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Bernoulli numbers and polynomials from a more general point of view

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Riassunto: *Si applica il metodo della funzione generatrice per introdurre nuove forme di numeri e polinomi di Bernoulli che vengono utilizzati per sviluppare e calcolare somme parziali che coinvolgono polimoni a più indici e a più variabili. Si sviluppano considerazioni analoghe per i polinomi ed i numeri di Eulero.*

Abstract: *We apply the method of generating function, to introduce new forms of Bernoulli numbers and polynomials, which are exploited to derive further classes of partial sums involving generalized many index many variable polynomials. Analogous considerations are developed for the Euler numbers and polynomials.*

1 – Introduction

In a previous paper [1] we have derived partial sums involving Hermite, Laguerre and Appèll polynomials in terms of generalized Bernoulli polynomials. The new and interesting possibilities offered by this class of polynomials are better illustrated by an example relevant to the derivations of a partial sum involving two-index Hermite polynomials.

To this aim we remind that:

$$\sum_{n=0}^{N-1}(x+ny)^r = \frac{y^r}{r+1}\Big[B_{r+1}\Big(N+\frac{x}{y}\Big) - B_{r+1}\Big(\frac{x}{y}\Big)\Big] \tag{1}$$

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where $B_n(x)$ are Bernoulli polynomials defined by the generating function:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x) \tag{2}$$

or in terms of Bernoulli numbers B_n as:

$$B_n(x)=\sum_{s=0}^{n}\binom{n}{s}B_{n-s}x^s\,. \tag{3}$$

Hermite polynomials with two variables and one parameter can be defined by means of the operational identity [2]:

$$e^{\tau\frac{\partial^2}{\partial x\partial y}}\{x^m y^n\}=h_{m,n}(x,y\mid\tau)\,, \tag{4}$$

and the $h_{m,n}(x,y\mid\tau)$ are defined by the double sum:

$$h_{m,n}(x,y\mid\tau)=m!n!\sum_{s=0}^{\min(m,n)}\frac{\tau^s x^{m-s}y^{n-s}}{s!(m-s)!(n-s)!}\,. \tag{5}$$

The identity (4) can be used to state that:

$$e^{\tau\frac{\partial^2}{\partial x\partial y}}\{(ax+b)^m(cy+d)^n\}=h_{m,n}(ax+b,cy+d\mid ac\tau)\,, \tag{6}$$

and to introduce the following two variable one parameter Bernoulli polynomials:

$${}_hB_{r,s}(x,y\mid\tau)=\sum_{q=0}^{r}\sum_{k=0}^{s}\binom{r}{q}\binom{s}{k}B_{r-q}B_{s-k}h_{q,k}(x,y\mid\tau)\,. \tag{7}$$

It is easy to note that from equation (4) we get:

$$e^{\tau\frac{\partial^2}{\partial x\partial y}}\{B_r(x)B_s(y)\}={}_hB_{r,s}(x,y\mid\tau)\,. \tag{8}$$

This new class of Bernoulli polynomials can be used to derive the following important result:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h_{r,s}(x+my, z+nw \mid \tau) = \frac{y^r w^s}{(r+1)(s+1)}\times$$

(9) $$\times\left\{ {}_hB_{r+1,s+1}\Big(M+\frac{x}{y}, N+\frac{z}{w}\Big|\frac{\tau}{yw}\Big) - {}_hB_{r+1,s+1}\Big(M+\frac{x}{y}, \frac{z}{w}\Big|\frac{\tau}{yw}\Big) + \right.$$
$$\left. - {}_hB_{r+1,s+1}\Big(\frac{x}{y}, N+\frac{z}{w}, \frac{\tau}{yw}\Big) + {}_hB_{r+1,s+1}\Big(M+\frac{x}{y}, \frac{z}{w}\Big|\frac{\tau}{yw}\Big)\right\}$$

which is a consequence of equations (1), (6), (7), (8). The generating function of this last class of Bernoulli polynomials can be shown to be provided by:

(10) $$\frac{uve^{ux+vy+\tau uv}}{(e^u-1)(e^v-1)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}\big({}_hB_{m,n}(x,y \mid \tau)\big)\,.$$

This introductory examples shows that a wealth of implications is offered by the use of generalized forms of Bernoulli polynomials. In the forthcoming sections we will develop a more systematic analysis which yields a deeper insight into the effectiveness of this type of generalizations.

