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The *b*-adic diaphony

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Riassunto: *Gli autori introducono una nuova misura numerica, denominata diafonia b-adica, per la distribuzione uniforme di successioni. Per la definizione della diafonia b-adica si usano le funzioni ortonormali di Chrestenson.*
Per un'ampia classe di successioni monodimensionali, le note successioni generalizzate di Van der Corput, si ottiene una utile formula per la somma trigonometrica di Chrestenson che permette di determinare limiti superiori ed inferiori per la diofania b-adica di queste successioni.
Si ottiene inoltre l'ordine esatto $\mathcal{O}(N^{-1}\sqrt{\log N})$ *della diofania b-adica delle successioni generalizzate di Van der Corput.*

Abstract: *The authors introduce a new numerical measure for uniform distribution of sequences, the so-called b-adic diaphony. For the definition of the b-adic diaphony we use a class of orthonormal system of functions, the so-called Chrestenson functions.*
For a very large class of one dimensional sequences, the well-known generalized Van der Corput sequences we obtain an useful formula for the Crestenson trigonometric sum. This result allows us to make upper and lower bounds of the b-adic diaphony of these sequences.
We obtain the exact order $\mathcal{O}(N^{-1}\sqrt{\log N})$ *of the b-adic diaphony of the generalized Van der Corput sequences.*

1 – Introduction

Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be a sequence in the s-dimensional unit cube $E^s = [0,1]^s$. For every subinterval $J \subset E^s$ and every positive integer N we

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denote $A_N(\xi; J) = \{\mathbf{x}_n : \mathbf{x}_n \in J, 0 \le n \le N-1\}$. The sequence ξ is called uniformly distributed in E^s if for every J we have $\lim_{N\to\infty} N^{-1} A_N(\xi; J) = \mu(J)$, where $\mu(J)$ is the Lebesgue measure of J.

For integrable function $f : [0,1)^s \to \mathbf{C}$ we define

$$S_N(f;\xi) = \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1)^s} f(\mathbf{x})d\mathbf{x}\,.$$

Let $\mathcal{T} = \{e_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbf{Z}^s, \quad \mathbf{x} \in E^s\}$, where $e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} \exp(2\pi \mathbf{i} k_j x_j)$, with $\mathbf{k} = (k_1, \dots, k_s)$ and $\mathbf{x} = (x_1, \dots, x_s)$ denotes the trigonometric functional system. The classical definition of the diaphony $F_N(\mathcal{T};\xi)$ of the first N elements of ξ is defined by Zinterhof [1] as

$$F_N(\mathcal{T};\xi) = \Big(\sum_{\mathbf{k}\ne\mathbf{0}} R^{-2}(\mathbf{k})|S_N(e_{\mathbf{k}};\xi)|^2\Big)^{\frac{1}{2}},$$

where $R(\mathbf{k}) = \prod_{j=1}^{s}\max(1,|k_j|)$. The diaphony is a numerical quantity of the irregularity of the distribution of sequences in E^s in sense that the sequence ξ is uniformly distributed mod 1 if and only if $\lim_{N\to\infty} F_N(\mathcal{T};\xi) = 0$.

The dyadic version of diaphony was introduced by Hellekalek and Leeb [2]. It is based on using the Walsh functional system [3] $\mathcal{W}(2) = \{w_k(x)\}_{k=0}^{\infty}$ $(2 \to x)$ in base 2. We will use the well-known functional system of Chrestenson [4] $\mathcal{W}(b) = \{w_k(x)\}_{k=0}^{\infty}$ $(b \to x)$, (see Definition 2) which includes the functional system of Walsh for obtaining a generalization of the dyadic diaphony. In [5] the authors have announced a b-adic version of the diaphony, the so-called b-adic diaphony. We will show that the b-adic diaphony is a numerical quantity of the irregularity of the distribution.

We will consider a very large class of sequences of rational elements, the so-called generalized. In the Theorems 3 and 4 we will obtain the exact order $\mathcal{O}(N^{-1}\sqrt{\log N})$ of the b-adic diaphony of the Van der Corput sequences [6].

Let $b \ge 2$ denote a fixed integer and put $\omega = \exp(\frac{2\pi\mathbf{i}}{b})$. Chrestenson [4] made a generalization of the Walsh functional system, exposed in the next two definitions:

Definition 1. The Rademacher functions of order b are defined by

$$r_0(x) = \omega^k, \quad \text{if} \quad \frac{k}{b} \le x < \frac{k+1}{b}, \quad k = 0, \dots, b-1, r_0(x+1) = r_0(x)$$

and for $n \ge 1 \quad r_n(x) = r_0(b^n x)$.

Definition 2. The Chrestenson functions of order b are defined by $w_0(x) = 1$, and if $n = a_1 b^{n_1} + \ldots + a_m b^{n_m}$, where for $1 \leq j \leq m$ $a_j \in \{1, \ldots, b-1\}$ and $n_1 > \ldots > n_m$, then

$$w_n(x) = r_{n_1}^{a_1}(x) \ldots r_{n_m}^{a_m}(x) .$$

We will use the signification $\mathcal{W}(b) = \{w_k(x)\}_{k=0}^{\infty}$ $(b \to x)$ for the Chrestenson functional system. If in the Definition 1 we put $b = 2$, the original Rademacher [7] functions are obtained, and if we put $b = 2$ in Definition 2 the original Walsh system is obtained in the Paley [8] terms.

If $\mathbf{k} = (k_1, \ldots, k_s)$ is a vector with nonnegative integer coordinates, then the $\mathbf{k}$-th Chrestenson function of order b $w_{\mathbf{k}}(\mathbf{x})$ on $[0,1)^s$ is defined as

$$w_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} w_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_s).$$

If f is an integrable function on $[0,1)^s$ and $\mathbf{k} = (k_1, \ldots, k_s)$ is a vector with nonnegative integer coordinates, then let $\widehat{f}(\mathbf{k})$ denote the $\mathbf{k}$-th Chrestenson-Fourier coefficient of f,

$$\widehat{f}(\mathbf{k}) = \int_{[0,1)^s} f(\mathbf{x}) \overline{w}_{\mathbf{k}}(\mathbf{x}) d\mathbf{x} ,$$

with respect to the Chrestenson function $w_{\mathbf{k}}$.

