The $b$-adic diaphony

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Abstract: The authors introduce a new numerical measure for uniform distribution of sequences, the so-called $b$-adic diaphony. For the definition of the $b$-adic diaphony we use a class of orthonormal system of functions, the so-called Chrestenson functions.

For a very large class of one dimensional sequences, the well-known generalized Van der Corput sequences we obtain an useful formula for the Chrestenson trigonometric sum. This result allows us to make upper and lower bounds of the $b$-adic diaphony of these sequences.

We obtain the exact order $O(N^{-1}\sqrt{\log N})$ of the $b$-adic diaphony of the generalized Van der Corput sequences.

1 – Introduction

Let $\xi = (x_n)_{n \geq 0}$ be a sequence in the $s$-dimensional unit cube $E^s = [0, 1]^s$. For every subinterval $J \subset E^s$ and every positive integer $N$ we

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denote \( A_N(\xi; J) = \{x_n : x_n \in J, 0 \leq n \leq N - 1\} \). The sequence \( \xi \) is called uniformly distributed in \( E^s \) if for every \( J \) we have \( \lim_{N \to \infty} N^{-1} A_N(\xi; J) = \mu(J) \), where \( \mu(J) \) is the Lebesgue measure of \( J \).

For integrable function \( f : [0, 1)^s \to \mathbb{C} \) we define
\[
S_N(f; \xi) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) - \int_{[0,1)^s} f(x)dx.
\]

Let \( T = \{e_k(x) : k \in \mathbb{Z}^s, x \in E^s\} \), where \( e_k(x) = \prod_{j=1}^s \exp(2\pi ik_j x_j) \), with \( k = (k_1, \ldots, k_s) \) and \( x = (x_1, \ldots, x_s) \) denotes the trigonometric functional system. The classical definition of the diaphony \( F_N(T; \xi) \) of the first \( N \) elements of \( \xi \) is defined by ZINTERHOF [1] as
\[
F_N(T; \xi) = \left( \sum_{k \neq 0} R^{-2}(k)|S_N(e_k; \xi)|^2 \right)^{\frac{1}{2}},
\]
where \( R(k) = \prod_{j=1}^s \max(1, |k_j|) \). The diaphony is a numerical quantity of the irregularity of the distribution of sequences in \( E^s \) in sense that the sequence \( \xi \) is uniformly distributed mod 1 if and only if \( \lim_{N \to \infty} F_N(T; \xi) = 0 \).

The dyadic version of diaphony was introduced by HELLEKALEK and LEEB [2]. It is based on using the Walsh functional system [3] \( W(2) = \{w_k(x)\}_{k=0}^\infty (2 \to x) \) in base 2. We will use the well-known functional system of CHRESTENSON [4] \( W(b) = \{w_k(x)\}_{k=0}^\infty (b \to x) \), (see Definition 2) which includes the functional system of Walsh for obtaining a generalization of the dyadic diaphony. In [5] the authors have announced a b-adic version of the diaphony, the so-called b-adic diaphony. We will show that the b-adic diaphony is a numerical quantity of the irregularity of the distribution.

We will consider a very large class of sequences of rational elements, the so-called generalized. In the Theorems 3 and 4 we will obtain the exact order \( O(N^{-1}\sqrt{\log N}) \) of the b-adic diaphony of the Van der Corput sequences [6].

Let \( b \geq 2 \) denote a fixed integer and put \( \omega = \exp(\frac{2\pi i}{b}) \). CHRESTENSON [4] made a generalization of the Walsh functional system, exposed in the next two definitions:

**Definition 1.** The Rademacher functions of order \( b \) are defined by
\[
r_0(x) = \omega^k, \quad \text{if } \frac{k}{b} \leq x < \frac{k+1}{b}, \quad k = 0, \ldots, b-1, r_0(x + 1) = r_0(x)
\]
and for \( n \geq 1 \)
\[
r_n(x) = r_0(b^n x).
\]
Definition 2. The Chrestenson functions of order $b$ are defined by $w_0(x) = 1$, and if $n = a_1b^{n_1} + \ldots + a_m b^{n_m}$, where for $1 \leq j \leq m$ $a_j \in \{1, \ldots, b-1\}$ and $n_1 > \ldots > n_m$, then

$$w_n(x) = r_{n_1}^{a_1}(x) \ldots r_{n_m}^{a_m}(x).$$

We will use the signification $\mathcal{W}(b) = \{w_k(x)\}_{k=0}^\infty (b \rightarrow x)$ for the Chrestenson functional system. If in the Definition 1 we put $b = 2$, the original Rademacher [7] functions are obtained, and if we put $b = 2$ in Definition 2 the original Walsh system is obtained in the Paley [8] terms.

If $k = (k_1, \ldots, k_s)$ is a vector with nonnegative integer coordinates, then the $k$-th Chrestenson function of order $b$ $w_k(x)$ on $[0,1)^s$ is defined as

$$w_k(x) = \prod_{j=1}^s w_{k_j}(x_j), \quad x = (x_1, \ldots, x_s).$$

If $f$ is an integrable function on $[0,1)^s$ and $k = (k_1, \ldots, k_s)$ is a vector with nonnegative integer coordinates, then let $\hat{f}(k)$ denote the $k$-th Chrestenson-Fourier coefficient of $f$,

$$\hat{f}(k) = \int_{[0,1)^s} f(x) \overline{w}_k(x) dx,$$

with respect to the Chrestenson function $w_k$.

The set $G(b) = \{0,1, \ldots, b-1\}$ becomes a group if for $\forall \alpha, \beta \in G(b)$ we define binary operations $\alpha \oplus \beta = \alpha + \beta \mod b$ and

$$\alpha \ominus \beta = \begin{cases} 
\alpha - \beta & \text{if } \alpha \geq \beta \\
\beta + \alpha - \beta & \text{if } \alpha < \beta.
\end{cases}$$

Let the reals $x, y \in [0,1)$ have a $b$-adic expansions in form $x = \sum_{j=0}^\infty x_j b^{-j-1}$ and $y = \sum_{j=0}^\infty y_j b^{-j-1}$, as we will use the finite expansions for rational numbers. We define $x+y = \sum_{j=0}^\infty (x_j \oplus y_j) b^{-j-1}$ and $x-y = \sum_{j=0}^\infty (x_j \ominus y_j) b^{-j-1}$. If $x, y \in [0,1)^s$ with $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, then $x+y = (x_1+y_1, \ldots, x_s+y_s)$ and $x-y = (x_1-y_1, \ldots, x_s-y_s)$.

