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On a class of variational integrals with linear growth satisfying the condition of μ-ellipticity

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Riassunto: *Si considerano gli integrali variazionali* $J(u) = \int_\Omega f(\nabla u)\, dx$ *con integrando* f *convesso con crescita lineare e soddisfacente alla condizione di ellitticità*

$$D^2 f(X)(Y,Y) \geq \lambda \big(1 + |X|^2\big)^{-\frac{\mu}{2}} |Y|^2$$

con esponente $1 < \mu < 1 + 2/n$, $n = \dim \Omega$, *dove nel caso vettoriale è assunta l'ipotesi* $f(\nabla u) = g(|\nabla u|^2)$. *Si dimostra la continuità di Hölder della soluzione duale del problema* $J \to \min$ *in* $u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$ *e si stabilisce, usando degli argomenti alla De Giorgi, la* $C^{1,\alpha}$*-regolarità dei limiti deboli delle successioni* J*-minimizzanti. Inoltre si dimostra che i punti di accumulazione deboli delle successioni minimizzanti sono unici a meno di una costante. Per domini* Ω *bi-dimensionali questi risultati sono estesi parzialmente al caso limite* $\mu = 2$.

Abstract: *We consider variational integrals* $J(u) = \int_\Omega f(\nabla u)\, dx$ *with convex integrand* f *of linear growth satisfying an ellipticity condition of the form*

$$D^2 f(X)(Y,Y) \geq \lambda \big(1 + |X|^2\big)^{-\frac{\mu}{2}} |Y|^2$$

with exponent $1 < \mu < 1 + 2/n$, $n = \dim \Omega$, *where in the vectorvalued case the structure condition* $f(\nabla u) = g(|\nabla u|^2)$ *is assumed. We prove Hölder continuity of the dual solution of the problem* $J \to \min$ *in* $u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$ *and establish, using De Giorgi type arguments,* $C^{1,\alpha}$*-regularity of weak limits of* J*-minimizing sequences. Moreover, it is shown that weak cluster points of minimizing sequences are unique up to a constant. For two-dimensional domains* Ω *these results are partially extended to the limit case* $\mu = 2$.

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1 – Introduction and statement of the main results

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and let

$$J(u) = \int_\Omega f(\nabla u)\, dx\,, \tag{1.1}$$

where $\nabla u = (\partial_\alpha u^i)$ denotes the Jacobi matrix of the vectorial function u: $\Omega \to \mathbb{R}^M$. We assume that $f \geq 0$ is of class $C^2(\mathbb{R}^{nM})$ (of course $f \geq c$ for some $c > -\infty$ is also sufficient) satisfying the following set of hypotheses:

$$a|X| - b \leq f(X)\,; \tag{1.2}$$

$$|\nabla f(X)| \leq A\,; \tag{1.3}$$

$$|D^2 f(X)| \leq \Lambda\big(1 + |X|^2\big)^{-\frac{1}{2}}\,; \tag{1.4}$$

$$D^2 f(X)(Y,Y) \geq \lambda\big(1 + |X|^2\big)^{-\frac{\mu}{2}}|Y|^2\,. \tag{1.5}$$

Here a, b, A, λ, Λ denote positive constants, $\mu > 1$ is some fixed exponent, and (1.2)-(1.5) are valid for any choice of X, $Y \in \mathbb{R}^{nM}$. Note that (1.3) immediately implies the linear growth condition

$$a|X| - b \leq f(X) \leq \tilde{a}|X| + \tilde{b} \tag{1.6}$$

with suitable constants $\tilde{a}$, $\tilde{b} > 0$. Combining (1.4) and (1.6) we see that f is "balanced" in the sense that

$$|D^2 f(X)||X|^2 \leq \mathrm{const}(f(X) + 1) \tag{1.7}$$

holds. Finally, the μ-ellipticity condition (1.5) gives strict convexity of our integrand f.

It is easily seen that

$$f(X) = \int_0^{|X|}\int_0^s (1+t^2)^{-\frac{\mu}{2}}\, dt\, ds\,, \qquad X \in \mathbb{R}^{nM}\,,$$

satisfies (1.3)-(1.5) (compare [7] for details). For (1.2) let us write $f(X) = \varphi(|X|)$ with convex function φ. Then, if $t = |X| \geq 1$, we get

$$f(X) = \varphi(t) \geq \varphi(1) + \varphi'(1)(|X| - 1)$$

with

$$\varphi'(1) = \int_0^1 (1+|t|^2)^{-\frac{\mu}{2}}\, dt > 0\,.$$

In the vectorial case $M > 1$ we suppose in addition to (1.2)-(1.5) that f is of "special structure" in the sense that

$$f(X) = g(|X|^2)\,, \qquad X \in \mathbb{R}^{nM}\,, \tag{1.8}$$

holds with g: $[0,\infty) \to [0,\infty)$ of class $C^2(\mathbb{R})$. Note that this implies

$$\frac{\partial^2 f}{\partial X^i_\alpha \partial X^j_\beta}(X) = 4g''(|X|^2)X^i_\alpha X^j_\beta + 2g'(|X|^2)\delta^{ij}\delta_{\alpha\beta}\,.$$

We also assume in case $M > 1$ that there are real numbers $\alpha \in (0,1]$, $K > 0$ satisfying(1)

$$|D^2 f(X) - D^2 f(\widetilde{X})| \le K|X - \widetilde{X}|^\alpha\,. \tag{1.9}$$

The above example is easily adjusted to (1.9) (and in fact to much stronger conditions) by letting

$$\tilde{f}(X) = \int_0^{\sqrt{\varepsilon+|X|^2}} \int_0^s (1+|t|^2)^{-\frac{\mu}{2}}\, dt\, ds\,, \qquad \varepsilon > 0\,.$$

Next consider a given function $u_0 \in W^1_p(\Omega, \mathbb{R}^M)$ for some $p > 1$. (By considering a suitable approximation it is also possible to include the limit case $p = 1$, we refer to [4].) As a role the variational problem

$$\text{(V)} \qquad \text{to minimize} \quad J(u) = \int_\Omega f(\nabla u)\, dx \quad \text{in } u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$$

in general may fail to have solutions in the non-reflexive space $W^1_1(\Omega, \mathbb{R}^M)$. For this reason one either studies suitable relaxations or passes to the dual variational problem. In our note we try to handle both aspects: first of all, due to (1.6), any minimizing sequence $u_m \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$ is bounded

(1) Although not explicitely mentioned, some Hölder condition of this type is also assumed in [11] if $M > 1$.

in the space $BV(\Omega, \mathbb{R}^M)$, hence there is a subsequence and a function u in $BV(\Omega, \mathbb{R}^M)$ such that $u_{m_k} \to u$ in $L^1(\Omega, \mathbb{R}^M)$, and we define the set of all generalized minimizers of problem (V) as

$$\mathcal{M} = \{u \in BV(\Omega, \mathbb{R}^M) : \quad u \text{ is the } L^1\text{-limit of a } J\text{-minimizing sequence from } u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)\} .$$

Now let us write (see [10])

$$J(u) = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^{nM})} \Big\{ \int_\Omega \tau : \nabla u \, dx - \int_\Omega f^*(\tau)\, dx \Big\} , \tag{1.10}$$

$u \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$, where f^* is the conjugate function of f. We define the Lagrangian $l(u, \tau)$ for $(u, \tau) = (u_0 + \varphi, \tau) \in (u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)) \times L^\infty(\Omega, \mathbb{R}^{nM})$ through the formula

$$l(u, \tau) := \int_\Omega \tau : \nabla u \, dx - \int_\Omega f^*(\tau)\, dx = l(u_0, \tau) + \int_\Omega \tau : \nabla \varphi \, dx .$$

Then the dual functional is by definition

$$R : \ L^\infty(\Omega, \mathbb{R}^{nM}) \to \overline{\mathbb{R}} ,$$
$$R(\tau) := \inf_{u \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)} l(u, \tau) = \begin{cases} -\infty , & \text{if } \operatorname{div} \tau \neq 0 \\ l(u_0, \tau) , & \text{if } \operatorname{div} \tau = 0 , \end{cases}$$

and the dual problem reads

(V*) to maximize R among all functions in $L^\infty(\Omega, \mathbb{R}^{nM})$.

