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Homogenization of the Stokes flow with small viscosity in a non-periodic porous medium

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RIASSUNTO: *Si considerano le equazioni stazionarie di Stokes con un piccolo coefficiente di viscosità in un dominio perforato in cui i buchi non sono della stessa grandezza e non sono distributiti in modo periodico. Ogni buco è perforato al centro di un cubo la cui grandezza è legata alla dimensione del cubo da una data condizione. Domini perforati in modo periodico sono un caso particolare dei domini di questo tipo. Si utilizza il lavoro Cioranescu-Murat e le funzioni test di Allaire. Si mostra che molti risultati esistenti in letteratura per domini perforati in modo periodico sono anche verificati nel constesto di questo lavoro.*

ABSTRACT: *Stationary Stokes equations with a small viscosity coefficient are considered in a perforated domain in which holes are not of the same size and are not distributed periodically. Each hole is perforated at the centre of a cube whose size relates to the size of the hole by a given condition. Periodically perforated domains are a particular case of domains of this type. The framework of D. Cioranescu and F. Murat is employed together with the test functions by G. Allaire. It is shown that many existing results in the literature for periodically perforated domains relating this framework hold in this setting.*

1 – Introduction

Homogenization of the Stokes equations plays an important role in understanding the macroscopic laws which govern fluid flow in porous media. Most of the works in the literature consider either a periodic dis-

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tribution of holes or a complete random medium. The two scale asymptotic expansion, the energy method and oscillating test functions (see Besoussans *et al.* [6]) were used to treat the periodic case where the holes' size and the period are of the same order. The homogenized equation is the Darcy's law (Tartar [20], Allaire [2], Lions [16], Zhikov [22]) or some of its time dependent versions (Lions [15], Allaire [4], Sandrakov [19]). The case where the holes' size is of a smaller order than the period is amenable to the general framework developed by Cioranescu and Murat [8] which is essentially a generalization of the energy method. In the case where holes are of a critical order of the period, the effective equation is the Brinkman law, first introduced in 1947 in [7]. When holes are smaller, they have no effect and the asymptotic limit equation is the Stokes equations. The homogenized equation is in the form of the Darcy's law when the holes' size is bigger. This is demonstrated in the two part paper by Allaire [3]. Random domains are studied in Rubinstein [17], Beliaev and Kozlov [5] and Wright [21].

In this paper, we consider the Stokes equations with a small viscosity coefficient in a porous domain where holes are not necessarily distributed periodically and may not be of the same size. We employed the Cioranescu-Murat framework and the test functions introduced in Allaire [3].

Let Ω^ε be the perforated domain in $\mathbb{R}^N$ $(N \geq 2)$. We consider the problem

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + \lambda u^\varepsilon + \nabla p^\varepsilon = f \\ \nabla \cdot u^\varepsilon = 0, \quad u^\varepsilon \in (H_0^1(\Omega^\varepsilon))^N, \end{cases} \tag{1}$$

which is the Laplace transform of the time dependent Stokes equations, where $\nabla . u^\varepsilon$ denotes the divergence of the vector u^ε i.e $\nabla . u^\varepsilon = \sum_{i=1}^N \partial u_i^\varepsilon / \partial x_i$. We will only consider the case $\lambda > 0$ as the case $\lambda = 0$ can be treated by making the transformation $v^\varepsilon = \varepsilon u^\varepsilon$ and using the results by Allaire [3]. Let Ω be a bounded convex domain (the assumption that Ω is convex is needed for proving the inequality (8)). The boundary of a convex domain is Lipschitz. For each $\varepsilon > 0$, Ω is a subset of the union of M^ε closed cubes P_i^ε $(i = 1, \ldots, M^\varepsilon)$ of size $2\varepsilon^{r_{i,\varepsilon}}$ $(r_{i,\varepsilon} > 0)$ whose interiors are pairwise disjoint. The cubes P_i^ε can intersect the boundary of Ω but none of these cubes do not intersect Ω.

Let T be a closed bounded set in $\mathbb{R}^N$ with a Lipschitz boundary locally located on one side of its boundary, which is contained entirely in the inside of the unit ball B_1 and contains a ball of radius $\rho < 1$ centred at the origin in its interior. Let x_i^ε be the centre of the cube P_i^ε. For each cube P_i^ε we consider the set T_i^ε inside P_i^ε: $T_i^\varepsilon = \{x : x - x_i^\varepsilon \in a_i^\varepsilon T\}$ where a_i^ε is the holes' size which is defined as

$$(2) \qquad a_i^\varepsilon = \varepsilon^{s_{i,\varepsilon}} \ (N \geq 3), \text{ and } a_i^\varepsilon = \varepsilon^{r_{i,\varepsilon}} \exp(-a\varepsilon^{-s_{i,\varepsilon}}) \ (N = 2),$$

where $r_{i,\varepsilon}$, $s_{i,\varepsilon}$ and a are positive constants; a does not depend on i and ε. We assume further that there exist constants α and r with $r > 0$ such that $r_{i,\varepsilon} \geq r$ for all i and ε and $Nr_{i,\varepsilon} - (N-2)s_{i,\varepsilon} = \alpha$ $(N \geq 3)$ and $2r_{i,\varepsilon} - s_{i,\varepsilon} = \alpha$ $(N = 2)$ for all i and ε; and moreover $2r > \alpha$. Since $s_{i,\varepsilon} - r_{i,\varepsilon} \geq (2r - \alpha)/(N-2) > 0$ when $N \geq 3$ and $s_{i,\varepsilon} \geq 2r - \alpha > 0$ when $N = 2$, all the sets T_i^ε are strictly contained in the interior of the cube P_i^ε. Removing from Ω all the sets T_i^ε that do not intersect $\partial\Omega$, the remainder we get is the perforated domain Ω^ε which has a Lipschitz boundary and is locally located on one side of its boundary. The condition that the perforated domain is locally located on one side of its boundary guarantees that all the well known results about the existence and uniqueness of the solution of the Stokes equations holds; it plays no role in our results.

Periodically perforated domains are a particular case of domains constructed above. The framework by Cioranescu and Murat [8] and Kacimi and Murat [12] employed by Allaire [3] for the Stokes equations works in this more general class of perforated domains. If we drop the viscosity coefficient ε, the limits obtained in Allaire [3] hold in this situation, except that in [3] $\lambda = 0$ but the results for $\lambda > 0$ are essentially similar. The estimates in $L^2(\Omega)$ and $H^1(\Omega)$ of the test functions can be easily seen to be true in our setting. However, the estimates in $H^{-1}(\Omega)$ is more complicated. Kacimi and Murat [12] and Allaire [3] use an estimate in $H^{-1}(\Omega)$ for periodic distributions of zero average in the period cube, which is originally due to Kohn and Vogelius [13]. We show here that a new approach not using this estimate can be used to obtain an estimate in $H^{-1}(\Omega)$ for the test functions; and all the results in the above papers hold in this more general setting. For the Laplace equation, as we do not require that the boundary of Ω^ε is Lipschitz, we can also allow holes to intersect the boundary of Ω. The Poincare inequality (8) always holds; and we can consider any domain Ω, not just convex as in this paper. For the Stokes equations, as the boundary of Ω^ε needs to be Lipschitz, holes should not intersect the boundary $\partial\Omega$. For a general domain, the inequality (8) does not hold.

Understanding that u^ε is zero in the holes, our purpose is to study the behaviour of u^ε when $\varepsilon \to 0$. The test functions and their properties are introduced in Section 2. In Section 3, we prove that u^ε (or $u^\varepsilon\varepsilon^{1-\alpha}$ when $\alpha > 1$) converges strongly in $(L^2(\Omega))^N$ to the velocity field of a Darcy law in Ω. The pressure function p^ε can be extended over the holes by an extending operator $P^\varepsilon : L^2(\Omega^\varepsilon) \to L^2(\Omega)$ such that $P^\varepsilon p^\varepsilon$ converges strongly in $L^2(\Omega)/\mathbb{R}$ to the pressure function of this Darcy's law (recall that two functions p_1 and p_2 are identified as the same element in $L^2(\Omega)/\mathbb{R}$ if $p_1 - p_2$ is a constant; the $L^2(\Omega)/\mathbb{R}$ norm of this element is $\|p_1 - \int_\Omega p_1/|\Omega|\|^{1/2}_{L^2(\Omega)}$). Let M be the $N \times N$ symmetric matrix whose k^{th} column is the vector $F_k/2^N$ when $N \geq 3$ (F_k is the drag force defined after equation (4) or $\pi e_k/a$ when $N = 2$ (e_k is the kth unit vector). Let $L = M$ if $\alpha > 1$, $L = M + \lambda I$ if $\alpha = 1$ and $L = \lambda I$ if $\alpha < 1$ (I is the identity matrix). Let $(u, p) \in (L^2(\Omega))^N \times H^1(\Omega)$ be the unique solution of the Darcy law

$$u = L^{-1}(f - \nabla p), \quad \nabla \cdot u = 0, \quad u \cdot n = 0 \text{ on } \partial\Omega.$$

The behaviour of p^ε and u^ε are as follows.

THEOREM 1. *The pressure function p^ε converges strongly to p in $L^2(\Omega)/\mathbb{R}$. When $\alpha > 1$, $u^\varepsilon\varepsilon^{1-\alpha} \to u$ in $(L^2(\Omega))^N$. Otherwise $u^\varepsilon \to u$ in $(L^2(\Omega))^N$.*

The error estimates are established when the limit function is in $(W^{1,\infty}(\Omega))^N \bigcap (H_0^1(\Omega))^N$. This regularity assumption is weaker than that required in ALLAIRE [3] which, following an idea of KACIMI and MURAT [12], requires the limit to be in $(W^{2,\infty}(\Omega))^N$. Let W^ε be the $N \times N$ matrix of test functions defined in Section 2; $\|W^\varepsilon - I\|_{L^2(\Omega)}$ is less than a positive order of ε (Lemma 1). We have the following results, which will be proved in Section 3.

