOSITION 4.1. *Let $\{f_k\}_{k\geq 1}$ be a sequence of functions from $B^\phi(\mathbb{R})$. We have the following:*

i) *suppose there exist $f \in B^\phi(\mathbb{R})$ such that $\lim_{k\to+\infty} \rho_{B^\phi}(f_k - f) = 0$ and $g \in B^\phi a.p.$ such that* $\max(|f_k(x)|, |f(x)|) \leq g$. *Then, $\lim_{k\to+\infty} \rho_{B^\phi}(f_k) = \rho_{B^\phi}(f)$.*

ii) *If $f \in B^\phi a.p.$ and $\{P_n\}$ is the associated sequence of Bochner-Fejér's polynomials, we have: $\lim_{n\to+\infty} \rho_{B^\phi}(P_n) = \rho_{B^\phi}(f)$ (and $\lim_{n\to+\infty} \| P_n\|_{B^\phi} = \|f\|_{B^\phi}$).*

iii) *If $f \in B^\phi a.p.$ is such that $\lim_{n\to+\infty} \rho_{B^\phi}(f_n - f) = 0$, then*
   a) $\underline{\lim}_{n\to+\infty} \rho_{B^\phi}(f_n) \geq \rho_{B^\phi}(f)$.
   b) *$\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to f.*

5 – Auxiliary results

LEMMA 5.1. *Let $f \in B^\phi a.p.$, $f \neq 0$ and $\{f_n\}_{n\geq 1}$ be a sequence modular convergent to f, then:*

there exist constants $\alpha_1, \beta_1, \theta_1$ with $\theta_1 \in]0, 1[, 0 < \alpha_1 < \beta_1$, $n_0 \in \mathbb{N}$, such that for the sets $G_n = \{t \in \mathbb{R}, \alpha_1 \leq |\, f_n(t)| \leq \beta_1\}$ we have $\overline{\mu}(G_n) \geq \theta_1$ $\forall n \geq n_0$.

PROOF. From [5], there exist α, β, θ with $\theta \in]0, 1[$, $0 < \alpha < \beta$ and $G = \{t \in \mathbb{R}, \alpha \leq |f(x)| \leq \beta\}$ such that $\overline{\mu}(G) \geq \theta$. Take $\alpha_1 = \frac{\alpha}{2}$ and $\beta_1 = \beta + \frac{\alpha}{2}$, $\theta_1 = \frac{\theta}{2}$.

Since $\{f_n\}_{n\geq 1}$is modular convergent to f, it is also $\overline{\mu}$-convergent to f (see iii)-b) of Proposition. 4.1) and then,

$$\overline{\mu}\Big\{t\in \mathbb{R}, |\ f_n(t)-f(t)|\geq \frac{\alpha}{2}\Big\}<\frac{\theta}{2}\quad \forall n\geq n_0\,.$$

Putting $G'_n = \{t \in G, |f_n(t) - f(t)| \geq \frac{\alpha}{2}\}$, it is easily seen that $G - G'_n \subset G_n\ \forall n \geq n_0$.

Finally

$$\overline{\mu}(G_n)\geq \overline{\mu}(G-G'_n)\geq \overline{\mu}(G)-\overline{\mu}(G'_n)\geq \theta-\frac{\theta}{2}=\theta_1\,. \qquad \square$$

LEMMA 5.2. *Let (ϕ,ψ) be a normalized pair of Orlicz functions.*

i) *if $f \in B^{\phi}a.p.$ then $\rho_{B^\phi}(\frac{f}{\|\,f\,\|_{B^\phi}}) = \phi(1)$ $(\|\,f\,\|_{B^\phi}\neq 0)$.*

Moreover if $f \in C^o a.p$ we have $\rho_{B^\phi}(f) = \phi(1)$ iff $\|\,f\,\|_{\phi} = 1$.

ii) $\mathbf{M}(|f\,g|)\leq\|f\|_{B^\phi}.\|g\|_{B^\psi}$; *$f\in B^\phi a.p$, $g\in B^\psi.a.p$ (Hölder's inequality).*

iii) *if $f \in B^\phi a.p, g \in B^\psi a.p$ then $fg \in B^1 a.p.$*

PROOF. i) Let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty}\varepsilon_n = 0$ and $f_n = \frac{f}{\|f\|_{B^\phi}+\varepsilon_n}$. We have $\rho_{B^\phi}(f_n) \leq \phi(1)$ and the sequence (f_n) is modular convergent to $\frac{f}{\|f\|_{B^\phi}}$.

Using Proposition 4.1 we deduce,

$$\lim_{n\to\infty}\rho_{B^\phi}\Big(\frac{f}{\|f\|_{B^\phi}+\varepsilon_n}\Big)=\rho_{B^\phi}\Big(\frac{f}{\|f\|_{B^\phi}}\Big)\leq\phi(1)\,.$$

On the other hand, by similar arguments for the sequence $g_n = \frac{f}{\|f\|_{B^\phi}-\varepsilon_n}$, we get $\rho_{B^\phi}(\frac{f}{\|f\|_{B^\phi}}) \geq \phi(1)$.

Finally we have $\rho_{B^\phi}(\frac{f}{\|f\|_{B^\phi}}) = \phi(1)$.

Suppose now that $f \in C^o a.p.$ and $\rho_{B^\phi}(\frac{f}{a}) = \phi(1)$ for some $a > 0$.

The function $\phi(\frac{|f|}{a})$ is also in $C^o a.p.$ From (3.2) we have $\phi(\frac{|f|}{a}) = \phi(\frac{|f|}{\|f\|_{B^\phi}})$.