It is well known (see, again [10]) that

$$\inf_{u \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)} J(u) = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^{nM})} R(\tau) ,$$

moreover, (V*) admits a unique maximizer σ (compare [3] for a uniqueness theorem valid under much more general conditions and not formulated in terms of the conjugate function). Let us first assume that

$$\mu < 1 + \frac{2}{n} . \tag{1.11}$$

Then our main results are summarized in

THEOREM 1.1. *Let* (1.2)-(1.5), (1.11) *hold and assume in addition that in case* $M > 1$ (1.8) *and* (1.9) *are valid.*

a) *The dual solution* σ *is of class* $C^{0,\alpha}(\Omega, \mathbb{R}^{nM})$ *for any* $0 < \alpha < 1$. *Moreover,* σ *has weak derivatives in the space* $L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^{nM})$.
b) *Any generalized minimizer* $u \in \mathcal{M}$ *is in the space* $C^{1,\alpha}(\Omega, \mathbb{R}^M)$, $0 < \alpha < 1$.
c) *For* u, $v \in \mathcal{M}$ *we have* $\nabla u = \nabla v$, *i.e. up to a constant uniqueness of generalized minimizers holds true.*

We like to remark that for functionals with linear growth arising in the theory of perfect plasticity partial regularity of σ was established in the papers [23], [25] (see also [12] for an exhaustive list of references), whereas the general vectorial setting of Theorem 1.1 with only partial regularity results was studied in [4]. (Note that in [4] μ-ellipticity is replaced by a much weaker condition.) The case of functionals with linear growth satisfying an ellipticity condition of minimal-surface type was investigated in [16], see also [5].
Let us now look at the limit case $\mu = 1 + 2/n$ which unfortunately we could include only for $n = 2$. An example satisfying (1.2)-(1.5) with $\mu = 2$ is given by

$$f(X) = \int_0^{|X|} \arctan s \, ds = |X| \arctan |X| - \frac{1}{2} \ln(1 + |X|^2) \,,$$

and we have

THEOREM 1.2. *Let* $n = 2$, *let* (1.2)-(1.5) *hold with* $\mu = 2$. *In case* $M > 1$ *we also assume that* (1.8) *and* (1.9) *are valid. Then there exists an at most countable subset* Σ *of* Ω *with no interior accumulation points such that the following is true:*

a) *The dual solution* σ *is of class* $C^{0,\alpha}(\Omega - \Sigma, \mathbb{R}^{2M})$ *for any* $0 < \alpha < 1$ (*still having weak derivatives in the space* $L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^{2M})$).
b) *Any* $u \in \mathcal{M}$ *is of class* $C^{1,\alpha}(\Omega - \Sigma, \mathbb{R}^M)$, $0 < \alpha < 1$.
c) *For* u, $v \in \mathcal{M}$ *we have* $u = v + \mathrm{const}$ *a.e. on* Ω.

Corollary 1.3. *Under the assumptions of Theorem* 1.2 *we have* $u \in W^1_{t,\mathrm{loc}}(\Omega, \mathbb{R}^M)$ *for* $u \in \mathcal{M}$ *and any* $t < \infty$. *In particular, u is locally Hölder continuous with any exponent* $\alpha < 1$.

Our paper is organized as follows: in Section 2 we first replace (V) by a sequence (V_δ) of approximate problems with regular solutions u_δ being convergent to some $u^* \in \mathcal{M}$. In Section 3 we apply De Giorgi type arguments to show that this particular generalized minimizer is smooth provided the assumptions of Theorem 1.1 hold. The regularity of σ then follows from the duality relation $\sigma = \nabla f(\nabla u^*)$. Section 5 contains the proof of the remaining results from Theorem 1.1, in Section 6 we discuss the case $n = 2$ together with $\mu = 2$.

2 – Regularization of the original problem and weak differentiability of the dual solution

Let the assumptions of Theorem 1.1 or 1.2 hold. We fix some real number $1 < q < 2$ satisfying $q < p$ and in addition for $n \geq 3$ (recall (1.11))

$$q < (2-\mu)\frac{n}{n-2}\,. \tag{2.1}$$

For any $0 < \delta \leq 1$ we define

$$J_\delta(w) := \delta \int_\Omega \big(1 + |\nabla w|^2\big)^{\frac{q}{2}}\, dx + J(w)\,, \quad w \in u_0 + \overset{\circ}{W}{}_q^1(\Omega, \mathbb{R}^M)\,,$$

and denote by u_δ the unique solution of

$$(\mathrm{V}_\delta) \qquad \text{to minimize} \quad J_\delta(w) \quad \text{in the class } \ u_0 + \overset{\circ}{W}{}_q^1(\Omega, \mathbb{R}^M)\,.$$

Thus, letting $f_\delta(\cdot) = \delta\big(1 + |\cdot|^2\big)^{\frac{q}{2}} + f(\cdot)$, we obtain

$$\int_\Omega \nabla f_\delta(\nabla u_\delta) : \nabla\varphi\, dx = 0 \quad \text{for all } \ \varphi \in \overset{\circ}{W}{}_q^1(\Omega, \mathbb{R}^M)\,. \tag{2.2}$$

Moreover, the following lemma is seen to be true:

Lemma 2.1. *There is a real number $c_1 > 0$ such that for any $\eta \in C_0^\infty(\Omega)$*

$$\begin{aligned} &\int_\Omega \eta^2 D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta)\, dx \leq \\ &\leq\ c_1 \|\nabla \eta\|_\infty^2 \int_\Omega |D^2 f_\delta(\nabla u_\delta)| |\nabla u_\delta|^2 \, dx\,, \end{aligned} \tag{2.3}$$

where we always take the sum w.r.t. $s = 1, \dots, n$.

Proof. The idea is to choose the test function $\varphi = \partial_s(\eta^2 \partial_s u_\delta)$ in equation (2.2). However, in the vectorial case $M > 1$ it is not immediately obvious if φ is admissible in (2.2). In the paper [6], Lemma 3.1, we overcame this difficulty by some technical approximation argument leading to inequality (2.3). We like to remark that now we do not need the full strength of the arguments used in [6], Lemma 3.1, since the structural condition (1.8) together with [1], proposition 2.7, already implies $u_\delta \in W^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^M)$ (compare (3.4)).