THEOREM 2. *Assume that the limit u in Theorem 1 belongs to $(H_0^1(\Omega))^N \bigcap (W^{1,\infty}(\Omega))^N$. The error estimates when $N \geq 3$ are as follows.*

When $\alpha > 1$,

$$\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} + \|u^\varepsilon \varepsilon^{1-\alpha} - W^\varepsilon u\|_{L^2(\Omega^\varepsilon)} < \\ < c\big(\varepsilon^{\alpha-1} + \varepsilon^{\alpha/2} + \varepsilon^{r-\alpha/2} + \varepsilon^{(2r-\alpha)/(N-2)}\big).$$

When $\alpha = 1$,

$$\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} + \|u^\varepsilon - W^\varepsilon u\|_{L^2(\Omega^\varepsilon)} < c\big(\varepsilon^{1/2} + \varepsilon^{r-1/2} + \varepsilon^{(2r-1)/(N-2)}\big).$$

When $0 < \alpha < 1$

$$\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} + \|u^\varepsilon - W^\varepsilon u\|_{L^2(\Omega^\varepsilon)} < \\ < c\big(\varepsilon^{\alpha/2} + \varepsilon^{1-\alpha} + \varepsilon^{r-\alpha/2} + \|W^\varepsilon - I\|^{1/2}\varepsilon^{1/2-\alpha/4}\big).$$

When $\alpha \leq 0$,

$$\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} + \|u^\varepsilon - W^\varepsilon u\|_{L^2(\Omega^\varepsilon)} < c\big(\|W^\varepsilon - I\| + \varepsilon^{1/2}\big).$$

When $N = 2$, *similar results hold except that the term* $\varepsilon^{(2r-\alpha)/(N-2)}$ *does not appear in the right hand side of the above inequalities.*

The behaviour of u^ε in $(H_0^1(\Omega))^N$ for a few cases where the holes T_i^ε are significantly small in comparison with the corresponding cube is touched upon in Section 4. We will prove the followings:

Theorem 3. *Assume that* $u \in (H_0^1(\Omega))^N \bigcap (W^{1,\infty}(\Omega))^N$. *When* $\alpha \leq 0$ *and either* $2r > \alpha + 1/2$ $(N = 2$ *or* $N = 3)$ *or* $2r > \alpha + (N-2)/N$ $(N \geq 4)$, *then* $u^\varepsilon - W^\varepsilon u \to 0$ *in* $(H_0^1(\Omega))^N$.

Theorem 4. *Let* r *be such that* $0 < \alpha < r$ *and*

$$r > \max\{\alpha/4 + 1/2, 3\alpha/4\} \quad \text{when } N = 2,$$
$$r > \max\{\alpha/4 + 1/2, N\alpha/4\} \quad \text{when } N = 3,$$
$$r > \max\{\alpha/N + (N-2)/N, N\alpha/4\} \quad \text{when } N \geq 4.$$

Let $K^\varepsilon = (\varepsilon^{1-\alpha}M + \lambda I)^{-1}L$. *If* $u \in (H_0^1(\Omega))^N \bigcap (W^{1,\infty}(\Omega))^N$ *then* $u^\varepsilon - K^\varepsilon W^\varepsilon u \rightharpoonup 0$ *in* $(H_0^1(\Omega))^N$. *If in addition,* $u \in (W^{2,\infty}(\Omega))^N$ *the convergence is strong.*

As expected, when the holes are much smaller than the cubes, they have no effect; the behaviour of u^ε is similar to that in the case of a non perforated domain considered in LIONS [14], except that further regularity assumptions are necessary in compensation for the appearance of the holes.

The problem (1) in periodically perforated domains in which the holes and the period are of the same size is considered in [19], in which the limits of u^ε when $\varepsilon \to 0$ are presented; error estimates are not touched upon. The approach presented in this paper to find error estimates fails in this case due to the weak convergence of the test functions. We do not know if a similar generalization exists for perforated domains of this type.

Related singularly perturbed Dirichlet problems for the Laplacian in perforated domains have recently been considered in [9] for the periodic case and in [10] and [11] for random domains.

Throughout the paper, we denote by c various constants which do not depend on i and ε and whose values may vary form one line to the next. The duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is denoted by $\langle\, ,\, \rangle$; $\|.\|$ denotes the norm in $(L^2(\Omega))^N$ or in $(L^2(\Omega))^{N\times N}$ according to the context unless the space is explicitly specified. By $\nabla\cdot\phi$ we denote the divergence of the vector $\phi = (\phi_1, \ldots, \phi_N)$ i.e. $\nabla\cdot\phi = \sum_{i=1}^N \partial\phi_i/\partial x_i$. By $\nabla\phi$ we mean the gradient of ϕ which is the N dimensional vector $(\partial\phi/\partial x_1, \ldots, \partial\phi/\partial x_N)$ if ϕ is a scalar function or is the $N\times N$ matrix with entries $\partial\phi_i/\partial x_j$ if ϕ is an N dimensional vector $(\phi_1, \ldots, \phi_N)$. Given two $N\times N$ matrices A and B with entries a_{ij} and b_{ij}, we denote the inner product $\sum_{i,j=1}^N a_{ij}b_{ij}$ between A and B by $A : B$. As usual, repeated indices indicate summation.

2 – Test functions

We now construct the test functions and we establish their properties. These functions are exactly the same as those introduced in ALLAIRE [3] for the periodic case; they are constructed differently for the case $N = 2$ and the case $N \geq 3$. Let B_i^ε be the ball of radius $\varepsilon^{r_{i,\varepsilon}}$ inside P_i^ε. Let $C_i^\varepsilon = B_i^\varepsilon \setminus T_i^\varepsilon$ and $K_i^\varepsilon = P_i^\varepsilon \setminus B_i^\varepsilon$. When $N = 2$, for each $k = 1, \ldots, N$ we let $(w_{ki}^\varepsilon, q_{ki}^\varepsilon)$ be such that

$$\begin{cases} w_{ki}^\varepsilon = e_k, \quad q_{ki}^\varepsilon = 0 & \text{in } K_i^\varepsilon \\ \nabla q_{ki}^\varepsilon - \Delta w_{ki}^\varepsilon = 0, \nabla\cdot w_{ki}^\varepsilon = 0 & \text{in } C_i^\varepsilon \\ w_{ki}^\varepsilon = 0,\; q_{ki}^\varepsilon = 0, & \text{in } T_i^\varepsilon, \end{cases}$$

where e_k denotes the kth unit vector.

When $N = 3$ we first consider the Stokes problem in $\mathbb{R}^N \setminus T$:

$$\text{(3)}\qquad \begin{cases} \nabla q_k - \Delta w_k = 0 & \text{in } \mathbb{R}^N \setminus T \\ \nabla \cdot w_k = 0 & \text{in } \mathbb{R}^N \setminus T \\ w_k = 0 & \text{on } \partial T \\ w_k = e_k & \text{at } \infty \\ q_k \in L^2(\mathbb{R}^N \setminus T), & \text{and } \nabla w_k \in (L^2(\mathbb{R}^N \setminus T))^N. \end{cases}$$

This problem has a unique solution whose asymptotic behaviour at infinity is as follows (the proof can be found in the appendix of ALLAIRE [3]).

$$\text{(4)}\qquad \begin{cases} w_k = e_k - \dfrac{1}{2S_N r^{N-2}}\left(\dfrac{F_k}{N-2} + (F_k \cdot e_r)e_r\right) + O\left(\dfrac{1}{r^{N-1}}\right) \\ q_k = -\dfrac{1}{S_N r^{N-1}}(F_k \cdot e_r) + O\left(\dfrac{1}{r^N}\right) \\ \nabla w_k = O\left(\dfrac{1}{r^{N-1}}\right) \\ \dfrac{\partial w_k}{\partial r} - q_k e_r = \dfrac{1}{2S_N r^{N-1}}\left(F_k + N(F_k \cdot e_r)e_r\right) + O\left(\dfrac{1}{r^N}\right), \end{cases}$$

where F_k is the drag force exerted on T by the flow defined as $F_k = \int_{\partial T}(\partial w_k/\partial n - q_k n)ds$, e_r is the radial vector.

Let B'^{ε}_i be the ball centred at the centre of P^{ε}_i with radius $\varepsilon^{r_{i,\varepsilon}/2}$. Let $C'^{\varepsilon}_i = B'^{\varepsilon}_i \setminus T^{\varepsilon}_i$, $D^{\varepsilon}_i = B^{\varepsilon}_i \setminus B'^{\varepsilon}_i$ and as before $K^{\varepsilon}_i = P^{\varepsilon}_i \setminus B^{\varepsilon}_i$. The functions $(w^{\varepsilon}_{ki}, q^{\varepsilon}_{ki}) \in (H^1(P^{\varepsilon}_i))^N \times L^2(P^{\varepsilon}_i)$ with $\int_{D^{\varepsilon}_i} q^{\varepsilon}_{ki} = 0$ is defined such that

$$\begin{cases} w^{\varepsilon}_{ki} = e_k,\ q^{\varepsilon}_{ki} = 0, \text{ in } K^{\varepsilon}_i \\ \nabla q^{\varepsilon}_{ki} - \Delta w^{\varepsilon}_{ki} = 0,\ \nabla \cdot w^{\varepsilon}_{ki} = 0 \text{ in } D^{\varepsilon}_i \\ w^{\varepsilon}_{ki} = w_k(x/a^{\varepsilon}_i),\ q^{\varepsilon}_{ki} = q_k(x/a^{\varepsilon}_i)/a^{\varepsilon}_i, \text{ in } C'^{\varepsilon}_i \\ w^{\varepsilon}_{ki} = 0,\ q^{\varepsilon}_{ki} = 0 \text{ in } T^{\varepsilon}_i. \end{cases}$$

The test functions $(w^{\varepsilon}_k, q^{\varepsilon}_k) \in (H^1(\Omega))^N \times (L^2(\Omega))^N$ are defined as $w^{\varepsilon}_k = w^{\varepsilon}_{ki}$, and $q^{\varepsilon}_k = q^{\varepsilon}_{ki}$ in P^{ε}_i; w^{ε}_k are bounded pointwise. When $N = 2$, we define in each cube P^{ε}_i the functions $(w^{\varepsilon}_{0ki}, q^{\varepsilon}_{0ki}) \in (H^1(P^{\varepsilon}_i))^N \times L^2(P^{\varepsilon}_i)$ whose definitions are similar to that of w^{ε}_{ki} and q^{ε}_{ki} except that the holes T^{ε}_i is replaced by the ball $B^{a^{\varepsilon}_i}_i$ with radius a^{ε}_i and centred at the centre of P^{ε}_i; $B^{a^{\varepsilon}_i}_i$ contains T^{ε}_i in its interior. We define $(w^{\varepsilon}_{0k}, q^{\varepsilon}_{0k}) \in (H^1(\Omega))^N \times L^2(\Omega)$ such that $w^{\varepsilon}_{0k} = w^{\varepsilon}_{0ki}$ and $q^{\varepsilon}_{0k} = q^{\varepsilon}_{0ki}$ in P^{ε}_i.

The behaviours of w_k^ε and q_k^ε are as follows.