Now, since ϕ is stricty increasing we get $\|f\|_{B^\phi} = a$

ii) Let $\|f\|_{B^\phi} \neq 0, \|g\|_{B^\psi} \neq 0$, then, from Young's inequality we have:

$$\begin{aligned}\mathbf{M}\Big(\frac{|f|}{\|f\|_{B^\phi}}\frac{|g|}{\|g\|_{B^\psi}}\Big) &\leq \rho_{B^\phi}\Big(\frac{f}{\|f\|_{B^\phi}}\Big) + \rho_{B^\psi}\Big(\frac{g}{\| g\|_{B^\psi}}\Big) \leq \\ &\leq \phi(1) + \psi(1) = 1\end{aligned}$$

and then

$$\mathbf{M}(|f\, g|) \leq \|f\|_{B^\phi}\|g\|_{B^\psi} .$$

iii) Let (P_n) and (Q_n) be the sequences of Bochner-Fejér's polynomials that converge to f and g in the respective norms. Using ii) of this lemma and i) of (3.3), we get,

$$\mathbf{M}(|f\, g - P_nQ_n|) \leq \|f\|_{B^\phi}\|g - Q_n\|_{B^\psi} + \|g\|_{B^\psi}\|f - P_n\|_{B^\phi}$$

and the desired result follows immediately. ∎

Lemma 5.3. *Let $P \in \mathcal{P}$. The function $F : [0,+\infty[\to [0,+\infty[$, $F(k) = \rho_{B^\psi}(\phi'(k|P|)$ is continuous on $[0,+\infty[$. Moreover it satisfies $F(0) = 0$ and $\lim_{k\to\infty} F(k) = +\infty$, so that $F(k_0) = 1$ for some $k_0 \in]0,+\infty[$.*

Proof. Since the functions ψ and ϕ' are continuous, we have $\psi(\phi'(k|P|)) \in C^o a.p.$ and then if $k > 0$ and $P \neq 0$, we get from (3.2), $\rho_{B^\psi}(\phi'(k|P|)) > 0$.

Let now x_0 be such that $P(x_0) \neq 0$ and let $\alpha > 0$ satisfies $|P(x_0)| > \alpha$. There exists $\delta > 0$ for which $|P(x)| > \frac{2}{3}\alpha$, $\forall x \in (x_0 - \delta, x_0 + \delta)$.

Let $l_{\frac{\alpha}{3}} > 2\delta$ be an inclusion length of P corresponding to $\varepsilon = \frac{\alpha}{3}$. Then, each intervall of length $l_{\frac{\alpha}{3}}$ contains at least one number of the form $x_0+\tau$, with $\tau \in \mathcal{E}\,(\frac{\alpha}{3}, P)$, where $\mathcal{E}(\frac{\alpha}{3}, P)$ is the set of translation numbers of P corresponding to $\varepsilon = \frac{\alpha}{3}$. Such an intervall (of length $l_{\frac{\alpha}{3}}$) contains at least one of the intervalls $(x_0 + \tau - \delta, x_0 + \tau)$ or $(x_0 + \tau, x_0 + \tau + \delta)$.

Consequently, in each intervall of length $l_{\frac{\alpha}{3}}$, there exists a subintervall of length $\delta > 0$ where $|P(x)| > \frac{\alpha}{3}$.

It follows,

$$\int_a^{a+l_{\frac{\alpha}{3}}} \psi[\phi'(k|P(t)|)]\,d\mu > \psi\Big[\phi'\Big(k\frac{\alpha}{3}\Big)\Big]\,\delta \text{ and then,}$$

$$\rho_{B^\psi}[\phi'(k|P|)] = \lim_{n\to\infty}\frac{1}{n\,l_{\frac{\alpha}{3}}}\int_0^{n\,l_{\frac{\alpha}{3}}} \psi[\phi'(k|P(t)|)]\,dt \geq$$
$$\geq \frac{\psi\Big[\phi'\Big(k\frac{\alpha}{3}\Big)\Big]}{l_{\frac{\alpha}{3}}}\delta$$

Now, since an Orlicz function increases to infinity with it's derivative (cf. [4]. [8]) we get $\lim_{k\to\infty} F(k) = +\infty$.

We now show that F is continuous.

For, let $k_0 \in]0,\infty[$ and $\{k_n\}$ be a sequence of scalars converging to k_0.

A trigonometric polynomials being uniformly bounded, we put $\|P\|_\infty = M$.

Using the uniform continuity of ϕ' on the intervall $[\frac{k_0}{2}, \frac{3\,k_0}{2}]$, we get,

$\forall\varepsilon > 0$, $\exists n_0$ such that $n \geq n_0 \Rightarrow |\phi'(k_n|P|) - \phi'(k_0|P|)| \leq \psi^{-1}(\varepsilon)$ and then,

$$(*) \qquad \rho_{B^\psi}[\phi'(k_n\,|P|) - \phi'(k_0|P|)] \leq \varepsilon$$

Let us put $f_n = \phi'(k_n\,|P|)$ and $f = \phi'(k_0|P|)$, then clearly $f_n \in C^o a.p.$ and $f \in C^o a.p.$ Since ϕ' is increasing we have $f_n \leqslant \phi'(2k_0|P|)$ and from $(*)$ it follows $\lim_{n\to\infty} \rho_{B^\psi}(f_n - f) = 0$. Finally in view of Proposition 4.1-i) we get $\lim_{n\to\infty} \rho_{B^\psi}(f_n) = \rho_{B^\psi}(f)$, which means that F is continuous at k_0.