The set $G(b) = \{0, 1, \ldots, b-1\}$ becomes a group if for $\forall \alpha, \beta \in G(b)$ we define binary operations $\alpha \oplus \beta = \alpha + \beta \pmod{b}$ and

$$\alpha \ominus \beta = \begin{cases} \alpha - \beta & \text{if } \alpha \geq \beta \\ b + \alpha - \beta & \text{if } \alpha < \beta . \end{cases}$$

Let the reals $x, y \in [0,1)$ have a b-adic expansions in form $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$ and $y = \sum_{j=0}^{\infty} y_j b^{-j-1}$, as we will use the finite expansions for rational numbers. We define $x \dot{+} y = \sum_{j=0}^{\infty} (x_j \oplus y_j) b^{-j-1}$ and $x \dot{-} y = \sum_{j=0}^{\infty} (x_j \ominus y_j) b^{-j-1}$. If $\mathbf{x}, \mathbf{y} \in [0,1)^s$ with $\mathbf{x} = (x_1, \ldots, x_s)$ and $\mathbf{y} = (y_1, \ldots, y_s)$, then $\mathbf{x} \dot{+} \mathbf{y} = (x_1 \dot{+} y_1, \ldots, x_s \dot{+} y_s)$ and $\mathbf{x} \dot{-} \mathbf{y} = (x_1 \dot{-} y_1, \ldots, x_s \dot{-} y_s)$.

Each of the following conditions:

$$(C1) \quad x \dot{+} y \ \text{ is not a b-adic rational},$$
$$(C2) \quad x \ \text{ and } \ y \ \text{ are b-adic rationals},$$

implies that for $\forall n \geq 0$

$$w_n(x \dot{+} y) = w_n(x) w_n(y) \quad \text{and} \quad w_n(x \dot{-} y) = w_n(x)\overline{w}_k(y)\,.$$

Let arbitrary real $x \in [0,1)$ have b-adic expansion $x = \frac{x_g}{b^{g+1}} + \frac{x_{g+1}}{b^{g+2}} + \ldots$ with $x_g \neq 0$ and $g \geq 0$ is an integer. Then the b-adic logarithm of x is $-g$ and we signify $\lfloor \log_b x \rfloor = -g$.

2 – Statements of the results

Defnition 3. The b-adic diaphony $F_N(\mathcal{W}(b);\xi)$ of the first N elements of the sequence ξ in $[0,1)^s$ is defined by

$$F_N(\mathcal{W}(b);\xi) = \Big(\frac{1}{(b+1)^s - 1} \sum_{\mathbf{k} \neq \mathbf{0}} \rho(\mathbf{k}) |S_N(w_{\mathbf{k}};\xi)|^2\Big)^{\frac{1}{2}}\,,$$

where for a vector $\mathbf{k} = (k_1, \ldots, k_s)$ with nonnegative integer coordinates $\rho(\mathbf{k}) = \prod_{j=1}^{s} \rho(k_j)$,

$$\rho(k) = \begin{cases} b^{-2g} & \text{for } \forall k, \quad b^g \leq k < b^{g+1}, \quad g \geq 0, \quad g \in \mathbf{Z} \\ 1 & \text{if} \quad k = 0\,, \end{cases}$$

and where $S_N(w_{\mathbf{k}};\xi) = \frac{1}{N}\sum_{n=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_n)$ is the Chrestenson sum of the sequence ξ.

When we put $b = 2$ in Definition 3, we obtain the dyadic diaphony introduced by Hellekalek and Leeb [2].

We will establish that $F_N(\mathcal{W}(b);\xi)$ is a measure for uniform distribution of the sequences. The next theorem holds:

Theorem 1. *The sequence ξ is uniformly distributed mod 1 if and only if*

$$\lim_{N \to \infty} F_N(\mathcal{W}(b);\xi) = 0\,.$$

Theorem 2. *Let*

$$\varphi(x) = \begin{cases} (b+1) - (b+1) b^{1+\lfloor \log_b x \rfloor}, & \text{if} \quad x \in (0,1) \\ b+1, & \text{if} \quad x = 0 \end{cases}$$

and $\phi : [0,1)^s \to \mathbf{R}$,

$$\phi(\mathbf{x}) = -1 + \prod_{j=1}^{s} \varphi(x_j), \quad \mathbf{x} = (x_1, \dots, x_s)\,.$$

Then for every sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$, *such that the coordinates of all points* $\mathbf{x}_n$ *satisfy the conditions* $(C1)$ *or* $(C2)$, *in particular the coordinates of all points are b-adic rationals, we have the equation*

$$F_N^2(\mathcal{W}(b);\xi) = \frac{1}{(b+1)^s - 1}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \phi(\mathbf{x}_n \dot{-} \mathbf{x}_m)\,.$$

Definition 4. For fixed base $b \geq 2$ an arbitrary generalized Van der Corput sequence $S_b^\Sigma = \{S_b^\Sigma(n)\}_{n\geq 0}$ is defined as follows: Let $\Sigma = (\sigma_j)_{j\geq 0}$ be a sequence of permutations of the set $\{0, 1, \dots, b-1\}$. If an arbitrary nonnegative integer n has b-adic expansion $n = \sum_{j=0}^{\infty} a_j(n) b^j$ we replace

$$S_b^\Sigma(n) = \sum_{j=0}^{\infty} \sigma_j(a_j(n)) b^{-j-1}\,.$$

The sequence S_b^Σ is defined by Faure [6]. If $\Sigma = I-$ the identity $\sigma_j(a) = a$, for every $a \in \{0, 1, \dots, b-1\}$ and every $j \geq 0$ we obtain the sequence S_b^I, the well-known sequence of Halton [9]. When $b = 2$, we obtain the original sequence S_2^I of Van der Corput [10].

Theorem 3. *Let* $S_b^\Sigma = \{S_b^\Sigma(n)\}_{n=0}^{\infty}$ *be an arbitrary generalized Van der Corput sequence. Then for arbitrary positive integer N the b-adic diaphony of the sequence* σ *satisfies the inequation*

$$F_N(\mathcal{W}(b); S_b^\Sigma) < \frac{1}{N}\sqrt{\frac{4b^3 - 2b^2 - 3b - 1}{(b+1)\log b}\log[(b-1)N + 1] + \frac{(b-1)^3}{b}}\,.$$

There is no low estimation of the b-adic diaphony yet, in contrast to the low estimation of the classic diaphony where there is one. In our paper we will show a direct low estimation of $F_N(\mathcal{W}(b); S_b^{\Sigma})$. This result will be a sufficient condition for the above estimation, obtained in Theorem 3 to have exact order $\mathcal{O}(N^{-1}\sqrt{\log N})$. The next theorem holds:

THEOREM 4. *Let $S_b^{\Sigma} = (S_b^{\Sigma}(n))_{n\geq 0}$ be an arbitrary generalized Van der Corput sequence. Then for infinitely many values of N the b-adic diaphony satisfies the low estimation*

$$F_N(\mathcal{W}(b); S_b^{\Sigma}) > \frac{1}{N}\sqrt{\frac{b^5+b^4-3b^3-4b^2+4b+4}{2b^2(b+1)^2(b-1)\log b}\log[(b^2-1)N+1] - C(b)},$$

where the constant $C(b)$ is calculated exactly.