LEMMA 1. *In $(L^2(\Omega))^N$, when ε is sufficiently small*

$$\|w_k^\varepsilon - e_k\| \le c \begin{cases} \varepsilon^{2r-\alpha} & \textit{if } N=2 \\ \varepsilon^{2r-\alpha} & \textit{if } N=3 \\ \varepsilon^{2r-\alpha}|\log\varepsilon|^{1/2} & \textit{if } N=4 \\ \varepsilon^{(2r-\alpha)N/(2(N-2))} & \textit{if } N>4. \end{cases} \tag{5}$$

Furthermore $\|q_k^\varepsilon\| \le c\varepsilon^{-\alpha/2}$ and $\|\nabla w_k^\varepsilon\| \le c\varepsilon^{-\alpha/2}$.

PROOF. From the results of ALLAIRE [3] we have that

$$\frac{1}{(2\varepsilon^{r_{i,\varepsilon}})^N}\|w_k^\varepsilon - e_k\|^2_{L^2(P_i^\varepsilon)} \le c \begin{cases} \varepsilon^{2s_{i,\varepsilon}} & \text{if } N=2 \\ \varepsilon^{2(2r_{i,\varepsilon}-\alpha)} & \text{if } N=3 \\ \varepsilon^{2(2r_{i,\varepsilon}-\alpha)}|\log\varepsilon^{r_{i,\varepsilon}-s_{i,\varepsilon}}| & \text{if } N=4 \\ \varepsilon^{(2r_{i,\varepsilon}-\alpha)N/(N-2)} & \text{if } N>4. \end{cases}$$

In addition, using $r_{i,\varepsilon} \ge r$ and $s_{i,\varepsilon} - r_{i,\varepsilon} \ge (2r-\alpha)/(N-2)$ when $N \ge 3$ and $s_{i,\varepsilon} \ge 2r-\alpha$ when $N=2$ we get the conclusion. Other results of the theorem can be obtained in a similar manner. □

Let

$$m_{ki}^\varepsilon = \begin{cases} (2\varepsilon^{r_{i,\varepsilon}})^{-N}\displaystyle\int_{\partial B'^\varepsilon_i}\left(\frac{\partial w_k^\varepsilon}{\partial n} - q_k^\varepsilon n\right)ds \text{ when } N \ge 3 \\ (2\varepsilon^{r_{i,\varepsilon}})^{-N}\displaystyle\int_{\partial B^\varepsilon_i}\left(\frac{\partial w_{0k}^\varepsilon}{\partial n} - q_{0k}^\varepsilon n\right)ds \text{ when } N = 2. \end{cases}$$

The behaviour of m_{ki}^ε is as follows.

LEMMA 2. *When $N \ge 3$, $|m_{ki}^\varepsilon - F_k/(2^N\varepsilon^\alpha)| \le c\varepsilon^{(2r-\alpha)/(N-2)-\alpha}$; and when $N=2$, $|m_{ki}^\varepsilon - \pi e_k/(a\varepsilon^\alpha)| \le c\varepsilon^{2r-2\alpha}|\log\varepsilon|$.*

Proof. When $N \geq 3$ from the last equality of (4) we have that on $\partial B'^{\varepsilon}_i$

$$\frac{\partial w^{\varepsilon}_{ki}}{\partial n} - q^{\varepsilon}_{ki} n_i = \frac{2^{N-2}(a^{\varepsilon}_i)^{N-2}}{\varepsilon^{(N-1)r_{i,\varepsilon}} S_N}[F_k + N(F_k \cdot e^i_r)e^i_r] + O\left(\frac{(a^{\varepsilon}_i)^{N-1}}{\varepsilon^{N r_{i,\varepsilon}}}\right), \tag{6}$$

where e^i_r is the radial unit vector in P^{ε}_i. Integrating over $\partial B'^{\varepsilon}_i$ we readily have

$$|m^{\varepsilon}_{ki} - F_k/(2^N \varepsilon^{\alpha})| \leq c\varepsilon^{s_{i,\varepsilon}-\alpha-r_{i,\varepsilon}} \leq c\varepsilon^{(2r-\alpha)/(N-2)-\alpha}.$$

When $N = 2$, the functions w^{ε}_{0ki} and q^{ε}_{0ki} can be written as

$$w^{\varepsilon}_{0ki} = r^2 f(r)(e_r \cdot e_k)e_r + g(r)e_k, \quad q^{\varepsilon}_{0ki} = rh(r)(e_r \cdot e_k),$$

where $f(r) = Ar^{-2} + Br^{-4} + C$, $g(r) = -A\log r - Br^{-2}/2 - 3Cr^2/2 + D$ and $h(r) = 2Ar^{-2} - 4C$. Therefore

$$\begin{aligned}\frac{\partial w^{\varepsilon}_{0ki}}{\partial n} - q^{\varepsilon}_{0ki} n = (-2Ar^{-1} - 2Br^{-3} + 6Cr)(e_r \cdot e_k)e_r + \\ + (-Ar^{-1} + Br^{-3} - 3Cr)e_k.\end{aligned}$$

Since the constants A, B and C satisfy

$$\begin{aligned}
A &= -\frac{1}{a}\varepsilon^{s_{i,\varepsilon}}(1 + \varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}} + o(\varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}}))\\
B &= \frac{1}{a}(a^{\varepsilon}_i)^2 \varepsilon^{s_{i,\varepsilon}}(1 + \varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}} + o(\varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}}))\\
C &= \frac{1}{a}\varepsilon^{s_{i,\varepsilon}-2r_{i,\varepsilon}}(1 + \varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}} + o(\varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}})),\\
\frac{\partial w^{\varepsilon}_{0ki}}{\partial n} - q^{\varepsilon}_{0ki} n &= \frac{2}{a}\varepsilon^{s_{i,\varepsilon}-r_{i,\varepsilon}}(-e_k + 4(e_r \cdot e_k)e_r) \times\\
&\quad \times (1 + \varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}} + o(\varepsilon^{s_{i,\varepsilon}} \log \varepsilon^{r_{i,\varepsilon}})).
\end{aligned}$$

On integrating we have

$$|m^{\varepsilon}_{ki} - \pi e_k/(a\varepsilon^{\alpha})| \leq c\varepsilon^{s_{i,\varepsilon}-\alpha}|\log \varepsilon|(\alpha + s_{i,\varepsilon}).$$

Since $s_{i,\varepsilon} \geq 2r - \alpha > 0$ we have that $s_{i,\varepsilon}\varepsilon^{s_{i,\varepsilon}} \leq (2r-\alpha)\varepsilon^{2q-\alpha}$, so $|m^{\varepsilon}_{ki} - \pi e_k/(a\varepsilon^{\alpha})| \leq c\varepsilon^{2q-2\alpha}|\log \varepsilon|$. □

For detailed calculation of w^ε_{0ki}, q^ε_{0ki}, A, B, C and D we refer to the thesis of Allaire [1].

Next we establish the behaviour of $\nabla q^\varepsilon_k - \Delta w^\varepsilon_k$ in $H^{-1}(\Omega)$. It can be easily seen that $\nabla q^\varepsilon_k - \Delta w^\varepsilon_k = \mu^\varepsilon_k - \gamma^\varepsilon_k$ where

$$\mu^\varepsilon_k = \sum_{i=1}^{M^\varepsilon} \left(\frac{\partial w^\varepsilon_{ki}}{\partial n} - q^\varepsilon_{ki} n \right) \delta_{B'^\varepsilon_i} + \nabla \cdot (\chi_{D^\varepsilon_i} (q^\varepsilon_{ki} I - \nabla w^\varepsilon_{ki}))$$

when $N \geq 3$ ($\chi_{D^\varepsilon_i}$ is the characteristic function of D^ε_i) and

$$\mu^\varepsilon_k = \sum_{i=1}^{M^\varepsilon} \left(\frac{\partial w^\varepsilon_{ki}}{\partial n} - q^\varepsilon_{ki} n \right) \delta_{B^\varepsilon_i}$$

when $N = 2$, and

$$\gamma^\varepsilon_k = \sum_{i=1}^{M^\varepsilon} \left(\frac{\partial w^\varepsilon_{ki}}{\partial n} - q^\varepsilon_{ki} n \right) \delta_{T^\varepsilon_i},$$

which has no effect as all the functions considered below are zero in T^ε_i for all i. The behaviour of μ^ε_k is as follows.

Lemma 3. *For all $\phi \in H^1_0(\Omega^\varepsilon)$, $|\langle \mu^\varepsilon_k - \sum_{i=1}^{M^\varepsilon} m^\varepsilon_{ki} \chi_{P^\varepsilon_i}, \phi \rangle| \leq c\varepsilon^{r-\alpha} \|\nabla \phi\|$ where the constant c does not depend on ε.*

In fact when $N \geq 3$ we can prove a stronger results that $\|\mu^\varepsilon_k - \sum m^\varepsilon_{ki} \chi_{P^\varepsilon_i}\|_{H^{-1}(\Omega)} \leq c\varepsilon^{r-\alpha}$.

Proof. We consider the case $N \geq 3$ first. We note that

$$\| \sum_{i=1}^{M^\varepsilon} \nabla \cdot (\chi_{D^\varepsilon_i} (q^\varepsilon_{ki} I - \nabla w^\varepsilon_{ki})) \|_{H^{-1}(\Omega)} \leq \| \sum_{i=1}^{M^\varepsilon} \chi_{D^\varepsilon_i} (q^\varepsilon_{ki} I - \nabla w^\varepsilon_{ki}) \|_{L^2(\Omega)}.$$

From (4), we have that $|w^\varepsilon_{ki} - e_k| \leq c\varepsilon^{2r_{i,\varepsilon} - \alpha}$ and $|\partial w^\varepsilon_{ki} / \partial n| \leq c\varepsilon^{r_{i,\varepsilon} - \alpha}$ on $\partial B'^\varepsilon_i$, so $\int_{D^\varepsilon_i} |\nabla w^\varepsilon_{ki}|^2 dx \leq \int_{\partial B'^\varepsilon_i} |w^\varepsilon_{ki} - e_k| |\partial w^\varepsilon_{ki} / \partial n| ds \leq c\varepsilon^{2r - 2\alpha + N r_{i,\varepsilon}}$. In D^ε_i, $\|q^\varepsilon_{ki}\|_{L^2(D^\varepsilon_i)/\mathbb{R}} \leq c \|\nabla q^\varepsilon_{ki}\|_{H^{-1}(D^\varepsilon_i)}$ where the constant c can be the constant appears in the same inequality for the domain between the two balls of radii 1 and 1/2. This can be shown by a simple scaling argument. From the definition of w^ε_{ki} and q^ε_{ki}, $\|\nabla q^\varepsilon_{ki}\|_{H^{-1}(D^\varepsilon_i)} = \|\nabla w^\varepsilon_{ki}\|_{L^2(D^\varepsilon_i)} \leq$

$c\varepsilon^{r-\alpha+Nr_{i,\varepsilon}/2}$. Thus $\|q^\varepsilon_{ki}I-\nabla w^\varepsilon_{ki}\|_{L^2(D^\varepsilon_i)}\le c\varepsilon^{r-\alpha+Nr_{i,\varepsilon}/2}$ and so $\|\sum_{i=1}^{M^\varepsilon}\nabla\cdot(\chi_{D^\varepsilon_i}(q^\varepsilon_{ki}I-\nabla w^\varepsilon_{ki}))\|_{H^{-1}(\Omega)}\le c\varepsilon^{r-\alpha}$.