Now, since $F(0) = 0$ and $\lim_{k\to\infty} F(k) = +\infty$, there exists $k_0 \in]0,\infty[$ for which $F(k_0) = \rho_{B^\psi}[\phi'(k_0|P|)] = 1$. □

Lemma 5.4. *Let $f \in B^\phi a.p.$, then,*

i) $|||f|||_{B^\phi} = \inf\{\frac{1}{k}(1 + \rho_{B^\phi}(k\,f));\ k > 0\}$.

ii) $\rho_{B^\phi}(\frac{f}{|||f|||_{B^\phi}}) \leq 1$.

iii) $\phi(1).\|f\|_{B^\phi} \leq |||f|||_{B^\phi} \leq \frac{1}{\psi(1)}\|f\|_{B^\phi}$.

Proof. i) The proof will be down in several steps.

a) From iii) of Lemma 5.2 we have,

$$\begin{aligned}|||f|||_{B^\phi} &= \sup\{\mathbf{M}(|f\,g|), \rho_{B^\psi}(g) \leq 1, g \in B^\psi.a.p\} \leq \\ &\leq \frac{1}{\psi(1)} \sup\{\mathbf{M}(|f\,h|), \rho_{B^\psi}(h) \leq \psi(1), h \in B^\psi.a.p\}\end{aligned}$$

Now, since $\rho_{B^\psi}(h) \leq \psi(1)$ implies $\|h\|_{B^\psi} \leq 1$, using Young's inequality we get

$$|||f|||_{B^\phi} \leq \frac{1}{\psi(1)} \|f\|_\phi \tag{5.1}$$

b) Let P$\in \mathcal{P}$ then, there exists $k_0 \in]0, \infty[$ such that,

$$|||P|||_{B^\phi} = \frac{1}{k_0}(1 + \rho_{B^\phi}(k_0 P))\,.$$

Indeed, from Young's inequality we have:

$$\mathbf{M}(|P\,g|) = \frac{1}{k}\mathbf{M}(\,|k\,P\,g|) \leq \frac{1}{k}[\rho_{_{B^\phi}}(k\,P) + \rho_{B^\psi}(g)] \quad \forall k > 0$$

and then, $|||P|||_{B^\phi} \leq \inf_{k>0} \frac{1}{k}[\rho_{B^\phi}(k\ P) + 1]$.

Now, considering the case of equality in the Young's inequality and using Lemma 5.3, we get,

$$\begin{aligned}|||P|||_{B^\phi} &\geq \frac{1}{k_0}\mathbf{M}(|k_0 P|\phi'(k_0|P|)) = \frac{1}{k_0}[\rho_{B^\phi}(k_0 P) + \rho_{B^\psi}(\phi'(k_0|P|))] = \\ &= \frac{1}{k_0}[\rho_{B^\phi}(k_0 P) + 1]\end{aligned}$$

finally,

$$|||P|||_{B^\phi} = \inf_{k>0} \frac{1}{k}[\rho_{B^\phi}(k\,P) + 1] = \frac{1}{k_0}[\rho_{B^\phi}(k_0 P) + 1]\,.$$

c) We now show that the result of b) remains true for $f \in B^\phi a.p.$

For, let $\{P_n\}$ be the sequence of Bochner-Fejér polynomials of the approximation of f. From b) we know that,

$$\forall n \geq 1, \exists k_n \in]0,\infty[\text{ such that } |||P_n|||_{B^\phi} = \frac{1}{k_n}[\rho_{B^\phi}(k_n P_n) + 1] \tag{5.2}$$

from (5.1) and the properties of the Bochner-Fejér's polynomials (see i) of (3.3)), we get:

$$\frac{1}{k_n} \leq |||P_n|||_{B^\phi} \leq \frac{1}{\psi(1)}\|P_n\|_{B^\phi} \leq \frac{1}{\psi(1)}\|f\|_{B^\phi}.$$

and thus $k_n \geq \frac{\psi(1)}{\|f\|_{B^\phi}} = C_1 > 0$.

We now show that $k_n \leq C_2$, $\forall n \geq 0$ for some constant C_2.

Indeed, if this is not the case, there will exists a subsequence denoted by$\{k_n\}$ increasing to infinity and then:

$$\begin{aligned}1 = \rho_{B^\psi}(\phi'(k_n|P_n|)) = \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\psi[\phi'(k_n|P_n|)]dt \geq \\ \geq \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{G_n}\psi[\phi'(k_n|P_n|)]dt \geq \\ \geq \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{G_n}\psi[\phi'(k_n\alpha_1)]dt \geq \theta_1.\psi[\phi'(k_n\alpha_1)] \to \infty \text{ as } n \to \infty\end{aligned}$$

where G_n, θ_1, α_1 are defined in Lemma 5.1. A contradiction.

Now, $\{k_n\}$ being bounded, there exists a subsequence denoted by $\{k_n\}$ converging to k_0, $0 < k_0 < +\infty$.