3 – Preliminary statements

An interval of the form

$$J(g,h) = \Big[\frac{h}{b^g}, \frac{h+1}{b^g}\Big), \quad 0 \leq h < b^g, g \geq 0\,,$$

h and g integers, is called an elementary b-adic interval of length b^{-g}.

It is easy to verify the following Lemma:

LEMMA 1. *Let $c = c_{g-1}b^{-g} + c_g b^{-(g+1)} + \ldots$ with $c_{g-1} \neq 0$ be an arbitrary real number.*

(i) *Let $g \geq 1$ and $h = h_0h_1\ldots h_{g-2}h_{g-1}$ be an arbitrary integers. Then $J(g,h)\dot{+}c = J(g,\overline{h})$ with $\overline{h} = h_0h_1\ldots h_{g-2}\overline{h}_{g-1}$ and $\overline{h}_{g-1} = h_{g-1} \oplus c_{g-1}$.*
(ii) *For arbitrary integers $g \geq 1$ and $0 \leq h < b^{g-1}$ let $I(g,h) = \{x : \frac{hb}{b^g} \leq x < \frac{(h+1)b}{b^g}\}$. Then $I(g,h)\dot{+}c = I(g,h)$.*

For $x \in [0,1)$ with b-adic expansion in the form $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$ and integer $g \geq 0$ we define $x(0) = 0$ and for $g \geq 1$ $\ x(g) = 0.x_0 x_1 \dots x_{g-1}$. Then $x(g) \in \{a.b^{-g} : 0 \leq a < b^g\}$.

We consider a number $\beta = 0.0(b-1)0(b-1)\dots$, i.e. $\beta = \frac{1}{b+1}$. For $g \geq 1$ we have $\beta(g) = \frac{a}{b^g}$. If g is an even then $a \equiv (b-1)(\mathrm{mod}\ \ b)$ and when g is an odd then $a \equiv 0 \ (\mathrm{mod}\ \ b)$.

For functions $\varphi, \psi \in L^1([0,1))$ and $x \in [0,1)$, we define the *convolution* $\varphi * \psi$ of φ and ψ as $\varphi * \psi(x) = \int_{[0,1)} \varphi(t)\psi(t \dot{+} x)dt$.

Lemma 2. *For $x \in [0,1)$ we define $f(x) = 1_{[0,\beta)}(x)$. Then*

$$f * f(x) = \begin{cases} \dfrac{1}{b+1} - \dfrac{1}{b+1} b^{1+\lfloor \log_b x \rfloor}, & x \in (0,1) \\ \dfrac{1}{b+1}, & x = 0\,. \end{cases}$$

Proof. We have

$$f * f(x) = \int_{[0,\frac{1}{b+1})} 1_{[0,\frac{1}{b+1})}(t \dot{+} x)dt = \lambda\Big(\Big[0, \frac{1}{b+1}\Big) \cap \Big(\Big[0, \frac{1}{b+1}\Big) \dot{+} x\Big)\Big)\,.$$

Let $x = c$ be a b-adic rational and $g = -\lfloor \log_b c \rfloor$. Let g be an even and $\beta(g) = \frac{pb+b-1}{b^g}$, for some integer p. We have the equation

$$(1)\quad [0,\beta) = \Big[0, \frac{pb}{b^g}\Big) \cup \Big[\frac{pb}{b^g}, \frac{pb+b-c_{g-1}}{b^g}\Big) \cup \Big[\frac{pb+b-c_{g-1}}{b^g}, \beta(g)\Big) \cup [\beta(g), \beta)\,.$$

Using Lemma 1, consecutively we obtain

$$\Big[0, \frac{pb}{b^g}\Big) \dot{+} c = (\cup_{h=0}^{p-1} I(g,h)) \dot{+} c = \cup_{h=0}^{p-1} (I(g,h) \dot{+} c) = \cup_{h=0}^{p-1} I(g,h) = \Big[0, \frac{pb}{b^g}\Big)\,;$$

$$\Big[\frac{pb}{b^g}, \frac{pb+b-c_{g-1}}{b^g}\Big) \dot{+} c = \Big[\frac{pb+c_{g-1}}{b^g}, \beta(g) + b^{-g}\Big)\,;$$

$$\Big[\frac{pb+b-c_{g-1}}{b^g}, \beta(g)\Big) \dot{+} c = \Big[\frac{pb}{b^g}, \frac{pb+c_{g-1}-1}{b^g}\Big)\,;$$

$$[\beta(g), \beta) \dot{+} c \subset \Big[\frac{pb+c_{g-1}-1}{b^g}, \frac{pb+c_{g-1}}{b^g}\Big)\,.$$

We have from the upper expressions

$$[0,\beta) \dot{+} c = [0, \beta + b^{-g}) \setminus \Big[\frac{pb+c_{g-1}-1}{b^g} + \beta - \beta(g), \frac{pb+c_{g-1}}{b^g}\Big)\,.$$

Hence

$$\lambda([0,\beta)\cap([0,\beta)\dot{+}c)) = \frac{1}{b+1} - \frac{b}{b+1}b^{-g}\,. \tag{2}$$

If g is an odd then $[0,\beta)\dot{+}c = [0,\beta(g))$. By analogical considerations we again obtain the eq. (2). A Lemma 2 is proved.