Each of the following conditions:

$(C1)$ $x+y$ is not a $b$-adic rational,
$(C2)$ $x$ and $y$ are $b$-adic rationals,
implies that for $\forall n \geq 0$

$$w_n(x+y) = w_n(x)w_n(y) \quad \text{and} \quad w_n(x-y) = w_n(x)\bar{w}_k(y).$$

Let arbitrary real $x \in [0, 1)$ have $b$-adic expansion $x = \frac{x_g}{b^g+1} + \frac{x_{g+1}}{b^{g+1}} + \ldots$ with $x_g \neq 0$ and $g \geq 0$ is an integer. Then the $b$-adic logarithm of $x$ is $-g$ and we signify $\lfloor \log_b x \rfloor = -g$.

\section*{2 – Statements of the results}

\textbf{Definition 3.} The $b$-adic diaphony $F_N(\mathcal{W}(b); \xi)$ of the first $N$ elements of the sequence $\xi$ in $[0, 1)^s$ is defined by

$$F_N(\mathcal{W}(b); \xi) = \left( \frac{1}{(b+1)^s - 1} \sum_{k \neq 0} \rho(k)|S_N(w_k; \xi)|^2 \right)^{\frac{1}{2}},$$

where for a vector $k = (k_1, \ldots, k_s)$ with nonnegative integer coordinates $\rho(k) = \prod_{j=1}^s \rho(k_j)$,

$$\rho(k) = \begin{cases} b^{-2g} & \text{for } \forall k, \ b^g \leq k < b^{g+1}, \ g \geq 0, \ g \in \mathbb{Z} \\ 1 & \text{if } \ k = 0, \end{cases}$$

and where $S_N(w_k; \xi) = \frac{1}{N} \sum_{n=0}^{N-1} w_k(x_n)$ is the Chrestenson sum of the sequence $\xi$.

When we put $b = 2$ in Definition 3, we obtain the dyadic diaphony introduced by Hellekalek and Leeb [2].

We will establish that $F_N(\mathcal{W}(b); \xi)$ is a measure for uniform distribution of the sequences. The next theorem holds:

\textbf{Theorem 1.} The sequence $\xi$ is uniformly distributed mod 1 if and only if

$$\lim_{N \to \infty} F_N(\mathcal{W}(b); \xi) = 0.$$ 

\textbf{Theorem 2.} Let

$$\varphi(x) = \begin{cases} (b+1) - (b+1)b^{1+\lfloor \log_b x \rfloor} & \text{if } x \in (0, 1) \\ b+1, & \text{if } x = 0 \end{cases}$$
and \( \phi : [0, 1)^s \to \mathbb{R}, \)

\[
\phi(x) = -1 + \prod_{j=1}^{s} \varphi(x_j), \quad x = (x_1, \ldots, x_s) .
\]

Then for every sequence \( \xi = (x_n)_{n \geq 0} , \) such that the coordinates of all points \( x_n \) satisfy the conditions \((C1)\) or \((C2)\), in particular the coordinates of all points are \( b \)-adic rationals, we have the equation

\[
F^2_N(\mathcal{W}(b); \xi) = \frac{1}{(b + 1)^s} - \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \phi(x_n - x_m) .
\]

**Definition 4.** For fixed base \( b \geq 2 \) an arbitrary generalized Van der Corput sequence \( S_{\Sigma}^b = \{ S_{\Sigma}^b(n) \}_{n \geq 0} \) is defined as follows: Let \( \Sigma = (\sigma_j)_{j \geq 0} \) be a sequence of permutations of the set \( \{ 0, 1, \ldots, b-1 \} \). If an arbitrary nonnegative integer \( n \) has \( b \)-adic expansion \( n = \sum_{j=0}^{\infty} a_j(n)b^j \) we replace

\[
S_{\Sigma}^b(n) = \sum_{j=0}^{\infty} \sigma_j(a_j(n)) b^{-j-1} .
\]

The sequence \( S_{\Sigma}^b \) is defined by Faure [6]. If \( \Sigma = I \)– the identity \( \sigma_j(a) = a \), for every \( a \in \{ 0, 1, \ldots, b-1 \} \) and every \( j \geq 0 \) we obtain the sequence \( S_1^b \), the well-known sequence of Halton [9]. When \( b = 2 \), we obtain the original sequence \( S_2^I \) of Van der Corput [10].

**Theorem 3.** Let \( S_{\Sigma}^b = \{ S_{\Sigma}^b(n) \}_{n = 0}^{\infty} \) be an arbitrary generalized Van der Corput sequence. Then for arbitrary positive integer \( N \) the \( b \)-adic diaphony of the sequence \( \Sigma \) satisfies the inequation

\[
F_N(\mathcal{W}(b); S_{\Sigma}^b) < \frac{1}{N} \sqrt{\frac{4b^3 - 2b^2 - 3b - 1}{(b + 1) \log b} \log[(b - 1)N + 1] + \frac{(b - 1)^3}{b}} .
\]
There is no low estimation of the b-adic diaphony yet, in contrast to the low estimation of the classic diaphony where there is one. In our paper we will show a direct low estimation of $F_N(W(b); S^S_b)$. This result will be a sufficient condition for the above estimation, obtained in Theorem 3 to have exact order $O(N^{-1}\sqrt{\log N})$. The next theorem holds:

**Theorem 4.** Let $S^S_b = (S^S_b(n))_{n \geq 0}$ be an arbitrary generalized Van der Corput sequence. Then for infinitely many values of $N$ the b-adic diaphony satisfies the low estimation

$$F_N(W(b); S^S_b) > \frac{1}{N} \sqrt{\frac{b^5 + b^4 - 3b^3 - 4b^2 + 4 + 4b}{2b^2(b+1)^2(b-1)\log b}} \log[(b^2 - 1)N + 1] - C(b),$$

where the constant $C(b)$ is calculated exactly.