2 – Finite sums and new classes of Bernoulli numbers

In ref. [1] we have touched on the following new class of numbers:

(11) $${}_HB_n = \sum_{s=0}^{[n/2]}\frac{n!B_{n-2s}B_s}{s!(n-2s)!}$$

which are recognized as an Hermite convolution of Bernoulli numbers on themselves. The generating functions of the ${}_HB_n$ is provided by:

(12) $$\frac{t^3}{(e^t-1)(e^{t^2}-1)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}({}_HB_n)$$

which suggests the following generalization ($(a,b)\neq 0$):

(13) $$\frac{t^3}{(e^{at}-1)(e^{bt^2}-1)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\big({}_HB^*_n(a,b)\big)$$

with:

$$ {}_H B_n^*(a,b) = \frac{1}{ab} \sum_{s=0}^{[n/2]} \frac{n! a^{n-2s} b^s B_{n-2s} B_s}{s!(n-2s)!}\,. \tag{14} $$

It is also evident that the generating function:

$$ \frac{t^3 e^{xt+yt^2}}{(e^{at}-1)(e^{bt^2}-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left({}_H B_n^*(a,b \mid x,y)\right) \tag{15} $$

can be exploited to define the polynomials:

$$ {}_H B_n^*(a,b \mid x,y) = \sum_{s=0}^{n} \binom{n}{s} \left({}_H B_{n-s}^*(a,b) H_s(x,y)\right) \tag{16} $$

as results from equations (13) and (15) which are also expressed as:

$$ {}_H B_n^*(a,b \mid x,y) = \frac{n!}{ab} \sum_{s=0}^{[n/2]} \frac{n! a^{n-2s} b^s}{s!(n-2s)!} B_{n-2s}\left(\frac{x}{a}\right) B_s\left(\frac{y}{b}\right) \tag{17} $$

yielding:

$$ {}_H B_n^*(a,b \mid 0,0) = {}_H B_n^*(a,b) \tag{18} $$

where $H_n(x,y) = n! \sum_{s=0}^{[n/2]} \frac{y^s x^{n-2s}}{s!(n-2s)!}$ are the Kampè de Fèriet polynomials. The use of polynomials (16) is suggested by partial sums of the type:

$$ \begin{aligned} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} H_r(x+my, z+nw) &= \frac{1}{(r+1)(r+2)(r+3)} \times \\ \times \Big\{ {}_H B_{r+3}^*(y,w \mid x+My, z+Nw) &- {}_H B_{r+3}^*(y,w \mid x+My, z) + \\ - {}_H B_{r+3}^*(y,w \mid x, z+Nw) &+ {}_H B_{r+3}^*(y,w \mid x,z) \Big\}. \end{aligned} \tag{19} $$

which provides one of the main results of the paper.

A further application of polynomials ${}_HB_n^*(a,b \mid x,y)$ is relevant to the multiplication theorems. We find indeed:

$$\begin{aligned}&{}_HB_n(mx,m^2y)=\\ (20)\qquad &=\frac{m^{n-2}}{(n+1)(n+2)}\sum_{k=0}^{m-1}\sum_{h=0}^{m^2-1}\Big({}_HB_{n+2}^*\Big(1,1 \mid x+\frac{k}{m},y+\frac{h}{m^2}+\frac{1}{m^2}\Big)\Big)+\\ &-\Big({}_HB_{n+2}^*\Big(1,1 \mid x+\frac{k}{m},y+\frac{h}{m^2}\Big)\Big)\end{aligned}$$

and

$$(21)\quad {}_HB_n^*(a,b \mid mx,m^2y)=m^{n-3}\sum_{k=1}^{m-1}\sum_{h=1}^{m^2-1}\Big({}_HB_n^*\Big(a,b \mid x+\frac{ak}{m},y+\frac{bh}{m^2}\Big)\Big)$$

and:

$$(22)\quad {}_hB_{m,n}(px,qy \mid \tau)=p^{m-1}q^{n-1}\sum_{k=0}^{p-1}\sum_{h=0}^{q-1}\Big({}_hB_{m,n}\Big(x+\frac{k}{p},y+\frac{h}{q}\Big|\frac{\tau}{pq}\Big)\Big)$$

can be proved by exploiting the procedure outlined in appendix.

3 – Euler polynomials

The examples we have provided yields an idea of the implications offered by this type of generalization.