Let us show that $\lim_{n\to\infty}\rho_{B^\phi}(k_nP_n) = \rho_{B^\phi}(k_0 f)$. Indeed, we have by i) of (3.3),

$$\begin{aligned}\rho_{B^\phi}(k_nP_n - k_0 f) \leq \frac{1}{2}\rho_{B^\phi}[2(k_n - k_0)P_n] + \frac{1}{2}\rho_{B^\phi}[2k_0(P_n - f)] \leq \\ \leq |k_n - k_0|\,\rho_{B^\phi}(f) + \frac{1}{2}\rho_{B^\phi}[2k_0(P_n - f)]\end{aligned}$$

and then $\lim_{n\to\infty}\rho_{B^\phi}(k_nP_n - k_0\,f) = 0$. Now, in view of Proposition 4.1-iii) it follows that

$$\underline{\lim}_{n\to+\infty}\rho_{B^\phi}(k_nP_n) \geq \rho_{B^\phi}(k_0 f)\,.$$

In the other hand, from the inequality $\rho_{B^\phi}(k_nP_n) \leq \rho_{B^\phi}(k_n f)$ (see-i) of (3.3)) we have also,

$$\overline{\lim_{n\to+\infty}}\rho_{B^\phi}(k_nP_n) \leq \overline{\lim_{n\to+\infty}}\rho_{B^\phi}(k_n f) = \lim_{n\to\infty} \rho_{B^\phi}(k_n f) = \rho_{B^\phi}(k_0 f)$$

and then,

$$\overline{\lim_{n\to+\infty}}\rho_{B^\phi}(k_nP_n) \leq \rho_{B^\phi}(k_0 f) \leq \underline{\lim}_{n\to+\infty} \rho_{B^\phi}(k_nP_n)$$

this proves the desired result.

Finally, letting $n \to \infty$ in (5.2) we get$|||f|||_{B^\phi} = \frac{1}{k_0}[\rho_{B^\phi}(k_0 f) + 1]$.

ii) Suppose first that $f \in C^o a.p.$, $f \neq 0$. Let $g \in B^\psi a.p.$ then,

a) if $\rho_{B^\psi}(g) \leq 1$, we have $\mathbf{M}(|f\,g|) \leq |||f|||_{B^\phi}$.

b) if $\rho_{B^\psi}(g) > 1$, $\rho_{B^\psi}(\frac{g}{\rho_{B^\psi}(g)}) \leq \frac{1}{\rho_{B^\psi}(g)}\rho_{B^\psi}(g) = 1$ and then $\mathbf{M}(|f\frac{g}{\rho_{B^\psi}(g)}|) \leq |||f|||_{B^\phi}$.

It follows that in all cases we have, $\mathbf{M}(|f\,g|) \leq \max(1, \rho_{B^\psi}(g))|||f|||_{B^\phi}$.

Suppose now that $g = \phi'(\frac{f}{|||f|||_{B^\phi}})$, then $g \in C^o a.p.$.

Using the case of equality in the Young's inequality and the fact that in this case the limits exist, we will have:

$$\mathbf{M}\Big(\Big|\frac{f}{|||f|||_{B^\phi}}.g\Big|\Big) = \rho_{B^\phi}\Big(\frac{f}{|||f|||_{B^\phi}}\Big) + \rho_{B^\psi}(g) \leq \max(1, \rho_{B^\psi}(g))\,.$$

So that we get, $\rho_{B^\phi}(\frac{f}{|||f|||_{B^\phi}}) \leq 1$.

Consider now the case of $f \in B^\phi a.p.$

Let P_n be the sequence of Bochner-Fejér's polynomials of the approximation of f, we have:

$$\rho_{B^\phi}\Big(\frac{P_n}{|||P_n|||_{B^\phi}}\Big) \leq 1\,, \qquad \forall n \geq 1\,.$$

But, in view of Lemma 5.4-i) and i) of (3.3), we can write:

$$|||P_n|||_{B^\phi} = \inf_{k>0} \frac{1}{k}(1 + \rho_{B^\phi}(k\,P_n)) \leq \inf_{k>0} \frac{1}{k}(1 + \rho_{B^\phi}(kf)) = |||f|||_{B^\phi}\,.$$

So that, $\rho_{B^\phi}(\frac{P_n}{|||f|||_{B^\phi}}) \leq \rho_{B^\phi}(\frac{P_n}{|||P_n|||_{B^\phi}}) \leq 1$ and then by -ii) of Proposition 4.1, $\rho_{B^\phi}(\frac{|f|}{|||f|||_{B^\phi}}) \leq 1$.

iii) We have, $\rho_{B^\phi}(\frac{\phi(1)f}{|||f|||_{B^\phi}}) \leq \phi(1)\rho_{B^\phi}(\frac{f}{|||f|||_{B^\phi}}) \leq \phi(1)$ and then

$$\|f\|_{B^\phi} \leq \frac{1}{\phi(1)}|||f|||_{B^\phi} .$$

Now, in view of (5.1), we get:

$$\phi(1).\|f\|_{B^\phi} \leq |||\,f|||_{B^\phi} \leq \frac{1}{\psi(1)}\|f\|_{B^\phi} .$$

∎

Lemma 5.5. *Let $f \in B^\psi a.p.$ Then:*

$$|||f|||_{B^\psi} = \sup\{|\mathbf{M}(\mathbf{fQ})|, \mathbf{Q} \in \mathcal{P}, \rho_{\mathbf{B}^\phi}(\mathbf{Q}) \leq \mathbf{1}\}$$

Proof. We consider first the case when $f = P \in \mathcal{P}$.