### 3 – Preliminary statements

An interval of the form

$$J(g, h) = \left[ \frac{h}{b^g}, \frac{h + 1}{b^g} \right), \quad 0 \leq h < b^g, g \geq 0,$$

$h$ and $g$ integers, is called an elementary b-adic interval of length $b^{-g}$.

It is easy to verify the following Lemma:

**Lemma 1.** Let $c = c_{g-1}b^{-g} + c_g b^{-(g+1)} + \ldots$ with $c_{g-1} \neq 0$ be an arbitrary real number.

(i) Let $g \geq 1$ and $h = h_0h_1 \ldots h_{g-2}h_{g-1}$ be an arbitrary integers. Then $J(g, h) + c = J(g, \overline{h})$ with $\overline{h} = h_0h_1 \ldots h_{g-2}h_{g-1}$ and $\overline{h}_{g-1} = h_{g-1} \oplus c_{g-1}$.

(ii) For arbitrary integers $g \geq 1$ and $0 \leq h < b^{g-1}$ let $I(g, h) = \{x : \frac{hb}{b^g} \leq x < \frac{(h+1)b}{b^g}\}$. Then $I(g, h) + c = I(g, h)$. 

For \( x \in [0,1) \) with b-adic expansion in the form \( x = \sum_{j=0}^{\infty} x_j b^{-j-1} \) and integer \( g \geq 0 \) we define \( x(0) = 0 \) and for \( g \geq 1 \) \( x(g) = 0.x_0x_1 \ldots x_{g-1} \). Then \( x(g) \in \{ a.b^{-g} : 0 \leq a < b^g \} \).

We consider a number \( \beta = 0.0(b-1)0(b-1) \ldots \), i.e. \( \beta = \frac{a}{b^g} \). If \( g \) is an even then \( a \equiv (b-1)(\text{mod } b) \) and when \( g \) is an odd then \( a \equiv 0 \) (mod \( b \)).

For functions \( \varphi, \psi \in L_1([0,1]) \) and \( x \in [0,1) \), we define the convolution \( \varphi * \psi \) of \( \varphi \) and \( \psi \) as \( \varphi * \psi(x) = \int_{[0,1)} \varphi(t)\psi(t+x)dt \).

**Lemma 2.** For \( x \in [0,1) \) we define \( f(x) = 1_{[0, \beta]}(x) \). Then

\[
 f * f(x) = \begin{cases} 
 \frac{1}{b+1} - \frac{1}{b+1} b^{1 + \lfloor \log_b x \rfloor}, & x \in (0,1) \\
 \frac{1}{b+1}, & x = 0.
\end{cases}
\]

**Proof.** We have

\[
f * f(x) = \int_{[0, \frac{1}{b+1})} 1_{[0, \frac{1}{b+1})}(t+x)dt = \lambda \left( \left[ 0, \frac{1}{b+1} \right) \cap \left( \left[ 0, \frac{1}{b+1} \right) + x \right) \right).
\]

Let \( x = c \) be a b-adic rational and \( g = -\lfloor \log_b c \rfloor \). Let \( g \) be an even and \( \beta(g) = \frac{pb+b-1}{b^g} \), for some integer \( p \). We have the equation

\[
[0, \beta] = \left[ 0, \frac{pb}{b^g} \right] \cup \left[ \frac{pb+b-c_g-1}{b^g}, \frac{pb+b-c_g-1}{b^g} + \beta(g) \right] \cup [\beta(g), \beta].
\]

Using Lemma 1, consecutively we obtain

\[
\left[ 0, \frac{pb}{b^g} \right] + c = \bigcup_{h=0}^{p-1} I(h, \beta(g)) + c = \bigcup_{h=0}^{p-1} I(h, g) + c = \bigcup_{h=0}^{p-1} I(g, h) = \left[ 0, \frac{pb}{b^g} \right]; \\
\left[ \frac{pb}{b^g}, \frac{pb+b-c_g-1}{b^g} \right] + c = \left[ \frac{pb+c_g-1}{b^g}, \beta(g) + b^{-g} \right]; \\
\left[ \frac{pb+b-c_g-1}{b^g}, \beta(g) \right] + c = \left[ \frac{pb}{b^g}, \frac{pb+b-c_g-1}{b^g} \right]; \\
[\beta(g), \beta] + c \subset \left[ \frac{pb+c_g-1}{b^g}, \frac{pb+b-c_g-1}{b^g} \right].
\]

We have from the upper expressions

\[
[0, \beta] + c = [0, \beta + b^{-g}) \setminus \left[ \frac{pb+c_g-1}{b^g} + \beta - \beta(g), \frac{pb+c_g-1}{b^g} \right].
\]
Hence
\begin{equation}
\lambda([0, \beta) \cap ([0, \beta) + c)) = \frac{1}{b+1} - \frac{b}{b+1} b^{-g}.
\end{equation}

If \( g \) is an odd then \([0, \beta) + c = [0, \beta(g))\). By analogical considerations we again obtain the eq. (2). A Lemma 2 is proved.