Lemma 3. (i) *Let* $\varphi : [0,1) \to \mathbf{R}$ *and* $\varphi(x) = (b+1)f * (b+1)f(x)$, *where* $f(x)$ *is defined in Lemma* 2. *Then*

$$\widehat{\varphi}(k) = \rho(k) = \begin{cases} b^{-2g} & \text{for } \forall k,\quad b^g \le k < b^{g+1},\quad g \ge 0,\quad g \in \mathbf{Z} \\ 1 & \text{if}\quad k = 0\,. \end{cases}$$

(ii) *Let* $\phi : [0,1)^s \to \mathbf{R}$,

$$\phi(\mathbf{x}) = -1 + \prod_{i=1}^{s} \varphi(x_i), \quad \mathbf{x} = (x_1, \dots, x_s)\,.$$

Then

$$\widehat{\phi}(\mathbf{k}) = \begin{cases} \rho(\mathbf{k}) & \text{if } \mathbf{k} \ne \mathbf{0} \\ 0 & \text{if } \mathbf{k} = \mathbf{0}\,. \end{cases}$$

Proof. (i) For arbitrary integers $g \ge 0$ and k, $b^g \le k < b^{g+1}$ from Lemma 2 we have the next

$$\begin{aligned} \widehat{\varphi}(k) &= (b+1)^2 \int_0^1 \Big(\frac{1}{b+1} - \frac{b}{b+1} b^{\lfloor \log_b x\rfloor}\Big)\overline{w}_k(x)dx = \\ &= (b+1)\int_0^1 \overline{w}_k(x)dx - b(b+1)\int_0^1 b^{\lfloor \log_b x\rfloor}\overline{w}_k(x)dx = \\ &= -b(b+1)\int_0^1 b^{\lfloor \log_b x\rfloor}\overline{w}_k(x)dx\,. \end{aligned} \tag{3}$$

We with use the presentation

$$\begin{aligned} \int_0^1 b^{\lfloor \log_b x\rfloor}\overline{w}_k(x)dx &= \int_0^{b^{-(g+1)}} b^{\lfloor \log_b x\rfloor}\overline{w}_k(x)dx + \\ &+ \int_{b^{-(g+1)}}^{b^{-g}} b^{\lfloor \log_b x\rfloor}\overline{w}_k(x)dx + \int_{b^{-g}}^{1} b^{\lfloor \log_b x\rfloor}\overline{w}_k(x)dx\,. \end{aligned} \tag{4}$$

We have obtained the next results

$$\int_0^{b^{-(g+1)}} b^{\lfloor \log_b x \rfloor} \overline{w}_k(x)dx = \frac{b^{-2g}}{b^2(b+1)}\,; \tag{5}$$

$$\int_{b^{-(g+1)}}^{b^{-g}} b^{\lfloor \log_b x \rfloor} \overline{w}_k(x)dx = -\frac{b^{-2g}}{b^2}\,; \tag{6}$$

$$\int_{b^{-g}}^{1} b^{\lfloor \log_b x \rfloor} \overline{w}_k(x)dx = 0\,. \tag{7}$$

Realy, for $\forall x \in [0, b^{-(g+1)})$ and for $\forall k, \quad b^g \leq k < b^{g+1} \quad w_k(x) = 1$. Then

$$\begin{aligned}\int_0^{b^{-(g+1)}} b^{\lfloor \log_b x \rfloor} \overline{w}_k(x)dx &= \sum_{\alpha=1}^{\infty} \int_{b^{-(g+1+\alpha)}}^{b^{-(g+\alpha)}} b^{\lfloor \log_b x \rfloor} dx = \\ &= \frac{b-1}{b^2} b^{-2g} \sum_{\alpha=1}^{\infty} b^{-2\alpha} = \frac{b^{-2g}}{b^2(b+1)}\,.\end{aligned}$$

The eqs. (6) and (7) can be proved in the same way.

From (3), (4), (5), (6) and (7) we obtain for $\forall k$, $b^g \leq k < b^{g+1}$ $\widehat{\varphi}(k) = b^{-2g}$.

We have the equations:

$$\widehat{\varphi}(0) = (b+1) - b(b+1) \sum_{\alpha=0}^{\infty} \int_{b^{-(\alpha+1)}}^{b^{-\alpha}} b^{\lfloor \log_b x \rfloor} dx = 1\,.$$

(ii) Let $\mathbf{k} \neq \mathbf{0}$ be an arbitrary vector with nonnegative integer coordinates and $\mathbf{k} = (k_1, \ldots, k_s)$. Then

$$\widehat{\phi}(\mathbf{k}) = -\prod_{i=1}^{s} \int_0^1 \overline{w}_{k_i}(x_i)dx_i + \prod_{i=1}^{s} \int_0^1 \varphi(x_i)\overline{w}_{k_i}(x_i)dx_i = \rho(\mathbf{k})\,,$$

and

$$\widehat{\phi}(\mathbf{0}) = -\prod_{i=1}^{s} \int_0^1 w_0(x_i)dx_i + \prod_{i=1}^{s} \int_0^1 \varphi(x_i)w_0(x_i)dx_i = 0\,.$$

A Lemma 3 is proved.

In the next lemma we will expose a formula for Chrestenson sum of the generalized Van der Corput sequence. This formula has main importance for above and low estimations of the b-adic diaphony of the generalized Van der Corput sequence.

Lemma 4. *Let* $k = k_1 b^{\alpha_1} + k_2 b^{\alpha_2} + \ldots + k_p b^{\alpha_p}$, *where for* $1 \leq j \leq p$ $k_j \in \{1, \ldots, b-1\}$, *and* $\alpha_1 > \alpha_2 > \ldots > \alpha_p \geq 0$ *be an arbitrary integer. Let for arbitrary integer* $\nu > 0$ $m \equiv 0 (\mathrm{mod} \quad b^\nu)$ *and* $0 \leq \tau \leq \nu$ *be an arbitrary integer. Then*

(i) *If* $\tau > \alpha_p$ *then* $\sum_{i=m}^{m+b^\tau-1} w_k(S_b^\Sigma(i)) = 0$;

(ii) *If* $\tau \leq \alpha_p$ *then* $|\sum_{i=m}^{m+b^\tau-1} w_k(S_b^\Sigma(i))| = b^\tau$;