Let us look at the scalar case. Then, from standard arguments (see, e.g. [20], Chapter 4, Theorem 5.2) we get $u_\delta \in W^2_{2,\mathrm{loc}}(\Omega) \cap W^1_{\infty,\mathrm{loc}}(\Omega)$ using also the fact that u_δ is locally bounded which is proved under very weak assumptions in [14]. Alternatively we may quote [8] or [9] to get $u_\delta \in C^{1,\bar\alpha}(\Omega)$ for some $\bar\alpha > 0$. Let us fix a subdomain $\Omega' \Subset \Omega$ and let $K = 2\|\nabla u_\delta\|_{L^\infty(\Omega')}$. Following [19], p. 97, we replace ∇f_δ by a coercive vectorfield A of class C^1 in such a way that $A(X) = \nabla f_\delta(X)$ for $|X| \leq 2K$. From (2.2) we get

$$\int_{\Omega'} A(\nabla u_\delta) \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^1(\Omega')\,,$$

and the well known difference quotient technique implies $u_\delta \in W^2_{2,\mathrm{loc}}(\Omega)$. Let us fix a coordinate direction $\gamma \in \{1, \dots, n\}$. Then

$$\int_{\Omega'} a_{\alpha\beta}(x) \partial_\alpha(\partial_\gamma u_\delta) \partial_\beta \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^1(\Omega')$$

with elliptic coefficients $a_{\alpha\beta} = \frac{\partial A^\beta}{\partial Q_\alpha}(\nabla u_\delta)$ of class $C^0(\overline{\Omega'})$. Quoting "L^p-theory" for equations with continuous coefficients (see [22], Theorem 5.5.3, or [13], Chapter 4.3, pp. 71) we get $\partial_\gamma u_\delta \in W^1_{t,\mathrm{loc}}(\Omega')$ for any finite t. Thus we have $u_\delta \in W^2_{t,\mathrm{loc}}(\Omega)$ and $\partial_s(\eta^2 \partial_s u_\delta)$ is admissible in (2.2), the claim follows after partial integration and using the Cauchy-Schwarz inequality for the bilinear form $D^2 f_\delta(\nabla u_\delta)$ (compare [7], Lemma 2.3, and [11] for similar calculations). $\square$

Remark 2.2. In the papers [7] and [6] the bound (1.11) was imposed but obviously the statement of Lemma 2.3 can be obtained for any positive μ.

For the study of the dual variational problem we let

$$\begin{aligned}\tau_\delta &:= \nabla f(\nabla u_\delta)\,, \quad \sigma_\delta := \delta X_\delta + \tau_\delta = \nabla f_\delta(\nabla u_\delta)\,,\\ X_\delta &:= q\big(1+|\nabla u_\delta|^2\big)^{\frac{q-2}{2}} \nabla u_\delta\,.\end{aligned}$$

Note that $\sigma_\delta \in W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^{nM})$ which for $M > 1$ follows from [1], proposition 2.7. Next we argue as in [4], where the case $q = 2$ was considered: since $J_\delta(u_\delta) \le J_\delta(u_0) \le J_1(u_0)$, the existence of a real number $c_2 > 0$ is ensured such that

$$\delta \int_\Omega \big(1+|\nabla u_\delta|^2\big)^{\frac{q}{2}}\, dx \le c_2\,, \quad \int_\Omega f(\nabla u_\delta)\, dx \le c_2\,, \quad \|\tau_\delta\|_\infty \le c_2\,. \tag{2.4}$$

The first inequality implies

$$\|\delta^{\frac{q-1}{q}} X_\delta\|_{L^{\frac{q}{q-1}}(\Omega,\mathbb{R}^{nM})} \le c_3\,, \quad \text{hence } \delta X_\delta \rightharpoondown 0 \text{ in } L^{\frac{q}{q-1}}(\Omega,\mathbb{R}^{nM}) \tag{2.5}$$

as $\delta \to 0$. Here and in the following we always pass to subsequences if necessary. By (2.4) and (2.5) it is possible to define $\sigma \in L^{\frac{q}{q-1}}(\Omega,\mathbb{R}^{nM})$ via

$$\tau_\delta\,,\ \sigma_\delta \rightharpoondown: \sigma \quad \text{in } L^{\frac{q}{q-1}}(\Omega,\mathbb{R}^{nM}) \text{ as } \delta \to 0\,. \tag{2.6}$$

In addition, $\operatorname{div}\sigma = 0$ (in the sense of distributions) is a consequence of $\operatorname{div}\sigma_\delta = 0$. We now claim that the weak limit σ is the unique maximizer of the dual problem (V^*). To prove this, we recall the duality relation (compare [10])

$$\tau_\delta : \nabla u_\delta - f^*(\tau_\delta) = f(\nabla u_\delta)\,,$$

which together with the definition of σ_δ and with $\operatorname{div}\sigma_\delta = 0$ gives

$$J_\delta(u_\delta) = \delta \int_\Omega (1+|\nabla u_\delta|^2)^{\frac{q}{2}}\,dx + \int_\Omega (\sigma_\delta : \nabla u_0 - f^*(\tau_\delta))\,dx - \\ - \delta q \int_\Omega (1+|\nabla u_\delta|^2)^{\frac{q-2}{2}} |\nabla u_\delta|^2\,dx\,.$$

This yields for any $\varkappa \in L^\infty(\Omega, \mathbb{R}^{nM})$

$$\begin{aligned} R(\varkappa) \le & \inf_{u\in u_0+\overset{\circ}{W}{}_1^1(\Omega,\mathbb{R}^M)} J(u) \le J(u_\delta) \le J_\delta(u_\delta) = \\ & = \int_\Omega (\tau_\delta : \nabla u_0 - f^*(\tau_\delta))\,dx + \delta \int_\Omega X_\delta : \nabla u_0\,dx + \\ & + (1-q)\delta \int_\Omega (1+|\nabla u_\delta|^2)^{\frac{q}{2}}\,dx + \delta q \int_\Omega (1+|\nabla u_\delta|^2)^{\frac{q-2}{2}}\,dx\,, \end{aligned} \tag{2.7}$$

and, passing to the limit $\delta \to 0$, the second and the last integral on the right-hand side vanish according to (2.5) and since $q < 2$. Finally, lower semicontinuity of $\int_\Omega f^*(\cdot)\,dx$ w.r.t. weak-$*$ convergence proves the claim $R(\varkappa) \le R(\sigma)$ as well as

$$\delta \int_\Omega (1+|\nabla u_\delta|^2)^{\frac{q}{2}}\,dx \to 0 \quad \text{as } \delta \to 0\,. \tag{2.8}$$

To proceed further, we observe that the left-hand side of (2.3) is estimated via Young's inequality

$$|\nabla \sigma_\delta|^2 \le c_4 D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta)\,,$$

whereas (1.7), (2.4) and (2.8) prove the right-hand side to be bounded by:

$$c_5\Big(\delta \int_\Omega (1+|\nabla u_\delta|^2)^{\frac{q}{2}}\,dx + \int_\Omega (f(\nabla u_\delta)+1)\,dx\Big) \le c_6\,.$$

Summarizing the results have established:

Lemma 2.3. *Let σ be the weak limit defined in* (2.6). *Then* $\operatorname{div}\sigma = 0$, *$\sigma$ is the unique maximizer of the dual variational problem* (V*) *and we have $\sigma \in W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^{nM})$.*

3 – Construction of a smooth generalized minimizer

In this section we again concentrate on the original problem (V): (2.7) proves $\{u_\delta\}$ to be a J-minimizing sequence, and, by definition, each L^1-cluster point of $\{u_\delta\}$ is seen to be a generalized minimizer.

LEMMA 3.1. *Let the assumptions of Theorem* 1.1 *hold.*

a) There is a real number $c_7 > 0$, independent of δ, such that

$$\|\nabla u_\delta\|_{L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^{nM})} \le c_7 \,.$$

b) Let u^* denote a L^1-cluster point of the sequence $\{u_\delta\}$. Then u^* is of class $C^{1,\alpha}(\Omega,\mathbb{R}^M)$ for any $0 < \alpha < 1$.

REMARK 3.2. With Lemma 3.1 the results of [7] formulated for the unconstrained case are also seen to be true in the vectorial setting $M > 1$ assuming the hypotheses (1.8) and (1.9).

