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Pseudohermitian geometry on contact Riemannian manifolds

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Riassunto: *A partire dai lavori di S. Tanno,* [39], *e E. Barletta et al.,* [3], *si studia la geometria delle quasi CR strutture* (*possibilmente non integrabili*) *su varietà riemanniane di contatto. Si caratterizzano le funzioni CR-pluriarmoniche in termini di operatori differenziali naturali associati alla struttura riemanniana di contatto data. Si mostra che la quasi CR struttura di una varietà riemanniana di contatto* (M,η) *il cui fibratto canonico ammette sezioni globali, non nulle e chiuse, è integrabile e* η *è una forma di contatto pseudo-Einstein. Si mostra che il gruppo di olonomia pseudohermitiano di una varietà sasakiana è contenuto in* $SU(n)\times 1$ *se e solo se la connessione di Tanaka-Webster è Ricci piatta. Inoltre, per ogni varietà sasakiana quaternionica* $(M^{4m+1},(F,T,\theta,g))$ *o la connessione di Tanaka-Webster di* (M^{4m+1},θ) *è Ricci piatta oppure* $m=1$ *e allora* (M^5,θ) *è pseudo-Einstein se e solo se* $4p+\rho^*\theta$ *è chiusa, dove* p *è una* 1*-forma locale su* M^5 *tale che* $\nabla G = p\otimes H$ *e* $\nabla H = -p\otimes G$ *per qualche riferimento* $\{F,G,H\}$, *e* ρ^* *è la curvatura scalare pseudohermitiana di* (M^5,θ). *Su ogni varietà sasakiana* M *esiste un sistema di Pfaff integrabile, invariante per* $\Psi(x)$ (*il gruppo di olonomia pseudohermitiana in* $x\in M$) *e contenente il flow di contatto come una sottofogliazione. Si costruiscono connessioni canoniche* (*che ramentano la connessione di Tanaka,* [38]) *in fibrati vettoriali complessi su varietà riemanniane di contatto, dotati di un un operatore pre-*$\overline{\partial}$ *e di una metrica Hermitiana. Come un'applicazione, si calcola la prima funzione di struttura della quasi CR struttura di una varietà riemanniana di contatto. Si mostra che la classe conforme ristretta* $[G_\eta]$ *della metrica di Fefferman* (*generalizzata*) *come pure certe connessioni canoniche* D (*con traccia* $\Lambda_g R^D = 0$) *sono invarianti di gauge.*

Abstract: *Starting from work by S. Tanno,* [39], *and E. Barletta et al.,* [3], *we study the geometry of* (*possibly non integrable*) *almost CR structures on contact Riemannian manifolds. We characterize CR-pluriharmonic functions in terms of differential operators naturally attached to the given contact Riemannian structure. We show that the almost CR structure of a contact Riemannian manifold* (M,η) *admitting global* *nonzero closed sections (with respect to which η is volume normalized) in the canonical bundle is integrable and η is a pseudo-Einstein contact form. The pseudohermitian holonomy of a Sasakian manifold M^{2n+1} is shown to be contained in $SU(n) \times 1$ if and only if the Tanaka-Webster connection is Ricci flat. Also, for any quaternionic Sasakian manifold $(M^{4m+1}, (F, T, \theta, g))$ either the Tanaka-Webster connection of (M^{4m+1}, θ) is Ricci flat or $m = 1$ and then (M^5, θ) is pseudo-Einstein if and only if $4p + \rho^* \theta$ is closed, where p is a local 1-form on M^5 such that $\nabla G = p \otimes H$ and $\nabla H = -p \otimes G$ for some frame $\{F, G, H\}$, and ρ^* is the pseudohermitian scalar curvature of (M^5, θ). On any Sasakian manifold M there is a smooth integrable Pfaffian system, invariant by $\Psi(x)$ (the pseudohermitian holonomy group at $x \in M$) containing the contact flow as a subfoliation. We build canonical connections (reminiscent of the Tanaka connection, [38]) on complex vector bundles over contact Riemannian manifolds, carrying a pre-$\overline{\partial}$-operator and a Hermitian metric. As an application, we compute the first structure function of the underlying almost CR structure of a contact Riemannian manifold. The restricted conformal class $[G_\eta]$ of the (generalized) Fefferman metric and certain canonical connections D (with trace $\Lambda_g R^D = 0$) are shown to be gauge invariants.*

Contents:

1 – Introduction

Let M be a real $(2n+1)$-dimensional C^∞ manifold. An *almost CR structure* is a rank n complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbf{C}$ so that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0)\,, \tag{1}$$

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where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugation) and a pair $(M, T_{1,0}(M))$ is *an almost CR manifold* (of CR dimension n). If $\{\xi_\alpha : 1 \leq \alpha \leq n\}$ is a local frame of $T_{1,0}(M)$, defined on some open neighborhood U, then $T_{0,1}(M)$ corresponds to the system of PDEs

$$\overline{\xi_\alpha}(u) = 0, \quad 1 \leq \alpha \leq n \,, \tag{2}$$

(the *tangential Cauchy-Riemann equations*) and a solution $u \in C^\infty(U)$ to (2) is a *CR function*. Almost CR structures appear for instance on smooth real hypersurfaces M in a complex manifold V

$$T_{1,0}(M) := [T(M) \otimes \mathbf{C}] \cap \mathrm{Span}\{\partial/\partial z^j : 1 \leq j \leq n+1\} \,,$$

where z^j are (local) complex coordinates on V and, as such, possess the following (formal) Frobenius integrability property

$$[\Gamma^\infty(T_{1,0}(M)), \Gamma^\infty(T_{1,0}(M))] \subseteq \Gamma^\infty(T_{1,0}(M)) \,, \tag{3}$$

or, in terms of the local generators ξ_α

$$[\xi_\alpha, \xi_\beta] = C^\gamma_{\alpha\beta}\, \xi_\gamma \,, \tag{4}$$

for some smooth complex valued functions $C^\gamma_{\alpha\beta}$ on U. An almost CR structure satisfying the integrability condition (3) is a *CR structure* and the pair $(M, T_{1,0}(M))$ is a *CR manifold*. On the other hand, a smooth real hypersurface M in an almost Hermitian manifold (e.g. S^6 with the canonical nearly Kähler structure, [12]) inherits an almost contact metric structure (φ, ξ, η, g) (cf. [11]) and $T_{1,0}(M) = \{X - i\varphi X : X \in \mathrm{Ker}(\eta)\}$ is an almost CR structure, not integrable, in general. The 1-form η is a section in the conormal bundle of $H(M) := \mathrm{Ker}(\eta)$ i.e. a *pseudohermitian structure* (cf. [48]) on M. Even if M is nondegenerate, that is η is a contact form, the tools of pseudohermitian geometry are unavailable, e.g. the construction of the *Tanaka-Webster connection*, (cf. [38] and [48]) is tied to the integrability property of the almost CR structure. Nevertheless, due to the work of S. Tanno, [39], a sort of pseudohermitian geometry, in many ways similar to that of S. Webster (cf. *op. cit.*), may be developed on a contact manifold (with a generally non integrable almost CR structure) in the presence of a fixed associated Riemannian metric (cf. also [40]-[42]). Let (M, η) be a *contact manifold*, that is a real $(2n+1)$-dimensional C^∞ manifold carrying a 1-form η such that $\Psi = \eta \wedge (d\eta)^n$ is a volume form on M. There is a unique vector field $\xi \in \mathcal{X}(M)$ such that $\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0$ (the *characteristic direction* of (M, η)). By a well known result (cf. [11]), given a contact manifold (M, η) there are a Riemannian metric g and a $(1,1)$-tensor field φ on M such that

$$g(X, \xi) = \eta(X), \quad \varphi^2 = -I + \eta \otimes \xi \,,$$
$$g(X, \varphi Y) = (d\eta)(X, Y) \,,$$

for any $X, Y \in \mathcal{X}(M)$. Such a metric g is said to be *associated* to the contact form η. Let us denote by $\mathcal{M}(\eta)$ the set of all associated Riemannian metrics. Of course, once $g \in \mathcal{M}(\eta)$ is fixed, φ is uniquely determined. Each $g \in \mathcal{M}(\eta)$ has the same volume form Ψ. S. Tanno considered (cf. *op. cit.*) the (1,2)-tensor field

$$Q(X, Y) = (\nabla_Y \varphi)X + \{(\nabla_Y \eta)\varphi X\}\xi + \eta(X)\varphi(\nabla_Y \xi) \tag{5}$$

(the *Tanno tensor* field) and the linear connection ∇^* (the (*generalized*) *Tanaka-Webster connection* of (M, η)) given by

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\varphi Y - \eta(Y)\nabla_X \xi + [(\nabla_X \eta)Y]\xi \tag{6}$$

where ∇ is the Levi-Civita connection of (M, g). The connection ∇^* may be described axiomatically (cf. Proposition 3.1 in [39], p. 354) as the unique linear connection obeying

$$\nabla^* \eta = 0, \quad \nabla^* \xi = 0 \,, \tag{7}$$

$$\nabla^* g = 0 \,, \tag{8}$$

$$\begin{cases} T^*(X, Y) = 2(d\eta)(X, Y)\xi, & X, Y \in H(M) \,, \\ T^*(\xi, \varphi Z) = -\varphi\, T^*(\xi, Z), & Z \in T(M) \,, \end{cases} \tag{9}$$

$$(\nabla^*_X \varphi)Y = Q(Y, X), \quad X, Y \in T(M) \,. \tag{10}$$

where T^* is the torsion of ∇^*. As $Q = 0$ if and only if (3) holds ([39], p. 353-354), the axioms (7)-(10) show that on any contact Riemannian manifold with integrable almost CR structure ∇^* is the Tanaka-Webster connection of $(M, -\eta)$.

The present paper continues the work in [3] as an attempt to apply the geometric methods devised by S. Tanno to the study of the eqs. (2), without the involutivity property (4). We consider the *Webster torsion*

$$\tau(X) = T^*(\xi, X), \ X \in T(M)\,.$$

It is both trace-less (and, as a geometric interpretation, for any associated Riemannian metric $g \in \mathcal{M}(\eta)$ the contact distribution $H(M)$ is minimal in (M, g)) and self-adjoint, a property playing a crucial role in the derivation of the structure equations

$$\begin{aligned} \Omega_\alpha{}^\beta = {} & R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\eta^\lambda \wedge \eta^{\overline{\mu}} - W^\beta_{\alpha\lambda}\eta^\lambda \wedge \eta + W^\beta_{\alpha\overline{\mu}}\eta^{\overline{\mu}} \wedge \eta + \\ & - \frac{i}{4} g^{\beta\overline{\sigma}}\{g_{\overline{\rho}\lambda}Q^{\overline{\rho}}_{\mu\alpha,\overline{\sigma}}\eta^\lambda \wedge \eta^\mu + g_{\rho\overline{\lambda}}Q^\rho_{\overline{\mu}\,\overline{\sigma},\alpha}\eta^{\overline{\lambda}} \wedge \eta^{\overline{\mu}}\}\,, \end{aligned} \tag{11}$$

the CR counterpart of which allows one to relate the existence of (local or global) pseudo-Einstein structures to CR-pluriharmonic functions and the existence of closed sections in the canonical bundle (cf. [29]). In particular (11) is shown to imply the symmetry property $R_{\alpha\overline{\beta}\lambda\overline{\mu}} = R_{\lambda\overline{\beta}\alpha\overline{\mu}}$. As a general strategy, we express the terms without a CR counterpart in terms of the Tanno tensor Q and its first order covariant derivatives with respect to ∇^* [e.g. $R_\mu{}^{\overline{\beta}}{}_{\overline{\sigma}\alpha}$, a term which vanishes on a contact manifold with (3) as (by (10)) the Tanaka-Webster connection parallelizes $T_{1,0}(M)$].

Given a contact manifold (M, η), a C^∞ function $u : M \to \mathbf{R}$ is *CR-pluriharmonic* if it is locally the real part of a CR function. We determine natural differential operators characterizing CR-pluriharmonic functions. Precisely, we show that a smooth real valued funcion u on M is CR-pluriharmonic if and only if

$$u_{\alpha\overline{\beta}} = \mu\, g_{\alpha\overline{\beta}}\,, \ \ (Q^{\overline{\gamma}}_{\alpha\beta} - Q^{\overline{\gamma}}_{\beta\alpha})u_{\overline{\gamma}} = 0\,,$$

for some complex-valued function μ, provided the CR dimension is $n \geq 2$. The complex Hessian $u_{\alpha\overline{\beta}}$ is computed with respect to ∇^*. Note that the first set of equations are second order and correspond to the equations found by J. M. Lee, [29], while the second set are first order and have no (integrable) CR counterpart. Of course, the almost CR structure of any 3-dimensional contact Riemannian manifold is integrable, hence a characterization of CR-pluriharmonic functions in the case $n = 1$ already exists, cf. [29] (the relevant differential operators are third order).

In [3] one built a contact Riemannian analogue of the Fefferman metric in CR geometry (cf. e.g. [28]) relying on the existence, for any nowhere vanishing section $\mathcal{Z}$ in the canonical bundle, of a unique positive function λ so that

$$2^n\, i^{n(n+2)}\, n!\, \eta \wedge (\xi \rfloor \mathcal{Z}) \wedge (\xi \rfloor \overline{\mathcal{Z}}) = \lambda\, \eta \wedge (d\eta)^n\,.$$

If $\lambda = 1$ then η is said to be *volume normalized* with respect to $\mathcal{Z}$. We show that, if there is a closed globally defined nowhere zero section in the canonical bundle, with respect to which the contact form is volume normalized, then the almost CR structure is integrable and M is a pseudo-Einstein manifold.

The *pseudohermitian holonomy groups* of a nondegenerate CR manifold are the holonomy groups of its Tanaka-Webster connection (for a fixed choice of contact form). We show (cf. Section 5) that a strictly pseudoconvex CR manifold M^{2n+1} with vanishing Webster torsion ($\tau = 0$) has pseudohermitian holonomy contained in $SU(n) \times 1$ if and only if its Tanaka-Webster connection is Ricci flat. In Section 5.1 we deal again with the integrable case and start a study of *quaternionic Sasakian manifolds*. These are pseudohermitian analogues of quaternion Kähler manifolds (and one may bring to CR geometry a result by M. Berger, [9], and S. Ishihara, [24]). In Section 5.2 we show that, on any Sasakian manifold, a $\Psi(x)$-invariant subspace $\mathcal{D}_x$ of the tangent space at x [where $\Psi(x)$ is the pseudohermitian holonomy group with reference point x] gives rise (by parallel translation with respect to the Tanaka-Webster connection) to a smooth integrable Pfaffian system. The key ingredient is parallel displacement along *parabolic geodesics* (cf. [26]).

Given a complex vector bundle E over a contact Riemannian manifold, endowed with a pre-$\overline{\partial}$-operator $\overline{\partial}_E$ and a Hermitian metric H, we build canonical connections in E extending $\overline{\partial}_E$, parallelizing H, and with a prescribed trace P of their curvature tensor, thus generalizing the Tanaka connection of a Hermitian CR-holomorphic vector bundle over a strictly pseudoconvex CR manifold M ([39]), a CR invariant of M. As an application, we show that on any contact Riemannian manifold the first structure function of the underlying almost CR structure [regarded as a $U(n) \times 1$-structure] is nonzero.

Let $(M, (\varphi, \xi, \eta, g))$ be a contact Riemannian manifold. An object built in terms of (φ, ξ, η, g) is a *gauge invariant* (cf. S. Tanno, [39], p. 362-363, and [40], p. 537) if it is invariant under a transformation $(\varphi, \xi, \eta, g) \mapsto (\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ where

$$
\begin{aligned}
&\hat{\varphi} = \varphi + \frac{1}{2\lambda}\,\eta \otimes \{\nabla\lambda - \xi(\lambda)\xi\}\,, \\
&\hat{\xi} = \frac{1}{\lambda}\,(\xi + \zeta)\,, \quad \hat{\eta} = \lambda\,\eta\,, \\
&\hat{g} = \lambda\,g - \lambda\,(\eta \otimes w + w \otimes \eta) + \lambda\,(\lambda - 1 + \|\zeta\|^2)\eta \otimes \eta\,,
\end{aligned}
\tag{12}
$$

with $\lambda \in C^\infty(M)$ such that $\lambda(x) > 0$ for any $x \in M$. Here ζ is given by $\zeta = (1/(2\lambda))\,\varphi\nabla\lambda$ and w is the dual 1-form $w(X) = g(X, \zeta)$, $X \in T(M)$. For instance, the *Bochner curvature* tensor given by (8) in [40], p. 537, is a gauge invariant [coinciding with the Chern-Moser tensor when (3) holds]. If the almost CR structure of M is integrable then the gauge invariants of M are precisely its CR invariants. Let G_η be the (*generalized*) *Fefferman metric* of the contact Riemannian manifold (M, η, g), cf. [3], p. 27. The Fefferman metric is a Lorentz metric defined on the total space of a certain principal circle bundle $S^1 \to F(M) \xrightarrow{\pi} M$ (cf. Section 4.2 for definitions). The *restricted conformal class* of G_η is $[G_\eta] := \{\exp(2u \circ \pi)\,G_\eta : u \in C^\infty(M)\}$. In Section 7 we show that the restricted conformal class of G_η is a gauge invariant. The proof relies on the explicit calculation of the Fefferman metric G_η in terms of pseudohermitian invariants (cf. Section 4.2). Also, each canonical connection D whose curvature has trace $\Lambda_g R^D = 0$ (cf. Theorem 9) is shown to be a gauge invariant.

2 – Contact Riemannian *versus* CR geometry

Let M^{2n+1} be a real $(2n+1)$-dimensional C^∞ manifold. An *almost contact Riemannian structure* (φ, ξ, η, g) on M^{2n+1} consists of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η, and a Riemannian metric g

such that

$$\varphi^2 = -I + \eta \otimes \xi, \ \ \eta \circ \varphi = 0, \ \ \varphi\xi = 0\,,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,,$$

for any $X, Y \in T(M^{2n+1})$. It is a *contact Riemannian structure* if it satisfies $\Omega = d\eta$ (the *contact condition*) where $\Omega(X,Y) = g(X, \varphi Y)$. Set $h := \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ is the Lie derivative. A contact metric structure is K-*contact* if the contact vector ξ is Killing (equivalently $h = 0$). The Riemannian metric g, underlying a contact Riemannian structure, is *Sasakian* if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X\,,$$

for any $X, Y \in T(M^{2n+1})$, where ∇ is the Levi-Civita connection of g. Any Sasakian metric is K-contact (the converse is not true, in general, cf. [11]). Another approach to Sasakian metrics is as Webster metrics on strictly pseudoconvex CR manifolds whose Webster (or pseudohermitian) torsion vanishes.

As we need to apply the result in [13] to CR submanifolds (in the sense of [8]) of a Hermitian manifold (cf. Section 5.1) the notion of CR manifold considered in the introduction is not sufficiently general. We recall here the relevant notions, emphasizing on the relationship between contact Riemannian and CR geometry. A CR structure *of type* (n, k) on a $(2n + k)$-manifold M is a complex rank n subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbf{C}$ satisfying (1) and (3), and a pair $(M, T_{1,0}(M))$ is a CR manifold of type (n, k). The integers n and k are respectively the *CR dimension* and *CR codimension* of $(M, T_{1,0}(M))$. *A posteriori*, a CR manifold of type $(n, 1)$ (cf. Section 1) is said to be of *hypersurface type*. The real rank $2n$ distribution $H(M) := Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ is the *Levi* (or *maximally complex*) distribution of $(M, T_{1,0}(M))$. The *Levi form* is $L(Z, W) = i\pi[Z, \overline{W}]$, $\ Z, W \in T_{1,0}(M)$, where $\pi : T(M) \otimes \mathbf{C} \to [T(M) \otimes \mathbf{C}]/[T_{1,0}(M) \oplus T_{0,1}(M)]$ is the natural bundle map. If $L = 0$ then $(M, T_{1,0}(M))$ is *Levi flat*. For a Levi flat CR manifold $H(M)$ is integrable, hence M is foliated by complex n-dimensional manifolds. $(M, T_{1,0}(M))$ is *nondegenerate* if the Levi form L is nondegenerate. Let $E \subset T^*(M)$ be the conormal bundle of $H(M)$. If M is an oriented CR manifold of hypersurface type then E is a trivial line bundle, hence admits globally defined nowhere zero sections θ (i.e. real 1-forms on M

such that $\mathrm{Ker}(\theta) = H(M)$), each of which is referred to as a *pseudohermitian structure* on M. The Levi form may be recast as $L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W})$, $Z, W \in T_{1,0}(M)$, [L, L_θ are easily seen to coincide up to a bundle isomorphism $[T(M) \otimes \mathbf{C}]/[T_{1,0}(M) \oplus T_{0,1}(M)] \approx E$]. M is *strictly pseudoconvex* if L_θ is positive-definite, for some θ. There is no obvious way to define strict pseudoconvexity in higher CR codimension ($k \geq 2$). If M is nondegenerate then each pseudohermitian structure is actually a contact form (i.e. $\theta \wedge (d\theta)^n$ is a volume form on M). For a fixed contact form θ there is a unique vector field T on M so that $\theta(T) = 1$ and $T \rfloor d\theta = 0$ (the *characteristic direction* of (M, θ)). If this is the case, i.e. M is nondegenerate, then the Levi form extends naturally to a semi-Riemannian metric g on M (the *Webster metric*) given by

$$g(X, Y) = (d\theta)(\pi_H X, \varphi \pi_H Y) + \theta(X)\theta(Y), \quad X, Y \in T(M)\,.$$

Here $\pi_H : T(M) \to H(M)$ is the projection associated with $T(M) = H(M) \oplus \mathbf{R}T$ and $\varphi(Z + \overline{Z}) = i(Z - \overline{Z})$, $Z \in T_{1,0}(M)$, is the complex structure in $H(M)$. Then $\Omega = -d\theta$, i.e. if M is strictly pseudoconvex and θ has been chosen so that L_θ is positive definite then $(\varphi, -T, -\theta, g)$ is a contact metric structure on M. Let ∇^* be the *Tanaka-Webster connection* of (M, θ), i.e. the linear connection uniquely determined by the axioms (7)-(10) with $\xi = -T$ and $Q = 0$. Let $\tau(X) = T^*(T, X)$ be its *Webster torsion*. Then $(\varphi, -T, -\theta, g)$ is normal (i.e. g is Sasakian) if and only if $\tau = 0$ (cf. [11], [18]).

2.1 – Basic formulae

Let (M, η) be a contact manifold and $g \in \mathcal{M}(\eta)$ an associated Riemannian metric. Let $\{\xi_\alpha\}$ be a local frame of the almost CR structure $T_{1,0}(M)$ and $\{\eta^\alpha\}$ the corresponding *admissible* coframe, i.e.