It is also evident that the considerations we have developed for Bernoulli polynomials can be extended to Euler polynomials [3]:

$$(23)\qquad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}E_n(x)\,.$$

In analogy of ref.[1] and the results of previous sections, we introduce the following classes of Euler polynomials:

$$(24)\qquad \frac{2e^{xt+yt^2}}{e^t+1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\big({}_HE_n(x,y)\big)$$

and:

$$\frac{2^2 e^{xu+yv+\tau uv}}{(e^u+1)(e^v+1)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}\big({}_hE_{m,n}(x,y\mid\tau)\big) \tag{25}$$

It is easily realized that:

$${}_HE_n(x,y) = \sum_{s=0}^{[n/2]}\frac{y^s E_{n-2s}(x)}{s!(n-2s)!} \tag{26}$$

and:

$${}_hE_{m,n}(x,y\mid\tau) = \sum_{s=0}^{m}\sum_{r=0}^{n}\binom{m}{s}\binom{n}{r}E_{m-s}E_{n-r}h_{s,r}(x,y\mid\tau) \tag{27}$$

where $E_n(x)$ are the ordinary Euler polynomials and that the following theorems hold:

$${}_HE_n(x+z,y+w) = \sum_{s=0}^{n}\binom{n}{s}\big({}_HE_{n-s}(x,y)\big)H_s(z,w) \tag{28}$$

and (see appendix):

$${}_HE_n(mx,py) = m^n\sum_{k=0}^{m-1}(-1)^k\Big({}_HE_n\Big(x+\frac{k}{m},\frac{py}{m^2}\Big)\Big). \tag{29}$$

As to the polynomials ${}_hE_{m,n}(x,y\mid\tau)$ wa can also state that:

$${}_hE_{m,n}(x+z,y+w)\mid\tau) = \sum_{s=0}^{m}\sum_{r=0}^{n}\binom{m}{s}\binom{n}{r}\big({}_hE_{m-s,n-r}(x,y\mid\tau)\big)z^s w^r \tag{30}$$

and:

$${}_hE_{m,n}(px,qy\mid\tau) = p^m q^n\sum_{k=0}^{p-1}\sum_{h=0}^{q-1}(-1)^{k+h}{}_hE_{m,n}\Big(x+\frac{k}{p},y+\frac{h}{q}\Big|\frac{\tau}{pq}\Big). \tag{31}$$

4 – Concluding remarks

The introduction of the Hermite-Euler polynomials given by equation (24) offers the possibility of speculating about alternative definitions as e.g. $(t < \sqrt{\pi})$:

$$\frac{4e^{xt+yt^2}}{(e^t+1)(e^{t^2}+1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(2)}(x,y)\,. \tag{32}$$

The polynomials $E_n^{(2)}(x,y)$ are defined as:

$$E_n^{(2)}(x,y) = n! \sum_{s=0}^{[n/2]} \frac{E_s(y)E_{n-2s}(x)}{s!(n-2s)!} \tag{33}$$

and are shown to satisfy the following differential equation:

$$\frac{\partial}{\partial y} E_n^{(2)}(x,y) = \frac{\partial^2}{\partial x^2} E_n^{(2)}(x,y) \tag{34}$$

and the multiplication formula:

$$E_n^{(2)}(mx, m^2 y) = m^n \sum_{k=0}^{m-1} \sum_{h=0}^{m^2-1} (-1)^{k+h} \Big(E_n^{(2)}\Big(x + \frac{k}{m}, y + \frac{h}{m^2} \Big)\Big)\,. \tag{35}$$

The more general theorem relevant to $E_n^{(2)}(mx, py)$ requires the introduction of a class of Euler polynomials analogous to those provided by equation (13) for the Bernoulli case.

Before concluding this paper we want to emphasize that most of the identities holding for the ordinary Bernoulli or Euler polynomials can be extended to the generalized case. For example the identity [4]:

$$B_n(x+1) - B_n(x) = nx^{n-1} \tag{36}$$

can be generalized as:

$${}_HB_n(x+1,y) - {}_HB_n(x,y) = nH_{n-1}(x,y) \tag{37}$$

which is a consequence of:

$$e^{y\frac{\partial^2}{\partial x^2}} B_n(x) = {}_H B_n(x,y)\,. \tag{38}$$

For analogous reasons, we find:

$$\begin{aligned} &{}_h B_{m,n}(x+1,y+1\mid\tau) - {}_h B_{m,n}(x,y+1\mid\tau) - {}_h B_{m,n}(x+1,y\mid\tau)+\\ &\quad + {}_h B_{m,n}(x,y\mid\tau) = mnh_{m-1,n-1}(x,y\mid\tau) \end{aligned} \tag{39}$$

and:

$${}_H B^*_n(a,b\mid x+a,y+b) - {}_H B^*_n(a,b\mid x,y) = n(n-1)(n-2)H_{n-3}(x,y) \tag{40}$$

and

$${}_H E^{(2)}_n(x+1,y) + {}_H E^{(2)}_n(x,y) = 2H_n(x,y)\,. \tag{41}$$

We can also define the further generalized form:

$$\prod_{i=1}^{N} \frac{t^{i+1}e^{x_i t^i}}{(e^{a_i t^i}-1)} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\big({}_H B^*_n(\{a_i\}|\{x_i\})\big) \tag{42}$$

thus getting $(n > N)$:

$${}_H B^*_n(\{a_i\}|\{x_i\}) - {}_H B^*_n(\{a_i\}|\{x_i\}) = \frac{n!}{(n-N)!} H_{n-N}(\{x_i\}) \tag{43}$$

with:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(\{x_i\}) = e^{\sum_{i=1}^{n} x_i t^i}\,. \tag{44}$$

The results if this paper show that the combination of operational rules and the properties of ordinary and generalized polynomials offer a wealth of possibilities to introduce new familes of Euler and Bernoulli polynomials which provides a powerful tool in applications.

– Appendix

The multiplication formulae are easily stated by exploiting the method of the generating functions and suitable manipulations. We note indeed that:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}\left({}_hB_{m,n}(px,qy\mid\tau)\right)=\frac{uve^{upx+vqy+\tau uv}}{(e^u-1)(e^v-1)}\,. \tag{A.1}$$

The r.h.s. of the above relation can be more conveniently rewritten as:

$$\frac{uve^{upx+vqy+\tau uv}}{(e^u-1)(e^v-1)}=\frac{(up)(vq)}{pq}\frac{uve^{upx+vqy+uvpq\frac{\tau}{pq}}}{(e^{up}-1)(e^{vq}-1)}\frac{(e^{up}-1)(e^{vq}-1)}{(e^u-1)(e^v-1)} \tag{A.2}$$

by nothing that:

$$\frac{e^{up}-1}{e^u-1}=\sum_{r=0}^{p-1}e^{ru} \tag{A.3}$$

we can rearrange (A.2) as:

$$\begin{aligned}\frac{uve^{upx+vqy+\tau uv}}{(e^u-1)(e^v-1)}&=\sum_{k=0}^{p-1}\sum_{h=0}^{q-1}\frac{1}{pq}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(up)^m}{m!}\frac{(vq)^n}{n!}\times\\&\times{}_hB_{m,n}\Big(x+\frac{k}{p},y+\frac{h}{q}\Big|\frac{\tau}{pq}\Big)\end{aligned} \tag{A.4}$$

which once confronted with (A.1) yields equation (22). A symilar procedure can be exploited to prove the multiplication formulae relevant to the Euler's generalized forms.

We note indeed:

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\left({}_HE_n(mx,py)\right)=\frac{2e^{mxt+pyt^2}}{e^t+1} \tag{A.5}$$

and handling the r.h.s. of the above equations, we find:

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\left({}_HE_n(mx,py)\right)=\frac{2e^{mxt}}{e^{mt}+1}\frac{e^{mt}+1}{e^t+1}e^{pyt^2}\,. \tag{A.6}$$

By noting that:

$$\text{(A.7)}\quad \frac{2e^{mxt}}{e^{mt}+1}\,\frac{e^{mt}+1}{e^{t}+1}e^{pyt^2} = \sum_{k=0}^{m-1}(-1)^k\sum_{q=0}^{\infty}\frac{t^q m^q}{q!}E_q\Big(x+\frac{k}{m}\Big)\sum_{r=0}^{\infty}\frac{t^{2r}p^r}{r!}y^r$$

we obtain:

$$\text{(A.8)}\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}\big({}_HE_n(mx,py)\big)=\sum_{n=0}^{\infty}t^r m^n\sum_{k=0}^{m-1}(-1)^k\sum_{r=0}^{[n/2]}\frac{E_{n-2r}(x+\frac{k}{m})}{(n-2r)!r!}\Big(\frac{py}{m^2}\Big)^r$$

and using the (26) we finally state the relation (29).

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