Recall that from Lemma 5.3, there exists $0 < k_0 < +\infty$ such that $\rho_{B^\phi}(\psi'(k_0|P|)) = 1$ and,

$$\begin{aligned} |||P|||_{B^\psi} &= \mathbf{M}(|P|\psi'(k_0|P|)) = \\ &= \mathbf{M}(P(x).\operatorname{sign} P(x).\psi'(k_0|P(x)|)) \end{aligned}$$

Now, since $\operatorname{sign} P(x).\psi'(k_0|P(x)|) \in C^o a.p.$, it follows from the definition of the Orlicz norm that,

$$|||P|||_{B^\psi} = \sup\{|\mathbf{M}(P\,Q)|, Q \in C^o a.p., \rho_{B^\phi}(Q) \leq 1\}$$

(To see that $\operatorname{sign} P(x).\,\psi'(k_0|\,P(x)|) \in C^o a.p.$, remark that the function $F(u) = u\frac{\psi'(k_0|u|}{|u|}$ if $u \neq 0$ and $F(0) = 0$ is continuous so that $F(P) \in C^o a.p.$ if $P \in C^o a.p.$).

In fact, using the properties (3.3) of the Bochner-Fejer's approximation polynomials, we can easily show the following,

$$|||P|||_{B^\psi} = \sup\{|\mathbf{M}(P\,Q)|, Q \in \mathcal{P}, \rho_{B^\phi}(Q) \leq 1\} .$$

Consider now the general case of $f \in B^\psi a.p.$

Let $\{P_n\}$ be the sequence of Bochner-Fejèr's polynomials that converge to f in $B^\psi a.p.$

Put $I(f) = \sup\{|\mathbf{M}(f\,Q)|, Q \in \mathcal{P}, \rho_{B^\phi}(Q) \leq 1\}$ then, clearly $I(f) \leq |||f|||_{B^\psi}$.

Moreover, given $\varepsilon > 0$, there is $n_0 \geq 0$ such that for $n \geq n_0$ we have, $|||f - P_n|||_{B^\psi} \leq \varepsilon$ and $|||f|||_{B^\psi} \leq |||P_n|||_{B^\psi} + \varepsilon$.

Then, using the particular case and Hölder's inequality it follows,

$$\begin{aligned} |||f|||_{B^\psi} - \varepsilon \leq |||P_n|||_{B^\psi} &= \sup\{|\mathbf{M}(P_n\,Q)|, Q \in \mathcal{P}, \rho_{B^\phi}(Q) \leq 1\} \leq \\ &\leq \sup\{\|f - P_n\|_{B^\psi} . \|Q\|_{B^\phi}, Q \in \mathcal{P}, \rho_{B^\phi}(Q) \leq 1\} + \\ &\quad + \sup\{|\mathbf{M}(f\,Q)|,\, Q \in \mathcal{P},\, \rho_{B^\phi}(Q) \leq 1\} \leq \\ &\leq I(f) + \varepsilon \end{aligned}$$

Finally, $I(f) \leq |||f|||_{B^\psi} \leq I(f) + 2\varepsilon$.

Now, since $\varepsilon > 0$ is arbitrary, we get $I(f) = |||f|||_{B^\psi}$. This is the desired result. ∎

6 – Main results

We can now state and prove the main result.

Theorem 6.1 (Hausdorff-Young). *Let (ϕ, ψ) be a complementary pair of normalized Young's functions. Suppose:*

i) $\phi \leq \phi_0$ *where* $\phi_0(x) = \frac{1}{2} x^2$.

ii) $\psi'(x) \leq a_0 x^r$ $0 \leq x < +\infty$ *for some* $a_0 > 0$ *and* $r \geq 1$.

Then:

$$\begin{aligned} \|\hat{f}\|_{l^\psi} &\leq K_0 \|f\|_{B^\phi} \qquad f \in B^\phi p.p. \\ \|f\|_{B^\psi} &\leq K_1 \|\hat{f}\|_{l^\phi} \qquad f \in B^\psi p.p. \end{aligned}$$

where K_0 and K_1 are constants that depend on ϕ and ψ but not on f.

Proof. We first prove the theorem in the class $\mathcal{P}$ of generalised trigonometric polynomials.

For, let $G = G(t) = \sum_{j=1}^{n} d_j\, e^{i\lambda_j t}, d_j \in \mathbb{C},\, \lambda_j \in \mathbb{R}$.

We will show that there exists a constant $\gamma \geq 1$ depending only on ϕ and ψ, such that

$$\|G\|_{B^\psi} \leq \gamma \|\widehat{G}\|_{l^\phi}\,.$$

Put $\gamma = \sup\{\frac{\|G\|_{B^\psi}}{\|\widehat{G}\|_{l^\phi}}$, for $d_j \in \mathbb{C}$, $j = 1, 2, \ldots, n\}$.

It is easily seen that $\gamma = \sup\{\|G\|_{B^\psi},\ \|\widehat{G}\|_{l^\phi} = 1; d_j \in \mathbb{C}, j = 1, 2, \ldots, n\}$.

The sup being taken over all polynomials G such that $\|\widehat{G}\|_{l^\phi} = 1$, i.e.over all coefficients $(d_j)_{j\leq n}$ for which $\|\widehat{G}\|_{l^\phi} = 1$, the λ_i's being fixed.

Since the set $A = \{(d_j)_{j\leq n}, \|\widehat{G}\|_{l^\phi} = 1\}$ is compact and by the continuity of the mapping $T : G \to \|G\|_{B^\psi}$, it follows that $\gamma = \sup T(d_j)_{j\leq n}$ exists and is finite, more precisely, there exists a polynomial G such that,

$$\gamma = \frac{\|G\|_{B^\psi}}{\|\widehat{G}\|_{l^\phi}}\,. \tag{6.1}$$

We now show that γ depends only on ϕ and ψ (and not on$n \in \mathbb{N}$ or $\lambda_j \in \mathbb{R}$).