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta,\ \eta^\alpha(\xi_{\overline{\beta}}) = 0,\ \eta^\alpha(\xi) = 0\,,$$

where $\xi_{\overline{\beta}} = \overline{\xi_\beta}$. Due to the contact condition the (local) components of the Levi form are $g_{\alpha\overline{\beta}} = g(\xi_\alpha, \xi_{\overline{\beta}})$. Then (cf. (44) in [3])

$$d\eta = -2i g_{\alpha\overline{\beta}}\, \eta^\alpha \wedge \eta^{\overline{\beta}}\,. \tag{13}$$

We collect a few elementary properties of the Webster torsion in the following

LEMMA 1. *Let* (M, η, g) *be a contact Riemannian manifold and* ξ *the characteristic direction of* (M, η). *Then* i) $\tau(\xi) = 0$, ii) $\tau \circ \varphi + \varphi \circ \tau = 0$, iii) $\tau T(M) \subseteq H(M)$, iv) $\tau T_{1,0}(M) \subseteq T_{0,1}(M)$ *and* $\tau T_{0,1}(M) \subseteq T_{1,0}(M)$, *and* v) $\mathrm{trace}(\tau) = 0$.

PROOF. Property (ii) follows from axiom (9). Next (by (ii)) $\tau X = \varphi\tau\varphi X \in H(M)$, i.e. τ is $H(M)$-valued. Also (again by (ii)) $\varphi(\tau Z) = -i\tau Z$ for any $Z \in T_{1,0}(M)$ hence (by (iii)) property (iv) is proved, and (v) is a corollary of (iv) and of the fact that the traces of an endomorphism and its complex linear extension coincide. □

As a consequence of Lemma 1 $\tau\xi_\alpha = A_\alpha^{\overline{\beta}}\xi_{\overline{\beta}}$, for some smooth complex valued functions $A_\alpha^{\overline{\beta}}$ on U. The connection 1-forms of the (generalized) Tanaka-Webster connection ∇^* are given by $\nabla^*\xi_A = \omega_A^B \otimes \xi_B$. Our convention as to the range of indices is $A, B, C, \cdots \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$ with $\xi^0 = \xi$. Note that $\omega_0^B = 0$ (by (7)). We shall need the following

LEMMA 2. *Let* $\{\eta^\alpha\}$ *be an admissible frame on a contact Riemannian manifold. Then*

$$d\eta^\alpha = \eta^\beta \wedge \omega_\beta^\alpha + \eta^{\overline{\beta}} \wedge \omega_{\overline{\beta}}^\alpha + \eta \wedge \tau^\alpha \tag{14}$$

where $\tau^\alpha := A_{\overline{\beta}}^\alpha \eta^{\overline{\beta}}$ *and* $A_{\overline{\beta}}^\alpha = \overline{A_\beta^{\overline{\alpha}}}$.

The proof of Lemma 2 follows from the identity

$$2(d\eta^\alpha)(X, Y) = (\nabla_X^*\eta^\alpha)Y - (\nabla_Y^*\eta^\alpha)X + \eta^\alpha(T^*(X, Y))\,.$$

Let Q_{BC}^A be the components of the Tanno tensor (with respect to $\{\xi_A\}$). We introduce connection coefficients (of ∇^*) by setting $\omega_B^A = \Gamma_{CB}^A\eta^C$. The Tanno tensor (5) may be written

$$Q(X, Y) = \pi_H(\nabla_Y\varphi)X - \eta(X)(\nabla_Y\varphi)\xi\,.$$

On the other hand (by Corollary 6.1 in [11])

$$2g((\nabla_X\varphi)Y,Z) = g(N^{(1)}(Y,Z),\varphi X) + 2(d\eta)(\varphi Y,X)\eta(Z)+ \\ -\,2(d\eta)(\varphi Z,X)\eta(Y)\,,$$

where $N^{(1)} = [\varphi,\varphi] + 2(d\eta)\otimes\xi$. Consequently

$$2g(Q(X,Y),Z) = g(N^{(1)}(X,Z) - \eta(X)N^{(1)}(\xi,Z)+ \\ -\,\eta(Z)N^{(1)}(X,\xi)\,,\,\varphi Y)\,, \tag{15}$$

for any $X,Y,Z \in T(M)$. Using (10) and (15) one may derive (cf. also (31)-(34) in [3])

$$Q^{\gamma}_{\beta\alpha} = 0, \quad \Gamma^{\overline{\gamma}}_{\alpha\beta} = -\frac{i}{2}Q^{\overline{\gamma}}_{\beta\alpha}, \qquad \Gamma^{0}_{\alpha\beta} = 0\,, \tag{16}$$

$$\Gamma^{\gamma}_{\alpha\overline{\beta}} = 0, \quad Q^{\gamma}_{\overline{\beta}\alpha} = 0, \quad Q^{\overline{\gamma}}_{\overline{\beta}\alpha} = 0, \quad \Gamma^{0}_{\alpha\overline{\beta}} = 0\,, \tag{17}$$

$$\Gamma^{\gamma}_{\alpha 0} = 0, \quad \Gamma^{\overline{\gamma}}_{\alpha 0} = 0, \quad \Gamma^{\overline{\gamma}}_{0\beta} = 0, \quad \Gamma^{0}_{0\beta} = 0\,. \tag{18}$$

Lemma 3. *Let (M,η,g) be a contact Riemannian manifold. The Webster teorsion τ is self-adjoint, i.e. $g(\tau X,Y) = g(X,\tau Y)$ for any $X,Y \in T(M)$. Locally, if $A_{\alpha\beta} := A^{\overline{\gamma}}_{\alpha}\, g_{\beta\overline{\gamma}}$ then $A_{\alpha\beta} = A_{\beta\alpha}$.*

Proof. As $\nabla^* g = 0$ (cf. (8))

$$dg_{\alpha\overline{\beta}} = \omega^{\gamma}_{\alpha}\, g_{\gamma\overline{\beta}} + g_{\alpha\overline{\gamma}}\,\omega^{\overline{\gamma}}_{\overline{\beta}} \tag{19}$$

Differentiating in (13)

$$0 = -2i\, dg_{\alpha\overline{\beta}} \wedge \eta^{\alpha} \wedge \eta^{\overline{\beta}} - 2ig_{\alpha\overline{\beta}}(d\eta^{\alpha} \wedge \eta^{\overline{\beta}} - \eta^{\alpha} \wedge d\eta^{\overline{\beta}})$$

and substituting $dg_{\alpha\overline{\beta}}$ from (19), respectively $d\eta^{\alpha}$, $d\eta^{\overline{\beta}}$ from (14) (and its complex conjugate) we have

$$\begin{aligned} 0 = (\omega^{\gamma}_{\alpha}\, g_{\gamma\overline{\beta}} + g_{\alpha\overline{\gamma}}\,\omega^{\overline{\gamma}}_{\overline{\beta}}) \wedge \eta^{\alpha} \wedge \eta^{\overline{\beta}}+ \\ + g_{\alpha\overline{\beta}}\{(\eta^{\rho} \wedge \omega^{\alpha}_{\rho} + \eta^{\overline{\rho}} \wedge \omega^{\alpha}_{\overline{\rho}} + \eta \wedge \tau^{\alpha}) \wedge \eta^{\overline{\beta}}+ \\ -\,\eta^{\alpha} \wedge (\eta^{\overline{\rho}} \wedge \omega^{\overline{\beta}}_{\overline{\rho}} + \eta^{\rho} \wedge \omega^{\overline{\beta}}_{\rho} + \eta \wedge \tau^{\overline{\beta}})\} \end{aligned}$$

or (by $\Gamma^{\alpha}_{0\overline{\rho}} = 0$)

$$0 = \eta \wedge \{g_{\alpha\overline{\beta}}\, A^{\alpha}_{\overline{\gamma}}\, \eta^{\overline{\gamma}} \wedge \eta^{\overline{\beta}} + g_{\alpha\overline{\beta}}\, A^{\overline{\beta}}_{\gamma}\, \eta^{\alpha} \wedge \eta^{\gamma}\} + \\ + g_{\alpha\overline{\beta}}\, \Gamma^{\overline{\beta}}_{\mu\rho}\, \eta^{\alpha} \wedge \eta^{\mu} \wedge \eta^{\rho} + g_{\alpha\overline{\beta}}\, \Gamma^{\alpha}_{\overline{\mu}\,\overline{\rho}}\, \eta^{\overline{\mu}} \wedge \eta^{\overline{\beta}} \wedge \eta^{\overline{\rho}}\,.$$

Therefore (by looking at types)

$$g_{\alpha\overline{\beta}}\, A^{\overline{\beta}}_{\gamma}\, \eta \wedge \eta^{\alpha} \wedge \eta^{\gamma} = 0\,. \tag{20}$$

The identity (20) leads to $g_{\alpha\overline{\beta}}\, A^{\overline{\beta}}_{\gamma} = g_{\gamma\overline{\beta}}\, A^{\overline{\beta}}_{\alpha}$. ∎

Remark 1. Note that, for any contact Riemannian structure, $\tau = -\varphi \circ h$ as a straightforward consequence of (6), thus providing an alternative proof of Lemmas 1 and 3 (cf. also Lemma 6.2 in [11]).

2.2 – Geometric interpretation of τ

Given a Riemannian manifold (M, g) and a smooth distribution $\mathcal{D}$ on M, we say $\mathcal{D}$ is *minimal* in (M, g) if $\mathrm{trace}_g\, B(\mathcal{D}) = 0$ with $B(\mathcal{D})$ given by

$$B(\mathcal{D})(X, Y) = \pi^{\perp}\, \nabla_X Y, \quad X, Y \in \mathcal{D}\,,$$

where ∇ is the Levi-Civita connection and $\pi^{\perp}$ the natural projection with respect to $T(M) = \mathcal{D} \oplus \mathcal{D}^{\perp}$ (orthogonal decomposition). If M and N are almost CR manifolds, a *CR map* is a smooth map $f : M \to N$ such that $(d_x f) T_{1,0}(M)_x \subseteq T_{1,0}(N)_{f(x)}$, for any $x \in M$. A *CR automorphism* of M is a C^{∞} diffeomorphism (of M in itself) and a CR map. A vector field $X \in \mathcal{X}(M)$ is an *infinitesimal CR automorphism* if the local 1-parameter group of X consists of (local) CR automorphisms. We shall prove the following

Theorem 1. *Let (M, η) be a contact manifold. Then for any associated Riemannian metric $g \in \mathcal{M}(\eta)$ the contact distribution $H(M) =$* Ker(η) *is minimal in (M, g). Moreover, for any fixed associated metric $g \in \mathcal{M}(\eta)$, the characteristic direction ξ of (M, η) is an infinitesimal CR automorphism of the underlying almost CR manifold if and only if the Webster torsion vanishes or, equivalently, g is a K-contact metric.*

Proof. Let $B := B(H(M))$. Then $B(X,Y)$ is the ξ-component of $\nabla_X Y$, for any $X, Y \in H(M)$. Now since $\nabla_\xi \xi = 0$ and ξ has vanishing divergence with respect to any associated metric, $\mathrm{trace}_g(B) = 0$. To prove the second statement in Theorem 1 we recall

$$\mathcal{L}_X = X \rfloor d + d\, X \rfloor$$

for any vector field X on M. By (14)

$$\begin{aligned}\mathcal{L}_\xi \eta^\alpha &= \xi \rfloor d\eta^\alpha + d(\xi \rfloor \eta^\alpha) = \xi \rfloor (\eta^\beta \wedge \omega^\alpha_\beta + \eta^{\overline{\beta}} \wedge \omega^\alpha_{\overline{\beta}} + \eta \wedge \tau^\alpha) = \\ &= -\frac{1}{2}\, \omega^\alpha_\beta(\xi)\, \eta^\beta - \frac{1}{2}\, \omega^\alpha_{\overline{\beta}}(\xi)\, \eta^{\overline{\beta}} + \frac{1}{2}\, \tau^\alpha\end{aligned}$$

so that (by $\omega^\alpha_{\overline{\beta}}(\xi) = \Gamma^\alpha_{0\overline{\beta}} = 0$) we get $\mathcal{L}_\xi \eta^\alpha \equiv \frac{1}{2}\, \tau^\alpha$, $\mod \eta^\beta$. Therefore $\mathcal{L}_\xi \eta^\alpha \equiv 0$, $\mod \eta^\beta$, if and only if $\tau \xi_\alpha = A^{\overline{\beta}}_\alpha \xi_{\overline{\beta}} = 0$ (i.e. $\tau = 0$). Also $\mathcal{L}_\xi \eta = \xi \rfloor d\eta + d\, \xi \rfloor \eta = 0$ hence the proof of Theorem 1 follows from

Lemma 4. *A tangent vector field X on a contact manifold (M, η) is an infinitesimal CR automorphism if and only if $\mathcal{L}_X \eta \equiv 0$, $\mod \eta$ and $\mathcal{L}_X \eta^\alpha \equiv 0$, $\mod \eta, \eta^\beta$, for any admissible (local) frame $\{\eta^\alpha\}$ of $T_{1,0}(M)^*$.*

The proof of Lemma 4 follows easily from Proposition 3.2 in [27], vol. I, p. 29.

For further use, note that the formula (6) may be also written

$$\nabla = \nabla^* - (\Omega + A) \otimes \xi + \tau \otimes \eta - 2\eta \odot \varphi \tag{21}$$

where $A(X,Y) = g(\tau X, Y)$ and $\odot$ is the symmetric tensor product.

2.3 – The tangent sphere bundle

As an example of contact Riemannian manifold with (generally) non-integrable almost CR structure we recall the *tangent sphere bundle* of a Riemannian manifold. Let (M^n, G) be a Riemannian manifold and $U(M^n)_x = \{v \in T_x(M^n) : G_x(v,v) = 1\}$, $x \in M^n$. The total space $U(M^n)$ of the corresponding sphere bundle $S^{n-1} \to U(M^n) \xrightarrow{\pi} M^n$ is a real hypersurface in the almost complex manifold $(T(M^n), J)$, where

$J\beta X = \gamma X$, $J\gamma X = -\beta X$, $X \in \mathcal{X}(M^n)$, is the standard almost complex structure on $T(M^n)$. Here $\beta_v : T_x(M^n) \to T_v(T(M^n))$ and $\gamma_v : T_x(M^n) \to T_v(T(M^n))$, $v \in T_x(M)$, are respectively the horizontal and vertical lifts with respect to the Levi-Civita connection of (M^n, G). Thus $U(M^n)$ carries the almost CR structure

$$\mathcal{H} = [T(U(M^n)) \otimes \mathbf{C}] \cap T^{1,0}(T(M^n))\,,$$

where $T^{1,0}(T(M^n)) = \{Y - iJY : Y \in T(T(M^n))\}$. Although J is rarely integrable (in fact only when (M^n, G) is locally Euclidean, cf. [16]) $\mathcal{H}$ turns out to be a CR structure in a number of geometrically interesting situations. If $n \geq 3$ then $\mathcal{H}$ is integrable if and only if (M^n, G) is a real space form (cf. Proposition 4.1 in [40], p. 540). $U(M^n)$ carries the contact form (locally) given by $\eta = \frac{1}{2} g_{ij} y^i dx^j$. Here (x^i) is a local coordinate system on M^n and (x^i, y^i) are the induced local coordinates on $T(M^n)$. Let $\tilde{g}$ be the *Sasaki metric* on $T(M^n)$, i.e.

$$\tilde{g}\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) = g_{ij}, \quad \tilde{g}\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = g_{ij}\,,$$

where $\delta/\delta x^i = \partial/\partial x^i - \Gamma^k_{ij}(x) y^j \partial/\partial y^k$ is the horizontal lift of $\partial/\partial x^i$ determined by Γ^i_{jk} [and Γ^i_{jk} are the Christoffel symbols of (M^n, G)]. The metric $g = \frac{1}{4} j^* \tilde{g}$ [where $j : U(M^n) \subset T(M^n)$] is an associated metric [i.e. $g \in \mathcal{M}(\eta)$] and the corresponding field of $(1,1)$-endomorphisms φ is given by $\varphi = tan \circ J$, where $tan_v : T_v(T(M^n)) \to T_v(U(M^n))$ is the projection associated to $T_v(T(M^n)) = T_v(U(M^n)) \oplus E(j)_v$, $v \in U(M^n)$, and $E(j) \to U(M^n)$ is the normal bundle of j. Set $\xi = 2y^i \delta/\delta x^i$. Then (φ, ξ, η, g) is a contact Riemannian structure on $U(M^n)$ (cf. e.g. [11]). g is a K-contact metric if and only if (M^n, G) has constant sectional curvature 1 (cf. [43]) and, if this is the case then g is actually a Sasakian metric; also $(U(M^n), \eta)$ is pseudo-Einstein (cf. our Section 5 for definitions) of positive pseudohermitian scalar curvature (cf. [2]).

3 – CR-pluriharmonic functions

A complex-valued q-form ω on a contact manifold (M, η) is *of type* $(0, q)$ (or a $(0, q)$-*form*) if $T_{1,0}(M) \rfloor \omega = 0$ and $\xi \rfloor \omega = 0$. Let $\Lambda^{0,q}(M) \to$

M be the bundle of all $(0,q)$-forms on M and $\Omega^{0,q}(M) = \Gamma^\infty(\Lambda^{0,q}(M))$. Associated to the almost CR structure, there is a natural differential operator

$$\overline{\partial}_H : \Omega^{0,q}(M) \to \Omega^{0,q+1}(M), \;\; q \geq 0\,,$$

(the *tangential Cauchy-Riemann operator*) so that the eqs. (2) may be written $\overline{\partial}_H u = 0$. If ω is a $(0,q)$-form then $\overline{\partial}_H \omega$ is the unique $(0, q+1)$-form coinciding with $d\omega$ on $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($q+1$ terms). Therefore, on functions $(\overline{\partial}_H f)\overline{Z} = \overline{Z}(f)$, for any $Z \in T_{1,0}(M)$. The sequence of $C^\infty(M)$-modules and differential operators

$$\cdots \to \Omega^{0,q-1}(M) \xrightarrow{\overline{\partial}_H} \Omega^{0,q}(M) \xrightarrow{\overline{\partial}_H} \Omega^{0,q+1}(M) \to \cdots \tag{22}$$

is only a *pseudocomplex* (in the sense of [46]) and $\overline{\partial}_H^2 = 0$ precisely when the almost CR structure of (M, η) is integrable (cf. [3]). Of course, one may associate with (22) a *twisted cohomology*

$$H^{0,q}_{\overline{\partial}_H}(M) = \frac{\mathrm{Ker}(\overline{\partial}_H : \Omega^{0,q}(M) \to \Omega^{0,q+1}(M))}{[\overline{\partial}_H \Omega^{0,q-1}(M)] \cap [\mathrm{Ker}(\overline{\partial}_H : \Omega^{0,q}(M) \to \cdot)]}\,,$$

(cf. also Theorem 4 in [3]) yet its study presents a number of difficulties. For instance the natural filtration of the de Rham complex

$$F^p\Omega^k(M) = \{\omega \in \Omega^k(M) : \omega(W_1, \cdots, W_{p-1}, \overline{V}_1, \cdots, \overline{V}_{k-p+1}) = 0\,,$$
$$\text{for any } W_i \in T(M) \otimes \mathbf{C} \text{ and } V_j \in T_{1,0}(M), 1 \leq i \leq p-1, 1 \leq j \leq k-p+1\}$$

is not stable under exterior differentiation, the problem of devising a contact Riemannian analogue of the *Frölicher spectral sequence* (cf. [38]) being open. Also, given a Riemannian manifold (M^n, G), the problem of computing the twisted cohomology groups $H^{0,q}_{\overline{\partial}_H}(U(M^n))$ is currently unsolved [even for the integrable case, e.g. compute $H^{0,1}_{\overline{\partial}_H}(U(S^2))$, the Kohn-Rossi cohomology of $U(S^2)$].

A smooth real valued function u on (M, η) is *CR-pluriharmonic* if for any $x \in M$ there is an open neighborhood $U \subset M$ and a real-valued function $v \in C^\infty(U)$ so that $u + iv$ is a CR function, i.e. $\overline{\partial}_H(u + iv) = 0$. Given a simply connected smooth domain $\Omega \subset \mathbf{C}^{n+1}$ and a holomorphic function F defined on an open neighborhood of the closure of Ω, the trace f of F on $M = \partial\Omega$ is a CR function ($\overline{\partial}_H f = 0$). As pluriharmonic functions on Ω are real parts of holomorphic functions, one may think of CR-pluriharmonic functions as boundary values of pluriharmonic functions. One of the antique problems in analysis (going back to H. Poincaré, T. Levi-Civita, and E. E. Levi, [34]) is to characterize boundary values of pluriharmonic functions in terms of tangential differential operators. L. Amoroso, [1], was the first to solve(1) the problem for $n = 1$, followed by work for arbitrary n and investigating ramifications of the phenomenon (cf. e.g. [5], [7]). Real parts of CR-functions on CR submanifolds of $\mathbf{C}^{n+1}$ were first studied in [6] and a characterization of CR-pluriharmonic functions in the abstract CR setting (employing the tools of pseudohermitian geometry) was obtained in [29]. Our purpose in the present paper is to characterize CR-pluriharmonic functions on contact Riemannian manifolds, very much in the spirit of [29].

Let (M, η) be a contact manifold. A complex-valued p-form ω on M is a $(p, 0)$-*form* if $T_{0,1}(M) \rfloor \omega = 0$. Unlike $(0, q)$-forms (which are locally sums of monomials of the form $f_{\alpha_1 \cdots \alpha_q} \eta^{\overline{\alpha}_1} \wedge \cdots \wedge \eta^{\overline{\alpha}_q}$) the exterior monomials entering the local manifestation of a $(p, 0)$-form may contain η [hence the top degree $(p, 0)$-forms are $\Lambda^{n+1,0}(M)$]. We define a differential operator

$$\partial_H : C^\infty(M) \to \Omega^{1,0}(M)\,,$$

by declaring $\partial_H f$ to be the unique $(1, 0)$-form on M such that $\xi \rfloor \partial_H f = 0$. For further use, set $d^c_H = i(\overline{\partial}_H - \partial_H)$. We shall need the following result of [29]

Lemma 5. *Let (M, η) be a contact manifold and $u \in C^\infty(M)$ a real-valued function. Then u is CR-pluriharmonic if and only if for any $x \in M$ there is an open neighborhood $U \subseteq M$ of x and a real-valued function $\lambda \in C^\infty(U)$ so that $d^c_H u + \lambda\eta$ is a closed 1-form. Moreover, there is a (globally defined) real-valued function $v \in C^\infty(M)$ such that $\overline{\partial}_H(u + iv) = 0$ if and only if $d^c_H u + \lambda\eta$ is exact, for some (real-valued) $\lambda \in C^\infty(M)$.*

(1) Although later F. Severi has shown (cf. [35]) that the conditions found by L. Amoroso were overdetermined, the work of L. Amoroso (cf. *op. cit.*) remains of great importance and, according to G. Fichera, [21], insufficient credit is given to L. Amoroso in the existing literature on functions of several complex variables.