Let $\phi\in H^1_0(\Omega)$. Then

$$\Big|\Big\langle\sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w^\varepsilon_{ki}}{\partial n}-q^\varepsilon_{ki}n\Big)\delta_{B'\varepsilon_i}-m^\varepsilon_{ki}\chi_{P^\varepsilon_i},\phi\Big\rangle\Big|=$$
$$=\Big|\sum_{i=1}^{M^\varepsilon}\int_{\partial B'^\varepsilon_i}\Big(\frac{\partial w^\varepsilon_{ki}}{\partial n}-q^\varepsilon_{ki}n\Big)\phi ds-\int_{\partial B'^\varepsilon_i}\Big(\frac{\partial w^\varepsilon_{ki}}{\partial n}-q^\varepsilon_{ki}n\Big)ds.(2\varepsilon^{r_{i,\varepsilon}})^{-N}\int_{P^\varepsilon_i}\phi dx\Big|\le$$
$$\le\sum_{i=1}^{M^\varepsilon}\Big\|\frac{\partial w^\varepsilon_{ki}}{\partial n}-q^\varepsilon_{ki}n\Big\|_{L^\infty(\partial B'^\varepsilon_i)}\int_{\partial B'^\varepsilon_i}\Big|\phi-(2\varepsilon^{r_{i,\varepsilon}})^{-N}\int_{P^\varepsilon_i}\phi dx\Big|.$$

From (6) we have that $\|\partial w^\varepsilon_{ki}/\partial n-q^\varepsilon_{ki}n\|_{L^\infty(\partial B'^\varepsilon_i)}\le c\varepsilon^{r_{i,\varepsilon}-\alpha}$. Making the rescale $x=x^\varepsilon_i+\varepsilon^{r_{i,\varepsilon}}y$ (x^ε_i is the centre of P^ε_i), the function $\phi(y)$ is then defined in the cube $P=[-1,1]^N$. Let $\Phi(y)=\phi(y)-2^{-N}\int_P\phi(z)dz$ and B be the ball centred at the origin and of radius $1/2$. Using the trace and the Poincaré inequalities, we have

$$\int_{\partial B}|\Phi|ds\le c\Big(\int_P|\nabla\Phi|^2dy\Big)^{1/2}.$$

Putting $y=\varepsilon^{-r_{i,\varepsilon}}(x-x^\varepsilon_i)$, we have

$$\int_{\partial B'^\varepsilon_i}|\phi-(2\varepsilon^{r_{i,\varepsilon}})^{-N}\int_{P^\varepsilon_i}\phi dx|ds\le c\varepsilon^{Nr_{i,\varepsilon}/2}\Big(\int_{P^\varepsilon_i}|\nabla\phi|^2dx\Big)^{1/2}.$$

Thus

$$\Big|\Big\langle\sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w^\varepsilon_{ki}}{\partial n}-q^\varepsilon_{ki}n\Big)\delta_{B'^\varepsilon_{ki}}-m^\varepsilon_{ki}\chi_{P^\varepsilon_i},\phi\Big\rangle\Big|\le$$
$$\le\varepsilon^{r-\alpha}\varepsilon^{Nr_{i,\varepsilon}/2}\|\nabla\phi\|_{L^2(P^\varepsilon_i)}\le c\varepsilon^{r-\alpha}\|\nabla\phi\|,$$

where we have used the Cauchy-Schwarz inequality.

The case $N=2$ is more complicated. We can only prove the above inequality for $\phi\in H^1_0(\Omega^\varepsilon)$. The distribution μ^ε_k is written as $\mu^\varepsilon_k=\mu^\varepsilon_{0k}+{\mu'}^\varepsilon_k$ where

$$\mu^\varepsilon_{0k}=\sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w^\varepsilon_{0ki}}{\partial n}-q^\varepsilon_{0ki}n\Big)\delta_{B^\varepsilon_i},\quad {\mu'}^\varepsilon_k=\sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial {w'}^\varepsilon_{ki}}{\partial n}-{q'}^\varepsilon_{ki}n\Big)\delta_{B^\varepsilon_i}$$

where $w'^{\varepsilon}_k = w^{\varepsilon}_k - w^{\varepsilon}_{0k}$ and $q'^{\varepsilon}_k = q^{\varepsilon}_k - q^{\varepsilon}_{0k}$. We remark that

$$\sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w'^{\varepsilon}_{ki}}{\partial n} - q'^{\varepsilon}_{ki} n\Big)\delta_{B^\varepsilon_i} = \nabla q'^{\varepsilon}_k - \Delta w'^{\varepsilon}_k - \sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w^{\varepsilon}_{0ki}}{\partial n} - q^{\varepsilon}_{0ki} n\Big)\delta_{B_i^{a_i^\varepsilon}} + \gamma^\varepsilon.$$

Let $\phi \in H^1_0(\Omega^\varepsilon)$. Then $\langle \gamma^\varepsilon, \phi\rangle = 0$. Following Allaire [3], we have that $\|q'^{\varepsilon}_k\|_{L^2(P^\varepsilon_i)} \le c\varepsilon^{2r_{i,\varepsilon}-\alpha}$ and $\|\nabla w'^{\varepsilon}_k\|_{L^2(P^\varepsilon_i)} \le c\varepsilon^{2r_{i,\varepsilon}-\alpha}$. Therefore $\|q'^{\varepsilon}_k\| \le c\varepsilon^{r-\alpha}$ and $\|\nabla w'^{\varepsilon}_k\| \le c\varepsilon^{r-\alpha}$ which implies $|\langle \nabla q'^{\varepsilon}_k - \Delta w'^{\varepsilon}_k, \phi\rangle| \le c\varepsilon^{r-\alpha}\|\nabla\phi\|$. Since $\|\partial w^\varepsilon_{0ki}/\partial n - q^\varepsilon_{0ki} n\|_{L^\infty(B_i^{a_i^\varepsilon})} \le c\varepsilon^{2r_{i,\varepsilon}-\alpha}/a^\varepsilon_i$,

$$\Big|\Big\langle \sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w^{\varepsilon}_{0ki}}{\partial n} - q^{\varepsilon}_{0ki} n\Big)\delta_{B_i^{a_i^\varepsilon}}, \phi\Big\rangle\Big| \le c\frac{\varepsilon^{2r_{i,\varepsilon}-\alpha}}{a^\varepsilon_i}\int_{\partial B_i^{a_i^\varepsilon}} |\phi| ds.$$

Using the trace inequality we have

$$\int_{\partial B_i^{a_i^\varepsilon}} |\phi| ds \le c\left(\int_{B_i^{a_i^\varepsilon}} |\phi|^2 dx\right)^{1/2} + c a^\varepsilon_i \left(\int_{B_i^{a_i^\varepsilon}} |\nabla\phi|^2 dx\right)^{1/2}.$$

Furthermore, as $\phi = 0$ in T^ε_i and T^ε_i contains a ball of radius ra^ε_i, so

$$\int_{B_i^{a_i^\varepsilon}} |\phi|^2 dx \le c(a^\varepsilon_i)^2 \int_{B_i^{a_i^\varepsilon}} |\nabla\phi|^2 dx.$$

Therefore

$$\Big|\Big\langle \sum_{i=1}^{M^\varepsilon}\Big(\frac{\partial w^{\varepsilon}_{0ki}}{\partial n} - q^{\varepsilon}_{0ki} n\Big)\delta_{B_i^{a_i^\varepsilon}}, \phi\Big\rangle\Big| \le c\varepsilon^{r-\alpha}\|\nabla\phi\|.$$

By carrying out a similar procedure as in the case $N \ge 3$, we have that

$$\Big|\Big\langle \mu^\varepsilon_{0k} - \sum_{i=1}^{M^\varepsilon} m^\varepsilon_{ki}\chi_{P^\varepsilon_i}, \phi\Big\rangle\Big| \le c\varepsilon^{r-\alpha}\|\nabla\phi\|.$$

The conclusion follows. ∎

3 – Behaviour of u^ε and p^ε in $L^2(\Omega)$

In variational formulation, equations (1) can be written as

$$\varepsilon\int_\Omega \nabla u^\varepsilon : \nabla\psi dx + \lambda\int_\Omega u^\varepsilon \psi dx - \int_\Omega p^\varepsilon \nabla\cdot\psi dx = \int_\Omega f\psi dx, \quad \forall\psi\in(H_0^1(\Omega^\varepsilon))^N$$
$$\int_\Omega q\nabla\cdot u^\varepsilon dx = 0, \quad \forall q\in L^2(\Omega^\varepsilon)/\mathbb{R}. \tag{7}$$

Since T contains a ball of radius ρ, in each cube P_i^ε whose interior does not intersect $\partial\Omega$, $\|u^\varepsilon\|^2_{L^2(P_i^\varepsilon)} \le c\varepsilon^\alpha\|\nabla u^\varepsilon\|^2_{L^2(P_i^\varepsilon)}$ (a proof of this can be found in Allaire [3] part II). For those cubes P_i^ε that are not perforated, the boundary $\partial\Omega$ intersects the set T_i^ε and so intersects the ball centred at x_i^ε with radius $\varepsilon^{r_{i,\varepsilon}}/2$. Choosing a point on $\partial\Omega$ that is inside this ball, as Ω is convex, there is a hyperplane passing this point that is completely outside Ω. This hyperplane divides P_i^ε into two parts, one of which has an empty intersection with Ω. It is easy to see that as the hyperplane passing through a point inside the ball centred at x_i^ε with radius $\varepsilon^{r_{i,\varepsilon}}/2$ the volume of this part is larger than $c_1|P_i^\varepsilon|$ and is smaller than $c_2|P_i^\varepsilon|$ for some constants c_1 and c_2 which do not depend on ε and the hyperplane. As u^ε can be extended to 0 outside Ω, $u^\varepsilon = 0$ on this part. Using a simple recalling argument and the Poincare inequality (see Ziemer [23], page 189), we see that

$$\|u^\varepsilon\|^2_{L^2(P_i^\varepsilon)} \le c\varepsilon^{2r_{i,\varepsilon}}\|\nabla u^\varepsilon\|^2_{L^2(P_i^\varepsilon)} \le c\varepsilon^\alpha\|\nabla u^\varepsilon\|^2_{L^2(P_i^\varepsilon)}$$

as $2r_{i,\varepsilon} \ge 2r \ge \alpha$. Summing up over the M^ε cubes, we have that

$$\|u^\varepsilon\| \le c\varepsilon^{\alpha/2}\|\nabla u^\varepsilon\|. \tag{8}$$