Define the function $g(x) = \psi'(\frac{|G(x)|}{\|G\|_{B^\psi}}).\,\mathrm{sign}(G(x))$.

Using -i) of Lemma 5.2 and (2.1), we can write,

$$\psi(1) = \rho_{B^\psi}\Big(\frac{|G|}{\|G\|_{B^\psi}}\Big) \leq \mathbf{M}\Big[\frac{|G|}{\|G\|_{B^\psi}}.\psi'\Big(\frac{|G|}{\|G\|_{B^\psi}}\Big)\Big] = \\ = \mathbf{M}\Big(g(x).\frac{G}{\|G\|_{B^\psi}}\Big)\,.$$

hence, $\mathbf{M}(gG) \geq \psi(1).\|G\|_{B^\psi}$ and then,

$$\psi(1).\|G\|_{B^\psi} \leq \mathbf{M}(gG) = \sum_{j=1}^{n} d_j\,\mathbf{M}(g(x)\,e^{i\lambda_j x}) = \sum_{j=1}^{n} a(G,\lambda_j)a(g,\lambda_j) \leq \\ \leq \|\widehat{G}\|_{l^\phi}.\|\widehat{g}\|_{l^\psi}\,.$$

Consequently, in view of (6.1), we get:

$$\|\widehat{g}\|_{l^\psi} \geq \psi(1)\frac{\|G\|_{B^\psi}}{\|\widehat{G}\|_{l^\phi}} \geq \gamma.\psi(1) \tag{6.2}$$

Recall that since G is a trigonometric polynomials and ψ' continuous, $|g|$ is a Bohr's almost periodic function and then, Bessel 's inequality holds for g (cf. [2]):

$$(\|\widehat{g}\|_{l^2})^2 \leq \mathbf{M}(g^2) \leq \|g^2\|_{B^\psi}.\|1\|_{B^\phi} = \|g^2\|_{B^\psi} \tag{6.3}$$

Take $\psi_1(x) = \psi(x^2)$, it is easily seen that ψ_1 is an Orlicz function such that $\psi \leq \psi_1$.

Let us put $a^2 = \|g^2\|_{B^\psi}$ then, since $\psi(g^2)$ is a Bohr's almost periodic function, we have $0 < a < \infty$ and then using -i) of Lemma 5.2,

$$\psi_1(1) = \psi(1) = \rho_{B^\psi}\Big(\frac{g^2}{a^2}\Big) = \rho_{B^{\psi_1}}\Big(\frac{g}{a}\Big)$$

and thus

$$a = \|g\|_{B^{\psi_1}} \tag{6.4}$$

From (6.3) and (6.4) it follows immediately:

$$\|\widehat{g}\|_{l^2} \leq \|g\|_{B^{\psi_1}} \tag{6.5}$$

Let now $\psi_2(x) = \psi_1(x^r)$. From (6.1), (6.4) and the hypothesis ii) of the theorem, we can write,

$$\begin{aligned} \psi_2(1) = \psi_1(1) = \psi(1) = \rho_{B^{\psi_1}}\Big(\frac{g}{a}\Big) = \rho_{B^{\psi_1}}\Big(\frac{1}{a}\,\psi'\Big(\frac{|G(t)|}{\|G\|_{B^\psi}}\Big)\Big) \leq \\ \leq \rho_{B^{\psi_1}}\Big[\frac{a_0}{a}\Big(\frac{|G(t)|}{\|G\|_{B^\psi}}\Big)^r\Big] \leq \rho_{B^{\psi_2}}\Big(\beta_1\frac{|G(t)|}{\|G\|_{B^\psi}}\Big) \end{aligned} \tag{6.6}$$

where

$$(\beta_1)^r = \frac{a_0}{a} \tag{6.7}$$

Consequently, we have $\|\beta_1\frac{|G(t)|}{\|G\|_{B^\psi}}\|_{B^{\psi_2}} \geq 1$ since in the opposite case we will have $\rho_{B^{\psi_2}}(\frac{\beta_1\frac{|G(t)|}{\|G\|_\Psi}}{\alpha}) \leq \psi_2(1)$ for some $0 < \alpha < 1$ and then G being a trigonometric polynomials, from (3.2) we will deduce $\rho_{B^{\psi_2}}(\beta_1 \frac{|G(t)|}{\|G\|_{B^\psi}}) < \psi_2(1)$, a contradiction.

Hence

$$\Big\|\beta_1 \frac{|G(t)|}{\|G\|_{B^\psi}}\Big\|_{B^{\psi_2}} \geq 1 \tag{6.8}$$

From (6.7) and (6.8) it follows,

$$\Big[\frac{\|G\|_{B^{\psi_2}}}{\|G\|_{B^\psi}}\Big]^r \geq \frac{a}{a_0} = \frac{1}{a_0}\|g\|_{B^{\psi_1}} \tag{6.9}$$

Considering now the hypothesis i) of the theorem, Proposition 3.1 implies the following inequalities and relations inclusion, $\psi_0 \leq \psi \leq \psi_1 \leq \psi_2$ and then $\phi_0 \geq \phi \geq \phi_1 \geq \phi_2$. From this it follows, $l^{\phi_2} \subseteq l^{\phi_1} \subseteq l^\phi \subseteq l^2 \subseteq l^\psi \subseteq l^{\psi_1} \subseteq l^{\psi_2}$.