Let us check that the arguments in [29] carry over to the case of a contact manifold with a possibly non integrable almost CR structure. For the sake of simplicity, we denote by $\mathcal{P}$ and CR^∞ respectively the sheaves of CR-pluriharmonic and CR functions on M. Assume that $u \in \mathcal{P}(M)$, i.e. in a neighborhood U of each point of M we may consider a function $v \in C^\infty(U)$ so that $u + iv \in CR^\infty(U)$ i.e.

$$i\, u_{\overline{\alpha}}\, \eta^{\overline{\alpha}} - v_{\overline{\alpha}}\, \eta^{\overline{\alpha}} = 0$$

(throughout we set $f_A = \xi_A(f)$, for any $f \in C^\infty(U)$), an identity which summed up with its complex conjugate gives $d_H^c u + v_0 \eta = dv$. Viceversa, if $d_H^c u + \lambda\eta$ is closed then (by the Poincaré lemma) $d_H^c u + \lambda\eta = dv$, for some $v \in C^\infty(V)$ and some open set $V \subseteq U$, hence (by looking at the $(0,1)$-components) $i\, u_{\overline{\alpha}} = v_{\overline{\alpha}}$, i.e. $u + iv \in CR^\infty(V)$. ∎

Lemma 6. *Let (M, η) be a contact manifold of dimension ≥ 5 (that is $n \geq 2$) and $\Xi \in \Omega^2(M)$ a closed 2-form ($d\,\Xi = 0$). If $\Xi|_{H(M)\otimes H(M)} = 0$ then $\Xi = 0$.*

The proof is similar to that of Lemma 3.2 in [29], p. 167. We shall prove the following

Theorem 2. *Let (M, η) be a contact manifold of CR dimension $n \geq 2$. Let $u \in C^\infty(M)$ be a real-valued function and $g \in \mathcal{M}(\eta)$ an associated Riemannian metric. Then u is CR-pluriharmonic if and only if for any (local) frame $\{\xi_\alpha\}$ of (the almost CR structure) $T_{1,0}(M)$ on $U \subseteq M$ there is a complex-valued function $\mu \in C^\infty(U)$ such that*

$$u_{\alpha\overline{\beta}} = \mu\, g_{\alpha\overline{\beta}} \tag{23}$$

and

$$(Q^{\overline{\gamma}}_{\alpha\beta} - Q^{\overline{\gamma}}_{\beta\alpha})u_{\overline{\gamma}} = 0\,. \tag{24}$$

If this is the case [i.e. $u \in \mathcal{P}(M)$] then

$$u_{\alpha 0} = A^{\overline{\beta}}_{\alpha}\, u_{\overline{\beta}} + \Big(i\mu - \frac{1}{2}\, u_0\Big)_{\alpha}\,. \tag{25}$$

Here $u_{AB} = (\nabla^2 u)(\xi_A, \xi_B)$ and $(\nabla^2 u)(X, Y) := (\nabla^*_X du)Y$, for $X, Y \in T(M)$.

PROOF. Assume that $u \in \mathcal{P}(M)$. By Lemma 5 we may consider, in the neighborhood U of each point of M, a real valued function $\lambda \in C^\infty(U)$ such that $d_H^c u + \lambda\eta$ is closed, i.e.

$$\begin{aligned}0 &= d(iu_{\overline{\alpha}}\,\eta^{\overline{\alpha}} - iu_\alpha\,\eta^\alpha + \lambda\,\eta) = \\ &= idu_{\overline{\alpha}} \wedge \eta^{\overline{\alpha}} + iu_{\overline{\alpha}}\,d\eta^{\overline{\alpha}} - idu_\alpha \wedge \eta^\alpha - iu_\alpha\,d\eta^\alpha + d\lambda \wedge \eta + \lambda\,d\eta\,.\end{aligned}$$

Let us substitute from (13)-(14)

$$\begin{aligned}&i\,\xi_\beta(u_{\overline{\alpha}})\,\eta^\beta \wedge \eta^{\overline{\alpha}} + i\,\xi_{\overline{\beta}}(u_{\overline{\alpha}})\,\eta^{\overline{\beta}} \wedge \eta^{\overline{\alpha}} + i\,\xi(u_{\overline{\alpha}})\,\eta \wedge \eta^{\overline{\alpha}} + \\ &\quad + iu_{\overline{\alpha}}(\eta^{\overline{\beta}} \wedge \omega_{\overline{\beta}}^{\overline{\alpha}} + \eta^\beta \wedge \omega_\beta^{\overline{\alpha}} + \eta \wedge \tau^{\overline{\alpha}}) + \\ &\quad - i\,\xi_\beta(u_\alpha)\,\eta^\beta \wedge \eta^\alpha - i\,\xi_{\overline{\beta}}(u_\alpha)\,\eta^{\overline{\beta}} \wedge \eta^\alpha - i\,\xi(u_\alpha)\,\eta \wedge \eta^\alpha + \\ &\quad - i\,u_\alpha(\eta^\beta \wedge \omega_\beta^\alpha + \eta^{\overline{\beta}} \wedge \omega_{\overline{\beta}}^\alpha + \eta \wedge \tau^\alpha) + \\ &\quad + \lambda_\alpha\,\eta^\alpha \wedge \eta + \lambda_{\overline{\alpha}}\,\eta^{\overline{\alpha}} \wedge \eta - 2\,i\,\lambda\,g_{\alpha\overline{\beta}}\,\eta^\alpha \wedge \eta^{\overline{\beta}} = 0\,.\end{aligned} \tag{26}$$

The $\eta^\alpha \wedge \eta^{\overline{\beta}}$-component in (26) is

$$\{\xi_\alpha(u_{\overline{\beta}}) - u_{\overline{\gamma}}\,\Gamma_{\alpha\overline{\beta}}^{\overline{\gamma}} + \xi_{\overline{\beta}}(u_\alpha) - u_\gamma\,\Gamma_{\overline{\beta}\alpha}^\gamma - 2\,\lambda\,g_{\alpha\overline{\beta}}\}\eta^\alpha \wedge \eta^{\overline{\beta}} = 0\,. \tag{27}$$

By (16)-(18) $u_{\overline{\beta}\alpha} = \xi_\alpha(u_{\overline{\beta}}) - \Gamma_{\alpha\overline{\beta}}^{\overline{\gamma}}\,u_{\overline{\gamma}}$ hence (27) may be written

$$u_{\overline{\beta}\alpha} + u_{\alpha\overline{\beta}} - 2\lambda\,g_{\alpha\overline{\beta}} = 0\,. \tag{28}$$

At this point, using the identity

$$(\nabla_X^* du)Y = (\nabla_Y^* du)X - T^*(X,Y)(u), \quad X, Y \in T(M)\,,$$

one may derive the (second order) commutation formulae

$$\begin{cases} u_{\alpha 0} = u_{0\alpha} - A_\alpha^{\overline{\beta}}\,u_{\overline{\beta}} \\ u_{\alpha\beta} = u_{\beta\alpha} \\ u_{\overline{\beta}\alpha} = u_{\alpha\overline{\beta}} + 2\,i\,g_{\alpha\overline{\beta}}\,u_0 \end{cases} \tag{29}$$

where $u_0 = \xi(u)$. Now substitution from (29) into (28) yields

$$u_{\alpha\overline{\beta}} + (iu_0 - \lambda)g_{\alpha\overline{\beta}} = 0\,,$$

which is (23) with $\mu := \lambda - i\, u_0$. The $\eta^\alpha \wedge \eta^\beta$-component in (26) is

$$\Big\{\xi_\beta(u_\alpha) - u_\gamma\, \Gamma^\gamma_{\beta\alpha} - \frac{i}{2}\, u_{\overline{\gamma}}\, Q^{\overline{\gamma}}_{\alpha\beta}\Big\}\, \eta^\alpha \wedge \eta^\beta = 0$$

and substitution from $u_{\alpha\beta} = \xi_\beta(u_\alpha) - \Gamma^\gamma_{\beta\alpha}\, u_\gamma + \frac{i}{2}\, Q^{\overline{\gamma}}_{\alpha\beta}\, u_{\overline{\gamma}}$ leads to

$$Q^{\overline{\gamma}}_{\alpha\beta}\, u_{\overline{\gamma}}\, \eta^\alpha \wedge \eta^\beta = 0$$

which is (24). Finally, the $\eta \wedge \eta^\alpha$-component in (26) is

$$\{i\, u_{\overline{\gamma}}\, \Gamma^{\overline{\gamma}}_{0\alpha} - i\, u_{\overline{\gamma}}\, A^{\overline{\gamma}}_\alpha + i\, \xi(u_\alpha) - i\, u_\gamma\, \Gamma^\gamma_{0\alpha} + \lambda_\alpha\}\eta^\alpha \wedge \eta = 0$$

and substitution from $u_{\alpha 0} = \xi(u_\alpha) - \Gamma^\gamma_{0\alpha}\, u_\gamma$ gives

$$u_{\alpha 0} = i\, \lambda_\alpha + A^{\overline{\beta}}_\alpha\, u_{\overline{\beta}}$$

which is (25).

Conversely, let us assume that u satisfies (23)-(24) for some complex-valued function $\mu \in C^\infty(U)$.

Claim. The function $\lambda \in C^\infty(U)$ given by $\lambda := \mu + i\, u_0$ is real-valued.

Summing (23) to its complex conjugate $u_{\overline{\beta}\alpha} = (\overline{\lambda} + i\, u_0) g_{\overline{\beta}\alpha}$ gives (by the commutation formulae (29)) $\lambda = \overline{\lambda}$. The claim is proved.

Let us differentiate $u_\alpha \eta^\alpha$, substitute from (13)-(14), and replace the ordinary derivatives in terms of covariant derivatives (with respect to ∇^*). We obtain the identity

$$\begin{aligned} d(u_\alpha \eta^\alpha) = {} & \frac{i}{2}(Q^{\overline{\gamma}}_{\alpha\beta}\, u_{\overline{\gamma}}\, \eta^\alpha \wedge \eta^\beta + Q^\gamma_{\overline{\alpha}\,\overline{\beta}}\, u_\gamma\, \eta^{\overline{\alpha}} \wedge \eta^{\overline{\beta}}) + \\ & - u_{\alpha\overline{\beta}}\, \eta^\alpha \wedge \eta^{\overline{\beta}} + \eta \wedge (u_{\alpha 0}\, \eta^\alpha + u_\alpha\, a^\alpha_{\overline{\beta}}\, \eta^{\overline{\beta}}) \end{aligned} \tag{30}$$

for any real-valued $u \in C^\infty(M)$. Next, for u satisfying (23)-(24) (by (30))

$$d(u_\alpha \eta^\alpha) = \eta \wedge (u_{\alpha 0}\, \eta^\alpha + u_\alpha\, A^\alpha_{\overline{\beta}}\, \eta^{\overline{\beta}}) - (\lambda - i\, u_0)\, g_{\alpha\overline{\beta}}\, \eta^\alpha \wedge \eta^{\overline{\beta}}\, . \tag{31}$$

Then

$$\begin{aligned}
d(d^c_H u + \lambda\eta) &= d(i\, u_{\overline{\alpha}}\, \eta^{\overline{\alpha}} - i\, u_\alpha\, \eta^\alpha + \lambda\eta) = \text{ (by (31))} \\
&= i\{\eta \wedge (u_{\overline{\alpha}0}\, \eta^{\overline{\alpha}} + u_{\overline{\alpha}}\, A^{\overline{\alpha}}_\beta\, \eta^\beta) - (\lambda + i\, u_0) g_{\overline{\alpha}\beta}\, \eta^{\overline{\alpha}} \wedge \eta^\beta\} + \\
&\quad - i\{\eta \wedge (u_{\alpha 0}\, \eta^\alpha + u_\alpha\, A^\alpha_{\overline{\beta}}\, \eta^{\overline{\beta}}) - (\lambda - i\, u_0)\, g_{\alpha\overline{\beta}}\, \eta^\alpha \wedge \eta^{\overline{\beta}}\} + \\
&\quad + d\lambda \wedge \eta - 2i\lambda\, g_{\alpha\overline{\beta}}\, \eta^\alpha \wedge \eta^{\overline{\beta}}
\end{aligned}$$

i.e.

$$d(d^c_H u + \lambda\eta) = \eta \wedge \phi$$

where

$$\phi := i\, u_{\overline{\alpha}0}\, \eta^{\overline{\alpha}} + i\, u_{\overline{\alpha}}\, A^{\overline{\alpha}}_\beta\, \eta^\beta - i\, u_{\alpha 0}\, \eta^\alpha - i\, u_\alpha\, A^\alpha_{\overline{\beta}}\, \eta^{\overline{\beta}} - \lambda_\alpha\, \eta^\alpha - \lambda_{\overline{\alpha}}\, \eta^{\overline{\alpha}}\,.$$

Applying Lemma 6 to the 2-form $\Xi = d(d^c_H u + \lambda\eta)$ gives $\Xi = 0$ hence (by Lemma 5) $u \in \mathcal{P}(M)$. ∎

For a given Riemannian manifold (M^n, G), no examples of functions in $CR^\infty(U(M^n))$ or $\mathcal{P}(U(M^n))$ are known. Of course, for any (almost) holomorphic function $f \in C^1(T(M^n))$ i.e.

$$\frac{\delta f}{\delta x^j} + i\, \frac{\partial f}{\partial y^j} = 0, \quad 1 \le j \le n\,, \tag{32}$$

the trace of f on $U(M^n)$ is a CR function, yet the existing information on the solutions to (32) is equally scarce. We recall the commutation formula

$$\Big[\frac{\delta}{\delta x^i}\,,\, \frac{\partial}{\partial y^j}\Big] = \Gamma^k_{ij}(x)\, \frac{\partial}{\partial y^k} \tag{33}$$

If $f = u + iv$ are the real and imaginary parts of f then (32) may be written

$$\frac{\delta u}{\delta x^j} = \frac{\partial v}{\partial y^j}\,, \quad \frac{\partial u}{\partial y^j} = -\frac{\delta v}{\delta x^j}\,. \tag{34}$$

Therefore, the holomorphic functions on $T(\mathbf{R}^n)$ are precisely the ordinary holomorphic functions on $\mathbf{C}^n$ and each element of $CR^\omega(U(\mathbf{R}^n))$ is the trace of a holomorphic function. Indeed, when $\Gamma^i_{jk} = 0$, (34) are the ordinary Cauchy-Riemann equations in $\mathbf{R}^{2n}$. Moreover $U(\mathbf{R}^n)$ is the real analytic hypersurface $\sum_{i=1}^n (y^i)^2 = 1$ hence, by a theorem of G. Tomassini (cf. [44]), any real analytic CR function on $U(\mathbf{R}^n)$ extends holomorphically to a neighborhood of $U(\mathbf{R}^n)$ in $\mathbf{C}^n$. Applying $\delta/\delta x^i$ to the first equation in (34), respectively $\partial/\partial y^i$ to the second, and adding the resulting equations gives (by (33))

$$\frac{\delta^2 u}{\delta x^i \delta x^j} + \frac{\partial^2 u}{\partial y^j \partial y^i} = \Gamma^k_{ij}(x)\,\frac{\delta u}{\delta x^k}\,. \tag{35}$$

We have

Proposition 1. *Let $F = u + iv \in C^2(M^n)$. If the vertical lift $F^v := F \circ \Pi$ is (almost) holomorphic then u, v are harmonic functions. In particular, if M^n is compact then $C^2(M^n)^v \cap \mathcal{O}(T(M^n)) = \mathbf{C}$.*

Proof. If $f = F^v$, i.e. f is a function of the positional arguments x^i alone, then (35) becomes

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \Gamma^k_{ij}(x)\frac{\partial u}{\partial x^k}$$

in a neighborhood of each point of M^n. Now contraction with g^{ij} gives $\Delta u = 0$, i.e. u is harmonic. The same arguments apply to v. $\square$

Remark 2. 1) Set $z^j = x^j + iy^j$ and $\delta/\delta\overline{z}^j = \frac{1}{2}(\delta/\delta x^j + i\partial/\partial y^j)$. Under a coordinate transformation $x'^j = x'^j(x^1, \cdots, x^n)$, $\det[\partial x'^j/\partial x^k] \neq 0$ in $U \cap U'$, one has $y'^j = (\partial x'^j/\partial x^k) y^k$ hence $z'^j = z'^j(z^1, \cdots, z^n)$ are (almost) holomorphic (i.e. $\delta z'^j/\delta\overline{z}^k = 0$ in $\Pi^{-1}(U \cap U')$) if and only if $\Delta x'^j = 0$ in $U \cap U'$.

2) If $f \in \mathcal{O}(T(M^n))$ then $\partial f/\partial\overline{z}^j = \frac{1}{2}\Gamma^k_{j\ell}(x) y^\ell\, \partial f/\partial y^k$ in $\Pi^{-1}(U)$ reminiscent of I.N. Vekua's generalized analytic functions (cf. [47], [23]).

4 – Curvature theory

Let (M, η, g) be a contact Riemannian manifold. Let R, R^* be respectively the curvature tensor fields of ∇, ∇^*. We consider the tensor field S defined by

$$S(X,Y) = (\nabla^*_X \tau)Y - (\nabla^*_Y \tau)X\,.$$

A straightforward calculation based on (21) leads to

$$\begin{aligned} R(X,Y)Z = R^*(X,Y)Z + (LX \wedge LY)Z + 2\Omega(X,Y)\varphi Z+ \\ - g(S(X,Y),Z)\xi + \eta(Z)S(X,Y)+ \\ - 2g((\eta \wedge \mathcal{O})(X,Y),Z)\xi + 2\eta(Z)(\eta \wedge \mathcal{O})(X,Y) \end{aligned} \tag{36}$$

for any $X,Y,Z \in T(M)$, where

$$L = \varphi - \tau, \quad \mathcal{O} = \tau^2 - 2\varphi\,\tau - I\,,$$

and $(X \wedge Y)Z = g(X,Z)Y - g(Y,Z)X$. In particular, if $X,Y,Z \in H(M)$

$$R(X,Y)Z = R^*(X,Y)Z + (LX \wedge LY)Z + 2\Omega(X,Y)\varphi Z - g(S(X,Y),Z)\xi$$

Take the inner product with $W \in H(M)$ to obtain

$$\begin{aligned} R(W,Z,X,Y) = g(R^*(X,Y)Z,W)+ \\ + g((LX \wedge LY)Z,W) - 2\Omega(X,Y)\Omega(Z,W) \end{aligned} \tag{37}$$

for any $X,Y,Z,W \in H(M)$, where $R(W,Z,X,Y) = g(R(X,Y)Z,W)$ is the Riemann-Christoffel 4-tensor of (M,g). Exploiting the well known symmetry

$$R(W,Z,X,Y) = R(X,Y,W,Z)$$

the identity (37) furnishes

$$\begin{aligned} g(R^*(X,Y)Z,W) = g(R^*(W,Z)Y,X)+ \\ + g((LW \wedge LZ)Y,X) - g((LX \wedge LY)Z,W)\,. \end{aligned} \tag{38}$$

We adopt the following convention as to the curvature components

$$R^*(\xi_B,\xi_C)\xi_A = R_A{}^D{}_{BC}\xi_D\,.$$

Then (by (7)) $R_0{}^D{}_{BC} = 0$. Moreover

$$
\begin{aligned}
& R^*(X,Y)\xi_A = 2(d\omega_A^B - \omega_A^C \wedge \omega_C^B)(X,)\xi_B\,, \\
(39) \qquad & 2(d\omega_A^B - \omega_A^C \wedge \omega_C^B) = 2R_A{}^B{}_{\lambda\overline{\mu}}\,\eta^\lambda \wedge \eta^{\overline{\mu}} + \\
& \quad + R_A{}^B{}_{\lambda\mu}\,\eta^\lambda \wedge \eta^\mu + R_A{}^B{}_{\overline{\lambda}\overline{\mu}} + 2\eta \wedge (R_A{}^B{}_{0\overline{\mu}}\,\eta^{\overline{\mu}} - R_A{}^B{}_{\lambda 0}\,\eta^\lambda)\,.
\end{aligned}
$$

We shall prove the following

THEOREM 3. *Let (M, η, g) be a contact Riemannian manifold. Consider the 2-forms*

$$
\begin{aligned}
\Pi_\alpha^\beta &= d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta - \omega_\alpha^{\overline{\gamma}} \wedge \omega_{\overline{\gamma}}^\beta\,, \\
\Omega_\alpha{}^\beta &= \Pi_\alpha^\beta + 2i\,\eta_\alpha \wedge \tau^\beta - 2i\,\tau_\alpha \wedge \eta^\beta\,, \\
\Omega_{\overline{\alpha}}{}^\beta &= d\omega_{\overline{\alpha}}^\beta - \omega_{\overline{\alpha}}^\gamma \wedge \omega_\gamma^\beta - \omega_{\overline{\alpha}}^{\overline{\gamma}} \wedge \omega_{\overline{\gamma}}^\beta\,,
\end{aligned}
$$

where $\eta_\alpha = g_{\alpha\overline{\beta}}\eta^{\overline{\beta}}$ and $\tau_\alpha = g_{\alpha\overline{\beta}}\tau^{\overline{\beta}}$, $\tau^{\overline{\beta}} = \overline{\tau^\beta}$. Then

$$
\begin{aligned}
& \Omega_\alpha{}^\beta = R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\,\eta^\lambda \wedge \eta^{\overline{\mu}} - W_{\alpha\lambda}^\beta\,\eta^\lambda \wedge \eta + W_{\alpha\overline{\mu}}^\beta\,\eta^{\overline{\mu}} \wedge \eta + \\
(40) \qquad & \qquad - \frac{i}{4}\,g^{\beta\overline{\sigma}}\,\{g_{\overline{\rho}\lambda}\,Q_{\mu\alpha,\overline{\sigma}}^{\overline{\rho}}\,\eta^\lambda \wedge \eta^\mu + g_{\rho\overline{\lambda}}\,Q_{\overline{\mu}\,\overline{\sigma},\alpha}^\rho,\eta^{\overline{\lambda}} \wedge \eta^{\overline{\mu}}\}
\end{aligned}
$$

$$
\begin{aligned}
& \Omega_{\overline{\alpha}}{}^\beta = -W_{\overline{\alpha}\lambda}^\beta\,\eta^\lambda \wedge \eta + W_{\overline{\alpha}\,\overline{\mu}}^\beta\,\eta^{\overline{\mu}} \wedge \eta + \\
(41) \qquad & \qquad + \frac{i}{2}\{Q_{\overline{\alpha}\,\overline{\mu},\lambda}^\beta\,\eta^\lambda \wedge \eta^{\overline{\mu}} - Q_{\overline{\alpha}\,\overline{\lambda},\overline{\mu}}^\beta\eta^{\overline{\lambda}} \wedge \eta^{\overline{\mu}}\}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
W_{\alpha\lambda}^\beta &= S_{\alpha\overline{\sigma}}^{\overline{\rho}}g_{\lambda\overline{\rho}}g^{\beta\overline{\sigma}}, \quad W_{\alpha\overline{\mu}}^\beta = -S_{\alpha\overline{\sigma}}^\rho g_{\rho\overline{\mu}}g^{\beta\overline{\sigma}} \\
W_{\overline{\alpha}\lambda}^\beta &= S_{\overline{\alpha}\,\overline{\sigma}}^{\overline{\rho}}g_{\lambda\overline{\rho}}g^{\beta\overline{\sigma}}, \quad W_{\overline{\alpha}\,\overline{\mu}}^\beta = -S_{\overline{\alpha}\,\overline{\sigma}}^\rho g_{\rho\overline{\mu}}g^{\beta\overline{\sigma}}
\end{aligned}
$$

and comas denote covariant derivatives with respect to the generalized Tanaka-Webster connection.