Letting $\psi = u^\varepsilon$ in (7), we have $\varepsilon\|\nabla u^\varepsilon\|^2 + \lambda\|u^\varepsilon\|^2 \le \|f\|.\|u^\varepsilon\|$ which implies $\|u^\varepsilon\| \le c$ and $\varepsilon^{1/2}\|\nabla u^\varepsilon\| \le c$. On using (8), we also have $\|\nabla u^\varepsilon\| \le c\varepsilon^{\alpha/2-1}$ so $\|u^\varepsilon\| \le c\varepsilon^{\alpha-1}$. Thus when $\alpha \le 1$, from a sequence u^ε, we can extract a subsequence which converges weakly to a limit u in $(L^2(\Omega))^N$ when $\varepsilon \to 0$. When $\alpha > 1$, a subsequence can be extracted such that $\varepsilon^{1-\alpha}u^\varepsilon$ converges weakly to u in $(L^2(\Omega))^N$.

Next we consider the pressure function p^ε. From Allaire [3], we know that there is a map R^ε from $(H_0^1(\Omega))^N$ into $(H_0^1(\Omega^\varepsilon))^N$ such that for all $\psi \in H_0^1(\Omega)$,

$$\|\nabla(R^\varepsilon \psi)\| \le c(\|\nabla\psi\| + \varepsilon^{-\alpha/2}\|\psi\|),$$

$R^\varepsilon\psi = \psi$ in K_i^ε and $R^\varepsilon\psi = 0$ in T_i^ε, and if $\nabla\cdot\psi = 0$ then $\nabla\cdot R^\varepsilon\psi = 0$. The pressure function p^ε can be extended over the holes T_i^ε to a function $P^\varepsilon p^\varepsilon$ such that $P^\varepsilon p^\varepsilon = p^\varepsilon$ in Ω^ε and $P^\varepsilon p^\varepsilon = \int_{C_i^\varepsilon} p^\varepsilon/|C_i^\varepsilon|$ in each hole T_i^ε. It can be shown that

$$\langle \nabla P^\varepsilon p^\varepsilon, \psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle p^\varepsilon, R^\varepsilon\psi\rangle_{H^{-1}(\Omega^\varepsilon),H_0^1(\Omega^\varepsilon)}$$

for all $\psi \in (H_0^1(\Omega))^N$. From (7), for each $\psi \in H_0^1(\Omega)$,

$$|\langle\nabla P^\varepsilon p^\varepsilon,\psi\rangle| \le \|f\|.\|R^\varepsilon\psi\| + \lambda\|u^\varepsilon\|.\|R^\varepsilon\psi\| + \varepsilon\|\nabla u^\varepsilon\|.\|\nabla(R^\varepsilon\psi)\|.$$

If $\alpha \ge 0$, from (8) and the above estimate for $\|\nabla(R^\varepsilon\psi)\|$, we have

$$|\langle\nabla P^\varepsilon p^\varepsilon,\psi\rangle| \le c\|f\|(\varepsilon^{\alpha/2}\|\nabla\psi\| + \|\psi\|) \le c\|f\|.\|\nabla\psi\|,$$

so $\|\nabla P^\varepsilon p^\varepsilon\|_{H^{-1}(\Omega)} \le c\|f\|$. If $\alpha < 0$, using $\|R^\varepsilon\psi\| \le \|\nabla(R^\varepsilon\psi)\|$ and the estimate for $\|\nabla(R^\varepsilon\psi)\|$, we get the same inequality. Thus $\|P^\varepsilon p^\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \le c$. We therefore can extract a subsequence such that $P^\varepsilon p^\varepsilon$ converges weakly to a function p in $L^2(\Omega)/\mathbb{R}$. Indeed we can show that the convergence is strong. We now prove Theorem 1.

PROOF OF THEOREM 1. When $\alpha > 1$ we can extract a subsequence such that $\varepsilon^{1-\alpha}u^\varepsilon$ converges weakly to a function u in $(L^2(\Omega))^N$. Let $\phi \in \mathcal{D}(\Omega)$. Letting $\psi = w_k^\varepsilon\phi$ and $q = q_k^\varepsilon\phi$ as test functions in (7), we have

$$\varepsilon\int_\Omega \nabla u^\varepsilon : \nabla w_k^\varepsilon \phi dx + \varepsilon\int_\Omega \nabla u^\varepsilon : \nabla\phi w_k^\varepsilon dx + \lambda\int_\Omega u^\varepsilon w_k^\varepsilon\phi dx - \int_\Omega p^\varepsilon w_k^\varepsilon\cdot\nabla\phi dx = \\ = \int_\Omega f w_k^\varepsilon \phi dx,$$

i.e.

$$-\varepsilon\langle\Delta w_k^\varepsilon - \nabla q_k^\varepsilon, u^\varepsilon\phi\rangle + \varepsilon\int_\Omega q_k^\varepsilon u^\varepsilon\cdot\nabla\phi dx - \varepsilon\int_\Omega \nabla w_k^\varepsilon : u^\varepsilon\nabla\phi dx + \\ + \varepsilon\int_\Omega \nabla u^\varepsilon : \nabla\phi w_k^\varepsilon dx + \lambda\int_\Omega u^\varepsilon w_k^\varepsilon\phi dx - \int_\Omega p^\varepsilon w_k^\varepsilon\cdot\nabla\phi dx = \\ = \int_\Omega f w_k^\varepsilon\phi dx.$$

As $w_k^\varepsilon \to e_k$ in $(L^2(\Omega))^N$, the right hand side converges to $\int_\Omega f e_k \phi dx$ when $\varepsilon \to 0$. In the left hand side, since $\varepsilon^{\alpha/2}\|q_k^\varepsilon\|$, $\varepsilon^{\alpha/2}\|\nabla w_k^\varepsilon\|, \varepsilon^{1-\alpha}\|u^\varepsilon\|$ are bounded, the second, third, fourth and fifth terms converges to 0, the final term converges to $-\int_\Omega p e_k \cdot \nabla\phi dx = \langle \nabla p \cdot e_k, \phi\rangle$. The first term can be written as

$$\varepsilon\Big\langle \mu_k^\varepsilon - \sum_{i=1}^{M^\varepsilon} m_{ki}^\varepsilon \chi_{P_i^\varepsilon}, u^\varepsilon\phi\Big\rangle + \varepsilon\sum_{i=1}^{M^\varepsilon}\Big(m_{ki}^\varepsilon - \frac{F_k}{2^N\varepsilon^\alpha}\Big)\int_{P_i^\varepsilon} u^\varepsilon\phi dx + \int \varepsilon^{1-\alpha} u^\varepsilon\phi\cdot F_k/2^N dx.$$

the last term converges to $\int_\Omega u\phi\cdot F_k/2^N dx$. From Lemmas 2 and 3 the other terms converges to 0. Therefore

$$\int_\Omega u\phi\cdot F_k/2^N dx + \langle \nabla p\cdot e_k, \phi\rangle = \int_\Omega f\cdot e_k\phi dx.$$

Since this is true for all $\phi \in \mathcal{D}(\Omega)$, we have $Mu + \nabla p = f$ i.e. $u = M^{-1}(f - \nabla p)$. As $\nabla\cdot u^\varepsilon = 0$ so $\nabla\cdot u = 0$. With the boundary condition $u\cdot n = 0$ (see for example Sanchez-Palencia [18]), this problem for (u, p) has a unique solution. Since the limit does not depend on the subsequence, the sequence $(u^\varepsilon, p^\varepsilon)$ converges weakly to (u, p) in $(L^2(\Omega))^N \times L^2(\Omega)/\mathbb{R}$.

Next we prove the strong convergence. The following proof for p^ε follows from that of Tartar [20] and Allaire [3]. Let ψ_ε be a sequence which converges weakly to 0 in $(H_0^1(\Omega))^N$. Since $\|\nabla(R^\varepsilon\psi^\varepsilon)\| \le c(\|\nabla\psi^\varepsilon\| + \varepsilon^{-\alpha/2}\|\psi^\varepsilon\|)$ and $\|R^\varepsilon\psi^\varepsilon\| \le c(\varepsilon^{\alpha/2}\|\nabla\psi^\varepsilon\| + \|\psi^\varepsilon\|)$ which converges to 0, we have

$$\langle \nabla P^\varepsilon p^\varepsilon, \psi^\varepsilon\rangle = \langle \nabla p^\varepsilon, R^\varepsilon\psi^\varepsilon\rangle = \int_\Omega (f - \lambda u^\varepsilon)R^\varepsilon\psi^\varepsilon dx - \varepsilon\int_\Omega \nabla u^\varepsilon\cdot\nabla(R^\varepsilon u^\varepsilon)dx$$

which converges to 0. Since this is true for all sequence $\psi^\varepsilon \rightharpoonup 0$ in $(H_0^1(\Omega))^N$, we have that $\nabla P^\varepsilon p^\varepsilon \to \nabla p$ in $H^{-1}(\Omega)$ strongly and so $P^\varepsilon p^\varepsilon \to p$ in $L^2(\Omega)/\mathbb{R}$.