On the other hand, from the hypothesis ii) of the theorem, we have,

$$\psi_2(x) = \psi_1(x^r) = \psi(x^{2\,r}) \leq x^{2\,r}\psi'(x^{2\,r}) \leq a_0 x^{2\,r}.(x^{2\,r})^r = a_0 x^{2\,r\,(r+1)}$$

let us put $\psi_3(x) = \frac{a_0}{r+1}|x|^{2\,r(r+1)}$, then $\psi_3(x) \geq \psi_2\ (x)\ \forall x \geq 0$.

From this and (6.1), (6.2), (6.5), (6.9) it follows,

$$\begin{aligned} \psi(1) &\leq \gamma.\psi(1) \leq \|\widehat{g}\|_{l^\psi} \leq K_2\|\widehat{g}\|_{l^2} \leq \\ &\leq K_2\|g\|_{B^{\psi_1}} \leq K_2.a_0\Big[\frac{\|G\|_{B^{\psi_3}}}{\|G\|_{B^\psi}}\Big]^r \leq \\ &\leq K_2.a_0\Big[\frac{M_{\phi_3}}{\gamma}\frac{\|\widehat{G}\|_{l^{\phi_3}}}{\|\widehat{G}\|_{l^\phi}}\Big]^r \leq K_2.a_0\Big[\beta_4\frac{M_{\phi_3}}{\gamma}\Big]^r \end{aligned} \tag{6.10}$$

where K_2 is the constant of the inequality $\psi(x) \geq \psi_0(x) = \frac{1}{2}x^2$ (i.e. $\|.\|_{l^\psi} \leq K_2\|.\|_{l^2}$).

M_{ϕ_3} is the constant of the Hausdorff-Young inequality in B^q with $q = 2r(r+1)$.

The constant β_4 is from the inequality $\|\widehat{G}\|_{l^{\phi_3}} \leq \beta_4\|\widehat{G}\|_{l^\phi}$ which is a consequence of an inequality $\phi_3(x) \leq \phi(x)\ \forall x, 0 \leq x \leq x_0$. This later may be obtained by a suitable restriction on ϕ near the origine.

Now, inequality (6.10) may be written in the form,

$$\psi(1) \leq \psi(1).\gamma^{r+1} \leq K_2 a_0.\beta_4 M_{\phi_3}^r$$

and then, $1 \leq \gamma \leq K_5$ where K_5 is a constant depending only on ϕ.

Finally,

$$\|P\|_{B^\psi} \leq \gamma \|\hat{P}\|_{l^\phi}, \qquad \forall P \in \mathcal{P}\,. \tag{6.11}$$

where γ is a constant depending only on ϕ.

To show the converse inequality, let $P(t) = \sum_{j=1}^n c_j\, e^{i\lambda_j t}$ be a trigonometric polynomial. Putting $G(t) = \sum_{j=1}^n d_j e^{-i\lambda_j t}$, we will have, $|\mathbf{M}(PG)| = |\sum_{j=1}^n c_j\, d_j|$ and $|\mathbf{M}(P\,G)| \leq \|P\|_{B^\phi}.\|G\|_{B^\psi}$, thus, by (6.11): $|\mathbf{M}(PG)| \leq \|P\|_{B^\phi}.\ \|G\|_{B^\psi} \leq \gamma \|P\|_{B^\phi}.\|\widehat{G}\|_{l^\phi}$ so that,

$$\tag{6.12} \begin{cases} |||\hat{P}|||_{l^\psi} = \sup\Big\{\Big|\sum_{j=1}^n c_j d_j\Big|, (d_j)_{j\geq 1}; \sum_{j=1}^\infty \phi(|d_j|) \leq 1\Big\} \leq \\ \leq \sup\Big\{\Big|\sum_{j=1}^n c_j d_j\Big|, (d_j)_{j\geq 1}; \sum_{j=1}^n \phi(|d_j|) \leq 1\Big\} \leq \\ \leq \sup\Big\{\Big|\sum_{j=1}^n c_j d_j\Big|, (d_j)_{j\geq 1}; \sum_{j=1}^n \phi(\phi(1)|d_j|) \leq \phi(1)\Big\} \leq \\ \leq \dfrac{1}{\phi(1)} \sup\Big\{\Big|\sum_{j=1}^n c_j h_j\Big|, (h_j)_{j\geq 1}; \sum_{j=1}^n \phi(|h_j|) \leq \phi(1)\Big\} \leq \\ \leq \dfrac{1}{\phi(1)} \sup\Big\{\Big|\sum_{j=1}^n c_j h_j\Big|, \|\widehat{G}\|_{l^\phi} \leq 1\Big\} \leq \dfrac{\gamma}{\phi(1)} \|P\,\|_\phi \end{cases}$$

Now, using (2.3), we get finally,

$$\|\hat{P}\|_{l^\psi} \leq \frac{\gamma}{\phi(1).\psi(1)} \|P\|_{B^\phi} \tag{6.13}$$

The theorem is then proved in the class $\mathcal{P}$ of trigonometric polynomials. To consider the general case, let $f \in B^\phi a.p.$ and $\{P_n\}$ be the associated sequence of Bochner-Fejér polynomials that converge to f in the norms $\|.\|_{B^\phi}$ and $|||.|||_{B^\phi}$ since the laters are equivalent.