**Lemma 3.**

(i) Let \( \varphi : [0, 1) \to \mathbb{R} \) and \( \varphi(x) = (b+1)f \ast (b+1)f(x) \), where \( f(x) \) is defined in Lemma 2. Then
\[ \hat{\varphi}(k) = \rho(k) = \begin{cases} 
 b^{-2g} & \text{for } \forall k, \ b^{g} \leq k < b^{g+1}, \ g \geq 0, \ g \in \mathbb{Z} \\
 1 & \text{if } k = 0.
\end{cases} \]

(ii) Let \( \phi : [0, 1)^s \to \mathbb{R} \),
\[ \phi(x) = -1 + \prod_{i=1}^{s} \varphi(x_i), \ x = (x_1, \ldots, x_s). \]

Then
\[ \hat{\phi}(k) = \begin{cases} 
 \rho(k) & \text{if } k \neq 0 \\
 0 & \text{if } k = 0.
\end{cases} \]

**Proof.** (i) For arbitrary integers \( g \geq 0 \) and \( k, \ b^{g} \leq k < b^{g+1} \) from Lemma 2 we have the next
\begin{align}
\hat{\varphi}(k) = (b+1)^2 \int_{0}^{1} \left( \frac{1}{b+1} - \frac{b}{b+1} b^{[\log_b x]} \right) \overline{w}_k(x) dx = \\
= (b+1) \int_{0}^{1} \overline{w}_k(x) dx - b(b+1) \int_{0}^{1} b^{[\log_b x]} \overline{w}_k(x) dx = \\
= -b(b+1) \int_{0}^{1} b^{[\log_b x]} \overline{w}_k(x) dx.
\end{align}

We with use the presentation
\begin{align}
\int_{0}^{1} b^{[\log_b x]} \overline{w}_k(x) dx &= \int_{0}^{b^{-(g+1)}} b^{[\log_b x]} \overline{w}_k(x) dx + \\
&+ \int_{b^{-(g+1)}}^{b^{-g}} b^{[\log_b x]} \overline{w}_k(x) dx + \int_{b^{-g}}^{1} b^{[\log_b x]} \overline{w}_k(x) dx.
\end{align}
We have obtained the next results

\[
\int_{0}^{b^{-(g+1)}} b^{[\log_b x]} \overline{w}_k(x) dx = \frac{b^{-2g}}{b^2(b + 1)};
\]

\[
\int_{b^{-(g+1)}}^{b^{-g}} b^{[\log_b x]} \overline{w}_k(x) dx = -\frac{b^{-2g}}{b^2};
\]

\[
\int_{b^{-g}}^{1} b^{[\log_b x]} \overline{w}_k(x) dx = 0.
\]

Really, for \( \forall x \in [0, b^{-(g+1)}) \) and for \( \forall k, \ b^g \leq k < b^{g+1} \) \( w_k(x) = 1 \). Then

\[
\int_{0}^{b^{-(g+1)}} b^{[\log_b x]} \overline{w}_k(x) dx = \sum_{\alpha=1}^{\infty} \int_{b^{-(g+1)+\alpha}}^{b^{-\alpha}} b^{[\log_b x]} dx = \frac{b - 1}{b^2} \frac{b^{-2g}}{b^{2(g+1)}} \sum_{\alpha=1}^{\infty} b^{-2\alpha} = \frac{b^{-2g}}{b^2(b + 1)}.
\]

The eqs. (6) and (7) can be proved in the same way.

From (3), (4), (5), (6) and (7) we obtain for \( \forall k, \ b^g \leq k < b^{g+1} \) \( \hat{\varphi}(k) = b^{-2g} \).

We have the equations:

\[
\hat{\varphi}(0) = (b + 1) - b(b + 1) \sum_{\alpha=0}^{\infty} \int_{b^{-(\alpha+1)}}^{b^{-\alpha}} b^{[\log_b x]} dx = 1.
\]

(ii) Let \( k \neq 0 \) be an arbitrary vector with nonnegative integer coordinates and \( k = (k_1, \ldots, k_s) \). Then

\[
\hat{\varphi}(k) = -\prod_{i=1}^{s} \int_{0}^{1} \overline{w}_{k_i}(x_i) dx_i + \prod_{i=1}^{s} \int_{0}^{1} \varphi(x_i) \overline{w}_{k_i}(x_i) dx_i = \rho(k),
\]

and

\[
\hat{\varphi}(0) = -\prod_{i=1}^{s} \int_{0}^{1} w_0(x_i) dx_i + \prod_{i=1}^{s} \int_{0}^{1} \varphi(x_i) w_0(x_i) dx_i = 0.
\]

A Lemma 3 is proved.

In the next lemma we will expose a formula for Chrestenson sum of the generalized Van der Corput sequence. This formula has main importance for above and low estimations of the b-adic diaphony of the generalized Van der Corput sequence.
Lemma 4. Let \( k = k_1 b^{\alpha_1} + k_2 b^{\alpha_2} + \ldots + k_p b^{\alpha_p} \), where for \( 1 \leq j \leq p \) \( k_j \in \{1, \ldots, b-1\} \), and \( \alpha_1 > \alpha_2 > \ldots > \alpha_p \geq 0 \) be an arbitrary integer. Let for arbitrary integer \( \nu > 0 \) \( m \equiv 0(\text{mod } b^\nu) \) and \( 0 \leq \tau \leq \nu \) be an arbitrary integer.

(i) If \( \tau > \alpha_p \) then \( \sum_{i=m}^{m+b^\tau-1} w_k(S^\Sigma_b(i)) = 0 \);
(ii) If \( \tau \leq \alpha_p \) then \( |\sum_{i=m}^{m+b^\tau-1} w_k(S^\Sigma_b(i))| = b^\tau \);
(iii) Let \( N \) be an arbitrary integer and \( N = a_1 b^{\nu_1} + a_2 b^{\nu_2} + \ldots + a_t b^{\nu_t} \), where for \( 1 \leq j \leq t \) \( a_j \in \{1, \ldots, b-1\} \), and \( \nu_1 > \nu_2 > \ldots > \nu_t \geq 0 \).