PROOF. As in [7], Lemma 2.4, we first use Lemma 2.1 to prove that we have local higher integrability of ∇u_δ which holds uniformly w.r.t. δ. For simplicity let us first assume that $n \ge 3$ and let $\chi = n/(n-2)$. Moreover, fix a ball B_r satisfying $B_{2r} \Subset \Omega$ and choose $\eta \in C_0^\infty(B_{2r})$, $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r. Then, by Sobolev's inequality,

$$\begin{aligned}\int_{B_r} (1+|\nabla u_\delta|^2)^{\frac{(2-\mu)\chi}{2}}\, dx \le \int_{B_{2r}} (\eta(1+|\nabla u_\delta|^2)^{\frac{(2-\mu)}{4}})^{2\chi}\, dx \le \\ \le c_8\Big(\int_{B_{2r}} |\nabla\eta|^2(1+|\nabla u_\delta|^2)^{\frac{(2-\mu)}{2}}\, dx + \\ + \int_{B_{2r}} \eta^2(1+|\nabla u_\delta|^2)^{-\frac{\mu}{2}}|\nabla^2 u_\delta|^2\, dx\Big)^\chi \,.\end{aligned}$$

Hence, Lemma 2.1 and (1.7) imply the bound (observe $(2-\mu)\chi > q$ by (2.1))

$$\int_{B_r} (1+|\nabla u_\delta|^2)^{\frac{(2-\mu)\chi}{2}}\, dx \le c_9(r)\,, \tag{3.1}$$

which corresponds to Lemma 2.4 of [7]. If $n = 2$, we let χ denote some number > 1 such that $(2-\mu)\chi > q$. Writing as before

$$\Big(\int_{B_r} (1+|\nabla u_\delta|^2)^{\frac{(2-\mu)\chi}{2}}\, dx\Big)^{\frac{1}{2\chi}} \le \Big(\int_{B_{2r}} (\eta(1+|\nabla u_\delta|^2)^{\frac{(2-\mu)}{4}})^{2\chi}\, dx\Big)^{\frac{1}{2\chi}}$$

we may estimate the right-hand side by Sobolev's inequality

$$\Big(\int_{B_{2r}} (\eta(1+|\nabla u_\delta|^2)^{\frac{(2-\mu)}{4}})^{2\chi}\,dx\Big)^{\frac{1}{2\chi}} \le$$
$$\le c_{10}\Big(\int_{B_{2r}} |\nabla(\eta(1+|\nabla u_\delta|^2)^{\frac{(2-\mu)}{4}})|^s\,dx\Big)^{\frac{1}{s}}\,,$$

s being defined through $2\chi = 2s/(2-s)$. Applying Hölder's inequality we get as before

$$\int_{B_r} (1+|\nabla u_\delta|^2)^{\frac{(2-\mu)\chi}{2}}\,dx \le c_{11}(r)\Big(\int_{B_{2r}} |\nabla(\eta(1+|\nabla u_\delta|^2)^{\frac{(2-\mu)}{4}})|^2\,dx\Big)^{\chi}$$

and (3.1) follows also in case $n=2$ with a different constant c_{12} depending also on the chosen value for χ.

In the scalar case we may now exactly follow the lines of [7], Lemma 2.5, Lemma 2.6 and step 5 (conclusion) to get $a)$ of Lemma 3.1. We have to show that Lemma 2.5 of [7] remains also valid in the vectorial case with the additional assumption (1.8), i.e. we claim that there is a real number $c_{13} < \infty$, depending only on the data and not on δ such that for any $k \ge 0$

$$\begin{aligned}&\int_{A(k,R)} (1+|\nabla u_\delta|^2)^{1-\frac{\mu}{2}}|\nabla\omega_\delta|^2\eta^2\,dx+\\ &\quad+\int_{A(k,R)} (1+|\nabla u_\delta|^2)^{-\frac{\mu}{2}}(\omega_\delta-k)^2\eta^2|\nabla^2 u_\delta|^2\,dx \le\\ &\le c_{13}\int_{A(k,R)} (1+|\nabla u_\delta|^2)^{\frac{q}{2}}|\nabla\eta|^2(\omega_\delta-k)^2\,dx\,.\end{aligned} \tag{3.2}$$

Here we have set

$$\omega_\delta = \ln(1+|\nabla u_\delta|^2)\,,\qquad A(h,r)=\{x\in B_r : \omega_\delta \ge h\}\,,\ \ h\ge 0\,,\ \ r\le R\,,$$

the ball B_R is chosen such that $B_{2R} \Subset \Omega$ and $\eta \in C_0^\infty(B_R)$, $0 \le \eta \le 1$. Assuming (3.2) for the moment, we see as in the scalar case that De Giorgi's technique works exactly as outlined in [7], i.e. the uniform bound (3.1) gives $a)$ of Lemma 3.1 and proves any L^1-cluster point u^* to be locally Lipschitz.

To verify part *b*), we quote the well known explicit formula for the relaxation of J (see, for instance, [18])

$$\hat{J}(u, B_r) := \int_{B_r} f(\nabla^a u)\, dx + \int_{B_r} f_\infty\Big(\frac{\nabla^s u}{|\nabla^s u|}\Big)\, d|\nabla^s u|\,, \quad u \in BV(B_r, \mathbb{R}^M)\,.$$

Here, f_∞ is the recession function of f,

$$f_\infty(X) = \limsup_{t \to +\infty} \frac{f(tX)}{t}\,,$$

the absolutely continuous part of ∇u with respect to the Lebesgue measure is denoted by $\nabla^a u$, the singular part by $\nabla^s u$ and $\nabla^s u/|\nabla^s u|$ is the Radon-Nikodym derivative.

Then, according to [4] (see the proof of Theorem 5.1), each generalized minimizer of problem (V) is seen to minimize $\hat{J}$ on a.a. balls $B_r \Subset \Omega$. In particular, this result can be applied to the sequence $\{u_\delta\}$, whose L^1-cluster points are already known to be locally Lipschitz. Thus, if we fix one of these cluster-points u^*, then the singular part of ∇u^* can be neglected and we obtain the Euler-equation

$$\int_\Omega \nabla f(\nabla u^*) : \nabla \varphi\, dx = 0 \quad \text{for any } \varphi \in C_0^1(\Omega, \mathbb{R}^M)\,. \tag{3.3}$$

In the scalar case, again by strict ellipticity of $D^2 f$ (see (1.5)) together with local boundedness of ∇u^*, the assertion of part *b*) immediately follows from known regularity results for weak solutions of (3.3) (compare the references quoted in the proof of Lemma 2.1). In the vectorial setting we follow the lines of [21] where an auxiliary integrand $\tilde{f}$ is constructed satisfying $\tilde{f}(Z) = f(Z)$ whenever $|Z| \le 2K$, $K := \|\nabla u^*\|_{L^\infty(\Omega', \mathbb{R}^{nM})}$ for some subdomain $\Omega' \Subset \Omega$. Then Theorem 3.1 of [15] can be applied (choosing $m = 6$ and recalling the Hölder condition (1.9)) to prove $C^{1,\alpha}$-regularity of u^* for any $0 < \alpha < 1$.

It remains to verify (3.2) for $M > 1$: the structure condition (1.8) allows us to cite [1], proposition 2.7. We obtain, as already remarked in the proof of Lemma 2.1

$$\nabla u_\delta \in W^1_{2,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{nM}) \tag{3.4}$$

and, as a consequence, for any $s = 1, \dots n$, and for any $\psi \in C_0^1(\Omega, \mathbb{R}^M)$

$$\int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \nabla \psi)\, dx = 0\,. \tag{3.5}$$

In addition to (3.4) we have $D^2 f_\delta \in L^\infty$, hence $\psi \in \overset{\circ}{W}{}_2^{\,1}(\Omega, \mathbb{R}^M)$ is admissible in (3.5) (by approximation arguments), in particular we may choose $\psi = \eta^2 \partial_s u_\delta \max\{\omega_\delta - k, 0\}$ with the result

$$\begin{aligned}
&\int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta)\eta^2(\omega_\delta - k)\, dx + \\
&\quad + \int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s u_\delta \otimes \nabla \omega_\delta)\eta^2\, dx = \\
&= -2\int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \nabla \eta \otimes \partial_s u_\delta)\eta(\omega_\delta - k)\, dx\,.
\end{aligned} \tag{3.6}$$