Now we show the strong convergence for $\varepsilon^{1-\alpha}u^\varepsilon$. Let W^ε be the $N\times N$ matrix whose k^{th} column is the vector w_k^ε. Let $\Phi \in (\mathcal{D}(\Omega))^N$. After some manipulation using (7), we have

$$\begin{aligned}
&\varepsilon^\alpha\int_\Omega |\nabla(u^\varepsilon\varepsilon^{1-\alpha} - W^\varepsilon\Phi)|^2 dx + \lambda\varepsilon^{\alpha-1}\int_\Omega |u^\varepsilon\varepsilon^{1-\alpha} - W^\varepsilon\Phi|^2 dx = \\
&= \int_\Omega f u^\varepsilon\varepsilon^{1-\alpha}dx - 2\int_\Omega fW^\varepsilon\cdot\Phi dx + 2\int_\Omega \nabla p^\varepsilon\cdot(W^\varepsilon\Phi)dx + \\
&+ \varepsilon^\alpha\int_\Omega |\nabla(W^\varepsilon\Phi)|^2 dx + \lambda\varepsilon^{\alpha-1}\int_\Omega |W^\varepsilon\Phi|^2 dx.
\end{aligned} \tag{9}$$

On the right hand side, the first, second, third and fifth terms converges to $\int_\Omega fudx$, $-2\int_\Omega f\Phi dx$, $-2\int_\Omega p\nabla\cdot\Phi dx$ and 0 respectively. The fourth term can be written as

$$\varepsilon^\alpha\int_\Omega \nabla(\phi_k w_k^\varepsilon):\nabla(\phi_l w_l^\varepsilon)dx = \varepsilon^\alpha\int_\Omega w_k^\varepsilon\nabla\phi_k : w_l^\varepsilon\nabla\phi_l dx + \\ + \varepsilon^\alpha\int_\Omega \phi_k\nabla w_k^\varepsilon : \phi_l\nabla w_l^\varepsilon dx + \\ + 2\varepsilon^\alpha\int_\Omega \phi_k\nabla w_k^\varepsilon : \nabla\phi_l w_l^\varepsilon dx,$$

where ϕ_k denotes the k^{th} component of the vector Φ. On the right hand side of this equation, the first term and the third term converges to 0. The second term can be written as

$$\varepsilon^\alpha\langle\mu^\varepsilon,\phi_k\phi_l w_l^\varepsilon\rangle + \varepsilon^\alpha\int_\Omega q_k^\varepsilon\nabla\cdot(\phi_k\phi_l)w_l^\varepsilon dx - \varepsilon^\alpha\int_\Omega \nabla w_k^\varepsilon:\nabla(\phi_k\phi_l)w_l^\varepsilon dx,$$

in which the last two terms tend to 0 as $\varepsilon\to 0$. The first term is written as

$$\varepsilon^\alpha\Big\langle\mu_k^\varepsilon - \sum_{i=1}^{M^\varepsilon} m_{ki}^\varepsilon\chi_{P_i^\varepsilon},\phi_k\phi_l w_l^\varepsilon\Big\rangle + \varepsilon^\alpha\sum_{i=1}^{M^\varepsilon}\int_{P_i^\varepsilon} m_{ki}^\varepsilon\phi_k\phi_l w_l^\varepsilon dx$$

which converges to $\int M\Phi\cdot\Phi$. Therefore the right hand side in (9) converges to $\int_\Omega fudx - 2\int_\Omega f\Phi dx - 2\int_\Omega p\nabla\cdot\Phi dx + \int_\Omega M\Phi\cdot\Phi dx$. Since $\nabla\cdot u = 0$ and $u\cdot n = 0$ on $\partial\Omega$, there exists a sequence $\{\Phi_i\}\subset(\mathcal{D}(\Omega))^N$ such that $\nabla\cdot\Phi_i = 0$ and $\Phi_i\to u$ in $(L^2(\Omega))^N$. We have

$$\int_\Omega fudx - 2\int_\Omega f\cdot\Phi_i dx + \int_\Omega M\Phi_i\cdot\Phi_i dx = \int_\Omega f(u-\Phi_i)dx + \int_\Omega M(u-\Phi_i)\cdot\Phi_i dx,$$

converges to 0 as $i\to\infty$. From (9), we have $\limsup_{\varepsilon\to 0}\|u^\varepsilon\varepsilon^{1-\alpha} - W^\varepsilon\Phi_i\| \le \int_\Omega(f + M\Phi_i)(u-\Phi_i)dx$ so

$$\limsup_{\varepsilon\to 0}\|u^\varepsilon\varepsilon^{1-\alpha} - W^\varepsilon u\| \le \int_\Omega(f + M\Phi_i)(u-\Phi_i)dx + \\ + \limsup_{\varepsilon\to 0}\|W^\varepsilon\|_{L^\infty(\Omega)}.\|u-\Phi_i\|.$$

for all i. Since $\|W^\varepsilon\|_{L^\infty(\Omega)}$ is bounded, letting $i\to\infty$, we have that $\lim_{\varepsilon\to 0}\|u^\varepsilon\varepsilon^{1-\alpha} - W^\varepsilon u\| = 0$. As $W^\varepsilon\to I$ in $(L^2(\Omega))^{N\times N}$, so $u^\varepsilon\varepsilon^{1-\alpha}\to u$.

The proof for $\alpha = 1$ is similar except that since $u^\varepsilon \rightharpoonup u$ in $(L^2(\Omega))^N$ so $\lambda \int_\Omega u^\varepsilon w_k^\varepsilon \phi dx \to \lambda \int_\Omega u_k \phi dx$.

Now we consider the case $\alpha \le 1$. Letting $w_k^\varepsilon \phi$ be the test function in (7), we have

$$\varepsilon\int_\Omega \nabla u^\varepsilon : \nabla w_k^\varepsilon \phi dx + \varepsilon\int_\Omega \nabla u^\varepsilon : \nabla\phi w_k^\varepsilon dx + \lambda\int_\Omega u^\varepsilon \cdot w_k^\varepsilon \phi dx - \int_\Omega p^\varepsilon w_k^\varepsilon \cdot \nabla\phi dx = \\ = \int_\Omega f w_k^\varepsilon \phi dx.$$

The first two terms on the left hand side converge to 0 due to the boundedness of $\varepsilon^{1/2}\|\nabla u^\varepsilon\|$ and $\varepsilon^{\alpha/2}\|\nabla w_k^\varepsilon\|$. The other terms converge to $\lambda \int u e_k \phi$ and $-\int p e_k \nabla\phi$. The right hand side converges to $\int f e_k \phi$. Hence $u = \lambda^{-1}(f - \nabla p)$. As $\nabla \cdot u^\varepsilon = 0$ so $\nabla \cdot u = 0$ and $u \cdot n = 0$ on $\partial\Omega$. From (7),

$$\varepsilon \int_\Omega |\nabla u^\varepsilon|^2 dx + \lambda \int_\Omega |u^\varepsilon - u|^2 dx = \lambda \int_\Omega |u|^2 dx - \lambda \int_\Omega u \cdot u^\varepsilon dx.$$

Since the right hand side converges to 0, $u^\varepsilon \to u$ in $(L^2(\Omega))^N$.

Next we show the strong convergence of $P^\varepsilon p^\varepsilon$ in $L^2(\Omega)/\mathbb{R}$. The case $\alpha > 0$ is shown in the same manner as in the previous case. We consider the case $\alpha \le 0$. Let ψ^ε be a sequence which converges weakly to 0 in $(H_0^1(\Omega))^N$. Then

$$\langle \nabla P^\varepsilon p^\varepsilon, \psi^\varepsilon \rangle = \langle \nabla p, R^\varepsilon \psi^\varepsilon \rangle + \lambda \int_\Omega (u - u^\varepsilon) R^\varepsilon \psi^\varepsilon dx - \varepsilon \int_\Omega \nabla u^\varepsilon : \nabla(R^\varepsilon \psi^\varepsilon) dx.$$

The last two terms on the right hand side converges to 0. From the construction of $R^\varepsilon \psi^\varepsilon$ (see Allaire [3]), it can be shown that $\langle \nabla p, R^\varepsilon \psi^\varepsilon \rangle$ also converges to 0. Therefore $p^\varepsilon \to p$ in $L^2(\Omega)/\mathbb{R}$. □

Next we show the results on error estimate.

Proof of Theorem 2. We only prove the theorem for $N \ge 3$. The case $N = 2$ is similar. For simplicity we will denote by μ_k the k^{th} column of the matrix M. When $\alpha > 1$, let $v^\varepsilon = u^\varepsilon \varepsilon^{1-\alpha} - W^\varepsilon u$ which belongs to

$(H_0^1(\Omega^\varepsilon))^N$. Using (7), for all $\nu^\varepsilon \in (H_0^1(\Omega^\varepsilon))^N$, we have

$$
\begin{aligned}
\varepsilon \int_\Omega \nabla v^\varepsilon : \nabla \nu^\varepsilon dx + \lambda \int_\Omega v^\varepsilon \cdot \nu^\varepsilon dx = {} & \varepsilon^{1-\alpha} \langle \nabla(p - p^\varepsilon), \nu^\varepsilon \rangle + \\
& -\lambda \int_\Omega W^\varepsilon u \nu^\varepsilon dx - \varepsilon \int_\Omega w_k^\varepsilon \nabla u_k : \nabla \nu^\varepsilon dx + \\
& + \varepsilon \int_\Omega \nabla w_k^\varepsilon : \nabla u_k \nu^\varepsilon dx + \varepsilon \langle \nabla q_k^\varepsilon, \nu^\varepsilon u_k \rangle + \\
& - \varepsilon \langle \mu_k^\varepsilon - \sum_{i=1}^{M^\varepsilon} m_{ki}^\varepsilon \chi_{P_i^\varepsilon}, \nu^\varepsilon u_k \rangle + \\
& - \varepsilon \sum_{i=1}^{M^\varepsilon} \int_{P_i^\varepsilon} (m_{ki}^\varepsilon - \mu_k/\varepsilon^\alpha) \nu^\varepsilon u_k dx,
\end{aligned} \tag{10}
$$

where u_k denotes the k^{th} component of u. Putting $\nu^\varepsilon = v^\varepsilon$, we have that

$$|\langle \nabla(p - p^\varepsilon) \cdot v^\varepsilon \rangle| = \Big| \int_\Omega (p - p^\varepsilon)(w_k^\varepsilon - e_k) \nabla u_k dx \Big| \le c \|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} \|W^\varepsilon - I\|,$$

and that

$$\Big| \int_\Omega \nabla q_k^\varepsilon (v^\varepsilon u_k) dx \Big| \le c \varepsilon^{-\alpha/2} \|v^\varepsilon\| + c \varepsilon^{-\alpha/2} \|W^\varepsilon - I\|.$$