(iii) *Let* N *be an arbitrary integer and* $N = a_1 b^{\nu_1} + a_2 b^{\nu_2} + \ldots + a_t b^{\nu_t}$, *where for* $1 \leq j \leq t$ $a_j \in \{1, \ldots, b-1\}$, *and* $\nu_1 > \nu_2 > \ldots > \nu_t \geq 0$. *Then we have the next equations*

$$\Big|\sum_{i=0}^{N-1} w_k(S_b^\Sigma(i))\Big| =$$

$$= \begin{cases} 0, & \text{if } \nu_t > \alpha_p \\ \Big|\sum_{h=0}^{a_t-1} \omega^{(\sigma_{\nu_t}(h)-\sigma_{\nu_t}(0))k_p}\Big| b^{\nu_t}, & \text{if } \nu_t = \alpha_p \\ \sum_{j=s+1}^{t} a_j b^{\nu_j}, & \text{if } \nu_s > \alpha_p > \nu_{s+1} \\ \Big|\sum_{h=0}^{a_s-1} \omega^{(\sigma_{\nu_s}(h)-\sigma_{\nu_s}(0))k_p} b^{\nu_s} + & \\ +\omega^{(\sigma_{\nu_s}(a_s)-\sigma_{\nu_s}(0))k_p} \sum_{j=s+1}^{t} a_j b^{\nu_j}\Big|, & \text{if } \nu_s = \alpha_p > \nu_{s+1} \\ N, & \text{if } \alpha_p > \nu_1 . \end{cases} \tag{8}$$

(iv) *For arbitrary integers* $\alpha, \nu \geq 0$ *we define*

$$\delta_{b^\alpha}(\nu) = \begin{cases} 1, & \text{if } \alpha \geq \nu \\ 0, & \text{if } \alpha < \nu . \end{cases}$$

We put $\alpha_p = \alpha$. *Then the following estimation holds*

$$\Big|\sum_{i=0}^{N-1} w_k(S_b^\Sigma(i))\Big| \leq \sum_{j=1}^{t} a_j b^{\nu_j} \delta_{b^\alpha}(\nu_j) . \tag{9}$$

Proof. (i) Let $\tau > \alpha_1$. Then we divide the interval $I(\tau) = \{S_b^{\Sigma}(i) : m \le i < m + b^{\tau}\}$ into $b^{\tau-\alpha_1}$ intervals in the form

$$I(\tau;\alpha_1;j_1;h_1) = \{S_b^{\Sigma}(i) : m + j_1 b^{\alpha_1+1} + h_1 b^{\alpha_1} \le i < m + j_1 b^{\alpha_1+1} + (h_1+1) b^{\alpha_1}; \\ 0 \le j_1 < b^{\tau-\alpha_1-1}, 0 \le h_1 < b\},$$

so that for

$$\forall i \in I(\tau;\alpha_1;j_1;h_1) \quad r_{\alpha_1}^{k_1}(S_b^{\Sigma}(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1}\,.$$

By analogy we divide $I(\tau;\alpha_1;j_1;h_1)$ into $b^{\alpha_1-\alpha_2}$ intervals in the form

$$I(\alpha_1;\alpha_2;j_2;h_2) = \{S_b^{\Sigma}(i) : m_{\alpha_1} + j_2 b^{\alpha_2+1} + h_2 b^{\alpha_2} \le \\ \le i < m_{\alpha_1} + j_2 b^{\alpha_2+1} + (h_2+1) b^{\alpha_2}\,; \\ 0 \le j_2 < b^{\alpha_1-\alpha_2-1}, 0 \le h_2 < b\}\,,$$

so that for

$$\forall i \in I(\alpha_1;\alpha_2;j_2;h_2) \quad r_{\alpha_1}^{k_1}(S_b^{\Sigma}(i)) r_{\alpha_2}^{k_2}(S_b^{\Sigma}(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1+\sigma_{\alpha_2}(h_2)k_2}\,.$$

We continue in this way and obtain intervals in the form

$$I(\alpha_{p-2};\alpha_{p-1};j_{p-1};h_{p-1}) = \{S_b^{\Sigma}(i) : m_{\alpha_{p-2}} + j_{p-1} b^{\alpha_{p-1}+1} + h_{p-1} b^{\alpha_{p-1}} \le \\ \le i < m_{\alpha_{p-2}} + j_{p-1} b^{\alpha_{p-1}+1} + (h_2+1) b^{\alpha_{p-1}}; \\ 0 \le j_{p-1} < b^{\alpha_{p-2}-\alpha_{p-1}-1}, 0 \le h_{p-1} < b\}\,,$$

so that for

$$\forall i \in I(\alpha_{p-1};\alpha_{p-2};j_{p-1};h_{p-1}) \prod_{j=1}^{p-1} r_{\alpha_j}^{k_j}(S_b^{\Sigma}(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1+\cdots+\sigma_{\alpha_{p-1}}(h_{p-1})k_{p-1}}\,.$$

We divide $I(\alpha_{p-2};\alpha_{p-1};j_{p-1};h_{p-1})$ into $b^{\alpha_{p-1}-\alpha_p}$ intervals in the form

$$I(\alpha_{p-1};\alpha_p;j_p;h_p) = \{S_b^{\Sigma}(i) : m_{\alpha_{p-1}} + j_p b^{\alpha_p+1} + \\ + h_p b^{\alpha_p} \le i < m_{\alpha_{p-1}} + j_p b^{\alpha_p+1} + (h_p+1) b^{\alpha_p}\,; \\ 0 \le j_p < b^{\alpha_{p-1}-\alpha_p-1}, 0 \le h_p < b\}\,,$$

so that for

$$\forall i \in I(\alpha_p;\alpha_{p-2};j_{p-1};h_{p-1}) \quad w_k(S_b^{\Sigma}(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1+\ldots+\sigma_{\alpha_p}(h_p)k_p}\,.$$

We put $\tau = \alpha_0$ and hence

$$\sum_{i=m}^{m+b^\tau-1} w_k(S_b^\Sigma(i)) = \sum_{s=1}^{p} \sum_{j_s=0}^{b^{\alpha_{s-1}-\alpha_s-1}-1} \sum_{h_s=0}^{b-1} \omega^{\sigma_{\alpha_1}(h_1)k_1+\ldots+\sigma_{\alpha_s}(h_s)k_s} =$$

$$= \sum_{s=1}^{p} b^{\alpha_{s-1}-\alpha_s-1} \omega^{\sigma_{\alpha_1}(h_1)k_1+\ldots+\sigma_{\alpha_{s-1}}(h_{s-1})k_{s-1}} \sum_{h_s=0}^{b-1} \omega^{\sigma_{\alpha_s}(h_s)k_s} = 0\,.$$

If some s, $1 \leq s \leq p$ exists so that $\alpha_1 > \ldots > \alpha_s \geq \tau > \ldots > \alpha_p$ then $\forall i \in I(\tau)$ $\prod_{j=1}^{s} r_{\alpha_j}^{k_j}(S_b^\Sigma(i))$ has a constant value. We continue in the same way.