The first integral I on the left-hand side is non-negative, the second one II is handled as follows (letting $f_\delta(Z) = g_\delta(|Z|^2) = \delta(1 + |Z|^2)^{q/2} + g(|Z|^2))$

$$\begin{aligned}
II &= \int_{A(k,R)} (4 g_\delta'' \partial_\alpha u_\delta^i \partial_s \partial_\alpha u_\delta^i \partial_\beta u_\delta^j \partial_s u_\delta^j \partial_\beta \omega_\delta + 2 g_\delta' \partial_s \partial_\alpha u_\delta^i \partial_s u_\delta^i \partial_\alpha \omega_\delta)\eta^2 dx = \\
&= \int_{A(k,R)} (2 g_\delta'' \partial_s |\nabla u_\delta|^2 \partial_\beta \omega_\delta \partial_\beta u_\delta^j \partial_s u_\delta^j + g_\delta' \partial_\alpha |\nabla u_\delta|^2 \partial_\alpha \omega_\delta)\eta^2\, dx = \\
&= \int_{A(k,R)} (2 g_\delta'' \partial_s \omega_\delta \partial_\beta \omega_\delta \partial_s u_\delta^j \partial_\beta u_\delta^j + g_\delta' \partial_\alpha \omega_\delta \partial_\alpha \omega_\delta)(1 + |\nabla u_\delta|^2)\eta^2\, dx = \\
&= \frac{1}{2}\int_{A(k,R)} \frac{\partial^2 f_\delta}{\partial X_\beta^j \partial X_s^j}(\nabla u_\delta)\partial_\beta \omega_\delta \partial_s \omega_\delta (1 + |\nabla u_\delta|^2)\eta^2\, dx\,.
\end{aligned} \tag{3.7}$$

If e_j denotes the j^{th} coordinate vector, then we have

$$\begin{aligned}
II &= \frac{1}{2}\int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(\nabla \omega_\delta \otimes e_j, \nabla \omega_\delta \otimes e_j)(1 + |\nabla u_\delta^2|)\eta^2\, dx \geq \\
&\geq c_{14}\int_{A(k,R)} \left(1 + |\nabla u_\delta|^2\right)^{1 - \frac{\mu}{2}} |\nabla \omega_\delta|^2 \eta^2\, dx\,.
\end{aligned} \tag{3.8}$$

Similar calculations show the right-hand side of (3.6) to be bounded from above by

$$c_{15}\Big|\int_{A(k,R)} \frac{\partial^2 f_\delta}{\partial X_\beta^j \partial X_s^j}(\nabla u_\delta)\partial_s \omega_\delta \partial_\beta \eta \eta (1 + |\nabla u_\delta|^2)(\omega_\delta - k)\, dx\Big|\,. \tag{3.9}$$

Finally, for each fixed $j \in \{1, \dots M\}$, the Cauchy-Schwarz inequality can be applied to the bilinear form

$$\mathbb{R}^n \ni \xi \to \frac{\partial^2 f_\delta}{\partial X^j_\beta \partial X^j_s}(\nabla u_\delta)\xi_\beta \xi_s$$

and, together with Young's inequality, (3.6)-(3.9) prove that the first term of the left-hand side of (3.2) is bounded in the desired way. (Note that from the growth assumptions imposed on f it follows that $|D^2 f_\delta(X)\|X|^2 \leq c_{16}(f_\delta(X) + 1) \leq c_{17}(1 + |X|^2)^{q/2}$ with suitable constants c_{16} and c_{17}.) The second integral on the left-hand side of (3.2) is studied by inserting $\psi = \eta^2 \partial_s u_\delta \max\{\omega_\delta - k, 0\}^2$ in (3.5). This gives

$$\begin{aligned}(3.10)\qquad & \int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta)\eta^2(\omega_\delta - k)^2\, dx + II = \\ & = -2\int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s u_\delta \otimes \nabla\eta)\eta(\omega_\delta - k)^2\, dx\,,\end{aligned}$$

where we have abbreviated

$$\begin{aligned}(3.11)\qquad II &= 2\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s u_\delta \otimes \nabla\omega_\delta)(\omega_\delta - k)\eta^2\, dx = \\ &= \int_{A(k,R)} \frac{\partial^2 f_\delta}{\partial X^j_\beta \partial X^j_s}(\nabla u_\delta)\partial_\beta \omega_\delta \partial_s \omega_\delta(\omega_\delta - k)(1 + |\nabla u_\delta|^2)\eta^2\, dx \geq \\ &\geq 0\,.\end{aligned}$$

Here the second equation in (3.11) again uses the special structure $f_\delta(X) = g_\delta(|X|^2)$. Given (3.10) and (3.11) the proof of (3.2) (and of Lemma 3.1) is completed by applying Young's inequality once more. □

4 – Hölder continuity of the dual solution

We apply the results of the previous section to get regularity of the maximizer σ, more precisely

LEMMA 4.1. *Let the assumptions of Theorem* 1.1 *hold.*

a) If u^* denotes a L^1-cluster point of $\{u_\delta\}$, then we have

$$\sigma = \nabla f(\nabla u^*)\,.$$

b) σ *is of class* $C^{0,\alpha}(\Omega, \mathbb{R}^{nM})$ *for any* $0 < \alpha < 1$.

Proof. Recalling (3.3), we choose $\varphi = \eta^2(u_\delta - u^*)$, $\eta \in C_0^1(\Omega)$, $0 \le \eta \le 1$. Then, together with (2.2), the counterpart of (6.4), [4], is established:

$$\begin{aligned}
&\int_\Omega \eta^2(\nabla f(\nabla u_\delta) - \nabla f(\nabla u^*)) : (\nabla u_\delta - \nabla u^*)\, dx + \\
&\quad + \delta \int_\Omega \eta^2 X_\delta : (\nabla u_\delta - \nabla u^*)\, dx = \\
&= -2 \int_\Omega \sigma_\delta : (\nabla \eta \otimes (u_\delta - u^*))\eta\, dx + \\
&\quad + 2 \int_\Omega \nabla f(\nabla u^*) : (\nabla \eta \otimes (u_\delta - u^*))\eta\, dx\,.
\end{aligned}$$

Clearly the second integral on the right-hand side vanishes as $\delta \to 0$ and by (2.5), (2.8) this is also true for the second one on the left-hand side. Since the definition of σ_δ gives the same result for the first integral on the right-hand side, it is proved that

$$\lim_{\delta \downarrow 0} \int_\Omega \eta^2(\nabla f(\nabla u_\delta) - \nabla f(\nabla u^*)) : (\nabla u_\delta - \nabla u^*)\, dx = 0\,.$$

On the other hand we have by (1.5)

$$\begin{aligned}
&\int_\Omega \eta^2(\nabla f(\nabla u_\delta) - \nabla f(\nabla u^*)) : (\nabla u_\delta - \nabla u^*)\, dx \ge \\
&\ge c_{18} \int_\Omega \int_0^1 \left(1 + |\nabla u^* + t(\nabla u_\delta - \nabla u^*)|^2\right)^{-\frac{\mu}{2}} |\nabla u_\delta - \nabla u^*|^2 \eta^2\, dt\, dx\,,
\end{aligned}$$

hence as $\delta \to 0$

$$(1 + |\nabla u^*|^2 + |\nabla u_\delta - \nabla u^*|^2)^{-\frac{\mu}{2}} |\nabla u_\delta - \nabla u^*|^2 \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ and a.e.}$$

Thus, on account of $\mu \le 2$, it immediately follows that a.e.

$$\limsup_{\delta \downarrow 0} |\nabla u_\delta(x)| < \infty\,, \qquad \text{which implies} \qquad \nabla u_\delta(x) \overset{\delta \downarrow 0}{\to} \nabla u^*(x)\,.$$

Hence $\nabla f(\nabla u_\delta(x)) \overset{\delta \downarrow 0}{\to} \nabla f(\nabla u^*(x))$ which together with the weak convergence (2.6) completes the proof of the first assertion. The second one follows from Lemma 3.1. ∎

5 – Local $C^{1,\alpha}$-regularity and uniqueness of generalized minimizers

So far we have proved Theorem 1.1 *a*). Now we fix any $u \in \mathcal{M}$ and use ideas of [26] to show that the pair (u,σ) satisfies an appropriate minimax inequality. A variation of the tensorial argument will finally give

Lemma 5.1. *Under the assumptions of Theorem* 1.1 *any generalized minimizer* $u \in \mathcal{M}$ *satisfies*

$$\nabla u = \nabla f^*(\sigma)\,,$$

where the right-hand side is of class $C^{0,\alpha}(\Omega, \mathbb{R}^{nM})$.