Then we have the next equations

\[
\left| \sum_{i=0}^{N-1} w_k(S^\Sigma_b(i)) \right| =
\begin{cases}
0, & \text{if } \nu_t > \alpha_p \\
\sum_{h=0}^{a_t-1} \omega(\sigma_{\nu_t}(h)-\sigma_{\nu_t}(0))k_p b^{\nu_t}, & \text{if } \nu_t = \alpha_p \\
\sum_{j=s+1}^{t} a_j b^{\nu_j}, & \text{if } \nu_s > \alpha_p > \nu_{s+1} \\
\sum_{h=0}^{a_s-1} \omega(\sigma_{\nu_s}(h)-\sigma_{\nu_s}(0))k_p b^{\nu_s} + 1, & \text{if } \nu_s = \alpha_p > \nu_{s+1} \\
\sum_{j=s+1}^{t} a_j b^{\nu_j} + \omega(\sigma_{\nu_s}(a_s)-\sigma_{\nu_s}(0))k_p \sum_{j=s+1}^{t} a_j b^{\nu_j}, & \text{if } \nu_s = \alpha_p > \nu_{s+1} \\
N, & \text{if } \alpha_p > \nu_1.
\end{cases}
\]

(iv) For arbitrary integers \( \alpha, \nu \geq 0 \) we define

\[
\delta_{b^\alpha}(\nu) = \begin{cases}
1, & \text{if } \alpha \geq \nu \\
0, & \text{if } \alpha < \nu.
\end{cases}
\]

We put \( \alpha_p = \alpha \). Then the following estimation holds

\[
\left| \sum_{i=0}^{N-1} w_k(S^\Sigma_b(i)) \right| \leq \sum_{j=1}^{t} a_j b^{\nu_j} \delta_{b^\alpha}(\nu_j).
\]
Proof. (i) Let $\tau > \alpha_1$. Then we divide the interval $I(\tau) = \{S^\Sigma_b(i) : m \leq i < m + b^\tau\}$ into $b^\tau - \alpha_1$ intervals in the form

$$I(\tau; \alpha_1; j_1; h_1) = \{S^\Sigma_b(i) : m + j_1 b^{\alpha_1 + 1} + h_1 b^\alpha \leq i < m + j_1 b^{\alpha_1 + 1} + (h_1 + 1) b^\alpha ;$$

$$0 \leq j_1 < b^{\tau - \alpha_1 - 1}, 0 \leq h_1 < b\},$$

so that for

$$\forall i \in I(\tau; \alpha_1; j_1; h_1) \quad r_{\alpha_1}^{k_1}(S^\Sigma_b(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1}.$$

By analogy we divide $I(\tau; \alpha_1; j_1; h_1)$ into $b^{\alpha_1 - \alpha_2}$ intervals in the form

$$I(\alpha_1; \alpha_2; j_2; h_2) = \{S^\Sigma_b(i) : m_{\alpha_1} + j_2 b^{\alpha_2 + 1} + h_2 b^{\alpha_2} \leq$$

$$\leq i < m_{\alpha_2} + j_2 b^{\alpha_2 + 1} + (h_2 + 1) b^{\alpha_2} ;$$

$$0 \leq j_2 < b^{\alpha_1 - \alpha_2 - 1}, 0 \leq h_2 < b\},$$

so that for

$$\forall i \in I(\alpha_1; \alpha_2; j_2; h_2) \quad r_{\alpha_1}^{k_1}(S^\Sigma_b(i)) r_{\alpha_2}^{k_2}(S^\Sigma_b(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1 + \sigma_{\alpha_2}(h_2)k_2}.$$

We continue in this way and obtain intervals in the form

$$I(\alpha_{p-2}; \alpha_{p-1}; j_{p-1}; h_{p-1}) = \{S^\Sigma_b(i) : m_{\alpha_{p-2}} + j_{p-1} b^{\alpha_{p-1} + 1} + h_{p-1} b^{\alpha_{p-1}} \leq$$

$$\leq i < m_{\alpha_{p-2}} + j_{p-1} b^{\alpha_{p-1} + 1} + (h_{p-1} + 1) b^{\alpha_{p-1}} ;$$

$$0 \leq j_{p-1} < b^{\alpha_{p-2} - \alpha_{p-1} - 1}, 0 \leq h_{p-1} < b\},$$

so that for

$$\forall i \in I(\alpha_{p-1}; \alpha_{p-2}; j_{p-1}; h_{p-1}) \prod_{j=1}^{p-1} r_{\alpha_j}^{k_j}(S^\Sigma_b(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1 + \cdots + \sigma_{\alpha_{p-1}}(h_{p-1})k_{p-1}}.$$

We divide $I(\alpha_{p-2}; \alpha_{p-1}; j_{p-1}; h_{p-1})$ into $b^{\alpha_{p-1} - \alpha_p}$ intervals in the form

$$I(\alpha_{p-1}; \alpha_p; j_p; h_p) = \{S^\Sigma_b(i) : m_{\alpha_{p-1}} + j_p b^{\alpha_p + 1} +$$

$$+ h_p b^\alpha \leq i < m_{\alpha_{p-1}} + j_p b^{\alpha_p + 1} + (h_p + 1) b^\alpha ;$$

$$0 \leq j_p < b^{\alpha_{p-1} - \alpha_p - 1}, 0 \leq h_p < b\},$$

so that for

$$\forall i \in I(\alpha_p; \alpha_{p-2}; j_{p-1}; h_{p-1}) \quad w_k(S^\Sigma_b(i)) = \omega^{\sigma_{\alpha_1}(h_1)k_1 + \cdots + \sigma_{\alpha_p}(h_p)k_p}.$$
We put $\tau = \alpha_0$ and hence
\[
\sum_{i=m}^{m+b^\tau-1} w_k(S_b^\Sigma(i)) = \sum_{s=1}^{p} \sum_{j_s=0}^{b^{-\alpha_s-1}-1} \sum_{h_s=0}^{b-1} \omega^{\alpha_1(h_1)k_1+\ldots+\sigma_{\alpha_s}(h_s)k_s} =
\]
\[
= \sum_{s=1}^{p} b^{-\alpha_s-1} \omega^{\alpha_1(h_1)k_1+\ldots+\sigma_{\alpha_s-1}(h_s)k_s-1} \sum_{h_s=0}^{b-1} \omega^{\sigma_{\alpha_s}(h_s)k_s} = 0.
\]

If some $s$, $1 \leq s \leq p$ exists so that $\alpha_1 > \ldots > \alpha_s \geq \tau > \ldots > \alpha_p$ then $\forall i \in I(\tau) \prod_{j=1}^{k_j} r_{\alpha_j}^{-1}(\Sigma_{b}^\Sigma(i))$ has a constant value. We continue in the same way.