We need the following

Lemma 7. *On any contact Riemannian manifold M the curvature of ∇^* is expressed as*

$$R_\alpha{}^\rho{}_{\lambda\mu} = 2i(A_{\alpha\lambda}\,\delta^\rho_\mu - A_{\alpha\mu}\,\delta^\rho_\lambda) - \frac{i}{2}\,g^{\rho\overline{\sigma}}g_{\overline{\beta}\lambda}\,Q^{\overline{\beta}}_{\mu\alpha,\overline{\sigma}} \tag{42}$$

$$R_\alpha{}^\rho{}_{\overline{\lambda}\,\overline{\mu}} = 2i(A^\rho_{\overline{\lambda}}\,g_{\alpha\overline{\mu}} - A^\rho_{\overline{\mu}}\,g_{\alpha\overline{\lambda}}) - \frac{i}{2}\,g^{\rho\overline{\sigma}}g_{\beta\overline{\lambda}}Q^\beta_{\overline{\mu}\,\overline{\sigma},\alpha} \tag{43}$$

$$R_\alpha{}^\beta{}_{0\overline{\mu}} = g^{\beta\overline{\sigma}}g_{\rho\overline{\mu}}S^\rho_{\alpha\overline{\sigma}} \tag{44}$$

$$R_\alpha{}^\beta{}_{\lambda 0} = g^{\beta\overline{\sigma}}g_{\lambda\overline{\rho}}S^{\overline{\rho}}_{\overline{\sigma}\alpha} \tag{45}$$

Proof. For instance, to establish (42) we set $X = \xi_\lambda$, $Y = \xi_\mu$, $Z = \xi_\alpha$ and $W = \xi_{\overline{\sigma}}$ in (38) and use

$$L\xi_\alpha = i\xi_\alpha - A^{\overline{\beta}}_\alpha\,\xi_{\overline{\beta}}$$

and Lemma 3 to obtain

$$R_\alpha{}^\beta{}_{\lambda\mu}g_{\beta\overline{\sigma}} = R_\mu{}^{\overline{\beta}}{}_{\overline{\sigma}\alpha}g_{\overline{\beta}\lambda} + 2i(A_{\alpha\lambda}g_{\mu\overline{\sigma}} - A_{\alpha\mu}g_{\lambda\overline{\sigma}})\,. \tag{46}$$

On the other hand a calculation based on (16)-(18) and the following decomposition

$$\eta([\xi_{\overline{\sigma}},\xi_\alpha]) = -2\,i\,g_{\alpha\overline{\sigma}} \tag{47}$$

$$[\xi_{\overline{\sigma}},\xi_\alpha]_{1,0} = \Gamma^\rho_{\overline{\sigma}\alpha}\,\xi_\rho \tag{48}$$

$$[\xi_{\overline{\sigma}},\xi_\alpha]_{0,1} = -\Gamma^{\overline{\rho}}_{\alpha\overline{\sigma}}\,\xi_{\overline{\rho}} \tag{49}$$

(where $V_{1,0}$ is the $T_{1,0}(M)$-component of $V \in T(M)\otimes\mathbf{C}$ with respect to $T(M)\otimes\mathbf{C} = T_{1,0}(M)\oplus T_{0,1}(M)\oplus\mathbf{C}\xi$ and $V_{0,1} = \overline{V_{1,0}}$) of commutators, leads to

$$R_\mu{}^{\overline{\beta}}{}_{\overline{\sigma}\alpha} = -\frac{i}{2}\,Q^{\overline{\beta}}_{\mu\alpha,\overline{\sigma}}$$

and then (46) yields (42). A similar approach leads to (43). The proof of (44) (respectively (45)) is a bit trickier. Let us take the inner product of (36) by $W \in T(M)$ and use the resulting identity and the symmetry of the 4-tensor R to obtain an identity of the form $g(R^*(X,Y)Z,W) = g(R^*(W,Z)Y,X) + \textit{other terms}$. Set $X = \xi$ and $Y, Z, W \in H(M)$. This leads to (as $L\xi = 0$ and $H(M)$ is ∇^*-parallel)

$$g(R^*(\xi, Y)Z, W) = g(Y, S(Z,W))\,. \tag{50}$$

Now (50) for $Y = \xi_{\overline{\mu}}$, $Z = \xi_\alpha$ and $W = \xi_{\overline{\sigma}}$ is (44) and (45) follows similarly. ∎

The substitution from (42)-(45) into (39) and the identity

$$(A_{\alpha\lambda}\,\delta^\beta_\mu - A_{\alpha\mu}\,\delta^\beta_\lambda)\eta^\lambda \wedge \eta^\mu = 2\,\tau_\alpha \wedge \eta^\beta$$

lead now to (40) in Theorem 3. The proof of (41) is similar and thus omitted.

The *Ricci curvature* of ∇^* is

$$\mathrm{Ric}^*(X,Y) := \mathrm{trace}\{V \mapsto R^*(V,Y)X\}, \quad X, Y \in T(M)\,,$$

and $R_{\alpha\overline{\beta}} = \mathrm{Ric}^*(\xi_\alpha, \xi_{\overline{\beta}})$ is the contact Riemannian analogue of the *pseudohermitian Ricci* curvature of [48] and [29]. From the very definition

$$R_{\alpha\overline{\beta}} = R_\alpha{}^\gamma{}_{\gamma\overline{\beta}} + R_\alpha{}^{\overline{\gamma}}{}_{\overline{\gamma}\overline{\beta}}\,.$$

Next, a calculation based on

$$[\xi_\lambda, \xi_\mu]_{1,0} = (\Gamma^\rho_{\lambda\mu} - \Gamma^\rho_{\mu\lambda})\xi_\rho \tag{51}$$

$$[\xi_\lambda, \xi_\mu]_{0,1} = \frac{i}{2}\,(Q^{\overline{\rho}}_{\lambda\mu} - Q^{\overline{\rho}}_{\mu\lambda})\xi_{\overline{\rho}} \tag{52}$$

leads to $R_{\overline{\alpha}}{}^\beta{}_{\lambda\mu} = 0$ and hence

$$R_{\alpha\overline{\beta}} = R_\alpha{}^\gamma{}_{\gamma\overline{\beta}}\,. \tag{53}$$

As a consequence of Theorem 3 we shall prove

Corollary 1. *Let (M, η, g) be a contact Riemannian manifold and set*

$$R_{ABCD} := g(R^*(\xi_C, \xi_D)\xi_A, \xi_B)\,,$$

for any (local) frame $\{\xi_\alpha\}$ *in* $T_{1,0}(M)$. *Then*

$$R_{\alpha\overline{\beta}\lambda\overline{\mu}} = R_{\lambda\overline{\beta}\alpha\overline{\mu}}\,, \tag{54}$$

and consequently the pseudohermitian Ricci tensor is given by

$$R_{\alpha\overline{\beta}} = R_{\gamma}{}^{\gamma}{}_{\alpha\overline{\beta}}\,. \tag{55}$$

Proof. Let us contract with $g_{\beta\overline{\gamma}}$ in (40) (respectively in (41)). We get

$$\begin{aligned}\Omega_{\alpha\overline{\gamma}} = {} & R_{\alpha\overline{\gamma}\lambda\overline{\mu}}\,\eta^\lambda\wedge\eta^{\overline{\mu}} + \lambda_{\alpha\overline{\gamma}}\wedge\eta + \\ & -\frac{i}{4}\{g_{\overline{\rho}\lambda}\,Q^{\overline{\rho}}_{\mu\alpha,\overline{\gamma}}\,\eta^\lambda\wedge\eta^\mu + g_{\rho\overline{\lambda}}\,Q^{\rho}_{\overline{\mu}\,\overline{\gamma},\alpha}\,\eta^{\overline{\lambda}}\wedge\eta^{\overline{\mu}}\}\,,\end{aligned} \tag{56}$$

$$\Omega_{\overline{\alpha}\,\overline{\gamma}} = \lambda_{\overline{\alpha}\,\overline{\gamma}}\wedge\eta + \frac{i}{2}\,g_{\beta\overline{\gamma}}\,Q^{\beta}_{\overline{\alpha}\,\overline{\mu},\lambda}\,\eta^\lambda\wedge\eta^{\overline{\mu}} + -\frac{i}{2}\,g_{\beta\overline{\gamma}}Q^{\beta}_{\overline{\alpha}\,\overline{\lambda},\overline{\mu}}\,\eta^{\overline{\lambda}}\wedge\eta^{\overline{\mu}}\,, \tag{57}$$

where

$$\begin{aligned}\lambda_{\alpha\overline{\gamma}} &= -W^{\beta}_{\alpha\lambda}g_{\beta\overline{\gamma}}\eta^\lambda + W^{\beta}_{\alpha\overline{\mu}}g_{\beta\overline{\gamma}}\eta^{\overline{\mu}}\,,\\ \lambda_{\overline{\alpha}\,\overline{\gamma}} &= -W^{\beta}_{\overline{\alpha}\lambda}g_{\beta\overline{\gamma}}\eta^\lambda + W^{\beta}_{\overline{\alpha}\,\overline{\mu}}g_{\beta\overline{\gamma}}\eta^{\overline{\mu}}\,.\end{aligned}$$

Differentiating in (14) we have

$$0 = d\eta^\beta\wedge\omega^\alpha_\beta - \eta^\beta\wedge d\omega^\alpha_\beta + d\eta^{\overline{\beta}}\wedge\omega^\alpha_{\overline{\beta}} - \eta^{\overline{\beta}}\wedge d\omega^\alpha_{\overline{\beta}} + d\eta\wedge\tau^\alpha - \eta\wedge d\tau^\alpha$$

and substituting from (13)-(14)

$$0 = \eta^\gamma\wedge(-\Pi^\alpha_\gamma - 2i\,\eta_\gamma\wedge\tau^\alpha) + \eta^{\overline{\gamma}}\wedge(-\Omega_{\overline{\gamma}}{}^{\alpha}) + \eta\wedge(\tau^\beta\wedge\omega^\alpha_\beta + \tau^{\overline{\beta}}\wedge\omega^\alpha_{\overline{\beta}} - d\tau^\alpha)$$

or, by observing that $\eta^\gamma\wedge\tau_\gamma = 0$

$$\eta^\gamma\wedge\Omega_\gamma{}^\alpha + \eta^{\overline{\gamma}}\wedge\Omega_{\overline{\gamma}}{}^\alpha + \eta\wedge\Omega^\alpha = 0 \tag{58}$$

where

$$\Omega^\alpha = d\tau^\alpha - \tau^\beta\wedge\omega^\alpha_\beta - \tau^{\overline{\beta}}\wedge\omega^\alpha_{\overline{\beta}}\,.$$

Let us contract with $g_{\alpha\overline{\beta}}$ in (58) and subsequently substitute from (56)-(57). The $\eta^\alpha\wedge\eta^\beta\wedge\eta^{\overline{\gamma}}$-component of the resulting identity is

$$R_{\alpha\overline{\gamma}\lambda\overline{\mu}}\,\eta^\alpha\wedge\eta^\lambda\wedge\eta^{\overline{\mu}} = 0\,,$$

which yields the first statement in Corollary 1. The second part is a consequence of the first and of (53). □

4.1 – Pseudo-Einstein contact forms

Let M be a nondegenerate CR manifold, of CR dimension n. A contact form η on M is *pseudo-Einstein* if the pseudohermitian Ricci tensor (of the Tanaka-Webster connection) of (M, η) is proportional to the Levi form. That is $R_{\alpha\overline{\beta}} = (\rho^*/n) g_{\alpha\overline{\beta}}$, where $\rho^* = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ is the *pseudohermitian scalar curvature* (cf. [29]). A CR manifold admitting a (globally defined) pseudo-Einstein contact form is a *pseudo-Einstein manifold.* Odd-dimensional spheres (with the standard Sasakian structure) and unit tangent bundles over real space forms (of sectional curvature 1, cf. [2]) are pseudo-Einstein manifolds. Also, the quotient of the Heisenberg group by the (discrete) group of dilations is a strictly pseudoconvex CR manifold admitting a pseudo-Einstein contact form (cf. [19]) with nonvanishing Webster torsion ($\tau = 0$ in the previous examples). The local existence of pseudo-Einstein contact forms on a nondegenerate CR manifold is related to the existence of closed sections in $K^0(M) := K(M) \setminus \{zero \;\; section\}$, where $K(M) = \Lambda^{n+1,0}(M)$ is the *canonical bundle*, and therefore to the local embedding problem for CR structures (cf. [29]).

Let (M, η) be a contact manifold and $\mathcal{Z} \in \Gamma^0(K^0(M))$. We say η is *volume normalized* with respect to $\mathcal{Z}$ if

$$2^n \, i^{n(n+2)} n! \, \eta \wedge (\xi \rfloor \mathcal{Z}) \wedge (\xi \rfloor \overline{\mathcal{Z}}) = \Psi \,, \tag{59}$$

where $\Psi = \eta \wedge (d\eta)^n$. In CR geometry, if such sections $\mathcal{Z}$ exist and $d\mathcal{Z} = 0$ then η is pseudo-Einstein (cf. [29]). Opposite to the integrable case, on a contact manifold with a non integrable almost CR structure there exist none. Precisely, we shall prove

THEOREM 4. *Any contact manifold* (M, η) *admitting a smooth, globally defined, nowhere vanishing, closed section in the canonical bundle has an integrable almost CR structure* (*and* η *is pseudo-Einstein*).

PROOF. Let $\mathcal{Z} \in \Gamma^\infty(K^0(M))$. Locally, with respect to an orthonormal frame $\{\xi_\alpha\}$ in $T_{1,0}(M)$ on U

$$\mathcal{Z} = f \, \eta \wedge \eta^1 \wedge \cdots \wedge \eta^n \,,$$

for some smooth function $f : U \to \mathbf{C} \setminus \{0\}$. As

$$\xi \rfloor \mathcal{Z} = \frac{f}{n+1} \, \eta^1 \wedge \cdots \wedge \eta^n \,,$$

substitution in (59) together with the identity (46) in [3] (with $\det(g_{\alpha\overline{\beta}}) = 1$)

$$\Psi = 2^n i^{n(n+2)} n!\, \eta \wedge \eta^1 \wedge \cdots \wedge \eta^n \wedge \eta^{\overline{1}} \wedge \cdots \wedge \eta^{\overline{n}}$$

lead to $|f| = n + 1$. Set

$$\hat{\xi}_\alpha = U_\alpha^\beta\, \xi_\beta, \quad [U_\beta^\alpha] = \operatorname{diag}(f, 1, \cdots, 1),$$

and let $\{\hat{\eta}^\alpha\}$ be the corresponding admissible coframe. Then

$$\mathcal{Z} = \eta \wedge \hat{\eta}^1 \wedge \cdots \wedge \hat{\eta}^n$$

and exterior differentiation gives

$$\begin{aligned} d\mathcal{Z} = {} & (d\eta) \wedge \hat{\eta}^1 \wedge \cdots \wedge \hat{\eta}^n + \\ & + \sum_{\alpha=1}^{n} (-1)^\alpha\, \eta \wedge \hat{\eta}^1 \wedge \cdots \wedge (d\hat{\eta}^\alpha) \wedge \cdots \wedge \hat{\eta}^n = \quad (by \;\; (13)\text{-}(14)) \\ = {} & \sum_{\alpha=1}^{n} (-1)^\alpha\, \eta \wedge \hat{\eta}^1 \wedge \cdots \wedge (\hat{\eta}^\beta \wedge \hat{\omega}_\beta^\alpha + \hat{\eta}^{\overline{\beta}} \wedge \hat{\omega}_{\overline{\beta}}^\alpha + \eta \wedge \hat{\tau}^\alpha) \wedge \cdots \wedge \hat{\eta}^n \end{aligned}$$

hence (by (16)-(18))

$$\begin{aligned} d\mathcal{Z} = {} & -\hat{\omega}_\alpha^\alpha \wedge \mathcal{Z} + \\ & + \frac{i}{2} \hat{Q}_{\overline{\beta}\,\overline{\mu}}^\alpha \hat{\eta}^{\overline{\beta}} \wedge \hat{\eta}^{\overline{\mu}} \sum_{\alpha=1}^{n} (-1)^\alpha \eta \wedge \hat{\eta}^1 \wedge \cdots \wedge \hat{\eta}^{\alpha-1} \wedge \hat{\eta}^{\alpha+1} \wedge \cdots \wedge \hat{\eta}^n . \end{aligned} \tag{60}$$

If $d\mathcal{Z} = 0$, the $(n, 2)$ component in (60) is (note that the first term is of type $(n + 1, 1)$)

$$\hat{Q}_{\overline{\beta}\,\overline{\mu}}^\alpha\, \hat{\eta}^{\overline{\beta}} \wedge \hat{\eta}^{\overline{\mu}} \sum_\alpha \eta \wedge \hat{\eta}^1 \wedge \cdots \wedge \hat{\eta}^{\alpha-1} \wedge \hat{\eta}^{\alpha+1} \wedge \cdots \wedge \hat{\eta}^n = 0$$

which yields $\hat{Q}_{\overline{\beta}\,\overline{\mu}}^\rho = \hat{Q}_{\overline{\mu}\,\overline{\beta}}^\rho$ and together with

$$[\hat{\xi}_{\overline{\lambda}}, \hat{\xi}_{\overline{\mu}}]_{1,0} = \frac{i}{2}\,(\hat{Q}_{\overline{\mu}\,\overline{\lambda}}^\rho - \hat{Q}_{\overline{\lambda}\,\overline{\mu}}^\rho)\hat{\xi}_\rho$$

completes the proof of Theorem 4. ∎

By a result of J.M. Lee (cf. [29]) on any nondegenerate CR manifold a contact form η is pseudo-Einstein if and only if the 1-form $\omega_\alpha^\alpha - \frac{i}{2n}\,\rho^*\,\eta$ is closed, for any (local) frame $\{\xi_\alpha\}$ in $T_{1,0}(M)$. We shall prove

Proposition 2. *Let (M,η) be a contact manifold, of CR dimension n, and $g \in \mathcal{M}(\eta)$ an associated Riemannian metric so that the 1-form $\omega_\alpha^\alpha - \frac{i}{2n}\,\rho^*\,\eta$ is closed, for some frame $\{\xi_\alpha\}$ in $T_{1,0}(M)$. Then*

$$R_{\lambda\overline{\mu}} = (\rho^*/n)\, g_{\lambda\overline{\mu}} - \frac{1}{4}\, Q^{\rho}_{\overline{\gamma}\,\overline{\mu}} Q^{\overline{\gamma}}_{\rho\lambda}\,. \tag{61}$$

Proof. Let us contract α and β in (40) of Theorem 3. As

$$\eta_\alpha \wedge \tau^\alpha = 0, \quad \omega_\gamma^\alpha \wedge \omega_\alpha^\gamma = 0, \;\; \omega_{\overline{\gamma}}^{\alpha} \wedge \omega_{\alpha}^{\overline{\gamma}} = -\frac{1}{4} Q^{\alpha}_{\overline{\gamma}\,\overline{\mu}} Q^{\overline{\gamma}}_{\alpha\lambda}\,\eta^\lambda \wedge \eta^{\overline{\mu}}$$

we obtain (by (55) in Corollary 1)

$$\begin{aligned} d\omega_\alpha^\alpha = {} & \Big(R_{\lambda\overline{\mu}} + \frac{1}{4}\, Q^{\alpha}_{\overline{\gamma}\,\overline{\mu}} Q^{\overline{\gamma}}_{\alpha\lambda}\Big)\,\eta^\lambda \wedge \eta^{\overline{\mu}} - W^{\alpha}_{\alpha\lambda}\eta^\lambda \wedge \eta + W^{\alpha}_{\alpha\overline{\mu}}\eta^{\overline{\mu}} \wedge \eta + \\ & - \frac{i}{2}\, g^{\alpha\overline{\sigma}} g_{\overline{\rho}\lambda} Q^{\overline{\rho}}_{\mu\alpha,\overline{\sigma}}\,\eta^\lambda \wedge \eta^\mu - \frac{i}{2}\, g^{\alpha\overline{\sigma}} g_{\rho\overline{\lambda}}\, Q^{\rho}_{\overline{\mu}\,\overline{\sigma},\alpha}\,\eta^{\overline{\lambda}} \wedge \eta^{\overline{\mu}}\,. \end{aligned} \tag{62}$$

Assume $\omega_\alpha^\alpha - \frac{i}{2n}\rho^*\eta$ to be closed i.e.

$$d\omega_\alpha^\alpha = \frac{i}{2n}(d\rho^* \wedge \eta + \rho^* d\eta)\,,$$

substitute into (62), and apply the resulting identity to the pair $(\xi_\lambda, \xi_{\overline{\mu}})$. This procedure yields (61) in Proposition 2 . ∎

Remark 3. The converse of Proposition 2 is not true in general. For if we assume (61) to hold for a pair (η, g) then (62) may be written

$$d\Big(\omega_\alpha^\alpha - \frac{i}{2n}\,\rho^*\,\eta\Big) = \phi \wedge \eta - \frac{i}{2}\,\phi_{\lambda\mu}\eta^\lambda \wedge \eta^\mu - \frac{i}{2}\,\phi_{\overline{\lambda}\,\overline{\mu}}\eta^{\overline{\lambda}} \wedge \eta^{\overline{\mu}}\,, \tag{63}$$

where

$$\phi = -W^{\alpha}_{\alpha\lambda}\eta^\lambda + W^{\alpha}_{\alpha\overline{\mu}}\eta^{\overline{\mu}} + (i/2n)\, d\rho^*\,,$$
$$\phi_{\lambda\mu} = g^{\alpha\overline{\sigma}} g_{\overline{\rho}\lambda} Q^{\overline{\rho}}_{\mu\alpha,\overline{\sigma}}\,, \qquad \phi_{\overline{\lambda}\overline{\mu}} = g^{\alpha\overline{\sigma}} g_{\rho\overline{\lambda}}\, Q^{\rho}_{\overline{\mu}\,\overline{\sigma},\alpha}\,,$$

hence Lemma 6 does not apply [due to the presence of terms of type $(2,0)$, respectively $(0,2)$ in (63), originating in the non integrability of $T_{1,0}(M)$].