(ii) Let now $\tau \leq \alpha_p$. For i, such that $m \leq i < m + b^\tau$ we use the presentation

$$S_b^\Sigma(i) = 0.s_0(i)s_1(i)\ldots s_{\tau-1}(i)s_\tau s_{\tau+1}\ldots\,,$$

where the digits $s_0(i), s_1(i), \ldots, s_{\tau-1}(i)$ depend on i and $s_\tau, s_{\tau+1}, \ldots$ are constants. For $\forall j$, $1 \leq j \leq p$ we have $r_{\alpha_j}(S_b^\Sigma(i)) = \omega^{s_{\alpha_j}}$, i.e. $w_k(S_b^\Sigma(i)) = \omega^{s_{\alpha_p}+s_{\alpha_{p-1}}+\ldots+s_{\alpha_1}}$. Hence

$$\sum_{i=m}^{m+b^\tau-1} w_k(S_b^\Sigma(i)) = \omega^{s_{\alpha_p}+s_{\alpha_{p-1}}+\ldots+s_{\alpha_1}} b^\tau\,.$$

(iii) We introduce the significations $m_0 = 0$ and for $1 \leq j \leq t$ $\quad m_j = \sum_{s=1}^{j} a_s b^{\nu_s}$. It is clearly that for $1 \leq j \leq t$ we have

$$m_j = m_{j-1} + a_j b^{\nu_j}\,. \tag{10}$$

We have the equation

$$\sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) = \sum_{j=1}^{t} \sum_{h=0}^{a_j-1} \sum_{i=m_{j-1}+hb^{\nu_j}}^{m_{j-1}+(h+1)b^{\nu_j}-1} w_k(S_b^\Sigma(i))\,. \tag{11}$$

The first and second formulae of (8) are direct consequence of (i) and (ii) of the lemma.

Let some s exists so that $\nu_s > \alpha_p > \nu_{s+1}$. From (11) and Lemma 4 (i) we have

$$\sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) = \sum_{j=s+1}^{t} \sum_{h=0}^{a_j-1} \sum_{i=m_{j-1}+hb^{\nu_j}}^{m_{j-1}+(h+1)b^{\nu_j}-1} w_k(S_b^\Sigma(i))\,. \tag{12}$$

We will prove the next equations:

For $\forall j, \quad s+1 \le j \le t$

$$w_k(S_b^{\Sigma}(m_j)) = w_k(S_b^{\Sigma}(m_{j-1}))\,. \tag{13}$$

For each fixed j, $s+1 \le j \le t$ and for $\forall h$, $0 \le h \le a_j - 1$

$$w_k(S_b^{\Sigma}(m_{j-1})) = w_k(S_b^{\Sigma}(m_{j-1} + hb^{\nu_j}))\,. \tag{14}$$

For every two fixed j, $s+1 \le j \le t$ and h, $0 \le h \le a_j - 1$, for $\forall i$ $m_{j-1} + hb^{\nu_j} \le i < m_{j-1} + (h+1)b^{\nu_j}$ we have

$$w_k(S_b^{\Sigma}(m_{j-1} + hb^{\nu_j})) = w_k(S_b^{\Sigma}(i))\,. \tag{15}$$

To prove (13), we note the next property of the sequence S_b^{Σ} : For arbitrary integer $\nu \ge 0$ let $m \equiv 0 (\mathrm{mod} \quad b^{\nu})$ and $0 \le n < b^{\nu}$. Then we have the equation

$$S_b^{\Sigma}(m+n) = S_b^{\Sigma}(m) + S_b^{\Sigma}(n) - S_b^{\Sigma}(0)\,. \tag{16}$$

From (10) and (16) we obtain

$$S_b^{\Sigma}(m_j) = S_b^{\Sigma}(m_{j-1}) + \frac{\sigma_{\nu_j}(a_j) - \sigma_{\nu_j}(0)}{b^{\nu_j+1}}\,. \tag{17}$$

We note the next "periodicity" of the Rademacher functions: For $\forall \alpha \ge 0$ and $\forall x \in [0, \frac{b^{\alpha}-1}{b^{\alpha}})$

$$r_{\alpha}\Big(x + \frac{1}{b^{\alpha}}\Big) = r_{\alpha}(x)\,. \tag{18}$$

For $\forall q$, $1 \le q \le p$ and $\forall j$, $s+1 \le j \le t$ from (17) and (18) we obtain

$$\begin{aligned} r_{\alpha_q}(S_b^{\Sigma}(m_j)) &= r_{\alpha_q}\Big(S_b^{\Sigma}(m_{j-1}) + \frac{b^{\alpha_q - \nu_j - 1}(\sigma_{\nu_j}(a_j) - \sigma_{\nu_j}(0))}{b^{\alpha_q}}\Big) = \\ &= r_{\alpha_q}(S_b^{\Sigma}(m_{j-1}))\,, \end{aligned}$$

i.e. $w_k(S_b^{\Sigma}(m_j)) = w_k(S_b^{\Sigma}(m_{j-1}))$. The eqs. (14) and (15) can be proved in the same way.

From (12), (13), (14) and (15) we obtain

$$\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i)) = w_k(S_b^{\Sigma}(m_s)) \sum_{j=s+1}^{t} a_j b^{\nu_j}\,.$$

The rest eqs. of (8) can be proved similarly. The estimation (9) is direct consequence of the eqs. (8). A Lemma 4 is proved.

It is obvious that the equations in the above lemma have a combinatory character of the links between the degrees of N ans k in their b-adic developments.

4 – Proofs of the main results

4.1 – Proof of Theorem 1

From [11, corollary 1.2] the sequence ξ of $[0,1)^s$ is uniformly distributed in E^s if and only if $\lim_{N\to\infty} S_N(w_{\mathbf{k}};\xi) = 0$, for every $\mathbf{k} \neq \mathbf{0}$. We have the following: If the sequence ξ is uniformly distributed in E^s, then $\lim_{N\to\infty} S_N(w_{\mathbf{k}};\xi) = 0$ and hence $\lim_{N\to\infty} F_N(\mathcal{W}(b);\xi) = 0$. If we have the equation $\lim_{N\to\infty} F_N(\mathcal{W}(b);\xi) = 0$, then from Definition 3 we obtain for $\forall \mathbf{k} \neq \mathbf{0}$ $\lim_{N\to\infty} S_N(w_{\mathbf{k}};\xi) = 0$ and the sequence ξ is uniformly distributed in E^s.