(ii) Let now $\tau \leq \alpha_p$. For $i$, such that $m \leq i < m + b^\tau$ we use the presentation
\[
S_b^\Sigma(i) = 0.s_0(i)s_1(i)\ldots s_{\tau-1}(i)s_{\tau}s_{\tau+1}\ldots,
\]
where the digits $s_0(i), s_1(i), \ldots, s_{\tau-1}(i)$ depend on $i$ and $s_{\tau}, s_{\tau+1}, \ldots$ are constants. For $\forall j$, $1 \leq j \leq p$ we have $r_{\alpha_j}(S_b^\Sigma(i)) = \omega^{s_{\alpha_j}}$, i.e. $w_k(S_b^\Sigma(i)) = \omega^{s_{\alpha_p}+s_{\alpha_{p-1}}+\ldots+s_{\alpha_1}b^\tau}$.

(iii) We introduce the significations $m_0 = 0$ and for $1 \leq j \leq t$ $m_j = \sum_{s=1}^{j} a_s b^{\nu_s}$. It is clearly that for $1 \leq j \leq t$ we have
\[
m_j = m_{j-1} + a_j b^{\nu_j}.
\]

We have the equation
\[
\sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) = \sum_{j=1}^{t} \sum_{h=0}^{a_j-1} \sum_{i=m_{j-1}+hb^{\nu_j}}^{a_j m_{j-1}+(h+1)b^{\nu_j}-1} w_k(S_b^\Sigma(i)).
\]

The first and second formulae of (8) are direct consequence of (i) and (ii) of the lemma.

Let some $s$ exists so that $\nu_s > \alpha_p > \nu_{s+1}$. From (11) and Lemma 4 (i) we have
\[
\sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) = \sum_{j=s+1}^{t} \sum_{h=0}^{a_j-1} \sum_{i=m_{j-1}+hb^{\nu_j}}^{a_j m_{j-1}+(h+1)b^{\nu_j}-1} w_k(S_b^\Sigma(i)).
\]
We will prove the next equations:

For $\forall j, \ s+1 \leq j \leq t$

\begin{equation}
    w_k(S_b^{\Sigma}(m_j)) = w_k(S_b^{\Sigma}(m_{j-1})).
\end{equation}

(13)

For each fixed $j, \ s+1 \leq j \leq t$ and for $\forall h, 0 \leq h \leq a_j - 1$

\begin{equation}
    w_k(S_b^{\Sigma}(m_{j-1})) = w_k(S_b^{\Sigma}(m_{j-1} + h b^{\nu_j})).
\end{equation}

(14)

For every two fixed $j, \ s+1 \leq j \leq t$ and $h, 0 \leq h \leq a_j - 1$, for $i$

\begin{equation}
    m_{j-1} + h b^{\nu_j} \leq i < m_{j-1} + (h + 1) b^{\nu_j}$ we have

\begin{equation}
    w_k(S_b^{\Sigma}(m_{j-1} + h b^{\nu_j})) = w_k(S_b^{\Sigma}(i)).
\end{equation}

(15)

To prove (13), we note the next property of the sequence $S_b^{\Sigma}$: For arbitrary integer $\nu \geq 0$ let $m \equiv 0 \pmod{b^{\nu}}$ and $0 \leq n < b^{\nu}$. Then we have the equation

\begin{equation}
    S_b^{\Sigma}(m+n) = S_b^{\Sigma}(m) + S_b^{\Sigma}(n) - S_b^{\Sigma}(0).
\end{equation}

(16)

From (10) and (16) we obtain

\begin{equation}
    S_b^{\Sigma}(m_j) = S_b^{\Sigma}(m_{j-1}) + \frac{\sigma_{\nu_j}(a_j) - \sigma_{\nu_j}(0)}{b^{\nu_j+1}}.
\end{equation}

(17)

We note the next “periodicity” of the Rademacher functions: For $\forall \alpha \geq 0$ and $\forall x \in [0, \frac{b^{\alpha}-1}{b^{\alpha}})$

\begin{equation}
    r_\alpha(x + \frac{1}{b^\alpha}) = r_\alpha(x).
\end{equation}

(18)

For $\forall q, 1 \leq q \leq p$ and $\forall j, \ s+1 \leq j \leq t$ from (17) and (18) we obtain

\begin{align*}
    r_\alpha_q(S_b^{\Sigma}(m_j)) &= r_\alpha_q(S_b^{\Sigma}(m_{j-1}) + \frac{b^{\alpha q-\nu_j-1}(\sigma_{\nu_j}(a_j) - \sigma_{\nu_j}(0))}{b^{\alpha q}}) = \\
    &= r_\alpha_q(S_b^{\Sigma}(m_{j-1})),
\end{align*}

i.e. $w_k(S_b^{\Sigma}(m_j)) = w_k(S_b^{\Sigma}(m_{j-1}))$. The eqs. (14) and (15) can be proved in the same way.
From (12), (13), (14) and (15) we obtain
\[ \sum_{i=0}^{N-1} w_k(S^\Sigma_b(i)) = w_k(S^\Sigma_b(m_s)) \sum_{j=s+1}^{t} a_j b^{\nu_j}. \]

The rest eqs. of (8) can be proved similarly. The estimation (9) is direct consequence of the eqs. (8). A Lemma 4 is proved.

It is obvious that the equations in the above lemma have a combinatorial character of the links between the degrees of \( N \) ans \( k \) in their b-adic developments.

4 – Proofs of the main results

4.1 – Proof of Theorem 1

From [11, corollary 1.2] the sequence \( \xi \) of \([0, 1)^s \) is uniformly distributed in \( E^s \) if and only if \( \lim_{N \to \infty} S_N(w_k; \xi) = 0 \) for every \( k \neq 0 \).