Let $\Lambda(f) = \{\lambda_1, \lambda_2, \dots\}$ be the set of Fourier-Bohr's exponents of f. It is known that $\Lambda(P_n) \subseteq \Lambda(f)$, $\forall n \geq 1$. Put $c_j = a(\lambda_j, f)$, $j \geq 1$ and denote by $(c_j^m)_j$ the finite sequence of Fourier-Bohr coefficients of P_m, $c_j^m = \mathbf{M}(P_m e^{-i\lambda_j t})$. (note that $c_j^m = 0$ if $\lambda_j \notin \Lambda(P_m)$).

Remark first that we have $|c_j - c_j^{(m)}| \le \mathbf{M}(|P_m - f|) \le \|P_m - f\|_{B^\phi} \to 0$ as $m \to +\infty$.

It follows then

$$\lim_{m\to\infty} \sup_j |c_j - c_j^{(m)}| = 0\,. \tag{6.14}$$

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Put $\alpha_0 = \phi^{-1}(1)$

From (6.14) there exists $m_0 = m_0(\varepsilon, n) > n$ such that $\forall m \ge m_0, |c_j| \le |c_j^{(m)}| + \frac{\varepsilon}{n}$ and then, for $m \ge m_0$; using (6.13) and - i) of (3.3),

$$\begin{aligned}
&\sup\Big\{\Big|\sum_{j=1}^{n} c_j\, d_j\Big|, (d_j)_{j\ge1}; \sum_{j=1}^{\infty}\phi(|d_j|) \le 1\Big\} = \\
&= \sup\Big\{\sum_{j=1}^{n} |c_j d_j|, (d_j)_{j\ge1}; \sum_{j=1}^{\infty}\phi(|d_j|) \le 1\Big\} \le \\
&\le \sup\Big\{\sum_{j=1}^{n} |c_j^m d_j|, (d_j)_{j\ge1}; \sum_{j=1}^{\infty}\phi(|d_j|) \le 1\Big\} + \varepsilon\alpha_0 \le \\
&\le |||\hat{P}_m|||_{l^\psi} + \varepsilon\alpha_0 \le \frac{1}{\phi(1)}\|\hat{P}_m\|_{l^\psi} + \varepsilon\alpha_0 \le \\
&\le \frac{\gamma}{[\phi(1)]^2.\psi(1)}\|P_m\|_{B^\phi} + \varepsilon\alpha_0 \le \frac{\gamma}{[\phi(1)]^2.\psi(1)}\|f\|_{B^\phi} + \varepsilon\alpha_0\,.
\end{aligned}$$

Since ε and n are arbitrary, we get finally,

$$\psi(1)\|\hat{f}\|_{l^\psi} \le |||\hat{f}|||_{l^\psi} \le \frac{\gamma}{[\phi(1)]^2.\psi(1)}\|f\|_{B^\phi},\ \text{i.e.}\|\hat{f}\|_{l^\psi} \le \frac{\gamma}{[\phi(1).\psi(1)]^2}\|f\|_{B^\phi}\,.$$

This proves the first inequality.

To prove the second inequality, let $f \in B^\psi a.p.$, then the hypothesis i) of the theorem implies $f \in B^2 a.p.$ and then, if we consider $P_n(x) = \sum_{j=1}^{n} a(\lambda_j, f)e^{i\lambda_j x}$, (the partial sums of the Fourier-Bohr's series of f), we have $\|f - P_n\|_{B^2} \to 0$ as $n \to \infty$ (cf. [1], [2]).

Moreover if $Q \in \mathcal{P}$ with $\Lambda(Q) \bigcap \Lambda(f) \neq \emptyset$, one has,

$$|\mathbf{M}(P_n Q)| = \Big|\sum_{j=1}^{n} a(\lambda_j, f)a(\lambda_j, Q)\Big| \le \|\hat{f}\|_{l^\psi}.\|\hat{Q}\|_{l^\psi}\,.$$

Now, using (6.13), we get,

$$|\mathbf{M}(P_nQ)| \leq \|\hat{f}\|_{l^\psi}.\|\hat{Q}\|_{l^\psi} \leq \frac{\gamma}{\phi(1).\psi(1)}.\|\hat{f}\|_{l^\psi}.\| Q\|_{B^\phi}\,.$$

and, since $\|f - P_n\|_{B^2} \to 0$ as $n \to \infty$, we have also,

$$|\mathbf{M}(PQ)| \leq \frac{\gamma}{\phi(1).\psi(1)}.\|\hat{f}\|_{l^\psi}.\|Q\|_{B^\phi}\,.$$

Finally, in view of Lemma 5.5 and -iii) of Lemma 5.4, we get,

$$\begin{aligned}|||f\,|||_{B^\psi} &= \sup\{|\mathbf{M}(f\,Q)|, Q \in \mathcal{P}; \rho_{B^\phi}(Q) \leq 1\} \leq \\ &\leq \sup\{\|Q\|_{B^\phi}, \rho_{B^\phi}(Q) \leq 1\}.\frac{\gamma}{\phi(1).\psi(1)}.\|\hat{f}\|_{l^\phi} \leq \\ &\leq \frac{\gamma}{[\phi(1)]^2.\psi(1)}.\|\hat{f}\|_{l^\phi}\,.\end{aligned}$$

Thus,

$$\psi(1)\|f\|_{B^\psi} \leq |||f|||_{B^\psi} \leq \frac{\gamma}{[\phi(1)]^2.\psi(1)}\|\hat{f}\|_{l^\phi},$$

i.e.

$$\|f\|_{B^\psi} \leq \frac{\gamma}{[\phi(1).\psi(1)]^2}.\|\hat{f}\|_{l^\phi}\,.$$

This proves the theorem. ∎

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