4.2 – The Fefferman metric

Let (M, η), $g \in \mathcal{M}(\eta)$, be a contact Riemannian manifold. Consider the principal S^1-bundle $F(M) := K^0(M)/\mathbf{R}_+ \xrightarrow{\pi} M$, where $\mathbf{R}_+ = GL^+(1, \mathbf{R})$. We shall need the *tautologous form* $\Xi \in \Gamma^\infty(\Lambda^{n+1} T^*(K(M)) \otimes \mathbf{C})$ given by

$$\Xi_\omega(Z_1, \cdots, Z_{n+1}) = \omega((d_\omega \pi_0) Z_1, \cdots, (d_\omega \pi_0) Z_{n+1}),$$
$$Z_1, \cdots, Z_{n+1} \in T_\omega(K(M)), \;\; \omega \in K(M),$$

where $\pi_0 : K(M) \to M$ is the projection. By Lemma 3 in [3], p. 19, for any $[\omega] \in F(M)$ with $\pi_0(\omega) = x$, there is a unique $\lambda \in (0, +\infty)$ such that

$$2^n \, i^{n(n+2)} \, n! \, \eta_x \wedge (\xi_x \rfloor \omega) \wedge (\xi_x \rfloor \overline{\omega}) = \lambda \, \Psi_x \, .$$

Consequently, there is a natural embedding

$$\iota_\eta : F(M) \to K(M), \;\; \iota_\eta([\omega]) = (1/\sqrt{\lambda}) \, \omega \, .$$

Next, we consider $\mathcal{Z} \in \Gamma^\infty(\Lambda^{n+1} T^*(F(M)) \otimes \mathbf{C})$ given by

$$\mathcal{Z} = \frac{1}{n+1} \, \iota_\eta^* \, \Xi \, .$$

Given a (local) frame $\{\xi_\alpha\}$ of $T_{1,0}(M)$ and the corresponding admissible coframe $\{\eta^\alpha\}$, we build a local form $\Xi_0 \in \Gamma^\infty(U, K^0(M))$ by setting

$$\Xi_0 = \det(g_{\alpha\overline{\beta}})^{1/2} \, \eta \wedge \eta^1 \wedge \cdots \wedge \eta^n$$

and consider $\mathcal{Z}_0 \in \Gamma^\infty(\pi^{-1}(U), \Lambda^{n+1} T^*(F(M)) \otimes \mathbf{C})$ given by $\mathcal{Z}_0 = \pi^* \Xi_0$. Let $\gamma : \pi^{-1}(U) \to \mathbf{R}$ be the natural fibre coordinate on $F(M)$ and $\Gamma \in \mathcal{X}(F(M))$ the tangent to the S^1-action. Then (by Lemma 4 in [3], p. 22) $\mathcal{Z} = e^{i\gamma} \mathcal{Z}_0$ and $(d\gamma)\Gamma = 1$. By Proposition 3 in [3], p. 23, there is a unique real 1-form $\sigma \in \Gamma^\infty(T^*(F(M)))$ such that

$$d\mathcal{Z} = i(n+2) \, \sigma \wedge \mathcal{Z} + e^{i\gamma} \, \pi^*[\det(g_{\alpha\overline{\beta}})^{1/2} \, \mathcal{W}] \tag{64}$$

$$\sigma \wedge (d\mathbf{r}) \wedge \overline{\mathbf{r}} = \operatorname{trace}(d\sigma) \, i \, \sigma \wedge (\pi^* \, \eta) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} \, . \tag{65}$$

Here $\mathbf{r}$ is the complex n-form on $F(M)$ such that $V \rfloor \mathbf{r} = 0$ and $\mathcal{Z} = (\pi^*\eta) \wedge \mathbf{r}$, for any lift V of ξ to $F(M)$, i.e. $\pi_* V = \xi$ (the existence and uniqueness of $\mathbf{r}$ follow from Lemma 5 in [3], p. 22). Also $\mathcal{W}$ is the complex $(n+2)$-form on M given by

$$\mathcal{W} = \frac{i}{2}\,\eta \wedge \sum_{\alpha=1}^{n} (-1)^\alpha\, \eta^1 \wedge \cdots \wedge (Q^\alpha_{\overline{\beta}\overline{\gamma}}\, \eta^{\overline{\beta}} \wedge \eta^{\overline{\gamma}}) \wedge \cdots \wedge \eta^n \,.$$

The (*generalized*) *Fefferman metric* of the contact Riemannian manifold (M, η, g) is the Lorentz metric G_η on $F(M)$ given by

$$G_\eta = \pi^* L_\eta + 2(\pi^*\eta) \odot \sigma\,,$$

where L_η is the (degenerate) bilinear form on $T(M)$ given by 1) $L_\eta(X,Y) = -g(X,Y)$, $X, Y \in H(M)$, and 2) $L_\eta(\xi, X) = 0$, $X \in T(M)$. Cf. [3], p. 27. When the almost CR structure is integrable G_η is the ordinary Fefferman metric (of [20] and [28]). Also, if this is the case (i.e. $Q = 0$) then σ may be explicitely calculated in terms of pseudohermitian invariants. In this section, we attack the similar problem for a contact Riemannian manifold with a nonintegrable almost CR structure. We shall show that

$$\sigma = \frac{1}{n+2}\Big\{d\gamma + \pi^*\Big[i\omega^\alpha_\alpha - \frac{i}{2} g^{\alpha\overline{\beta}} dg_{\alpha\overline{\beta}} + \frac{1}{4(n+1)}\Big(\rho^* + \frac{1}{4} g^{\lambda\overline{\mu}} Q^{\overline{\beta}}_{\alpha\lambda} Q^\alpha_{\overline{\beta}\overline{\mu}}\Big)\eta\Big]\Big\}\,.$$

First, note that

$$\mathbf{r} = e^{i\gamma}\pi^*(G\,\eta^1 \wedge \cdots \wedge \eta^n)\,, \tag{66}$$

where $G = \det(g_{\alpha\overline{\beta}})^{1/2}$. Indeed, if V is a lift of ξ to $F(M)$ then

$$\begin{aligned}\mathbf{r} &= (n+1)\, V \rfloor \mathcal{Z} = (n+1)\, V \rfloor (e^{i\gamma}\mathcal{Z}_0) = (n+1)\, e^{i\gamma}\, V \rfloor \pi^*\Xi_0 = \\ &= (n+1) e^{i\gamma}\, \pi^*(\xi \rfloor \Xi_0) = e^{i\gamma}\pi^*[\det(g_{\alpha\overline{\beta}})^{1/2}\eta^{1\cdots n}]\,,\end{aligned}$$

where $\eta^{1\cdots n}$ is short for $\eta^1 \wedge \cdots \wedge \eta^n$. A calculation similar to that in the previous section shows that

$$d(\eta \wedge \eta^{1\cdots n}) = -\omega^\alpha_\alpha \wedge \eta \wedge \eta^{1\cdots n} + \mathcal{W}$$

and then

$$d\,\Xi_0 = (d\log G - \omega^\alpha_\alpha) \wedge \Xi_0 + G\,\mathcal{W}\,.$$

Consider the $(1,0)$-form

$$\omega = (h_\beta - \Gamma^{\overline{\alpha}}_{\beta\overline{\alpha}})\eta^\beta + \frac{1}{2}\,\Gamma^\alpha_{0\alpha}\,\eta\,,$$

where $h = \log G$ and $h_\beta = \xi_\beta(h)$. Then

$$d\,\Xi_0 = (h_{\overline{\beta}} - \Gamma^\alpha_{\overline{\beta}\alpha})\,\eta^{\overline{\beta}} \wedge \Xi_0 + G\,\mathcal{W} = (\overline{\omega} - \omega) \wedge \Xi_0 + G\,\mathcal{W}\,.$$

As $\overline{\omega} - \omega$ is pure imaginary, there is a real 1-form σ_0 on M such that $\overline{\omega} - \omega = i(n+2)\sigma_0$ hence

$$d\,\Xi_0 = i(n+2)\sigma_0 \wedge \Xi_0 + G\,\mathcal{W}\,. \tag{67}$$

At this point, differentiating $\mathcal{Z} = e^{i\gamma}\mathcal{Z}_0$ and using (67) leads to (64) with σ given by

$$\sigma = \frac{1}{n+2}d\gamma + \pi^*\sigma_0\,.$$

We wish to compute σ_0 in terms of pseudohermitian invariants. To this end, note that

$$h_\mu = \frac{1}{2}g^{\alpha\overline{\beta}}\xi_\mu(g_{\alpha\overline{\beta}})\,.$$

Indeed

$$\xi_\mu(h) = \xi_\mu(\log G) = \frac{1}{2G^2}\xi_\mu(G^2) = \frac{1}{2G^2}\frac{\partial G^2}{\partial g_{\alpha\overline{\beta}}}\xi_\mu(g_{\alpha\overline{\beta}}) = \frac{1}{2}g^{\alpha\overline{\beta}}\xi_\mu(g_{\alpha\overline{\beta}})\,.$$

On the other hand $g^{\alpha\overline{\beta}}dg_{\alpha\overline{\beta}} = \omega^\alpha_\alpha + \omega^{\overline{\alpha}}_{\overline{\alpha}}$ hence $h_\mu = \frac{1}{2}(\Gamma^\alpha_{\mu\alpha} + \Gamma^{\overline{\alpha}}_{\mu\overline{\alpha}})$ thus leading to

$$\sigma_0 = \frac{i}{2(n+2)}(\omega^\alpha_\alpha - \omega^{\overline{\alpha}}_{\overline{\alpha}}) \tag{68}$$

or

$$\sigma_0 = \frac{i}{n+2}\Big(\omega^\alpha_\alpha - \frac{1}{2}\,g^{\alpha\overline{\beta}}d\,g_{\alpha\overline{\beta}}\Big)\,.$$

For $f \in C^\infty(M)$ set

$$\sigma_f = \sigma + \pi^*(f\eta)\,.$$

Note that, should one replace σ by σ_f, (64) still holds. We wish to determine f such that (65) holds as well (with σ replaced by σ_f). Only the existence of such f has been proved in [3], p. 26. To accomplish our task we need to compute trace($d\sigma_0$). Recall that, for any complex 2-form Ω on M, if $\Omega \equiv i\,\Omega_{\alpha\overline{\beta}}\,\eta^\alpha \wedge \eta^{\overline{\beta}}$, $\mod \eta^\alpha \wedge \eta^\beta$, $\eta^{\overline{\alpha}} \wedge \eta^{\overline{\beta}}$, then $\mathrm{trace}(\Omega) = -\frac{1}{2} g^{\alpha\overline{\beta}}\Omega_{\alpha\overline{\beta}}$. Also $\mathrm{trace}(\pi^*\Omega) := \mathrm{trace}(\Omega)$. Differentiating in (68) we get

$$d\sigma_0 = \frac{i}{2(n+2)}(d\omega^\alpha_\alpha - d\omega^{\overline{\alpha}}_{\overline{\alpha}})\,.$$

By (62)

$$d\omega^\alpha_\alpha \equiv \Big(R_{\lambda\overline{\mu}} + \frac{1}{4} Q^{\overline{\gamma}}_{\alpha\lambda} Q^\alpha_{\overline{\gamma}\,\overline{\mu}}\Big)\,\eta^\lambda \wedge \eta^{\overline{\mu}}\,, \;\; \mod \eta \wedge \eta^\lambda,\ \eta \wedge \eta^{\overline{\mu}},\ \eta^\lambda \wedge \eta^\mu,\ \eta^{\overline{\lambda}} \wedge \eta^{\overline{\mu}}\,,$$

hence

$$\mathrm{trace}(i\,d\omega^\alpha_\alpha) = -\frac{1}{2}\rho^* - \frac{1}{8} g^{\lambda\overline{\mu}}\,Q^{\overline{\gamma}}_{\alpha\lambda}\,Q^\alpha_{\overline{\gamma}\,\overline{\mu}}\,,$$
$$\mathrm{trace}(i\,d\omega^{\overline{\alpha}}_{\overline{\alpha}}) = \frac{1}{2}\rho^* + \frac{1}{8} g^{\lambda\overline{\mu}}\,Q^{\overline{\alpha}}_{\gamma\lambda}\,Q^\gamma_{\overline{\alpha}\,\overline{\mu}}\,.$$

We conclude that

$$\mathrm{trace}(d\sigma_0) = -\frac{1}{2(n+2)}\Big\{\rho^* + \frac{1}{4}\,g^{\lambda\overline{\mu}}\,Q^{\overline{\gamma}}_{\alpha\lambda}\,Q^\alpha_{\overline{\gamma}\,\overline{\mu}}\Big\}\,. \tag{69}$$

Now we wish to solve for f in the equation

$$\sigma_f \wedge (d\mathbf{r}) \wedge \overline{\mathbf{r}} = \mathrm{trace}(d\sigma_f)\,i\,\sigma_f \wedge (\pi^*\eta) \wedge \mathbf{r} \wedge \overline{\mathbf{r}}\,.$$

By the very definition of σ_f

$$d\sigma_f = d\sigma + (df) \wedge \eta + f\,d\eta \equiv d\sigma_0 - 2i\,f\,g_{\alpha\overline{\beta}}\,\eta^\alpha \wedge \eta^{\overline{\beta}}\,,$$

hence

$$\mathrm{trace}(d\sigma_f) = n\,f + \mathrm{trace}(d\sigma_0)\,.$$

Therefore, we must solve for f in

$$(\sigma + f\eta) \wedge (d\mathbf{r}) \wedge \overline{\mathbf{r}} = \{nf + \mathrm{trace}(d\sigma_0)\}\, i\, \sigma \wedge (\pi^*\eta) \wedge \mathbf{r} \wedge \overline{\mathbf{r}}\,.$$

Differentiating in (66)

$$d\mathbf{r} = i\, e^{i\gamma}\, d\gamma \wedge \pi^*(G\eta^{1\cdots n}) + e^{i\gamma}\, \pi^*(dG \wedge \eta^{1\cdots n} + G\, d\eta^{1\cdots n})$$

we get

$$(d\mathbf{r}) \wedge \overline{\mathbf{r}} = (i\, d\gamma + d\log G) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} + G^2(d\eta^{1\cdots n}) \wedge \eta^{\overline{1}\cdots\overline{n}}\,, \tag{70}$$

where $\eta^{\overline{1}\cdots\overline{n}} = \overline{\eta^{1\cdots n}}$ and π^* is omitted for simplicity. Yet

$$(d\eta^{1\cdots n}) \wedge \eta^{\overline{1}\cdots\overline{n}} = -\omega_\alpha^\alpha\, \eta^{1\cdots n} \wedge \eta^{\overline{1}\cdots\overline{n}}$$

hence (70) becomes

$$(d\mathbf{r}) \wedge \overline{\mathbf{r}} = i\, d\gamma \wedge \mathbf{r} \wedge \overline{\mathbf{r}} + (d\log G - \omega_\alpha^\alpha) \wedge \mathbf{r} \wedge \overline{\mathbf{r}}$$

and then [as $\eta \wedge (d\log G - \omega_\alpha^\alpha) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = 0$]

$$\begin{aligned} &\eta \wedge (d\mathbf{r}) \wedge \overline{\mathbf{r}} = i\, \eta \wedge (d\gamma) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = \\ &= i(n+2)\, \eta \wedge (\sigma - \sigma_0) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = \ \ (\text{as } \eta \wedge \sigma_0 \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = 0) \\ &= i(n+2)\, \eta \wedge \sigma \wedge \mathbf{r} \wedge \overline{\mathbf{r}}\,. \end{aligned}$$

One is led to solve for f in

$$\sigma \wedge (d\mathbf{r}) \wedge \overline{\mathbf{r}} = i\,\{\mathrm{trace}(d\sigma_0) + 2(n+1)\, f\}\, \sigma \wedge \eta \wedge \mathbf{r} \wedge \overline{\mathbf{r}}\,.$$

Yet, the left hand member vanishes:

$$\begin{aligned} &\sigma \wedge (d\mathbf{r}) \wedge \overline{\mathbf{r}} = i\, \sigma \wedge (d\gamma) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} + \sigma \wedge (d\log G - \omega_\alpha^\alpha) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = \\ &= i\, \sigma_0 \wedge (d\gamma) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} + \sigma \wedge (h_0 - \Gamma_{0\alpha}^\alpha)\eta \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = \\ &= \frac{1}{2(n+2)}(\omega_{\overline{\alpha}}^{\overline{\alpha}} - \omega_\alpha^\alpha) \wedge (d\gamma) \wedge \mathbf{r} \wedge \overline{\mathbf{r}} + (h_0 - \Gamma_{0\alpha}^\alpha)\, \sigma \wedge \eta \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = \\ &= \Big\{h_0 - \frac{1}{2}(\Gamma_{0\overline{\alpha}}^{\overline{\alpha}} + \Gamma_{0\alpha}^\alpha)\Big\}\, \sigma \wedge \eta \wedge \mathbf{r} \wedge \overline{\mathbf{r}} = 0 \end{aligned}$$

because $h_0 = \frac{1}{2} g^{\alpha\overline{\beta}} \xi(g_{\alpha\overline{\beta}}) = \frac{1}{2}(\Gamma^{\alpha}_{0\alpha} + \Gamma^{\overline{\alpha}}_{0\overline{\alpha}})$. Moreover $\sigma \wedge \eta \wedge \mathbf{r} \wedge \overline{\mathbf{r}}$ is a volume form on $F(M)$ and hence

$$f = -\frac{1}{2(n+1)} \, \mathrm{trace}(d\sigma_0) \, ,$$

or (by (69))

$$f = \frac{1}{4(n+1)(n+2)} \Big\{ \rho^* + \frac{1}{4} \, g^{\lambda\overline{\mu}} \, Q^{\overline{\beta}}_{\alpha\lambda} \, Q^{\alpha}_{\overline{\beta}\,\overline{\mu}} \Big\} \, . \qquad \square$$

5 – Pseudohermitian holonomy

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n. Let θ be a contact form on M such that the Levi form L_θ be positive definite. Let T be the characteristic direction of (M,θ). Let $GL(2n+1, \mathbf{R}) \to \mathcal{L}(M) \longrightarrow M$ be the principal bundle of all linear frames tangent to M. For each $x \in M$, let $B(\theta)_x$ consist of all $\mathbf{R}$-linear isomorphisms $u : \mathbf{R}^{2n+1} \to T_x(M)$ such that

$$u(e_0) = T_x, \;\; u(e_\alpha) \in H(M)_x, \;\; u(e_{\alpha+n}) = \varphi_x \, u(e_\alpha) \, ,$$
$$g_x(u(e_\alpha), u(e_\beta)) = \delta_{\alpha\beta}, \;\; g_x(u(e_\alpha), u(e_{\beta+n})) = 0 \, ,$$

where g is the Webster metric of (M,θ). Also $\{e_0, e_\alpha, e_{\alpha+n}\} \subset \mathbf{R}^{2n+1}$ is the canonical linear basis. Then $B(\theta) \to M$ is a $U(n) \times 1$-structure on M [i.e. a principal $U(n) \times 1$-subbundle of $\mathcal{L}(M)$]. On a strictly pseudoconvex CR manifold, there are two natural families of holonomy groups one may consider, the holonomy of the Levi-Civita connection of (M,g) and the holonomy of the Tanaka-Webster connection. The Tanaka-Webster connection ∇^* of (M,θ) gives rise to a connection Γ in $B(\theta)$. Let $\Phi^0(u)$ be the restricted holonomy group of Γ, with reference point $u \in B(\theta)$. We call $\Phi^0(u)$ the *pseudohermitian holonomy group* of (M,θ) at u. A systematic study of the (pseudohermitian) holonomy of a CR manifold is still missing in the present day mathematical literature. In the present note we establish a pseudohermitian analogue of a result by H. Iwamoto, [25] (cf. also [27], Vol. II, p. 151)

THEOREM 5. *Let $(M, (\varphi, \xi, \eta, g))$ be a real $(2n+1)$-dimensional Sasakian manifold. The pseudohermitian holonomy groups of (M, η) are contained in $SU(n) \times 1$ if and only if the Tanaka-Webster connection of (M, η) is Ricci flat* ($\mathrm{Ric}^* = 0$).