Corollary 5.2. *Of course Lemma* 5.1 *implies uniqueness of generalized minimizers up to a constant.*

Proof. Consider a J-minimizing sequence $\{u_m\}$ in $u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$ such that

$$u_m \overset{L^{n/(n-1)}}{\rightharpoonup} u\,, \quad u_m \overset{L^1}{\to} u\,.$$

Then let for any $w \in \mathrm{BV}(\Omega, \mathbb{R}^M)$ and $\varkappa \in \mathcal{U} := \{\tau \in L^\infty(\Omega, \mathbb{R}^{nM}) : \operatorname{div}\tau \in L^n(\Omega, \mathbb{R}^M)\}$

$$\tilde{l}(w,\varkappa) = \int_\Omega \operatorname{div}\varkappa(u_0 - w)\,dx - \int_\Omega f^*(\varkappa)\,dx + \int_\Omega \varkappa : \nabla u_0\,dx\,.$$

The representation formula (1.10) implies

$$J(u_m) = \sup_{\lambda \in L^\infty(\Omega, \mathbb{R}^{nM})} l(u_m, \lambda) \geq l(u_m, \varkappa) = \tilde{l}(u_m, \varkappa)$$

for any $\varkappa \in \mathcal{U}$. Passing to the limit $m \to \infty$ we obtain

$$\inf_{w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)} J(w) \geq \sup_{\varkappa \in \mathcal{U}} \tilde{l}(u, \varkappa)\,.$$

On the other hand, given $\varkappa \in \mathcal{U}$, $v \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)$, we observe (recalling that $\operatorname{div} \sigma = 0$)

$$\tilde{l}(u, \varkappa) \leq \inf_{w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)} J(w) = R(\sigma) = \inf_{w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^M)} l(w, \sigma) \leq$$
$$\leq l(v, \sigma) = \tilde{l}(v, \sigma) = \int_\Omega \sigma : \nabla u_0 \, dx - \int_\Omega f^*(\sigma) \, dx =: \tilde{l}(\sigma) \, .$$

Thus, for any $u \in \mathcal{M}$ and $\varkappa \in \mathcal{U}$ it is proved that

$$\tilde{l}(u, \varkappa) \leq \tilde{l}(\sigma) \, . \tag{5.1}$$

To proceed further, fix $\lambda \in C_0^\infty(\Omega, \mathbb{R}^{nM})$; by Lemma 4.1, σ is a continuous function taking values in $\operatorname{Im} \nabla f$ and hence there is a real number $\gamma > 0$ such that $\operatorname{dist}(\sigma(x), \partial \operatorname{Im} \nabla f) > \gamma$ for any $x \in \operatorname{spt} \lambda$. If $|t|$ is suffienciently small, then the same is true if we replace σ by $\sigma_t := \sigma + t\lambda$ and γ by $\gamma/2$. (5.1) implies

$$\int_{\operatorname{spt} \lambda} \operatorname{div} \sigma_t \cdot (u_0 - u) \, dx + \int_{\operatorname{spt} \lambda} \sigma_t : \nabla u_0 \, dx \leq$$
$$\leq \int_{\operatorname{spt} \lambda} (f^*(\sigma_t) - f^*(\sigma)) \, dx + \int_{\operatorname{spt} \lambda} \sigma : \nabla u_0 \, dx \, .$$

If we observe that

$$\int_{\operatorname{spt} \lambda} \operatorname{div} \sigma_t \cdot u_0 \, dx + \int_{\operatorname{spt} \lambda} \sigma_t : \nabla u_0 \, dx - \int_{\operatorname{spt} \lambda} \sigma : \nabla u_0 \, dx =$$
$$= \int_{\operatorname{spt} \lambda} t \operatorname{div} \lambda \cdot u_0 \, dx + \int_{\operatorname{spt} \lambda} t\lambda : \nabla u_0 \, dx = 0 \, ,$$

then we obtain

$$- \int_{\operatorname{spt} \lambda} t \operatorname{div} \lambda \cdot u \, dx \leq \int_{\operatorname{spt} \lambda} (f^*(\sigma_t) - f^*(\sigma)) \, dx \, .$$

Dividing through $t > 0$ and passing to the limit $t \to 0$ we get

$$- \int_{\operatorname{spt} \lambda} \operatorname{div} \lambda \cdot u \, dx \leq \int_{\operatorname{spt} \lambda} \nabla f^*(\sigma) : \lambda \, dx \, ,$$

i.e., by definition, the first weak derivative of u is given by $\nabla f^*(\sigma)$ which is a function of class $C^{0,\alpha}(\widetilde{\Omega}, \mathbb{R}^{nM})$ on account of Lemma 4.1 and the fact that $\operatorname{dist}(\sigma(x), \partial \operatorname{Im} \nabla f) \geq c(\widetilde{\Omega}) > 0$ on $\widetilde{\Omega} \Subset \Omega$. ∎

6 – The limit case n = 2, $\mu = 2$

Let the assumptions of Theorem 1.2 hold. For simplicity we just consider the scalar case $M = 1$. With notation from Section 2 and Section 3 it is easy to check that (3.2) extends to the case $\mu = 2$, i.e. we have for any $k \geq 0$

$$\tag{6.1} \begin{aligned} &\int_{A(k,R)} |\nabla \omega_\delta|^2 \eta^2 \, dx + \int_{A(k,R)} \left(1 + |\nabla u_\delta|^2\right)^{-1} (\omega_\delta - k)^2 \eta^2 |\nabla^2 u_\delta|^2 \, dx \leq \\ &\leq c_{19} \int_{A(k,R)} \left(1 + |\nabla u_\delta|^2\right)^{\frac{q}{2}} |\nabla \eta|^2 (\omega_\delta - k)^2 \, dx \end{aligned}$$

being valid for all discs $B_{2R} \Subset \Omega$ and any $\eta \in C_0^1(B_R)$, $0 \leq \eta \leq 1$. Note that estimate (6.1) is true in any dimension $n \geq 2$. In order to get a variant of Lemma 2.6 in [7] we let

$$\begin{aligned} a(k,r) &:= \int_{A(k,r)} \left(1 + |\nabla u_\delta|^2\right)^{\frac{q}{2}} dx \,, \\ \tau(k,r) &:= \int_{A(k,r)} \left(1 + |\nabla u_\delta|^2\right)^{\frac{q}{2}} (\omega_\delta - k)^2 \, dx \,. \end{aligned}$$

Lemma 6.1. *There is a constant c_{20} independent of δ such that*

$$\tag{6.2} a(h,r) \leq (h-k)^{-2} \tau(k,r) \,,$$

$$\tag{6.3} \tau(k,r) \leq c_{20} (R-r)^{-2} \tau(k,R) a(k,R)$$

valid for $h > k$ and $r < R \leq R_0$, $B_{2R_0} \Subset \Omega$.