We have the following: If the sequence \( \xi \) is uniformly distributed in \( E^s \), then \( \lim_{N \to \infty} S_N(w_k; \xi) = 0 \) and hence \( \lim_{N \to \infty} F_N(W(b); \xi) = 0. \) If we have the equation \( \lim_{N \to \infty} F_N(W(b); \xi) = 0 \), then from Definition 3 we obtain for \( \forall k \neq 0 \) \( \lim_{N \to \infty} S_N(w_k; \xi) = 0 \) and the sequence \( \xi \) is uniformly distributed in \( E^s \).

4.2 – Proof of Theorem 2.

From Lemma 3 we have \( \phi(x) = \sum_{k \neq 0} \rho(k) w_k(x) \) and the last series is uniformly converge. Then using the condition \((C1)\) or \((C2)\) we obtain
\[
\frac{1}{(b+1)^s - 1} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \phi(x_n - x_m) =
\]
\[
= \frac{1}{(b+1)^s - 1} \sum_{k \neq 0} \rho(k) \sum_{n=0}^{N-1} w_k(x_n) \frac{1}{N} \sum_{m=0}^{N-1} w_k(x_m) =
\]
\[
= \frac{1}{(b+1)^s - 1} \sum_{k \neq 0} \rho(k) |S_N(w_k; \xi)|^2 = F^2_N(W(b); \xi). \]
4.3 – Proof of Theorem 3

Let $N$ be an arbitrary integer and $N = a_1 b^{\nu_1} + a_2 b^{\nu_2} + \ldots + a_t b^{\nu_t}$, where for $1 \leq j \leq t$ $a_j \in \{1, \ldots, b-1\}$, and $\nu_1 > \nu_2 > \ldots > \nu_t \geq 0$. From Definition 3 we have

\begin{equation}
NF_N(W(b); S_b^\Sigma))^2 = \frac{1}{b} \sum_{g=0}^{\infty} b^{-2g} \sum_{k=b^g}^{b^{g+1}-1} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2.
\end{equation}

Let $g \geq 0$ be a fixed integer. For fixed integer $\alpha$, $0 \leq \alpha \leq g$ we consider the set

$$A_\alpha = \{k_\alpha \in \mathbb{Z}; 1 \leq k_\alpha \leq b - 1\},$$

for $\alpha + 1 \leq j \leq g - 1$ we put

$$A_j = \{k_j \in \mathbb{Z}; 0 \leq k_j \leq b - 1\}, \quad \text{and} \quad A_g = \{k_g \in \mathbb{Z}; 1 \leq k_g \leq b - 1\}.$$

We define the set $A(g, \alpha) = A_g \times A_{g-1} \times \ldots \times A_\alpha$. It is obvious that

$$|A(g, \alpha)| = \begin{cases} (b-1)^2 b^{g-1-\alpha}, & 0 \leq \alpha \leq g-1 \\ b-1, & \alpha = g. \end{cases}$$

For arbitrary $k \in A(g, \alpha)$ in the form $k = (k_g k_{g-1} \ldots k_\alpha)$ we will use the presentation $k = k_g b^g + k_{g-1} b^{g-1} + \ldots + k_{\alpha+1} b^{\alpha+1} + k_\alpha b^\alpha$. Using the introduced sets $A(g, \alpha)$ for (19) we have

\begin{multline}
NF_N(W(b); S_b^\Sigma))^2 = \frac{1}{b} \sum_{k \in A(0,0)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2 + \\
+ \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{k \in A(g,\alpha)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2 + \\
+ \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{k \in A(g,g)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2 = \Sigma_1 + \Sigma_2 + \Sigma_3.
\end{multline}

We have obtained the next estimations:

\begin{align}
\Sigma_1 & \leq \frac{(b-1)^3}{b} ; \\
\Sigma_2 & < \frac{2 b^3 - 2 b - 1}{b + 1} t ; \\
\Sigma_3 & < \frac{b(2b^2 - 2b - 1)}{b + 1} t .
\end{align}
We will show the demonstrations of (22) and (23). We note that if 
\((a_{i,j})_{i,j=1}^{n}\) is a symmetric matrix, then

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} = 2 \sum_{i=1}^{n} a_{i,j} - \sum_{i=1}^{n} a_{i,i}.
\]

From the estimation (9) we have

\[
\Sigma_2 = \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{\alpha=0}^{g-1} \sum_{k \in A(g, \alpha)} \left| \sum_{i=0}^{N-1} w_k(S^\Sigma_b(i)) \right|^2 \leq \nabla \nabla
\]

\[
\leq \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{\alpha=0}^{g-1} (b-1)^2 b^{\nu_i+\nu_j} \nu_i+\nu_j (\nu_i) \delta_B(\nu_i) = \nabla \nabla
\]

\[
= \frac{(b-1)^2}{b^2} \sum_{g=1}^{\infty} b^{-g} b^{\nu_i+\nu_j} (2 \sum_{i=1}^{t} a_{i,j} b^{\nu_i+\nu_j} (\nu_i) \delta_B(\nu_i) - \nabla \nabla
\]

\[
- \sum_{i=1}^{t} a_{i,j}^2 b^{2\nu_i} (\nu_i) ) < \nabla \nabla
\]

\[
< \frac{2(b-1)^4}{b^2} \sum_{i=1}^{t} b^{\nu_i} \sum_{j=1}^{\nu_j} b^{\nu_j} \sum_{g=0}^{\nu_j} b^{-\alpha} \delta_B(\nu_i) \delta_B(\nu_j) - \nabla \nabla
\]

\[
- \frac{(b-1)^2}{b^2} \sum_{g=1}^{\infty} b^{-g} \sum_{\alpha=0}^{g-1} b^{-\alpha} \delta_B(\nu_i) = \nabla \nabla
\]

\[
= \frac{2(b-1)^4}{b^2} \sum_{i=1}^{t} b^{\nu_i} \sum_{j=1}^{\nu_j} b^{\nu_j} \sum_{g=0}^{\nu_j} b^{-\alpha} - \nabla \nabla
\]

\[
- \frac{(b-1)^2}{b^2} \sum_{g=1}^{\infty} b^{-g} \sum_{\alpha=0}^{g-1} b^{-\alpha} < \nabla \nabla
\]