4.2 – Proof of Theorem 2.

From Lemma 3 we have $\phi(\mathbf{x}) = \sum_{\mathbf{k}\neq\mathbf{0}} \rho(\mathbf{k}) w_{\mathbf{k}}(\mathbf{x})$ and the last series is uniformly converge. Then using the condition $(C1)$ or $(C2)$ we obtain

$$\begin{aligned}
&\frac{1}{(b+1)^s-1}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\phi(\mathbf{x}_n \dot{-} \mathbf{x}_m) = \\
&= \frac{1}{(b+1)^s-1}\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sum_{\mathbf{k}\neq\mathbf{0}}\rho(\mathbf{k}) w_{\mathbf{k}}(\mathbf{x}_n \dot{-} \mathbf{x}_m) = \\
&= \frac{1}{(b+1)^s-1}\sum_{\mathbf{k}\neq\mathbf{0}}\rho(\mathbf{k})\frac{1}{N}\sum_{n=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_n)\frac{1}{N}\sum_{m=0}^{N-1}\overline{w}_{\mathbf{k}}(\mathbf{x}_m) = \\
&= \frac{1}{(b+1)^s-1}\sum_{\mathbf{k}\neq\mathbf{0}}\rho(\mathbf{k})|S_N(w_{\mathbf{k}};\xi)|^2 = F_N^2(\mathcal{W}(b);\xi)\,.
\end{aligned}$$

4.3 – Proof of Theorem 3

Let N be an arbitrary integer and $N = a_1 b^{\nu_1} + a_2 b^{\nu_2} + \ldots + a_t b^{\nu_t}$, where for $1 \le j \le t \quad a_j \in \{1, \ldots, b-1\}$, and $\nu_1 > \nu_2 > \ldots > \nu_t \ge 0$. From Definition 3 we have

$$(NF_N(\mathcal{W}(b); S_b^{\Sigma}))^2 = \frac{1}{b} \sum_{g=0}^{\infty} b^{-2g} \sum_{k=b^g}^{b^{g+1}-1} \Big| \sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i)) \Big|^2 . \tag{19}$$

Let $g \ge 0$ be a fixed integer. For fixed integer α, $0 \le \alpha \le g$ we consider the set

$$A_\alpha = \{k_\alpha \in \mathbf{Z}; 1 \le k_\alpha \le b-1\},$$

for $\alpha + 1 \le j \le g-1$ we put

$$A_j = \{k_j \in \mathbf{Z}; 0 \le k_j \le b-1\}, \quad \text{and} \quad A_g = \{k_g \in \mathbf{Z}; 1 \le k_g \le b-1\}.$$

We define the set $A(g,\alpha) = A_g \times A_{g-1} \times \ldots \times A_\alpha$. It is obvious that

$$|A(g,\alpha)| = \begin{cases} (b-1)^2 b^{g-1-\alpha}, & 0 \le \alpha \le g-1 \\ b-1, & \alpha = g . \end{cases}$$

For arbitrary $k \in A(g,\alpha)$ in the form $k = (k_g k_{g-1} \ldots k_\alpha)$ we will use the presentation $k = k_g b^g + k_{g-1} b^{g-1} + \ldots + k_{\alpha+1} b^{\alpha+1} + k_\alpha b^\alpha$. Using the introdused sets $A(g,\alpha)$ for (19) we have

$$\begin{aligned} (NF_N(\mathcal{W}(b); S_b^{\Sigma}))^2 &= \frac{1}{b} \sum_{k \in A(0,0)} \Big| \sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i)) \Big|^2 + \\ &+ \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{\alpha=0}^{g-1} \sum_{k \in A(g,\alpha)} \Big| \sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i)) \Big|^2 + \\ &+ \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{k \in A(g,g)} \Big| \sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i)) \Big|^2 = \Sigma_1 + \Sigma_2 + \Sigma_3 . \end{aligned} \tag{20}$$

We have obtained the next estimations:

$$\Sigma_1 \le \frac{(b-1)^3}{b} ; \tag{21}$$

$$\Sigma_2 < \frac{2b^3 - 2b - 1}{b+1} t ; \tag{22}$$

$$\Sigma_3 < \frac{b(2b^2 - 2b - 1)}{b+1} t . \tag{23}$$

We will show the demonstrations of (22) and (23). We note that if $(a_{i,j})_{i,j=1}^{n}$ is a symmetric matrix, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} = 2\sum_{i=1}^{n}\sum_{j=1}^{i} a_{i,j} - \sum_{i=1}^{n} a_{i,i}\,.$$

From the estimation (9) we have

$$\begin{aligned}
\Sigma_2 &= \frac{1}{b}\sum_{g=1}^{\infty} b^{-2g}\sum_{\alpha=0}^{g-1}\sum_{k\in A(g,\alpha)}\Big|\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i))\Big|^2 \leq \\
&\leq \frac{1}{b}\sum_{g=1}^{\infty} b^{-2g}\sum_{\alpha=0}^{g-1}(b-1)^2 b^{g-1-\alpha}\sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j b^{\nu_i+\nu_j}\delta_{b^\alpha}(\nu_i)\delta_{b^\alpha}(\nu_j) = \\
&= \frac{(b-1)^2}{b^2}\sum_{g=1}^{\infty} b^{-g}\sum_{\alpha=0}^{g-1} b^{-\alpha}\Big(2\sum_{i=1}^{t}\sum_{j=1}^{i} a_i a_j b^{\nu_i+\nu_j}\delta_{b^\alpha}(\nu_i)\delta_{b^\alpha}(\nu_j) - \\
&\quad - \sum_{i=1}^{t} a_i^2 b^{2\nu_i}\delta_{b^\alpha}(\nu_i)\Big) < \\
&< \frac{2(b-1)^4}{b^2}\sum_{i=1}^{t} b^{\nu_i}\sum_{j=1}^{i} b^{\nu_j}\sum_{g=0}^{\infty} b^{-g}\sum_{\alpha=0}^{g} b^{-\alpha}\delta_{b^\alpha}(\nu_i)\delta_{b^\alpha}(\nu_j) - \\
&\quad - \frac{(b-1)^2}{b^2}\sum_{i=1}^{t} b^{2\nu_i}\sum_{g=1}^{\infty} b^{-g}\sum_{\alpha=0}^{g-1} b^{-\alpha}\delta_{b^\alpha}(\nu_i) = \\
&= \frac{2(b-1)^4}{b^2}\sum_{i=1}^{t} b^{\nu_i}\sum_{j=1}^{i} b^{\nu_j}\sum_{g=\nu_j}^{\infty} b^{-g}\sum_{\alpha=\nu_j}^{\infty} b^{-\alpha} - \\
&\quad - \frac{(b-1)^2}{b^2}\sum_{i=1}^{t} b^{2\nu_i}\sum_{g=\nu_i+1}^{\infty} b^{-g}\sum_{\alpha=\nu_i}^{g-1} b^{-\alpha} < \\
&< \frac{2(b-1)^3}{b}\sum_{i=1}^{t} b^{\nu_i}\sum_{j=1}^{i}\sum_{g=\nu_j}^{\infty} b^{-g} - \frac{b-1}{b}\sum_{i=1}^{t} b^{\nu_i}\sum_{g=\nu_i+1}^{\infty} b^{-g} + \\
&\quad + \frac{b-1}{b}\sum_{i=1}^{t} b^{2\nu_i}\sum_{g=\nu_i+1}^{\infty} b^{-2g} < \\
&< 2(b-1)^2\sum_{i=1}^{t} b^{\nu_i}\sum_{j=\nu_i}^{\infty} b^{-j} - \frac{1}{b+1}t = \frac{2b^3-2b-1}{b+1}t\,,
\end{aligned}$$