PROOF. As indicated in Section 2, M is thought of as a strictly pseudoconvex CR manifold carrying a contact form with vanishing Webster torsion ($\tau = 0$) and g is its Webster metric. For any $u \in B(\theta)$, $\Phi^0(u) \subset U(n) \times 1$. Let Ω be the curvature 2-form of Γ. By Lemma 1 in [27] , Vol. II, p. 151, given an ideal $\mathbf{h}$ of $L(U(n) \times 1)$, $L(\Phi^0(u)) \subset \mathbf{h}$ if and only if Ω is $\mathbf{h}$-valued. Throughout $L(G)$ is the Lie algebra of the Lie group G. Let $E^i_j \in \mathbf{gl}(2n+1, \mathbf{R})$ be the matrix with 1 in the j-th row and i-th column and 0 at all other entries. Then $\Omega = \Omega^i_j \otimes E^j_i$. A basis of $L(U(n) \times 1)$ is

$$\{E^{\beta+1}_{\alpha+1} - E^{\alpha+1}_{\beta+1} + E^{\beta+n+1}_{\alpha+n+1} - E^{\alpha+n+1}_{\beta+n+1},\ \ E^{\beta+1}_{\alpha+n+1} - E^{\alpha+n+1}_{\beta+1} + E^{\alpha+1}_{\beta+n+1} - E^{\beta+n+1}_{\alpha+1}\}\,,$$

hence

$$\Omega^i_1 = 0, \quad \Omega^1_j = 0\,,$$
$$\Omega^{\alpha+1}_{\beta+1} = \Omega^{\alpha+n+1}_{\beta+n+1} = \Phi^\alpha_\beta - \Phi^\beta_\alpha\,,$$
$$\Omega^{\alpha+n+1}_{\beta+1} = -\Omega^{\alpha+1}_{\beta+n+1} = \Psi^\alpha_\beta + \Psi^\beta_\alpha\,,$$

for some scalar 2-forms Φ^α_β, Ψ^α_β on $B(\theta)$. As $SU(n) = O(2n) \cap SL(n, \mathbf{C})$ it follows that Ω is $L(SU(n) \times 1)$-valued if and only if $\Psi^\alpha_\alpha = 0$. On the other hand, using the identity

$$2u(\Omega(X^\Gamma, Y^\Gamma)_u\, u^{-1}(Z_x)) = (R^*(X,Y)Z)_x,\ \ u \in B(\theta)_x\,, \tag{71}$$

we may compute the forms Ψ^α_α in terms of $R_A{}^B{}_{CD}$. Here X, Y, Z are vector fields on M and X^Γ is the Γ-horizontal lift of X. Let $x \in M$ and $\{X_\alpha, JX_\alpha, T\}$ be a cross section in $B(\theta)$, defined on some open neighborhood U of x. Set $\xi_\alpha = \frac{1}{\sqrt{2}}(X_\alpha - iJX_\alpha)$ (hence $g_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$). Let $u = (x, \{X_{\alpha,x}, \varphi_x X_{\alpha,x}, T_x\})$ and note that $u^{-1}(\xi_{\gamma,x}) = \frac{1}{\sqrt{2}}(e_\gamma - i\, e_{\gamma+n})$. Then (71) leads to

$$(R^*(X,Y)\xi_\gamma)_x = 2\{\Phi^\alpha_\gamma - \Phi^\gamma_\alpha + i(\Psi^\alpha_\gamma + \Psi^\gamma_\alpha)\}(X^\Gamma, Y^\Gamma)_u \xi_{\alpha,x}\,,$$

because of $E^i_j e_k = \delta^i_k e_j$. Take the inner product with $\xi_{\overline{\alpha}}$ and contract α and γ in the resulting identity. We obtain

$$4i\,\Psi^\alpha_\alpha(X^\Gamma, X^\Gamma)_u = \sum_{\alpha=1}^n g(R^*(X,Y)\xi_\alpha, \xi_{\overline{\alpha}})_x\,. \tag{72}$$

The curvature form Ω is horizontal, hence $L(U(n)\times 1)^* \rfloor \psi^\alpha_\alpha = 0$, where A^* is the fundamental vertical vector field associated to the left invariant vector field A. Also (by (42) with $Q=0$)

$$4i\,\Psi^\alpha_\alpha(\xi^\Gamma_\lambda, \xi^\Gamma_\mu)_u = \sum_\alpha R_\alpha{}^\sigma{}_{\lambda\mu} g_{\sigma\overline{\alpha}} = R_\alpha{}^\alpha{}_{\lambda\mu} = 0\,.$$

Similarly (by (45))

$$4i\,\Psi^\alpha_\alpha(T^\Gamma, \xi^\Gamma_\lambda) = R_\alpha{}^\alpha{}_{0\lambda} = \sum_\alpha S^{\overline{\lambda}}_{\alpha\overline{\alpha}} = 0\,.$$

Finally (again by (72))

$$R_{\lambda\overline{\mu}}(x) = 4i\Psi^\alpha_\alpha(\xi^\Gamma_\lambda, \xi^\Gamma_{\overline{\mu}})_u\,. \tag{73}$$

As Ψ^α_α is a real form, (73) shows that $\Psi^\alpha_\alpha = 0$ if and only if $R_{\lambda\overline{\mu}} = 0$. Yet when $\tau = 0$ the only nonzero components of Ric^* are $R_{\lambda\overline{\mu}}$ (cf. also Lemma 8 in Section 5.1). ∎

Note that the hypothesis $\tau = 0$ was not fully used in the proof of Theorem 5 (only $S = 0$ was actually needed). Therefore, we obtained the following result

Theorem 6. *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and θ a contact form with parallel Webster torsion ($\nabla^*\tau = 0$). Then the Tanaka-Webster connection ∇^* of (M,θ) has pseudohermitian holonomy contained in $SU(n)\times 1$ if and only if the pseudohermitian Ricci tensor of (M,θ) vanishes ($R_{\alpha\overline{\beta}} = 0$).*

5.1 – Quaternionic Sasakian manifolds

The closest odd dimensional analogue of Kählerian manifolds seem to be Sasakian manifolds (cf. [11]). On the other hand, real $4m$-dimensional Riemannian manifolds whose holonomy group is contained in $Sp(m)$ (*hyperkählerian* manifolds) or in $Sp(m)Sp(1)$ (*quaternionic-Kähler* manifolds) are quaternion analogues and, by a well known result of M. Berger, [9], any hyperkählerian manifold is Ricci flat, while any quaternion-Kähler manifold is Einstein (provided that $m \geq 2$). Cf. also S. Ishihara, [24]. It is a natural question whether a Sasakian counterpart of quaternionic-Kähler manifolds may be devised, with the expectation of producing new examples of pseudo-Einstein contact forms (cf. [29]). Evidence on the existence of such a notion may be obtained as follows. Recall (cf. e.g. [10], p. 403) that a Riemannian manifold (M^{4m}, g) is quaternionic-Kähler if and only if there is a covering of M^{4m} by open sets U_i and, for each i, two almost complex structures F and G on U_i so that a) g is Hermitian with respect to F and G on U_i, b) $FG = -GF$, c) the covariant derivatives (with respect to the Levi-Civita connection of (M^{4m}, g)) of F and G are linear combinations of F, G and $H := FG$, and d) for any $x \in U_i \cap U_j$ the linear space of endomorphisms of $T_x(M^{4m})$ spanned by F, G and H is the same for both i and j. In an attempt to unify the treatment of *quaternionic* submanifolds, and of *totally real* submanifolds of a quaternionic-Kähler manifold (cf. S. Funbashi, [22], S. Marchiafava, [30], A. Martinez, [31], A. Martinez & J.D. Pérez & F.G. Santos, [32], G. Pitis, [33], Y. Shibuya, [36]) M. Barros & B-Y. Chen & F. Urbano introduced (cf. [4]) the notion of *quaternionic CR submanifold* of a quaternionic-Kähler manifold, as follows. Let N be a real submanifold of a quaternionic-Kähler manifold M^{4m}. A C^∞ distribution $H(N)$ on N is a *quaternionic distribution* if for any $x \in N$ and any i such that $x \in U_i \subseteq M^{4m}$ one has $F(H(N)_x) \subseteq H(N)_x$, $G(H(N)_x) \subseteq H(N)_x$ [and then, of course, $H_x(H(N)_x) \subseteq H(N)_x$]. A submanifold N of a quaternionic-Kähler manifold is a *quaternionic CR submanifold* if it is endowed with a quaternionic distribution $H(N)$ such that its orthogonal complement $H(N)^\perp$ in $T(N)$ satisfies $F(H(N)^\perp_x) \subseteq T(N)^\perp_x$, $G(H(N)^\perp_x) \subseteq T(N)^\perp_x$ and $H(H(N)^\perp_x) \subseteq T(N)^\perp_x$ for any $x \in U_i$ and any i. Here $T(N)^\perp \to N$ is the normal bundle (of the given immersion of N in M^{4m}). Let us also recall (cf. e.g. [10],

p. 398) that a Riemannian manifold (M^{4m}, g) is hyperkählerian if and only if there exist on M^{4m} two complex structures F and G compatible with g and such that a) F and G are parallel, i.e. g is a Kählerian metric for both F and G, and b) $FG = -GF$. Given a quaternionic CR submanifold $(N, H(N))$ of a hyperkähler manifold (M^{4m}, g, F, G), by a theorem of D.E. Blair & B-Y. Chen, [13], the complex structures F and G induce two CR structures on N (provided N is *proper*, i.e. $H(N) \neq 0$ and $H(N)^{\perp} \neq 0$) so that $H(N)$ is the Levi distribution for both. Taking this situation as a model one may produce the following notion of abstract (i.e. not embedded) *hyper CR manifold*. Let $(M, T_{1,0}(M))$ be a CR manifold of type (n, k) where $n = 2m$ (hence $\dim_{\mathbf{R}} M = 4m + k$) and $k \geq 1$. Let $H(M)$ be its Levi distribution and

$$F : H(M) \to H(M), \;\; F(Z + \overline{Z}) = i(Z - \overline{Z}), \;\; Z \in T_{1,0}(M)\,,$$

its complex structure. We say $(M, T_{1,0}(M))$ is a *hyper CR manifold* if it pssesses two additional CR structures, say $T_{1,0}(M)'$ and $T_{1,0}(M)''$, with the same Levi distribution $H(M)$, such that the corresponding complex structures $G, H : H(M) \to H(M)$ satisfy

$$\begin{cases} F^2 = G^2 = H^2 = -I \\ FG = -GF = H \\ GH = -HG = F \\ HF = -FH = G\,. \end{cases} \tag{74}$$

More generally, a *quaternionic CR manifold* is a real $(4m+k)$-dimensional manifold M, $k \geq 1$, endowed with a real rank $4m$ subbundle $H(M) \subset T(M)$ and a real rank 3 subbundle $E \to M$ of $H(M)^* \otimes H(M) \to M$ such that for any $x \in M$ there is an open neighborhood U of x and a local frame $\{F, G, H\}$ of E on U satisfying the identities (74). *A priori*, the notions of a hyper CR manifold, or a quaternionic CR manifold, seem not to be direct analogues of the notions of hyperkähler and quaternionic-Kähler manifolds, as there is no counterpart of the metric structure there. However, in complex analysis one is interested in the metric structure arising from the complex structure, e.g. the Levi form of a given CR manifold, extending (for nondegenerate CR structures) to a semi-Riemannian metric (the Webster metric).

Let M^{4m+1} be a hyper CR manifold, such that $(M^{4m+1}, T_{1,0}(M))$ is nondegenerate, and let θ be a fixed contact form. We say θ is a *hyper contact form* if $\nabla^* G = 0$ and $\nabla^* H = 0$, where ∇^* is the Tanaka-Webster connection of (M^{4m+1}, θ). More generally, let $(M^{4m+1}, H(M), E)$ be a quaternionic CR manifold of the following sort: M^{4m+1} carries a nondegenerate CR structure $T_{1,0}(M)$ whose Levi distribution is $H(M)$ and for any $x \in M^{4m+1}$ there is an open neighborhood U and a local frame of E on U of the form $\{F, G, H\}$ where F is the (restriction to U of the) complex structure in $H(M)$ associated to $T_{1,0}(M)$ and satisfying the identities (74). Such a local frame of E at x will be referred hereafter as an *F-frame*. A contact form θ on $(M^{4m+1}, T_{1,0}(M))$ is said to be a *quaternionic contact form* if for any $x \in M^{4m+1}$ there is an open neighborhood U and an F-frame $\{F, G, H\}$ of E on U such that

$$\begin{cases} (d\theta)(FX, FY) + (d\theta)(X, Y) = 0\,, \\ (d\theta)(GX, GY) + (d\theta)(X, Y) = 0\,, \\ (d\theta)(HX, HY) + (d\theta)(X, Y) = 0\,, \end{cases} \tag{75}$$

for any $X, Y \in H(M)$, and moreover

$$\begin{cases} \nabla^*_X F = 0\,, \\ \nabla^*_X G = p(X) H\,, \\ \nabla^*_X H = -p(X) G\,, \end{cases} \tag{76}$$

for some 1-form p on U and any $X \in T(M)$, where ∇^* is the Tanaka-Webster connection of (M^{4m+1}, F, θ). Note that the first row identities in (75)-(76) are written for uniformity sake (and are automatically satisfied, one as a consequence of the formal integrability property of $T_{1,0}(M)$, and the other by the very construction of ∇^*). A quaternionic CR manifold carrying a quaternionic contact form θ is said to be a *quaternionic Sasakian manifold*. This is motivated by Theorem 7 below, according to which the Webster torsion of θ vanishes ($\tau = 0$), i.e. the underlying Riemannian metric is indeed Sasakian.

THEOREM 7. *Let (M^{4m+1}, θ) be a quaternionic Sasakian manifold. Then $\tau = 0$, i.e. the Webster metric g of (M^{4m+1}, θ) is a Sasakian metric. Moreover, either the Tanaka-Webster connection of (M^{4m+1}, θ) is Ricci* *flat, or* $m = 1$*. If this is the case (i.e.* $m = 1$*) then* (M^5, θ) *is pseudo-Einstein if and only if* $4p + \rho^* \theta$ *is a closed* 1*-form on* U*, for any* F*-frame of* E *on* U *obeying* (75)-(76)*.*

REMARK 4.

1) By Theorem 5 any quaternionic Sasakian manifold M^{4m+1} of dimension ≥ 9 has pseudohermitian holonomy contained in $SU(2m) \times 1$.
2) By a result of J.M. LEE, [29], the first Chern class of the CR structure of M^{4m+1} must vanish ($c_1(T_{1,0}(M)) = 0$).
3) Let $M^n(1)$ be a real space form of sectional curvature 1. By a result in [2] the pseudohermitian Ricci tensor of $U(M^n(1))$ is given by $R_{\alpha\overline{\beta}} = [\frac{1}{2} + 2(n + \|\mu\|)] g_{\alpha\overline{\beta}}$, where μ is the mean curvature vector of $U(M^n(1))$ in $T(M^n(1))$. Therefore (by Theorem 7) $U(M^{2m+1}(1))$ admits no quaternionic Sasakian structure for $m \geq 2$.

To prove Theorem 7, let (M^{4m+1}, θ) be a quaternionic Sasakian manifold and $\{F, G, H\}$ a (local) F-frame on U, satisfying (74)-(76). Let g be the Webster metric of (M^{4m+1}, θ). Then

$$\begin{cases} g(FX, FY) = g(X, Y)\,, \\ g(GX, GY) = g(X, Y)\,, \\ g(HX, HY) = g(X, Y)\,, \end{cases} \tag{77}$$

for any $X, Y \in H(M)$. The first identity is obvious. The second, for instance, follows from

$$\begin{aligned} g(GX, GY) &= (d\theta)(GX, FGY) = (d\theta)(GX, HY) = \\ &= -(d\theta)(G^2X, GHY) = (d\theta)(X, FY) = g(X, Y)\,, \end{aligned}$$

by the very definition of the Webster metric g. We shall need the following curvature identities

$$[R^*(X, Y), F] = 0 \tag{78}$$

$$[R^*(X, Y), G] = \alpha(X, Y) H \tag{79}$$

$$[R^*(X, Y), H] = -\alpha(X, Y) G \tag{80}$$

for any $X, Y \in T(M)$, where $\alpha := 2\, dp$. The first identity is a consequence of $\nabla^* F = 0$. The second, for instance, follows from

$$[R^*(X,Y), G]Z = R^*(X,Y)GZ - G\, R^*(X,Y)Z =$$
$$= \nabla^*_X(\nabla^*_Y G)Z - \nabla^*_Y(\nabla^*_X G)Z - (\nabla^*_{[X,Y]}G)Z + (\nabla^*_X G)\nabla^*_Y Z - (\nabla^*_Y G)\nabla^*_X Z =$$
$$= 2(dp)(X,Y)HZ + p(Y)(\nabla^*_X H)Z - p(X)(\nabla^*_Y H)Z = \alpha(X,Y)HZ\,,$$

for any $X, Y \in T(M)$ and $Z \in H(M)$.

Let us take the inner product of (80)

$$[R^*(X,Y), H]Z = -\alpha(X,Y)\, GZ, \quad Z \in H(M)\,,$$

with GZ to obtain

$$\alpha(X,Y)\|Z\|^2 = g(HZ, R^*(X,Y)GZ) + g(R^*(X,Y)Z, FZ)\,. \tag{81}$$

Consider a local orthonormal frame of $H(M)$ on U of the form

$$\{X_i : 1 \le i \le 4m\} = \{X_a, FX_a, GX_a, HX_a : 1 \le a \le m\}\,.$$

Set $Z = X_i$ in (81) and sum over i

$$4m\, \alpha(X,Y) = \sum_{i=1}^{4m} \{g(HX_i, R^*(X,Y)GX_i) + g(R^*(X,Y)X_i, FX_i)\}\,. \tag{82}$$

Since

$$\{(GX_i, HX_i) : 1 \le i \le 4m\} = \{(\epsilon_i X_i, \epsilon_i FX_i) : 1 \le i \le 4m\}\,,$$

where $\epsilon_i \in \{\pm 1\}$, the equation (82) becomes

$$2m\, \alpha(X,Y) = \sum_{i=1}^{4m} g(R^*(X,Y)X_i, FX_i)\,. \tag{83}$$

We shall need the first Bianchi identity

$$\sum_{XYZ} \{R^*(X,Y)Z + T^*(T^*(X,Y), Z) + (\nabla^*_X T^*)(Y,Z)\} = 0$$

for any $X, Y, Z \in T(M)$. Throughout $\sum_{XYZ}$ denotes the cyclic sum over X, Y, Z. Also, we recall (cf. [18])

$$T^*(X, Y) = 2(d\theta)(X, Y)T\,,$$

for any $X, Y \in H(M)$. Therefore (by $\nabla^* T = 0$ and $\nabla^* \Omega = 0$)

$$\sum_{XYZ} \{R^*(X, Y)Z - 2\Omega(X, Y)\,\tau Z\} = 0\,. \tag{84}$$

for any $X, Y, Z \in H(M)$. Set $Z = FX_i$ in (84), and take the inner product with X_i in the resulting identity. Next, sum over i so that to yield

$$\begin{aligned} -2m\alpha(X,Y) + \sum_{i=1}^{4m}\{g(X_i, R^*(Y, FX_i)X) + g(X_i, R^*(FX_i, X)Y)\} = \\ = 2\Omega(X,Y)\ \mathrm{trace}(\tau F) + 2\Omega(Y, F\tau X) + 2\Omega(F\tau Y, X)\,. \end{aligned} \tag{85}$$

Note that $\mathrm{trace}(\tau F) = 0$, because $\tau\, T_{1,0}(M) \subseteq T_{0,1}(M)$, and

$$\Omega(Y, F\tau X) + \Omega(F\tau Y, X) = 0\,,$$

by the symmetry property of $A(X, Y) = g(\tau X, Y)$, with the corresponding simpler from of (85). Let us replace (X, Y, Z, W) by (FX_i, X, Y, X_i) in (38). We obtain

$$\begin{aligned} \sum_{i=1}^{4m} g(R^*(FX_i, X)Y, X_i) = \sum_{i=1}^{4m} g(R^*(X_i, Y)X, FX_i) + g(X, LY)\mathrm{trace}(FL) + \\ - g(X, LFLY) + g(LX, Y)\mathrm{trace}(LF) - g(LFLX, Y)\,. \end{aligned}$$

Substitution into (85) gives

$$\begin{aligned} 2m\,\alpha(X,Y) = {} & \sum_{i=1}^{4m}\{g(R^*(Y, FX_i)X, X_i) - g(R^*(Y, X_i)X, FX_i)\} + \\ & + g(X, LY)\,\mathrm{trace}(FL) - g(X, LFLY) + \\ & + g(LX, Y)\,\mathrm{trace}(LF) - g(LFLX, Y) \end{aligned}$$

and, by observing that

$$\{(FX_i, X_i) : 1 \le i \le 4m\} = \{(\lambda_i X_i, \mu_i FX_i) : 1 \le i \le 4m\}\,,$$
$$\lambda_i,\ \mu_i \in \{\pm 1\},\quad \lambda_i \mu_i = -1\,,$$

we obtain

$$2m\,\alpha(X,Y) = 2\sum_i g(R^*(X_i,Y)X, FX_i) + \quad \text{terms}$$

or (replacing X by FX)

$$\begin{aligned} 2\,\mathrm{Ric}^*(X,Y) = {} & 2m\alpha(FX,Y) + g(FX, LFLY) + \\ & - g(FX, LY)\,\mathrm{trace}(FL) + \\ & + g(LFLFX, Y) - g(LFX, Y)\,\mathrm{trace}(LF)\,, \end{aligned} \tag{86}$$

for any $X, Y \in H(M)$. Next, let us take the inner product of (78),

$$[R^*(X,Y), F]Z = 0, \quad X, Y, Z \in H(M)\,,$$

with GZ so that

$$g(R^*(X,Y)FZ, GZ) + g(R^*(X,Y)Z, HZ) = 0\,.$$

Set $Z = X_i$ and sum over i

$$\sum_{i=1}^{4m} \{g(R^*(X,Y)FX_i, GX_i) + g(R^*(X,Y)X_i, HX_i)\} = 0$$

and observe that

$$\{(FX_i, GX_i) : 1 \le i \le 4m\} = \{(\epsilon_i X_i, \epsilon_i HX_i) : 1 \le i \le 4m\},\ \ \epsilon_i \in \{\pm 1\}\,.$$

Therefore

$$\sum_{i=1}^{4m} g(R^*(X,Y)X_i, HX_i) = 0\,. \tag{87}$$

Set $Z = HX_i$ in (84) and take the inner product with X_i in the resulting identity. Then (by (87))

$$\sum_{i=1}^{4m}\{g(R^*(Y, HX_i)X, X_i) + g(R^*(HX_i, X)Y, X_i)\} = \\ = 2\Omega(X, Y)\ \mathrm{trace}(\tau H) + 2\Omega(Y, H\tau X) + 2\Omega(H\tau Y, X)\,. \tag{88}$$

Now replace (X, Y, Z, W) by (HX_i, X, Y, X_i) in (38)

$$g(R^*(HX_i, X)Y, X_i) = g(R^*(X_i, Y)X, HX_i) + \\ + g((LX_i \wedge LY)X, HX_i) - g((LHX_i \wedge LX)Y, X_i)$$

and substitute into (88). Also observe that

$$\{(HX_i, X_i) : 1 \le i \le 4m\} = \{(\lambda_i X_i, \mu_i HX_i) : 1 \le i \le 4m\}\,, \\ \lambda_i, \mu_i \in \{\pm 1\}, \quad \lambda_i \mu_i = -1\,,$$

hence

$$2\sum_{i=1}^{4m} g(R^*(X_i, Y)X, HX_i) - g(LHLY, X) + g(LY, X)\ \mathrm{trace}(HL) + \\ - g(LHLX, Y) + g(LX, Y)\ \mathrm{trace}(LH) = \\ = 2\Omega(X, Y)\ \mathrm{trace}(\tau H) + 2\Omega(Y, H\tau X) + 2\Omega(H\tau Y, X)\,. \tag{89}$$

The inner product of (80),

$$R^*(X, Y)HZ = HR^*(X, Y)Z - \alpha(X, Y)GZ\,,$$

with X_i gives

$$g(R^*(X, Y)HZ, X_i) = g(HR^*(X, Y)Z, X_i) - \alpha(X, Y)g(GZ, X_i)$$

or, replacing (X, Z) by (X_i, X)

$$g(R^*(X_i, Y)HX, X_i) = g(H\, R^*(X_i, Y)X, X_i) - \alpha(X_i, Y)g(GX, X_i)$$

and, taking the sum over i we have

$$\sum_{i=1}^{4m} g(R^*(X_i, Y)X, HX_i) = -\alpha(GX, Y) - \mathrm{Ric}^*(HX, Y)$$

i.e. (by (89) and replacing X by HX)

$$\begin{aligned} 2\ \mathrm{Ric}^*(X,Y) = 2\,\alpha(FX,Y) + &\\ + g(LHLY, HX) - g(LY, HX)\ &\mathrm{trace}(HL) + \\ + g(LHLHX, Y) - g(LHX, Y)\ &\mathrm{trace}(LH) + \\ + 2\Omega(HX, Y)\ \mathrm{trace}(\tau H) + &2\Omega(Y, H\tau HX) + \\ + 2\Omega(H\tau Y, HX)\,, & \end{aligned} \tag{90}$$

for any $X, Y \in H(M)$. To compute the pseudohermitian Ricci curvature, set first $X = \xi_\lambda$ and $Y = \xi_{\overline{\mu}}$ in (86). We obtain

$$R_{\lambda\overline{\mu}} = i\,m\,\alpha_{\lambda\overline{\mu}} + \frac{1}{2}(A_\lambda^{\overline{\beta}}\,A_{\overline{\beta}\overline{\mu}} - A_{\alpha\lambda}\,A_{\overline{\mu}}^{\alpha})\,,$$

The torsion terms vanish (by $A_{\alpha\beta} = A_{\beta\alpha}$) hence

$$R_{\lambda\overline{\mu}} = i\,m\,\alpha_{\lambda\overline{\mu}}\,. \tag{91}$$

Set also $X = \xi_\lambda$ and $Y = \xi_\mu$ in (86) and note that

$$\begin{gathered} g(F\xi_\lambda, LFL\xi_\mu) = -2iA_{\lambda\mu} = g(LFLF\xi_\lambda, \xi_\mu)\,, \\ g(F\xi_\lambda, L\xi_\mu) = iA_{\lambda\mu} = g(LF\xi_\lambda, \xi_\mu)\,, \\ \mathrm{trace}(FL) = -4m = \mathrm{trace}(LF)\,, \end{gathered}$$

i.e.

$$R_{\lambda\mu} = im\,\alpha_{\lambda\mu} + 2i(2m-1)\,A_{\lambda\mu}\,. \tag{92}$$

Taking traces in (36) we obtain

LEMMA 8. *Let M^{2n+1} be a nondegenerate CR manifold, on which a contact form θ has been fixed. Let g be the Webster metric of (M^{2n+1}, θ). Then*

$$\begin{aligned} R_{\alpha\overline{\beta}} &= 2(g_{\alpha\overline{\beta}} - R^g_{\alpha\overline{\beta}}), & R_{\alpha\beta} &= i(n-1)\,A_{\alpha\beta}\,, \\ R_{0\beta} &= S^{\overline{\alpha}}_{\overline{\alpha}\beta}, & R_{\alpha 0} &= R_{00} = 0\,, \end{aligned}$$

for any local frame $\{\xi_\alpha\}$ *in* $T_{1,0}(M)$. *Here* $R^g_{\alpha\overline{\beta}} = \mathrm{Ric}(\xi_\alpha, \xi_{\overline{\beta}})$ *and* Ric *is the Ricci tensor of* (M^{2n+1}, g). *Also* $S^A_{BC}\xi_A := S(\xi_B, \xi_C)$ *with* $A, B, \cdots \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$ *and* $T_0 = T$.