\[
< \frac{2(b-1)^3}{b} \sum_{i=1}^{t} b^{\nu_i} \sum_{j=1}^{\nu_j} b^{-g} - \frac{b-1}{b} \sum_{i=1}^{t} b^{\nu_i} \sum_{g=0}^{\nu_j+1} b^{-g} + \nabla \nabla
\]

\[
+ \frac{b-1}{b} \sum_{i=1}^{t} b^{2\nu_i} \sum_{g=0}^{\nu_j+1} b^{-2g} < \nabla \nabla
\]

\[
< 2(b-1)^2 \sum_{i=1}^{t} b^{\nu_i} \sum_{j=\nu_i}^{\infty} b^{-j} - \frac{1}{b+1} t = \frac{2b^3 - 2b - 1}{b + 1} t , \nabla \nabla
so that the estimation (22) is proved. From the estimation (9) we have

\[
\Sigma_3 = \frac{1}{b} \sum_{g=1}^{\infty} b^{-2g} \sum_{k \in A(g,g)} \left| \sum_{i=0}^{N-1} w_k(S_b^G(i)) \right|^2 \leq \\
\leq \frac{b-1}{b} \sum_{g=1}^{\infty} b^{-2g} \left( 2 \sum_{i=1}^{t} \sum_{j=1}^{i} a_{i,j} b^{\nu_{i,j}} \delta_b(\nu_{i}) \delta_b(\nu_{j}) - \sum_{i=1}^{t} a_i^2 b^{2\nu_i} \delta_b(\nu_{i}) \right) \leq \\
\leq 2(b-1)^3 \sum_{i=1}^{t} b^{\nu_i} \sum_{j=1}^{i} b^{\nu_j} \sum_{g=1}^{\infty} b^{-2g} \delta_b(\nu_{i}) \delta_b(\nu_{j}) - \frac{b-1}{b} \sum_{i=1}^{t} b^{2\nu_i} \sum_{g=1}^{\infty} b^{-2g} \delta_b(\nu_{i}) = \\
= 2(b-1)^3 \sum_{i=1}^{t} b^{\nu_i} \sum_{j=1}^{i} b^{\nu_j} \sum_{g=1}^{\infty} b^{-2g} - \frac{b-1}{b} \sum_{i=1}^{t} b^{2\nu_i} \sum_{g=1}^{\infty} b^{-2g} < \\
< \frac{2b(b-1)^2}{b+1} \sum_{i=1}^{t} b^{\nu_i} \sum_{j=\nu_i}^{\infty} b^{-j} - \frac{b}{b+1} t = \frac{b(2b^2 - 2b - 1)}{b+1} t,
\]

so that the estimation (22) is proved.

From (20), (21), (22) and (23) we obtain

\[
(\text{NF}_N(W(b); S_b^G))^2 < \frac{4b^3 - 2b^2 - 3b - 1}{b+1} t + \frac{(b+1)^3}{b}.
\]

From the presentation of \( N \) in the form \( N = \sum_{j=1}^{t} a_{j} b^{\nu_j} \) with the conditions \( \nu_1 > \nu_2 > \ldots > \nu_t \geq 0 \) we obtain \( N \geq \frac{b^t-1}{b-1} \), so that \( t \leq \frac{1}{\log b} \log[(b-1)N+1] \). From (24) and the last estimation we have

\[
(\text{NF}_N(W(b); S_b^G))^2 < \frac{4b^3 - 2b^2 - 3b - 1}{(b+1) \log b} \log[(b-1)N+1] + \frac{(b+1)^3}{b}.
\]

Theorem 3 is proved.

4.4 – Proof of Theorem 4.

We will consider a special form of \( N \), so

\[
N = 1010 \ldots 101,
\]

where the number of ones is exact \( r \). From (25) we obtain that

\[
r = \frac{1}{2 \log b} \log[(b^2 - 1)N+1].
\]
From Definition 3 we have

\[
(\text{NF}_N(W(b); S_b^2))^2 \geq \frac{1}{b} \sum_{g=1}^{\infty} \frac{b^{-2g}}{g} \sum_{\alpha=0}^{g-1} \sum_{k \in A(g, \alpha)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2 + \\
+ \frac{1}{b} \sum_{g=1}^{\infty} \frac{b^{-2g}}{g} \sum_{k \in A(g,g)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2.
\]

Using the results of Lemma 4 and the special form of \( N \), we obtain the next low estimations:

\[
\frac{1}{b} \sum_{g=1}^{\infty} \frac{b^{-2g}}{g} \sum_{\alpha=0}^{g-1} \sum_{k \in A(g, \alpha)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2 > \\
\frac{4(b^2 - 2)^2 + b^2}{4b^2 (b + 1)^2 (b - 1)^2} - \frac{4(b^2 - 2)^2 (b^2 - b + 1) + b^3}{4b^2 (b + 1)^3 (b - 1)^2};
\]

\[
\frac{1}{b} \sum_{g=1}^{\infty} \frac{b^{-2g}}{g} \sum_{k \in A(g,g)} \left| \sum_{i=0}^{N-1} w_k(S_b^\Sigma(i)) \right|^2 > \\
\frac{b^4 - 3b^2 + 4}{b(b + 1)^2 (b - 1)^2} - \frac{2(b^4 - b^2 + 1)}{b(b + 1)^3 (b - 1)^2}.
\]

The demonstrations of (28) and (29) are similar to the demonstration of (22) and (23). From (26), (27), (28) and (29) we obtain

\[
(\text{NF}_N(W(b); S_b^2))^2 > b^5 + b^4 - 3b^2 - 4b^2 + 4b + 4 \frac{\log_b[(b^2 - 1)N+1] - C(b)}{2b^2 (b + 1)^2 (b - 1)^2},
\]

where \( C(b) = \frac{4b^6 + 4b^5 - 12b^4 + 96^3 - 8b + 16}{4b^2 (b + 1)^3 (b - 1)^2} \). A Theorem 4 is proved.

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