so that the estimation (22) is proved. From the estimation (9) we have

$$\Sigma_3 = \frac{1}{b}\sum_{g=1}^{\infty} b^{-2g} \sum_{k\in A(g,g)} \Big|\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i))\Big|^2 \leq$$

$$\leq \frac{b-1}{b}\sum_{g=1}^{\infty} b^{-2g}\Big(2\sum_{i=1}^{t}\sum_{j=1}^{i} a_i a_j b^{\nu_i+\nu_j}\delta_{b^g}(\nu_i)\delta_{b^g}(\nu_j) - \sum_{i=1}^{t} a_i^2 b^{2\nu_i}\delta_{b^g}(\nu_i)\Big) \leq$$

$$\leq \frac{2(b-1)^3}{b}\sum_{i=1}^{t} b^{\nu_i}\sum_{j=1}^{i} b^{\nu_j}\sum_{g=1}^{\infty} b^{-2g}\delta_{b^g}(\nu_i)\delta_{b^g}(\nu_j) - \frac{b-1}{b}\sum_{i=1}^{t} b^{2\nu_i}\sum_{g=1}^{\infty} b^{-2g}\delta_{b^g}(\nu_i) =$$

$$= \frac{2(b-1)^3}{b}\sum_{i=1}^{t} b^{\nu_i}\sum_{j=1}^{i} b^{\nu_j}\sum_{g=\nu_j}^{\infty} b^{-2g} - \frac{b-1}{b}\sum_{i=1}^{t} b^{2\nu_i}\sum_{g=\nu_i}^{\infty} b^{-2g} <$$

$$< \frac{2b(b-1)^2}{b+1}\sum_{i=1}^{t} b^{\nu_i}\sum_{j=\nu_i}^{\infty} b^{-j} - \frac{b}{b+1}t = \frac{b(2b^2-2b-1)}{b+1}t\,,$$

so that the estimation (22) is proved.

From (20), (21), (22) and (23) we obtain

$$(NF_N(\mathcal{W}(b);S_b^{\Sigma}))^2 < \frac{4b^3-2b^2-3b-1}{b+1}t + \frac{(b+1)^3}{b}\,. \tag{24}$$

From the presentation of N in the form $N = \sum_{j=1}^{t} a_j b^{\nu_j}$ with the conditions $\nu_1 > \nu_2 > \ldots > \nu_t \geq 0$ we obtain $N \geq \frac{b^t-1}{b-1}$, so that $t \leq \frac{1}{\log b}\log[(b-1)N+1]$. From (24) and the last estimation we have

$$(NF_N(\mathcal{W}(b);S_b^{\Sigma}))^2 < \frac{4b^3-2b^2-3b-1}{(b+1)\log b}\log[(b-1)N+1] + \frac{(b+1)^3}{b}.$$

Theorem 3 is proved.

4.4 – Proof of Theorem 4.

We will consider a special form of N, so

$$N = 1010\ldots101\,, \tag{25}$$

where the number of ones is exact r. From (25) we obtain that

$$r = \frac{1}{2\log b}\log[(b^2-1)N+1]\,. \tag{26}$$

From Definition 3 we have

$$(NF_N(\mathcal{W}(b); S_b^{\Sigma}))^2 \geq \frac{1}{b}\sum_{g=1}^{\infty} b^{-2g} \sum_{\alpha=0}^{g-1} \sum_{k\in A(g,\alpha)} \Big|\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i))\Big|^2 + \tag{27}$$
$$+\frac{1}{b}\sum_{g=1}^{\infty} b^{-2g} \sum_{k\in A(g,g)} \Big|\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i))\Big|^2 .$$

Using the results of Lemma 4 and the special form of N, we obtain the next low estimations:

$$\frac{1}{b}\sum_{g=1}^{\infty} b^{-2g} \sum_{\alpha=0}^{g-1} \sum_{k\in A(g,\alpha)} \Big|\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i))\Big|^2 > \tag{28}$$
$$> \frac{4(b^2-2)^2+b^2}{4b^2(b+1)^2(b-1)} r - \frac{4(b^2-2)^2(b^2-b+1)+b^3}{4b^2(b+1)^3(b-1)^2} ;$$

$$\frac{1}{b}\sum_{g=1}^{\infty} b^{-2g} \sum_{k\in A(g,g)} \Big|\sum_{i=0}^{N-1} w_k(S_b^{\Sigma}(i))\Big|^2 > \tag{29}$$
$$> \frac{b^4-3b^2+4}{b(b+1)^2(b-1)} r - \frac{2(b^4-b^2+1)}{b(b+1)^3(b-1)^2} .$$

The demonstrations of (28) and (29) are similar to the demonstration of (22) and (23). From (26), (27), (28) and (29) we obtain

$$(NF_N(\mathcal{W}(b); S_b^{\Sigma}))^2 > \frac{b^5+b^4-3b^3-4b^2+4b+4}{2b^2(b+1)^2(b-1)\log b} \log_b[(b^2-1)N+1] - C(b) ,$$

where $C(b) = \frac{4b^6+4b^5-12b^4+9b^3-8b+16}{4b^2(b+1)^3(b-1)^2}$. A Theorem 4 is proved.

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