Proof. (6.2) is immediate. Let $\Gamma := 1 + |\nabla u_\delta|^2$ and choose $\eta \in C_0^1(B_R)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_r. Then, by Sobolev's inequality,

$$\begin{aligned} \tau(k,r) &\leq \int_{A(k,R)} \{\eta \Gamma^{\frac{q}{4}} (\omega_\delta - k)\}^2 \, dx \leq \\ &\leq c_{21} \Big(\int_{A(k,R)} |\nabla \{\eta \Gamma^{\frac{q}{4}} (\omega_\delta - k)\}| \, dx \Big)^2 . \end{aligned}$$

On the right-hand side three different terms arise which may be estimated as follows:

$$\Big(\int_{A(k,R)} |\nabla\eta\Gamma^{\frac{q}{4}}(\omega_\delta - k)|\,dx\Big)^2 \le |A(k,R)|\int_{A(k,R)} |\nabla\eta|^2\Gamma^{\frac{q}{2}}(\omega_\delta - k)^2\,dx \le$$
$$\le c_{22}a(k,R)(R-r)^{-2}\tau(k,R)\,,$$
$$\Big(\int_{A(k,R)} |\eta\nabla\omega_\delta\Gamma^{\frac{q}{4}}|\,dx\Big)^2 \le \Big(\int_{A(k,R)} \eta^2|\nabla\omega_\delta|^2\,dx\Big)\Big(\int_{A(k,R)} \Gamma^{\frac{q}{2}}\,dx\Big) \le$$
$$\overset{(6.1)}{\le} c_{23}(R-r)^{-2}\tau(k,R)a(k,R)\,,$$
$$\Big(\int_{A(k,R)} \eta\Gamma^{\frac{q}{4}-1}|\nabla\Gamma|(\omega_\delta - k)\,dx\Big)^2 \le$$
$$\le c_{24}\Big(\int_{A(k,R)} \eta\Gamma^{\frac{q}{4}-\frac{1}{2}}|\nabla^2 u_\delta|(\omega_\delta - k)\,dx\Big)^2 \le$$
$$\le c_{24}\Big(\int_{A(k,R)} |\nabla^2 u_\delta|^2\eta^2\Gamma^{-1}(\omega_\delta - k)^2\,dx\Big)\cdot\Big(\int_{A(k,R)} \Gamma^{\frac{q}{2}}\,dx\Big) \le$$
$$\overset{(6.1)}{\le} c_{25}(R-r)^{-2}\tau(k,R)a(k,R)\,.$$

∎

Let $h > k$ and $R > r$. Then we get from Lemma 6.1 ($\alpha > 1$ being specified later)

$$\begin{aligned}\tau(h,r)^\alpha a(h,r) &\overset{(6.2)}{\le} \tau(h,r)^\alpha (h-k)^{-2}\tau(k,r) \le\\ &\le \tau(k,r)^\alpha (h-k)^{-2}\tau(k,R) \le\\ &\overset{(6.3)}{\le} c_{26}(R-r)^{-2\alpha}\tau(k,R)^\alpha a(k,R)^\alpha (h-k)^{-2}\tau(k,R) =\\ &= c_{26}(R-r)^{-2\alpha}(h-k)^{-2}[a(k,R)\tau(k,R)^{\frac{1+\alpha}{\alpha}}]^\alpha\,.\end{aligned}$$

For $\alpha = (\sqrt{5}+1)/2$ we see that $(1+\alpha)/\alpha = \alpha$, hence

$$\psi(h,r) := \tau(h,r)^\alpha a(h,r)$$

satisfies the growth estimate

$$\psi(h,r) \le c_{26}(R-r)^{-2\alpha}(h-k)^{-2}\psi(k,R)^\alpha\,, \qquad h > k \ge 0\,,\ \ r < R \le R_0\,,$$

and from [27], Lemma 5.1, we deduce

$$\psi(d, R_0/2) = 0\,,$$

where the number d is determined by R_0 and the quantity $\psi(0, R_0)$. Here R_0 is any radius such that $B_{2R_0} \Subset \Omega$. Clearly this implies

$$(6.4) \qquad |\nabla u_\delta|^2 \le e^d \quad \text{on } B_{R_0/2}\,,$$

and as usual (6.4) turns into a locally uniform gradient bound as soon as we can estimate the quantity d. In order to get such a bound, we observe first that the functions $\omega_\delta = \ln(1+|\nabla u_\delta|^2)$ satisfy

$$(6.5) \qquad \|\nabla \omega_\delta\|_{L^2(\Omega')} \le c_{27}(\Omega')$$

for any subdomain $\Omega' \Subset \Omega$. In fact, (1.5) implies in the case $\mu = 2$

$$|\nabla \omega_\delta|^2 \le c_{28} D^2 f_\delta(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta)\,,$$

thus we may quote Lemma 2.1 by observing that the right–hand side of (2.3) is bounded independent of δ.

According to (6.5) there is a Radon measure ν on Ω such that

$$(6.6) \qquad \nu_\delta := |\nabla \omega_\delta|^2 \overset{\delta\downarrow 0}{\rightharpoondown} \nu$$

in the sense of measures (at least for a subsequence). For $\varepsilon > 0$ being determined later let

$$\Sigma_\varepsilon := \{x \in \Omega :\, \lim_{r\downarrow 0} \nu(\overline{B}_r(x)) = \nu(\{x\}) \ge \varepsilon\}\,.$$

Being a subset of the atoms of ν the set Σ_ε is at most countable with no interior accumulation points.

Lemma 6.2 (compare [17], Theorem 7.21). *Let G denote a disc in $\mathbb{R}^2$. Then there are constants c_{29}, $c_{30} > 0$ as follows: if $\omega \in W_1^1(G)$ satisfies for some $K > 0$*

$$(6.7) \qquad \int_{G\cap B_R(z)} |\nabla \omega|\, dx \le KR \quad \textit{for all } B_R(z) \subset \mathbb{R}^2\,,$$

then we have

$$(6.8) \qquad \int_G \exp\Big(\frac{c_{29}}{K}|\omega - (\omega)_G|\Big)\, dx \le c_{30}(\mathrm{diam} G)^2\,.$$

Let us now choose $x_0 \notin \Sigma_\varepsilon$. Then $\nu(\overline{B}_t(x_0)) < 2\varepsilon$ for $t \le t_\varepsilon$, thus

$$\limsup_{\delta \downarrow 0} \nu_\delta(\overline{B}_t(x_0)) \le \nu(\overline{B}_t(x_0)) < 2\varepsilon\,,$$

and in conclusion $\nu_\delta(\overline{B}_t(x_0)) \le 3\varepsilon$ for all δ small enough. Let $G := B_t(x_0)$. Then

$$\int_{G\cap B_R(z)} |\nabla \omega_\delta|\,dx \le \sqrt{3\varepsilon}\sqrt{\pi}R\,,$$

thus we have (6.7) with $K = \sqrt{\pi}\sqrt{3\varepsilon}$, and (6.8) implies

$$\int_{B_t(x_0)} \exp\Big(\frac{c_{29}}{\sqrt{\pi}\sqrt{3\varepsilon}}|\omega_\delta - (\omega_\delta)_{B_t(x_0)}|\Big)\,dx \le c_{31}(t)$$

for δ small enough. We select ε according to $c_{29}/\sqrt{3\pi\varepsilon} = 1$, thus

$$\int_{B_t(x_0)} \exp|\omega_\delta - (\omega_\delta)_{B_t(x_0)}|\,dx \le c_{31}(t)\,.$$

Observing $(\omega_\delta)_{B_t(x_0)} = \fint_{B_t(x_0)} \ln(1+|\nabla u_\delta|^2)\,dx \le c_{32}(t)$ we have shown that

$$\int_{B_t(x_0)} |\nabla u_\delta|^2\,dx \le c_{33}(t) \tag{6.9}$$

holds for all δ small enough.