On using (8) we have

$$
\begin{aligned}
\varepsilon \|\nabla v^\varepsilon\|^2 + \lambda \|v^\varepsilon\|^2 \le {} & c \|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} \|W^\varepsilon - I\| \varepsilon^{1-\alpha} + \\
& + c(\varepsilon^{\alpha/2} + \varepsilon + \varepsilon^{1+r-\alpha} + \\
& \quad + \varepsilon^{1+(2r-\alpha)/(N-2)-\alpha/2}) \|\nabla v^\varepsilon\| + c \varepsilon^{1-\alpha/2} \|W^\varepsilon - I\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\nabla v^\varepsilon\| \le {} & c \|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}^{1/2} \|W^\varepsilon - I\|^{1/2} \varepsilon^{-\alpha/2} + \\
& + c(\varepsilon^{\alpha/2 - 1} + 1 + \varepsilon^{r-\alpha} + \varepsilon^{(2r-\alpha)/(N-2)-\alpha/2}),
\end{aligned}
$$

and so

$$\|v^\varepsilon\| \le c \|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}^{1/2} \|W^\varepsilon - I\|^{1/2} + c(\varepsilon^{\alpha-1} + \varepsilon^{\alpha/2} + \varepsilon^{r-\alpha/2} + \varepsilon^{(2r-\alpha)/(N-2)}).$$

From (10) and the inequality $\|\nabla(R^\varepsilon\nu)\| \le c\varepsilon^{-\alpha/2}\|\nabla\nu\|$, we have that if $\nu \in (H_0^1(\Omega))^N$ is such that $\nabla\cdot\nu = 0$, then

$$|\langle\nabla P^\varepsilon(p-p^\varepsilon),\nu\rangle| \le c(\varepsilon^{\alpha/2}\|\nabla v^\varepsilon\| + \varepsilon^{\alpha-1} + \varepsilon^{\alpha/2} + \varepsilon^{r-\alpha/2} + \varepsilon^{(2r-\alpha)/(N-2)})\|\nabla\nu\|.$$

From these inequalities and the fact that every function $g \in L_0^2(\Omega)$ can be represented in the form $g = \nabla\cdot\nu$ for some function $\nu \in (H_0^1(\Omega))^N$ such that $\|\nabla\nu\| \le c\|g\|$ where c is independent of g, we deduce that

$$\|P^\varepsilon(p-p^\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \le c(\varepsilon^{\alpha-1} + \varepsilon^{\alpha/2} + \varepsilon^{r-\alpha/2} + \varepsilon^{(2r-\alpha)/(N-2)}).$$

The results then follow.

Next we consider the case $\alpha = 1$. Let $v^\varepsilon = u^\varepsilon - W^\varepsilon u$. For all $\nu^\varepsilon \in (H_0^1(\Omega^\varepsilon))^N$, we have

$$\begin{aligned} &\varepsilon\int_\Omega \nabla v^\varepsilon : \nabla\nu^\varepsilon dx + \lambda\int_\Omega v^\varepsilon\cdot\nu^\varepsilon dx = \langle\nabla(p-p^\varepsilon),\nu^\varepsilon\rangle - \varepsilon\int_\Omega w_k^\varepsilon\nabla u_k : \nabla\nu^\varepsilon dx + \\ &+\varepsilon\int_\Omega \nabla w_k^\varepsilon : \nabla u_k\nu^\varepsilon dx - \varepsilon\langle\mu_k^\varepsilon - \sum_{i=1}^{M^\varepsilon} m_{ki}^\varepsilon\chi_{P_i^\varepsilon},\nu^\varepsilon u_k\rangle + \\ &-\varepsilon\sum_{i=1}^{M^\varepsilon}\int_{P_i^\varepsilon}(m_{ki}^\varepsilon - \mu_k/\varepsilon)\nu^\varepsilon u_k dx + \varepsilon\langle\nabla q_k^\varepsilon,\nu^\varepsilon u_k\rangle + \lambda\int_\Omega (I-W^\varepsilon)u\cdot\nu^\varepsilon dx. \end{aligned} \tag{11}$$

Letting $\nu^\varepsilon = v^\varepsilon$, we have

$$\begin{aligned} &\varepsilon\|\nabla v^\varepsilon\|^2 + \|v^\varepsilon\|^2 \le c\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\| + \\ &\quad + c(\varepsilon + \varepsilon^r + \varepsilon^{(2r-1)/(N-2)+1/2} + \|W^\varepsilon - I\|\varepsilon^{1/2})\|\nabla v^\varepsilon\| + c\varepsilon^{1/2}\|W^\varepsilon - I\|. \end{aligned}$$

Thus

$$\|\nabla v^\varepsilon\| \le c\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}^{1/2}\|W^\varepsilon - I\|^{1/2}\varepsilon^{-1/2} + c(1 + \varepsilon^{r-1} + \varepsilon^{(2r-1)/(N-2)-1/2}),$$

and

$$\|v^\varepsilon\| \le c\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}^{1/2}\|W^\varepsilon - I\|^{1/2} + c(\varepsilon^{1/2} + \varepsilon^{r-1/2} + \varepsilon^{(2r-1)/(N-2)}).$$

From (11), we have that for all $\nu \in (H_0^1(\Omega))^N$ such that $\nabla.\nu = 0$,

$$|\langle\nabla P^\varepsilon(p-p^\varepsilon),\nu\rangle| \le c(\varepsilon^{1/2}\|\nabla v^\varepsilon\| + \varepsilon^{1/2} + \varepsilon^{r-1/2} + \varepsilon^{(2r-1)/(N-2)})\|\nabla\nu\|.$$

Thus

$$\|P^\varepsilon(p-p^\varepsilon)\|^{1/2}_{L^2(\Omega)/\mathbb{R}} \le c(\varepsilon^{1/2}+\varepsilon^{r-1/2}+\varepsilon^{(2r-1)/(N-2)})$$

which implies the results.

When $\alpha < 1$, we have

$$\begin{aligned}
&\varepsilon\int_\Omega \nabla(u^\varepsilon - W^\varepsilon u) : \nabla\nu^\varepsilon dx + \int_\Omega v^\varepsilon\cdot\nu^\varepsilon dx = \\
&= \langle\nabla(p-p^\varepsilon),\nu^\varepsilon\rangle - \lambda\int_\Omega (W^\varepsilon - I)u\cdot\nu^\varepsilon dx - \varepsilon\int_\Omega w_k^\varepsilon \nabla u_k : \nabla\nu^\varepsilon dx + \\
&\quad + \varepsilon\int_\Omega \nabla w_k^\varepsilon : \nabla u_k \nu^\varepsilon dx - \varepsilon\Big\langle \mu_k^\varepsilon - \sum_{i=1}^{M^\varepsilon} m_{ki}^\varepsilon \chi_{P_i^\varepsilon}, \nu^\varepsilon u_k\Big\rangle + \\
&\quad + \varepsilon\int_\Omega \langle q_k^\varepsilon, \nu^\varepsilon u_k\rangle - \varepsilon\sum_{i=1}^{M^\varepsilon}\int_{P_i^\varepsilon}(m_{ki}^\varepsilon - \mu_k/\varepsilon^\alpha)\nu^\varepsilon u_k dx + \\
&\quad - \varepsilon^{1-\alpha}\int_\Omega M u\nu^\varepsilon dx.
\end{aligned} \tag{12}$$

From this we have

$$\begin{aligned}
\varepsilon\|\nabla v^\varepsilon\|^2 + \|v^\varepsilon\|^2 \le{}& c\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\| + \\
&+ c(\|W^\varepsilon - I\| + \varepsilon^{1-\alpha})\|v^\varepsilon\| + \\
&+ c(\varepsilon + \varepsilon^{1+r-\alpha})\|\nabla v^\varepsilon\| + c\varepsilon^{1-\alpha/2}\|W^\varepsilon - I\|.
\end{aligned} \tag{13}$$

We consider the case $0<\alpha<1$ first. If

$$\begin{aligned}
\|v^\varepsilon\|^2 \le{}& c\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\| + c(\|W^\varepsilon - I\| + \varepsilon^{1-\alpha})\|v^\varepsilon\| + \\
&+ c\varepsilon^{1-\alpha/2}\|W^\varepsilon - I\|,
\end{aligned}$$

then

$$\begin{aligned}
\|v^\varepsilon\| \le{}& c\|p-p^\varepsilon\|^{1/2}_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\|^{1/2} + c\varepsilon^{1/2-\alpha/4}\|W^\varepsilon - I\|^{1/2} + \\
&+ c\|W^\varepsilon - I\| + c\varepsilon^{1-\alpha}.
\end{aligned}$$

Otherwise, $\|\nabla v^\varepsilon\| \le c(1+\varepsilon^{r-\alpha})$ so $\|v^\varepsilon\| \le c(\varepsilon^{\alpha/2}+\varepsilon^{r-\alpha/2})$. Thus we always have

$$\begin{aligned}
\|v^\varepsilon\| \le{}& c(\varepsilon^{\alpha/2} + \varepsilon^{r-\alpha/2} + \varepsilon^{1-\alpha} + \|W^\varepsilon - I\|^{1/2}\varepsilon^{1/2-\alpha/4}) + \\
&+ c\|p-p^\varepsilon\|^{1/2}_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\|^{1/2}.
\end{aligned}$$

Putting this into (13), we get

$$\|\nabla v^\varepsilon\| \le c\varepsilon^{-1/2}\|p-p^\varepsilon\|^{1/2}_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\|^{1/2} + c(\varepsilon^{\alpha/2-1}\|W^\varepsilon - I\| + \varepsilon^{-\alpha/2}).$$

From (12), we have

$$|\langle \nabla P^\varepsilon(p-p^\varepsilon), \nu\rangle| \le c(\varepsilon^{1-\alpha/2}\|\nabla v^\varepsilon\| + \|W^\varepsilon - I\| + \varepsilon^{1-\alpha} + \|v^\varepsilon\|)\|\nabla \nu\|,$$

for all $\nu \in (H^1_0(\Omega))^N$. Therefore

$$\|P^\varepsilon(p-p^\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \le c(\varepsilon^{1-\alpha} + \varepsilon^{\alpha/2} + \varepsilon^{r-\alpha/2} + \varepsilon^{1/2-\alpha/4}\|W^\varepsilon - I\|^{1/2}).$$

The conclusion follows.