Cf. also a result in [17]. Combining (92) and Lemma 8 (with $n = 2m$) gives $m\,\alpha_{\lambda\mu} + (2m-1)\,A_{\lambda\mu} = 0$. Yet α is skew, while the Webster torsion is symmetric, hence $\alpha_{\lambda\mu} = 0$ and $A_{\lambda\mu} = 0$. Thus g is a Sasakian metric. As another consequence of $\tau = 0$, (90) becomes

$$2\ \mathrm{Ric}^*(X, Y) = 2\,\alpha(FX, Y) + g(FHFY, HX) + g(FHFHX, Y)$$

(as $\mathrm{trace}(HL) = \mathrm{trace}(HF) = \mathrm{trace}(G) = 0$) and then (by (77))

$$\mathrm{Ric}^*(X, Y) = \alpha(FX, Y)\,,$$

for any $X, Y \in H(M)$. Consequently $R_{\lambda\overline{\mu}} = i\,\alpha_{\lambda\overline{\mu}}$ and by (91) we get $(m-1)\,R_{\lambda\overline{\mu}} = 0$, hence either $m = 1$ or the pseudohermitian Ricci curvature vanishes. Therefore, if $m \geq 2$ then (by Lemma 8) $\mathrm{Ric}^* = 0$. Let us look now at the case $m = 1$. As a consequence of (81) we may write

$$\alpha(FX, X)\|Z\|^2 = g(R^*(FX, X)Z, FZ) + g(R^*(FX, X)GZ, HZ)$$
$$\alpha(HX, GX)\|Z\|^2 = g(R^*(HX, GX)Z, FZ) + g(R^*(HX, GX)GZ, HZ)$$

Summing up the last two identities we get

$$\begin{aligned}&\alpha(FX, X)\|Z\|^2 + \alpha(HX, GX)\|Z\|^2 = \\ &= g(R^*(FX, X)Z, FZ) + g(R^*(FX, X)GZ, HZ) + \\ &\quad + g(R^*(HX, GX)Z, FZ) + g(R^*(HX, GX)GZ, HZ)\end{aligned}$$

Note that the right hand member of this last identity is symmetric in X, Z. Hence

$$\{\alpha(FX, X) + \alpha(HX, GX)\}\|Z\|^2 = \{\alpha(FZ, Z) + \alpha(HZ, GZ)\}\|X\|^2$$

or

$$\{\mathrm{Ric}^*(X, X) + \mathrm{Ric}^*(GX, GX)\}\|Z\|^2 = \{\mathrm{Ric}^*(Z, Z) + \mathrm{Ric}^*(GZ, GZ)\}\|X\|^2$$

Set $Z = X_i$ and sum over i. We have

$$4\{\mathrm{Ric}^*(X,X) + \mathrm{Ric}^*(GX,GX)\} = \sum_{i=1}^{4}\{\mathrm{Ric}^*(X_i,X_i) + \mathrm{Ric}^*(GX_i,GX_i)\}\|X\|^2$$

or, again due to the particular form of our frame

$$2\{\mathrm{Ric}^*(X,X) + \mathrm{Ric}^*(GX,GX)\} = \sum_{i=1}^{4} \mathrm{Ric}^*(X_i,X_i)\|X\|^2$$

Finally, by a result in [17], $\mathrm{trace}(\mathrm{Ric}^*) = 2\rho^*$ hence (by $T \rfloor \mathrm{Ric}^* = 0$, cf. Lemma 8 with $\tau = 0$)

$$\mathrm{Ric}^*(X,Y) + \mathrm{Ric}^*(GX,GY) = \rho^*\, g(X,Y) \tag{93}$$

for any $X, Y \in H(M)$. Note that use was made of the symmetry of Ric^* on $H(M) \otimes H(M)$, a consequence of lemma [8], as well. It remains to be shown that θ is pseudo-Einstein if and only if

$$d(4p + \rho^*\,\theta) = 0 \tag{94}$$

Due to $FG = -GF$, $G^2 = -I$, and $g(GX,GY) = g(X,Y)$ for any $X, Y \in H(M)$, one has

$$G\xi_\alpha = G_\alpha^{\overline{\beta}}\xi_{\overline{\beta}}, \qquad G_\alpha^{\overline{\beta}} G_{\overline{\beta}}^{\lambda} = -\delta_\alpha^\lambda, \qquad g_{\alpha\overline{\beta}} = G_\alpha^{\overline{\mu}} G_{\overline{\beta}}^{\lambda} g_{\lambda\overline{\mu}}\,,$$

for some smooth functions $G_\alpha^{\overline{\beta}} : U \to U$, where $G_{\overline{\alpha}}^{\beta} = \overline{G_\alpha^{\overline{\beta}}}$. Set $\alpha_{AB} = \alpha(\xi_A, \xi_B)$ and note that $\alpha_{\lambda\mu} = 0$, $\alpha_{\overline{\lambda}\overline{\mu}} = 0$. The identity (93) may be written

$$R_{\alpha\overline{\beta}} + i\, G_\alpha^{\overline{\mu}} G_{\overline{\beta}}^{\lambda}\, \alpha_{\lambda\overline{\mu}} = \rho^*\, g_{\alpha\overline{\beta}}\,.$$

Consequently θ is pseudo-Einstein, i.e. $R_{\alpha\overline{\beta}} = (\rho^*/2) g_{\alpha\overline{\beta}}$, if and only if

$$i\,\alpha_{\lambda\overline{\beta}} = (\rho^*/2)\, g_{\alpha\overline{\beta}}\,. \tag{95}$$

As $d(\rho^*\theta) = (d\rho^*) \wedge \theta + \rho^*\, d\theta$ and θ vanishes on $H(M)$, (95) may be written

$$4(dp)(\xi_\alpha, \xi_{\overline{\beta}}) + d(\rho^*\theta)(\xi_\alpha, \xi_{\overline{\beta}}) = 0\,,$$

i.e. $d(4p + \rho^*\theta) = 0$ on $H(M) \otimes H(M)$. Finally, by Lemma 6 if a closed 2-form Ξ vanishes on $H(M) \otimes H(M)$ then $\Xi = 0$, hence (94) holds. $\square$

5.2 – Reducible CR manifolds

Let M be a nondegenerate CR manifold of CR dimension n, θ a contact form on M, g the Webster metric, and ∇^* the Tanaka-Webster connection of (M,θ). Let $\Psi(x)$ the holonomy group of ∇^* with reference point $x \in M$. Let T be the characteristic direction of (M,θ). Since $\nabla^* T = 0$, $\Psi(x)$ is reducible, as a linear group acting on $T_x(M)$. Let $\mathcal{D}_x$ be a subspace of $T_x(M)$ which is invariant by $\Psi(x)$ and such that $T_x \in \mathcal{D}_x$. Let $\mathcal{D}$ be the distribution on M obtained by parallel displacement of $\mathcal{D}_x$, with respect to ∇^*, along curves issuing at x. We shall prove the following

Theorem 8. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M with L_θ positive definite. Then $\mathcal{D}$ is a smooth Pfaffian system on M. Moreover, if g is a Sasakian metric (i.e. $\tau = 0$) then $\mathcal{D}$ is integrable.*

Examples (of strictly pseudoconvex CR manifolds) with $\dim_{\mathbf{R}} \mathcal{D} \geq 2$ do exist. For instance let $M = \mathbf{R}^3$ with the contact form $\theta = \frac{1}{2}(y\,dx - dz)$. The characteristic direction is $T = -2\partial/\partial z$. An associated Riemannian metric is

$$g \;:\; \frac{1}{4}\begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}$$

[and the corresponding $(1,1)$-tensor field is $\varphi(\partial/\partial y) = \partial/\partial x + y\partial/\partial z$, $\varphi(\partial/\partial x + y\partial/\partial z) = -\partial/\partial y$ (and $\varphi(\partial/\partial z) = 0$)]. The Levi-Civita connection of $(\mathbf{R}^3, g)$ is given by

$$\nabla_{\frac{\partial}{\partial y}}\Big(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\Big) = \frac{1}{2}\frac{\partial}{\partial z}\,,\ \nabla_{\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = -\frac{1}{2}\frac{\partial}{\partial z}\ ,$$

$$\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = 0,\ \ \nabla_{\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}}\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} = 0\,,$$

$$\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = -\frac{1}{2}\Big(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\Big)\,,$$

$$\nabla_{\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = \nabla_{\frac{\partial}{\partial z}}\Big(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\Big) = \frac{1}{2}\frac{\partial}{\partial y}\,.$$

A calculation (based on (6)) shows that the distribution $\mathcal{D}$ spanned by $\{\partial/\partial x + y\,\partial/\partial z\,,\, \partial/\partial z\}$ is ∇^*-parallel (hence $\mathcal{D}$ is invariant by pseudohermitian holonomy). The following simple philosophy underlies the preceeding example. Given a contact Riemannian manifold M^{2n+1} let Σ be an integral manifold of the contact distribution $H(M)$, of real dimension $\leq n$ [although $H(M)$ is nonintegrable, it possesses lower dimensional integral manifolds (of dimension not higher than n, cf. [11]. Then $T(\Sigma) \oplus \mathbf{R}\xi$ may be expected to be $\Psi(x)$-invariant. Using this approach, we may build a non-Sasakian example. Precisely, we shall prove

PROPOSITION 3. *Let $H^2(-1)$ be the 2-dimensional hyperbolic space of constant sectional curvature -1 and assume S^1 is embedded in $U(H^2(-1))$ as a fibre over some $x \in H^2(-1)$. Then $T(S^1) \oplus \mathbf{R}\xi$ is invariant by the pseudohermitian holonomy of $U(H^2(-1))$.*

PROOF. Let M^n be a Riemannian manifold. Let (x^i) be a local coordinate system on M^n and (x^i, y^i) the induced local coordinates on $T(M^n)$. The Levi-Civita connection of the Sasaki metric $\tilde{g}$ [with the notations and conventions in Section 2.3] is given by (cf. (13) in [40], p. 539)

$$\begin{aligned} \tilde{\nabla}_{\delta_i}\delta_j &= \Gamma^k_{ij}\delta_k - \frac{1}{2}R^k_{ij0}\dot{\partial}_k \\ \tilde{\nabla}_{\delta_i}\dot{\partial}_j &= \Gamma^k_{ij}\dot{\partial}_k - \frac{1}{2}R^k_{j0i}\delta_k \\ \tilde{\nabla}_{\dot{\partial}_i}\delta_j &= -\frac{1}{2}R^k_{i0j}\delta_k\,, \quad \tilde{\nabla}_{\dot{\partial}_i}\dot{\partial}_j = 0\,. \end{aligned} \tag{96}$$

Here $R^i_{jk0} = R^i_{jk\ell}y^\ell$ (throughout an index 0 denotes contraction with the supporting element y^i). Also we set $\delta_i = \delta/\delta x^i$ and $\dot{\partial}_i = \partial/\partial y^i$ for simplicity. Let ∇ and h be the induced connection on $U(M^n)$ and the second fundamental form of $j : U(M^n) \subset T(M^n)$, respectively. By (96) and the Gauss formula

$$\widetilde{\nabla}_{j_*X}j_*Y = j_*\nabla_X Y + h(X,Y), \quad X, Y \in T(U(M^n))\,,$$

we obtain

$$\begin{aligned} \nabla_{\delta_i}\delta_j &= \Gamma^k_{ij}\delta_k - \frac{1}{2}R^k_{ij0}\dot{\partial}_k\,, & \nabla_{\delta_i}Y &= Y^j{}_{|i}\dot{\partial}_j - \frac{1}{2}Y^j R^k_{j0i}\delta_k\,, \\ \nabla_X\delta_j &= -\frac{1}{2}X^i R^k_{i0j}\delta_k\,, & \nabla_X Y &= X^i\frac{\partial Y^j}{\partial y^i}\dot{\partial}_j + g(X,Y)\nu\,, \end{aligned}$$

for any $X, Y \in \operatorname{Ker}(\pi_*)$, where

$$Y^j{}_{|i} = \frac{\delta Y^j}{\delta x^i} + \Gamma^j_{ik} Y^k$$

and $\nu = y^i \dot{\partial}_i$ is a unit normal section on $U(M^n)$. In particular

$$\nabla_{\delta_i} \xi = -R^j_{i00} \dot{\partial}_j \,, \quad \nabla_X \xi = -2\varphi X - X^i R^j_{i00} \delta_j \,.$$

We ought to recall that the $\{\delta_i\}$ span a (globally defined) distribution N on $T(M^n)$ (the *nonlinear connection* associated with the Levi-Civita connection of M^n) orthogonal to $\operatorname{Ker}(\Pi_*)$ (with respect to the Sasaki metric $\tilde{g}$). Here $\Pi : T(M^n) \to M^n$ is the projection (and $\pi = \Pi_{|M^n}$). While $N_v \subset T_v(U(M^n))$, $v \in U(M^n)$, i.e. any horizontal vector is tangential, for the vertical vectors one has only $\operatorname{Ker}(d_v \pi) = \operatorname{Ker}(d_v \Pi) \cap T_v(U(M^n))$, $v \in U(M^n)$, and $X \in \operatorname{Ker}(\Pi_*)$ is tangential if and only if $g_{ij} X^i y^j = 0$, where $X = X^i \dot{\partial}_i$. It should be noticed that in general the components X^i depend on the directional arguments y^i, i.e. in general X is not the vertical lift of some vector field on M^n [but rather of a section in the pullback bundle $\Pi^{-1} T(M^n) \to T(M^n)$ of $T(M^n)$ via Π, a point of view which will not be needed in this paper]. As $\dot{\partial}_i$ decomposes [with respect to $T(T(M^n)) = j_* T(U(M^n)) \oplus E(j)$] as

$$\dot{\partial}_i = (\delta^j_i - g_{ik} y^k y^j) \dot{\partial}_j + g_{ij} y^j \nu$$

it follows that

$$\varphi(\delta_i) = (\delta^j_i - g_{ik} y^k y^j) \dot{\partial}_j \,.$$

Then (by (6)) the (generalized) Tanaka-Webster connection of $U(M^n)$ is

$$\begin{aligned}
\nabla^*_{\delta_i} \delta_j &= \Gamma^k_{ij} \delta_k + \frac{1}{2} \{ g_{j0} R^k_{i00} - R^k_{ij0} + g_{i0} (\delta^k_j - g_{j0} y^k) \} \dot{\partial}_k \\
\nabla^*_{\delta_i} Y &= \frac{1}{2} Y^j \{ R^\ell_{j0i} g_{\ell 0} y^k - R^k_{j0i} - g_{i0} \delta^k_j \} \delta_k + Y^j{}_{|i} \dot{\partial}_j \\
\nabla^*_X \delta_j &= X^i \Big\{ -\frac{1}{2} R^k_{i0j} + \frac{1}{2} R^k_{i00} g_{j0} + \frac{1}{2} R^\ell_{i0j} g_{\ell 0} y^k - g_{j0} \delta^k_i + g_{ij} y^k \Big\} \delta_k \\
\nabla^*_X Y &= X^i \Big\{ \frac{\partial Y^k}{\partial y^i} + g_{ij} Y^j y^k \Big\} \dot{\partial}_k \,.
\end{aligned}$$

Now assume that S^{n-1} is embedded in $U(M^n)$ as some fibre of π. Then $\nabla^*_X Y \in T(S^{n-1})$, for any $X, Y \in T(S^{n-1})$, and $\nabla^*_{\delta_i} Y \in T(S^{n-1}) \oplus \mathbf{R}\xi$ if and only if

$$R^k_{j0i} + g_{io}\delta^k_j = (R^\ell_{j0i} g_{\ell 0} + g_{i0} g_{j0}) y^k \,. \tag{97}$$

As a byproduct, if $M^n = \mathbf{R}^n$, $n \geq 2$, then $T(S^{n-1}) \oplus \mathbf{R}\xi$ is not ∇^*-parallel. Indeed, if this is the case (97) yields $y^i \delta^k_j = y^i y^j y^k$ hence (by contracting j and k) $(n-1)y^i = 0$, i.e. $n = 1$, a contradiction. Let us contract j and k in (97). We obtain

$$y^j (R_{ji} + (n-1) g_{ji}) = 0 \,,$$

which is clearly satisfied if M^n is an Einstein manifold of scalar curvature $\rho = -(n-1)n$. Finally, when $n = 2$, M^2 has constant curvature -1 and then (97) holds. $\square$

To prove Theorem 8 we need some preparation. Recall (cf. [26] that a smooth curve $\gamma(t)$ in M is a *parabolic geodesic* if it satisfies the ODE

$$\Big(\nabla^*_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} \Big)_{\gamma(t)} = 2c T_{\gamma(t)} \,, \tag{98}$$

for some $c \in \mathbf{R}$ and any value of the parameter t. Let $x \in M$ and $W \in H(M)_x$. By standard theorems on ODEs, there is $\delta > 0$ so that whenever $g_x(W, W)^{1/2} < \delta$ the unique solution $\gamma_{W,c}(t)$ to (98) of initial data (x, W) may be uniquely continued to an interval containing $t = 1$ and the map $\psi_x : B(0, \delta) \subset T_x(M) \to M$ given by $\psi_x(W + cT_x) = \gamma_{W,c}(1)$ (the *parabolic exponential map*) is a diffeomorphism of a sufficiently small neighborhood of $0 \in T_x(M)$ onto a neighborhood of $x \in M$. If $\gamma(t)$ is a parabolic geodesic, a C^1 diffeomorphism $t = \phi(s)$ such that $\hat{\gamma}(s) = \gamma(\phi(s))$ is a parabolic geodesic is called an *Eulerian parameter* for $\gamma(t)$ (the terminology is motivated by Lemma 9 below).

Now let $\{\xi_\alpha\}$ be a local orthonormal frame of $T_{1,0}(M)$, defined on a neighborhood of a point $x \in M$. It determines an isomorphism $\lambda : T_x(M) \to \mathbf{H}_n$ given by $\lambda_x(v) = (\eta^\alpha_x(v) e_\alpha, \theta_x(v))$, for any $v \in T_x(M)$. Here $\mathbf{H}_n = \mathbf{C}^n \times \mathbf{R}$ is the *Heisenberg group* and $\{\eta^\alpha\}$ is the admissible coframe corresponding to $\{\xi^\alpha, T\}$. The resulting local coordinates $(z, x) := \lambda_x \circ$

ψ_x^{-1}, defined in some neighborhood of x, are the *pseudohermitian normal coordinates* at x determined by $\{\xi_\alpha\}$. See also Proposition 2.5 in [26], p. 313. We shall need the following

LEMMA 9. *If $\gamma(t)$ is a solution to (98) such that $(d\gamma/dt)_{t=0} \in H(M)_{\gamma(0)}$, then $\hat\gamma(s) = \gamma(\phi(s))$ with*

$$\phi(s) = \sqrt{\alpha\, s^2 + a\, s + b}, \quad \alpha > 0, \quad a, b \in \mathbf{R}, \quad a^2 - 4\alpha b \geq 0\,, \tag{99}$$

satisfies $(\nabla^*_{d\hat\gamma/ds} d\hat\gamma/ds)_{\hat\gamma(s)} = 2\hat c T_{\hat\gamma(s)}$ with $\hat c = \alpha c$. Viceversa, any Eulerian parameter $t = \phi(s)$ for $\gamma(t)$ is of the form (99). In particular, for any point x on γ and any $W \in H(M)_x$, there is a unique Eulerian parameter $t = \phi(s)$ for γ such that $\hat\gamma(0) = x$ and $(d\hat\gamma/ds)_{s=0} = W$. Moreover, the parabolic geodesic $\gamma = \gamma_{W,c}$ is locally expressed, with respect to a pseudohermitian normal coordinate system, as

$$\gamma : \begin{cases} z^\alpha = W^\alpha t, \quad 1 \leq \alpha \leq n\,, \\ x = ct^2 \end{cases}$$

where $W = W^\alpha \xi_\alpha + W^{\overline\alpha} \xi_{\overline\alpha}$, $W^{\overline\alpha} = \overline{W^\alpha}$. Conversely, any local coordinate system (z,t) with this property is the pseudohermitian normal coordinate system determined by $\{\xi_\alpha\}$.