Let us suppose that in the beginning a sequence $\delta \downarrow 0$ has been chosen such that $u_\delta \to u^*$ in $L^1(\Omega)$. By the choice of q and the definition of $\psi(0,t)$ we see that (6.9) provides a uniform bound for d, and from (6.4) we get $\nabla u^* \in L^\infty(B_{t/2}(x_0))$. Altogether we have shown

Lemma 6.3. *Let u^* denote a L^1-cluster point of the sequence $\{u_\delta\}$. Then u^* is locally Lipschitz* (*and hence of class $C^{1,\alpha}$*) *on the open set $\Omega - \Sigma_\varepsilon$.*

Next we let $\Omega_0 = \Omega - \Sigma_\varepsilon$ (which is an open set) and consider $\eta \in C^1_0(\Omega_0)$. Then, as in Section 4, we get $\nabla u_\delta \to \nabla u^*$ a.e. on Ω_0, thus $\sigma = \nabla f(\nabla u^*)$ on Ω_0 which proves part $a)$ of Theorem 1.2. Now let u denote some generalized minimizer, i.e. $u \in \mathcal{M}$, and repeat the proof of Lemma 5.1 with $\lambda \in C^\infty_0(\Omega_0, \mathbb{R}^2)$. We get $\nabla u = \nabla f^*(\sigma)$ now on Ω_0, thus $\nabla u = \nabla u^*$ on Ω_0. But since Σ_ε has no interior accumulation points, we see that Ω_0 is connected, hence $u = u^* + \text{const}$ on Ω_0 and therefore a.e. on Ω. $\overset{Theorem 1.2}{\square}$

So far we have shown that any generalized minimizer u is of class $C^{1,\alpha}$ except for a discrete set of possible singular points forming a subset of Σ_ε. Let us have a closer look at the behaviour near isolated singularities x_0. We fix a disc $B_t(x_0)$ such that no other singular point occurs in $\overline{B}_t(x_0)$ and decompose the vector-measure ∇u as

$$\nabla u = (\nabla u)^a + (\nabla u)^s$$

with absolutely continuous part $(\nabla u)^a$ in the space $L^1(B_t(x_0), \mathbb{R}^2)$. Obviously $(\nabla u)^s$ has the form $\xi\delta_{x_0}$ for some $\xi \in \mathbb{R}^2$, and we claim $\xi = 0$. In fact, this is a consequence of the fine properties for BV–functions stated for example in [2], Lemma 3.76, saying that the measure $|\nabla u|$ vanishes on the set $\{x_0\}$ which is of $\mathcal{H}^1$ measure zero. Alternatively we may assume that $\xi \neq 0$ and recall that u minimizes $\hat{J}(\cdot, B_t(x_0))$ w.r.t. its boundary values. Let $v := \eta_r u$ with $\eta_r \in C^1(\overline{B}_t(x_0))$, $0 \le \eta_r \le 1$, $\eta_r \equiv 1$ on $B_t(x_0) - B_{2r}(x_0)$, $\eta_r \equiv 0$ on $B_r(x_0)$, $|\nabla \eta_r| \le c_{34} r^{-1}$, where $r \ll t$. Then

$$\begin{aligned}\int_{B_t(x_0)} f(\eta_r \nabla u + \nabla \eta_r u)\,dx = \hat{J}(v, B_t(x_0)) \ge \hat{J}(u, B_t(x_0)) = \\ = \int_{B_t(x_0)} f((\nabla u)^a)\,dx + f_\infty\Big(\frac{\xi}{|\xi|}\Big)\end{aligned}$$

with $f_\infty(\xi/|\xi|) > 0$ on account of (1.2). By convexity of f we have

$$f(\eta_r \nabla u) \ge f(\eta_r \nabla u + u \nabla \eta_r) - u\nabla\eta_r \cdot \nabla f\,\underbrace{(\eta_r \nabla u + u\nabla \eta_r)}_{=:\varkappa},$$

hence

$$\begin{aligned}
\hat{J}(v, B_t(x_0)) &\le \int_{B_t(x_0)} f(\eta_r \nabla u)\, dx + \int_{B_t(x_0)} u \nabla \eta_r \cdot \nabla f(\varkappa)\, dx = \\
&= \int_{B_t(x_0)-B_{2r}(x_0)} f((\nabla u)^a)\, dx + \int_{B_{2r}(x_0)-B_r(x_0)} f(\eta_r (\nabla u)^a)\, dx + \\
&\quad + \int_{B_t(x_0)} u \nabla \eta_r \cdot \nabla f(\varkappa)\, dx \le \\
&\le \int_{B_t(x_0)} f((\nabla u)^a)\, dx + \int_{B_{2r}(x_0)-B_r(x_0)} f(\eta_r (\nabla u)^a)\, dx + \\
&\quad + c_{35} \frac{1}{r} \int_{B_{2r}(x_0)} |u|\, dx\,,
\end{aligned}$$

where we have used (1.3). Recalling $n = 2$ and $u \in BV(\Omega)$, we see $u \in L^2(\Omega)$, hence $r^{-1} \int_{B_{2r}(x_0)} |u|\, dx \to 0$ as $r \downarrow 0$. The linear growth of f implies

$$\int_{B_{2r}(x_0)-B_r(x_0)} f(\eta_r (\nabla u)^a)\, dx \le c_{36} \int_{B_{2r}(x_0)} (1 + |(\nabla u)^a|)\, dx \overset{r\downarrow 0}{\to} 0\,,$$

and we finally get

$$\begin{aligned}
\int_{B_t(x_0)} f((\nabla u)^a)\, dx + O(r) &\ge \hat{J}(v, B_t(x_0)) \ge \\
&\ge \int_{B_t(x_0)} f((\nabla u)^a)\, dx + f_\infty\Big(\frac{\xi}{|\xi|}\Big)\,.
\end{aligned}$$

From (1.2) we deduce $f_\infty(\xi/|\xi|) \ge a$ which is a contradiction if r is small enough. Consequentely $\nabla u = (\nabla u)^a \in L^1(B_t(x_0), \mathbb{R}^2)$, and therefore $u \in W^1_{1,\mathrm{loc}}(\Omega)$. To prove the corollary we recall that according to (6.5) we may assume $\omega_\delta \rightharpoondown\colon \omega$ in $W^1_{2,\mathrm{loc}}(\Omega)$. On the other hand, $\nabla u_\delta \to \nabla u^*$ a.e. on Ω_0, and since $\nabla u^* \in L^1_{\mathrm{loc}}(\Omega)$ we have pointwise convergence a.e. on Ω, hence $\omega = \ln(1 + |\nabla u^*|^2) \in W^1_{2,\mathrm{loc}}(\Omega)$. Suppose that $t > 1$ is given, and let x_0 denote a point of Σ_ε. Then we have

$$\int_{B_r(x_0)\cap B_R(z)} |\nabla \omega|\, dx \le \|\nabla \omega\|_{L^2(B_r(x_0))} \sqrt{\pi} R\,,$$

and (6.8) implies

$$\int_{B_r(x_0)} \exp\Big(\frac{c_{29}}{\sqrt{\pi}\|\nabla \omega\|_{L^2(B_r(x_0))}} |\omega - (\omega)_{B_r(x_0)}|\Big)\, dx \le c_{30} (2r)^2\,.$$

Let us choose r small enough in such a way that

$$\frac{c_{29}}{\sqrt{\pi}\|\nabla\omega\|_{L^2(B_r(x_0))}} \geq \frac{t}{2}\,.$$

Then we get

$$\int_{B_r(x_0)} |\nabla u^*|^t\, dx < +\infty$$

which proves the claim of the corollary for u^* and hence for any $u \in \mathcal{M}$. Note that we can bound the mean value of ω on $B_r(x_0)$ in terms of the L^1-norm of ∇u^* due to the fact that we already know that u^* is of class $W^1_{1,\mathrm{loc}}(\Omega)$. ∎

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