If $\alpha \le 0$, from (13) we have

$$\begin{aligned}\varepsilon\|\nabla v^\varepsilon\|^2 + \|v^\varepsilon\|^2 \le c\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\| + \\ + c(\|W^\varepsilon - I\| + \varepsilon^{1-\alpha})\|v^\varepsilon\| c\varepsilon^{1-\alpha/2}\|W^\varepsilon - I\| + \\ + c\varepsilon\|\nabla v^\varepsilon\|.\end{aligned}$$

From this we deduce $\|v^\varepsilon\| \le c\|p-p^\varepsilon\|^{1/2}_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\|^{1/2} + c(\|W^\varepsilon - I\| + \varepsilon^{1/2})$. Therefore

$$\|\nabla v^\varepsilon\| \le c\|p-p^\varepsilon\|^{1/2}_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\|^{1/2}\varepsilon^{-1/2} + c + c\|W^\varepsilon - I\|\varepsilon^{-1/2}. \tag{14}$$

From (12), for all $\nu \in (H^1_0(\Omega))^N)$ we have

$$\begin{aligned}|\langle \nabla P^\varepsilon(p-p^\varepsilon), \nu\rangle| \le \varepsilon\|\nabla v^\varepsilon\|\|\nabla\nu\| + \\ + \|v^\varepsilon\|\|\nabla\nu\| + c(\|W^\varepsilon - I\| + \varepsilon)\|\nabla\nu\|.\end{aligned} \tag{15}$$

This equation together with the above estimates for $\|v^\varepsilon\|$ and $\|\nabla v^\varepsilon\|$ show that $\|p-p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}$ and $\|v^\varepsilon\|$ are smaller than $c(\|W^\varepsilon - I\| + \varepsilon^{1/2})$.

If $2r > \alpha + 1/2$ ($N = 2$ or $N = 3$), $2r > \alpha + (N-2)/N$ ($N \ge 4$) we have $\|W^\varepsilon - I\| < c\varepsilon^{1/2}$. Therefore $\|u^\varepsilon - u\| < c\varepsilon^{1/2}$. This is similar to the result found in Lions [14] for singularly perturbed Dirichlet problems. The holes are now too small in comparison with the cubes to have effect.

4 – Behaviour of u^ε in $H_0^1(\Omega)$

In this section, we prove Theorems 3 and 4 on the behaviour of u^ε in $H_0^1(\Omega)$ when the holes are sufficiently small.

Proof of Theorem 3. We prove the theorem for the case $N \geq 3$; the proof for $N = 2$ is similar. Let $v^\varepsilon = u^\varepsilon - W^\varepsilon u$. From (14), we have

$$\|\nabla v^\varepsilon\| \leq c(\|W^\varepsilon - I\|\varepsilon^{-1/2} + \|W^\varepsilon - I\|^{1/2}\varepsilon^{-1/4} + 1).$$

From (5) and the hypothesis of the theorem, we have $\|\nabla v^\varepsilon\| < c$ so we can extract a subsequence v^ε which converges weakly in $(H_0^1(\Omega))^N$. Since $v^\varepsilon \to 0$ in $(L^2(\Omega))^N$, the weak limit is 0. From (12) we have

$$\begin{aligned}\varepsilon\|\nabla v^\varepsilon\|^2 + \|v^\varepsilon\|^2 \leq{}& c\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\| + c(\|W^\varepsilon - I\| + \varepsilon^{1-\alpha/2})\|v^\varepsilon\| + \\ &+ c\varepsilon^{1-\alpha}\|\nabla v^\varepsilon\| + c\varepsilon^{1-\alpha/2}\|W^\varepsilon - I\| + \varepsilon\int_\Omega \nabla w_k^\varepsilon : \nabla u_k v^\varepsilon dx + \\ &- \varepsilon\int_\Omega w_k^\varepsilon \nabla u_k : \nabla v^\varepsilon dx.\end{aligned}$$

Thus

$$\begin{aligned}\|\nabla v^\varepsilon\|^2 \leq{}& c\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}\|W^\varepsilon - I\|\varepsilon^{-1} + c\|W^\varepsilon - I\|^2\varepsilon^{-1} + \\ &+ c\|W^\varepsilon - I\|\varepsilon^{-1/2} + c\varepsilon^{-\alpha/2} + \\ &+ \int_\Omega \nabla w_k^\varepsilon : \nabla u_k v^\varepsilon dx - \int_\Omega w_k^\varepsilon \nabla u_k : \nabla v^\varepsilon dx.\end{aligned}$$

It is simple to see that the first four terms converge to 0 as $\varepsilon \to 0$. We also have

$$\Big|\int_\Omega \nabla w_k^\varepsilon : \nabla u_k v^\varepsilon dx\Big| \leq c\|\nabla w_k^\varepsilon\|\|v^\varepsilon\| \leq c\varepsilon^{-\alpha/2}\|v^\varepsilon\| \to 0,$$

and

$$\int_\Omega w_k^\varepsilon \nabla u_k : \nabla v^\varepsilon dx = \int_\Omega (w_k^\varepsilon - 1)\nabla u_k : \nabla v^\varepsilon dx + \int_\Omega \nabla u_k : \nabla v^\varepsilon dx \to 0,$$

as $w_k^\varepsilon - 1 \to 0$, $\|\nabla v^\varepsilon\| < c$ and $v^\varepsilon \rightharpoonup 0$ in $(H_0^1(\Omega))^N$. Therefore $v^\varepsilon \to 0$ in $(H_0^1(\Omega))^N$. □

Proof of Theorem 4. We show this theorem for $N \geq 3$; the case $N = 2$ is similar. Let $v^\varepsilon = u^\varepsilon - K^\varepsilon W^\varepsilon u$. On using (7) when $\psi = v^\varepsilon$ we have

$$
\begin{aligned}
\varepsilon\|\nabla v^\varepsilon\|^2 + \lambda\|v^\varepsilon\|^2 = &\int_\Omega (f - \nabla p^\varepsilon) v^\varepsilon dx - \varepsilon \int_\Omega K^\varepsilon w_k^\varepsilon \nabla u_k : \nabla v^\varepsilon dx + \\
&+ \varepsilon \int_\Omega K^\varepsilon \nabla w_k^\varepsilon : \nabla u_k v^\varepsilon dx + \\
&- \varepsilon \langle K^\varepsilon (\mu_k^\varepsilon - \sum_{i=1}^{M^\varepsilon} m_{ki}^\varepsilon \chi_{P_i^\varepsilon}), v^\varepsilon u_k \rangle + \\
&- \varepsilon \sum_{i=1}^{M^\varepsilon} \int_{P_i^\varepsilon} K^\varepsilon (m_{ki}^\varepsilon - \mu_k/\varepsilon^\alpha) v^\varepsilon u_k dx + \\
&- \varepsilon^{1-\alpha} \int_\Omega K^\varepsilon \mu_k \cdot v^\varepsilon u_k dx - \lambda \int_\Omega K^\varepsilon w_k^\varepsilon \cdot u_k v^\varepsilon dx \leq \\
\leq &\int_\Omega (L - \varepsilon^{1-\alpha} K^\varepsilon M - \lambda K^\varepsilon W^\varepsilon) u v^\varepsilon dx + \langle \nabla (p - p^\varepsilon), v^\varepsilon \rangle + \\
&+ c\varepsilon \|\nabla v^\varepsilon\| + c\varepsilon^{1+(2r-\alpha)/(N-2)-\alpha} \|v^\varepsilon\|.
\end{aligned} \tag{16}
$$

Since $L - \varepsilon^{1-\alpha} K^\varepsilon M - \lambda K^\varepsilon W^\varepsilon = \lambda K^\varepsilon (I - W^\varepsilon)$ we have

$$
\begin{aligned}
\varepsilon\|\nabla v^\varepsilon\|^2 + \lambda\|v^\varepsilon\|^2 \leq\, & c\|W^\varepsilon - I\| \|v^\varepsilon\| + \\
& + c\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} \|W^\varepsilon - I\| + \\
& + c\varepsilon \|\nabla v^\varepsilon\| + c\varepsilon^{1+(2r-\alpha)/(N-2)-\alpha} \|v^\varepsilon\|,
\end{aligned}
$$

so

$$
\begin{aligned}
\|\nabla v^\varepsilon\| \leq\, & c\|W^\varepsilon - I\| \varepsilon^{\alpha/2-1} + c\|p - p^\varepsilon\|^{1/2}_{L^2(\Omega^\varepsilon)/\mathbb{R}} \|W^\varepsilon - I\|^{1/2} \varepsilon^{-1/2} + \\
& + (1 + \varepsilon^{(2r-\alpha)/(N-2)-\alpha/2}).
\end{aligned}
$$

With the conditions of the theorem, and the estimate for $\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}}$ in the previous section, we have that $\|\nabla v^\varepsilon\| < c$ so v^ε converges weakly in $(H_0^1(\Omega))^N$. Furthermore, since $\|v^\varepsilon\| \leq c\varepsilon^{\alpha/2}$, the weak limit is 0.

To show the strong convergence, we have again from (16) that

$$
\begin{aligned}
\|\nabla v^\varepsilon\|^2 \leq\, & c\|p - p^\varepsilon\|_{L^2(\Omega^\varepsilon)/\mathbb{R}} \|W^\varepsilon - I\| \varepsilon^{-1} + c\varepsilon^{r-\alpha} \|\nabla v^\varepsilon\| + \\
& + c\varepsilon^{(2r-\alpha)/(N-2)-\alpha/2} \|\nabla v^\varepsilon\| - \int_\Omega K^\varepsilon w_k^\varepsilon \nabla u_k : \nabla v^\varepsilon dx + \\
& + \int_\Omega K^\varepsilon \nabla w_k^\varepsilon : \nabla u_k v^\varepsilon dx.
\end{aligned}
$$

The first three terms on the right hand side converge to 0. We also have

$$\int_\Omega K^\varepsilon w_k^\varepsilon \nabla u_k : \nabla v^\varepsilon dx = \int_\Omega K^\varepsilon (w_k^\varepsilon - e_k)\nabla u_k : \nabla v^\varepsilon dx + \int_\Omega K e_k \nabla u_k : \nabla v^\varepsilon dx$$

converges to 0 since $w_k^\varepsilon \to e_k$ in $(L^2(\Omega))^N$ and $v^\varepsilon \rightharpoonup 0$ in $(H_0^1(\Omega))^N$. Furthermore

$$\int_\Omega K^\varepsilon \nabla w_k^\varepsilon : \nabla u_k v^\varepsilon dx = -\int_\Omega K^\varepsilon w_k^\varepsilon \cdot \Delta u_k v^\varepsilon dx - \int_\Omega K^\varepsilon w_k^\varepsilon \cdot \nabla v^\varepsilon \nabla u_k dx$$

which converges to 0 since $w_k^\varepsilon \to e_k$ in $(L^2(\Omega))^N$, $|\nabla u_k|$ and $|\Delta u_k|$ are bounded and $v^\varepsilon \rightharpoonup 0$ in $(H_0^1(\Omega))^N$. Therefore $v^\varepsilon \to 0$ in $(H_0^1(\Omega))^N$. □

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