PROOF. A local coordinate calculation shows that $\hat\gamma(s) = \gamma(\phi(s))$ is a parabolic geodesic, with the constant $\hat c \in \mathbf{R}$, if and only if

$$2[\hat c - c\,\phi'(s)^2]\, T_{\gamma(t)} = \phi''(s)\, \frac{d\gamma}{dt}(t)\,. \tag{100}$$

On the other hand, applying θ to both members of (98) gives (by (7)-(8) in Section 1)

$$2c = \theta\Big(\nabla^*_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt}\Big)_{\gamma(t)} = g\Big(T, \nabla^*_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt}\Big) = \frac{d}{dt}\Big[\theta\Big(\frac{d\gamma}{dt}\Big)\Big]\,,$$

hence

$$\theta\Big(\frac{d\gamma}{dt}\Big)_{\gamma(t)} = 2ct\,.$$

Therefore, by applying θ to (100) we obtain the ODE

$$2[\hat{c} - c\,\phi'(s)^2] = 2ct\phi''(s)\,,$$

with the obvious solution $\phi(s)^2 = (\hat{c}/c)s^2 + as + b$, with $a, b \in \mathbf{R}$ such that $a^2 - 4(\hat{c}/c)b \leq 0$ (and $\hat{c}/c > 0$). To check the second statement in Lemma 9 we reparametrize $\gamma(t)$ as $\hat{\gamma}(s) = \gamma(ts)$ (i.e. we set $a = b = 0$ and $\alpha = t^2$, $t > 0$, in (99)). Then $\psi_x^{-1}(\gamma(t)) = \psi_x^{-1}(\hat{\gamma}(1)) = tW + ct^2 T_x$. ∎

At this point we may prove Theorem 8. Let $y_0 \in M$ and $(z^1, \cdots, z^n, x)$ a pseudohermitian normal coordinate system at y_0, defined on an open set U. Let $\{X_1, \cdots, X_k\}$ be a linear basis of $\mathcal{D}_{y_0}$. Let $y \in U$ and consider the parabolic geodesic expressed locally (with respect to (z, x)) by

$$\gamma \;:\; z^\alpha = z^\alpha(y)t, \;\; x = x(y)t^2\,.$$

Set $X^*_{i,y} = T_\gamma(X_i) \in T_\gamma(\mathcal{D}_{y_0}) = \mathcal{D}_y$, where T_γ is the parallel displacement operator (associated with ∇^* and γ). Clearly $\{X^*_{i,y}\}$ is a basis of $\mathcal{D}_y$ and the vector fields X^*_i are smooth [because the parallel displacement depends differentiably on $(z(y), x(y))$]. To see that $\mathcal{D}$ is involutive, note first that for any $X, Y \in \mathcal{D}$ (by a standard argument, cf. [27], Vol. I, p. 181) $\nabla^*_X Y \in \mathcal{D}$. On the other hand, as $\tau = 0$, the torsion T^* has nozero components only along ξ. Therefore $[X, Y] \in \mathcal{D}$. ∎

It is an open problem to study the geometry of the leaves of $\mathcal{D}$. Note that each leaf of $\mathcal{D}$ is foliated by real curves (the integral lines of the contact vector ξ), i.e. the contact flow of M is a subfoliation ([15]) of $\mathcal{D}$.

6 – Canonical connections

Let c_1 be the first Chern class of $T_{1,0}(M)$. By a result of J. M. Lee, [29], a strictly pseudoconvex CR manifold admitting a globally defined pseudo-Einstein contact form satisfies $c_1 = 0$. For any connection D in $T_{1,0}(M)$, c_1 is represented by the 2-form $-(1/2\pi i)\, {R^D}_\alpha{}^\alpha{}_{AB}\eta^A \wedge \eta^B$. If D is the Tanaka-Webster connection, the components ${R^D}_\alpha{}^\alpha{}_{AB}$ may be computed in terms of ρ^* and its first order derivatives, a procedure which leads to $c_1 = 0$. On a contact Riemannian manifold, Tanno's connection ∇^* does not descend to a connection in $T_{1,0}(M)$. However, canonical connections in $T_{1,0}(M)$ may be built, by taking into account its *almost CR holomorphic* structure (arising from the presence of a natural pre-$\overline{\partial}$-operator induced by ∇^*) and its Hermitian structure (induced by a fixed associated Riemannian metric g).

Let E be a complex vector bundle over an almost CR manifold $(M, T_{1,0}(M))$. A *pre-$\overline{\partial}$-operator* is a differential operator

$$\overline{\partial}_E : \Gamma^\infty(E) \to \Gamma^\infty(T_{0,1}(M)^* \otimes E)$$

such that

$$\overline{\partial}_E(fu) = f\,\overline{\partial}_E u + (\overline{\partial}_H f) \otimes u$$

for any $f \in C^\infty(M)$ and $u \in \Gamma^\infty(E)$.

Examples. 1) Let (M, η) be a contact manifold and ξ the characteristic direction of (M, η). Then

$$\widehat{T}(M) := T_{1,0}(M) \oplus \mathbf{C}\xi \approx (T(M) \otimes \mathbf{C})/T_{0,1}(M)$$

is a complex vector bundle over M (the *holomorphic tangent bundle*) and

$$\overline{\partial}_{\hat{T}(M)} : \Gamma^\infty(\hat{T}(M)) \to \Gamma^\infty(T_{0,1}(M)^* \otimes \hat{T}(M)),$$
$$(\overline{\partial}_{\hat{T}(M)} W)\overline{Z} := [\overline{Z}, W]_{\hat{T}(M)}, \quad Z \in T_{1,0}(M), \;\; W \in \hat{T}(M),$$

is a pre-$\overline{\partial}$-operator on $\hat{T}(M)$. Here $X_{\hat{T}(M)}$ denotes the $\hat{T}(M)$ component of $X \in T(M) \otimes \mathbf{C} = \hat{T}(M) \oplus T_{0,1}(M)$.

2) Let $g \in \mathcal{M}(\eta)$ be an associated Riemannian metric and ∇^* the Tanno connection of (M, η, g). Then

$$\overline{\partial}_{T_{1,0}(M)} : \Gamma^\infty(T_{1,0}(M)) \to \Gamma^\infty(T_{0,1}(M)^* \otimes T_{1,0}(M)),$$
$$(\overline{\partial}_{T_{1,0}(M)} V)\overline{Z} := \pi_{1,0}\, \nabla^*_{\overline{Z}} V, \;\; Z, V \in T_{1,0}(M),$$

is a pre-$\overline{\partial}$-operator on $T_{1,0}(M)$. Here $\pi_{1,0} : T(M) \otimes \mathbf{C} \to T_{1,0}(M)$ is the natural projection associated with $T(M) \otimes \mathbf{C} = T_{1,0}(M) \oplus T_{0,1}(M) \oplus \mathbf{C}\xi$.

Let $(M, T_{1,0}(M))$ be a CR manifold. A pre-$\overline{\partial}$-operator $\overline{\partial}_E$ satisfying the *integrability condition*

$$\overline{Z} \cdot \overline{W} \cdot u - \overline{W} \cdot \overline{Z} \cdot u = [\overline{Z}, \overline{W}] \cdot u,$$
$$\overline{Z} \cdot u := (\overline{\partial}_E u)\overline{Z}, \quad Z, W \in T_{1,0}(M), \;\; u \in \Gamma^\infty(E),$$

is a $\overline{\partial}$-*operator* and $(E, \overline{\partial}_E)$ is a *CR-holomorphic* vector bundle. When the almost CR structure of (M, η) is integrable $(\hat{T}(M), \overline{\partial}_{\hat{T}(M)})$ is CR-holomorphic. Also, if $T_{1,0}(M)$ is integrable, ∇^* is the Tanaka-Webster connection of (M, η) and then (cf. e.g. [45], p. 569) $\overline{\partial}_{T_{1,0}(M)}$ is a $\overline{\partial}$-operator.

6.1 – P-connections

We shall prove the following

THEOREM 9. *Let (M, η) be a real $(2n+1)$-dimensional contact manifold and $g \in \mathcal{M}(\eta)$ an associated Riemannian metric. Let $(E, H) \to M$ be a Hermitian vector bundle, where H is the Hermitian metric, $\overline{\partial}_E$ a pre-$\overline{\partial}$-operator on E, and $P \in \Gamma^\infty(End(E, H))$ a skew-symmetric bundle endomorphism, i.e. $H(Pu, v) + H(u, Pv) = 0$ for any $u, v \in \Gamma^\infty(E)$. There is a unique connection D in E so that* i) $D^{0,1} = \overline{\partial}_E$, ii) $DH = 0$, *and* iii) $\Lambda_g R^D = 2nP$.

Given a connection D in E, $D^{0,1}u$ is the restriction of Du to $T_{0,1}(M)$. Also R^D denotes the curvature of D. The *trace* of R^D is given by

$$i\,(\Lambda_g R^D)u = \sum_{\alpha=1}^{n} R^D(\xi_\alpha, \xi_{\overline{\alpha}})u\,,$$

where $\{\xi_\alpha\}$ is a (local) orthonormal ($g(\xi_\alpha, \xi_{\overline{\beta}}) = \delta_{\alpha\beta}$) frame of $T_{1,0}(M)$. We establish first uniqueness. Let D be a connection in E satisfying (i)-(iii) in Theorem 9. Then

$$D_{\overline{Z}}u = (\overline{\partial}_E u)\overline{Z}\,, \tag{101}$$

$$H(D_Z u, v) = Z(H(u, v)) - H(u, (\overline{\partial}_E v)\overline{Z}) \tag{102}$$

hence it remains to be shown that $D_\xi u$ is uniquely determined. To this end, set

$$(D^2u)(X, Y) := D_X D_Y u - D_{\nabla^*_X Y} u\,,$$
$$B(X, Y)u := (D^2u)(X, Y) - (D^2u)(Y, X)$$

where ∇^* is the Tanno connection of (M, η, g). Note that

$$B(X, Y)u = R^D(X, Y)u - D_{T^*(X,Y)}u \tag{103}$$

for any $X, Y \in T(M)$. Taking traces in (103) and noting that

$$\sum_{\alpha=1}^{n} T^*(\xi_\alpha, \xi_{\overline{\alpha}}) = -2\, n\, i\, \xi$$

we obtain

$$D_\xi u = \frac{1}{2n}(\Lambda_g B)u - P(u)\,. \tag{104}$$

Claim. $\Lambda_g B$ is determined by (101)-(102). Therefore, by (101)-(102) and (104) D is uniquely determined.

Note that $Q(X, Y) \in \Gamma^\infty(H(M))$ for any $X, Y \in T(M)$. Indeed, applying η to (5) gives

$$\eta(Q(X,Y)) = g((\nabla_Y \varphi)X, \xi) + (\nabla_Y \eta)\varphi X = g(\nabla_Y \varphi X, \xi) - \eta(\nabla_Y \varphi X) = 0\,.$$

Moreover (by (10))

$$-i\nabla^*_Z \overline{Z} - \varphi \nabla^*_Z \overline{Z} = Q(\overline{Z}, Z)$$

hence $\nabla^*_Z \overline{Z} \in H(M) \otimes \mathbf{C}$, for any $Z \in T_{1,0}(M)$ and then (by (101)-(102))

$$B(Z, \overline{Z})u = D_Z D_{\overline{Z}} u - D_{\overline{Z}} D_Z u - D_{\nabla^*_Z \overline{Z}} u + D_{\nabla^*_{\overline{Z}} Z} u$$

is determined. The Claim is proved. Let us prove the existence statement in Theorem 9. Define $D : \Gamma^\infty(E) \to \Gamma^\infty(T(M)^* \otimes E)$ as the (real) differential operator given by (101)-(102) and (104). *Then D is a connection in E*. Let us check for instance that D_ξ is a derivation in $\Gamma^\infty(E)$ (as a $C^\infty(M)$-module). Note that

$$B(X, Y)(fu) = fB(X, Y)u - T^*(X, Y)(f)u$$

hence

$$(\Lambda_g B)(fu) = f\,(\Lambda_g B)u + 2\,n\,\xi(f)u\,.$$

Next (by (104))

$$\begin{aligned} D_\xi(fu) &= \frac{1}{2n}(\Lambda_g B)(fu) - P(fu) = \\ &= \frac{1}{2n} f(\Lambda_g B)u + \xi(f)u - fP(u) = f\, D_\xi u + \xi(f)u\,. \end{aligned}$$

Clearly (101) yields $D^{0,1} = \overline{\partial}_E$. Also (by (103))

$$i(\Lambda_g R^D)u = \sum_{\alpha=1}^{n} B(\xi_\alpha, \xi_{\overline{\alpha}})u - inD_\xi u = i(\Lambda_g B)u - 2inD_\xi u = 2inP(u)\,.$$

Finally, we wish to check that $DH = 0$. We have already

$$V(H(u,v)) = H(D_V u, v) + H(u, D_{\overline{V}} v) \tag{105}$$

for any $V \in H(M) \otimes \mathbf{C}$. As a consequence of (105)

$$T^*(V,W)(H(u,v)) = -H(B(V,W)u,v) - H(u, B(\overline{V},\overline{W})v) \tag{106}$$

for any $V, W \in H(M) \otimes \mathbf{C}$. Taking traces in (106) gives (as B is skew)

$$n\xi(H(u,v)) = H((\Lambda_g B)u, v) + H(u, (\Lambda_g B)v)$$

hence (by (104))

$$\xi(H(u,v)) = H(D_\xi u + P(u), v) + H(u, D_\xi v + P(v))\,. \qquad \square$$

Corollary 2. *Let (M,η) be a contact manifold of CR dimension n, $g \in \mathcal{M}(\eta)$ an associated Riemannian metric. The canonical connection D in $T_{1,0}(M)$ extending $\overline{\partial}_{T_{1,0}(M)}$, parallelizing g, and such that $\Lambda_g\, R^D = 0$ is given by*

$$D = \pi_{1,0}\, \nabla^* + \frac{1}{2n}\, \eta \otimes \pi_{1,0}\, \Lambda_\eta R^* - \frac{i}{4n}\, g^{\alpha\overline{\beta}} Q^{\overline{\rho}}_{\mu\alpha} Q^{\gamma}_{\overline{\rho}\,\overline{\beta}}\ \eta \otimes \eta^\mu \otimes \xi_\gamma\,,$$

for any (*local*) *frame $\{\xi_\alpha\}$ in $T_{1,0}(M)$.*

Using the connection D of Corollary 2 one may express the Chern classes of $T_{1,0}(M)$ in terms of $R_\alpha{}^\beta{}_{\lambda\overline{\mu}}$ and the Tanno tensor (and its first order covariant derivatives). It is an open problem whether the first Chern class of $T_{1,0}(M)$ vanishes on contact Riemannian manifolds admitting global pseudo-Einsten contact forms.

6.2 – Almost CR structures as G-structures

Let G_0 be the Lie subgroup of $GL(2n+1, \mathbf{R})$ consisting of all matrices of the form

$$\begin{pmatrix} u & 0 & 0 \\ a^\alpha & a^\alpha_\beta & b^\alpha_\beta \\ b^\alpha & -b^\alpha_\beta & a^\alpha_\beta \end{pmatrix}, \quad u \in \mathbf{R} \setminus \{0\}, \;\; a^\alpha, b^\alpha \in \mathbf{R}, \;\; [a^\alpha_\beta + i\, b^\alpha_\beta] \in GL(n, \mathbf{C}) .$$

Any G_0-structure (in the sense of [37], p. 309) on a real $(2n+1)$-dimensional manifold M determines an almost CR structure on M and conversely. Originally, S.S. Chern & J.K. Moser (cf. [14]) regarded almost CR structures as principal subbundles of the principal $GL(2n+1, \mathbf{C})$-bundle of linear frames in $T^*(M) \otimes \mathbf{C}$, rather than G_0-structures on M. The two points of view are equivalent due to the obvious group monomorphism $G_0 \to GL(2n+1, \mathbf{C})$

$$\begin{pmatrix} u & 0 & 0 \\ a^\alpha & a^\alpha_\beta & b^\alpha_\beta \\ b^\alpha & -b^\alpha_\beta & a^\alpha_\beta \end{pmatrix} \longmapsto \begin{pmatrix} u & 0 & 0 \\ a^\alpha + ib^\alpha & a^\alpha_\beta + ib^\alpha_\beta & 0 \\ a^\alpha - ib^\alpha & 0 & a^\alpha_\beta - ib^\alpha_\beta \end{pmatrix} .$$

Let $B \to M$ be a G_0-structure of M. If B is locally flat (in the sense of [37], p. 315) then the corresponding almost CR structure is Levi flat. Therefore, the G_0-structure arising from the almost CR structure of a contact Riemannian manifold is not locally flat. Let

$$c : B \to \mathrm{Hom}(\mathbf{R}^{2n+1} \wedge \mathbf{R}^{2n+1}, \mathbf{R}^{2n+1}) / \partial\, \mathrm{Hom}(\mathbf{R}^{2n+1}, L(G_0))$$

be the first structure function of B (cf. [37], p. 318). As is well known, $c = 0$ is only a necessary condition for locally flatness of the given G-structure. It is a natural question whether $c = 0$ for the almost CR structure of a contact Riemannian manifold. As an application of Theorem 9 we shall prove the following

Theorem 10. *Let (M, η, g) be a contact Riemannian manifold. The $U(n) \times 1$-structure $B(\eta) \to M$ has a nonzero first structure function.*

Proof. The connection D furnished by Corollary 2 extends to a linear connection on M by setting $D_X\overline{Z} = \overline{D_X Z}$ and $D_X\xi = 0$ for any $X \in T(M)$ and $Z \in T_{1,0}(M)$. As D parallelizes $T_{1,0}(M)$ and g, it gives rise to a connection Γ in $B(\eta)$. Consequently $c(u) = [c_\Gamma(u)]$ (the class of $c_\Gamma(u)$ modulo ∂ Hom$(\mathbf{R}^{2n+1}, L(U(n)\times 1)))$ for any $u \in B(\eta)$, where $c_\Gamma(u)(x \wedge y) = (d\omega)_u(X,Y)$ and $\omega \in \Gamma^\infty(T^*(B(\eta)) \otimes \mathbf{R}^{2n+1})$ is the canonical 1-form. Also $X, Y \in \Gamma_u$ are such that $\omega_u(X) = x$ and $\omega_u(Y) = y$, with $x, y \in \mathbf{R}^{2n+1}$. Let $\{X_\alpha, \varphi X_\alpha, \xi\}$ be a local section in $B(\eta)$, defined on an open set U. Of course, the torsion form of Γ may may be express in terms of the torsion tensor T^D of D, hence

$$2c_\Gamma(u)(e_\alpha \wedge e_{\beta+n}) = u^{-1}(T^D(X_\alpha, \varphi X_\beta)_p)\,,$$

where $u = (p, \{X_{\alpha,p}, \varphi_p X_{\alpha,p}, \xi_p\},\ \ p \in U$. Set $\xi_\alpha = (1/\sqrt{2})(X_\alpha - i\varphi X_\alpha)$. Using (47)-(52) we obtain

$$T^D(\xi_\alpha, \xi_\beta) = \frac{i}{2}(Q^{\overline{\rho}}_{\beta\alpha} - Q^{\overline{\rho}}_{\alpha\beta})\xi_{\overline{\rho}},\ \ T^D(\xi_\alpha, \xi_{\overline{\beta}}) = -i\,\delta_{\alpha\beta}\,\xi\,,$$

hence

$$\langle c_\Gamma(e_\alpha \wedge e_{\beta+n}), e_0\rangle = -\delta_{\alpha\beta}\,,$$

where $\langle\, ,\, \rangle$ is the Euclidean inner product in $\mathbf{R}^{2n+1}$. ∎

7 – Gauge invariants

Let (M, η, g) be a contact Riemannian manifold. As as a consequence of axioms (8)-(9)

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= 2\,g(\nabla^*_X Y, Z) + \\ + 2g(X,\varphi Z)\eta(Y) + 2g(Y,\varphi Z))\eta(X) &- 2g(X,\varphi Y)\eta(Z) + \\ + g([X,Z],Y) + g([Y,Z],X) &- g([X,Y],Z)\,, \end{aligned}$$

for any $X, Y, Z \in T(M)$. This leads to the explicit expressions of the connection coefficients

$$\Gamma^\rho_{\alpha\beta}\, g_{\rho\overline{\gamma}} = \xi_\alpha(g_{\beta\overline{\gamma}}) - g([\xi_\alpha, \xi_{\overline{\gamma}}], \xi_\beta) \tag{107}$$

$$\Gamma^\rho_{\overline{\alpha}\beta}\, g_{\rho\overline{\gamma}} = g([\xi_{\overline{\alpha}}, \xi_\beta], \xi_{\overline{\gamma}}) \tag{108}$$

$$\Gamma^\rho_{0\beta}\, g_{\rho\overline{\gamma}} = \xi(g_{\beta\overline{\gamma}}) - g([\xi, \xi_{\overline{\gamma}}], \xi_\beta)\,, \tag{109}$$

and then to the transformation laws in

Lemma 10. *Under a gauge transformation* (12) *the connection coefficients of the Tanaka-Webster connection and the Webster torsion change as*

$$\widehat{\Gamma}^{\rho}_{\alpha\beta} = \Gamma^{\rho}_{\alpha\beta} + 2(u_\alpha\,\delta^{\rho}_{\beta} + u_\beta\,\delta^{\rho}_{\alpha}) \tag{110}$$

$$\widehat{\Gamma}^{\rho}_{\overline{\alpha}\beta} = \Gamma^{\rho}_{\overline{\alpha}\beta} - 2\,u^{\rho}\,g_{\overline{\alpha}\beta} \tag{111}$$

$$e^{2u}\widehat{\Gamma}^{\rho}_{\hat{0}\beta} = \Gamma^{\rho}_{0\beta} + 2u_0\delta^{\rho}_{\beta} + 2iu_\beta u^{\rho} - iu_\beta{}^{\rho} + iu^{\alpha}\Gamma^{\rho}_{\alpha\beta} - iu^{\overline{\alpha}}\Gamma^{\rho}_{\overline{\alpha}\beta} \tag{112}$$

$$\widehat{A}_{\alpha\beta} = A_{\alpha\beta} - iu_{\alpha\beta} + 2iu_\alpha u_\beta + \frac{1}{2}(Q^{\overline{\gamma}}_{\rho\alpha} - Q^{\overline{\gamma}}_{\alpha\rho})u^{\rho}g_{\beta\overline{\gamma}} \tag{113}$$

where $u := \log\sqrt{\lambda}$. *Consequently, the connection* 1*-forms change as*

$$\begin{aligned}\hat{\omega}^{\alpha}_{\beta} &= \omega^{\alpha}_{\beta} + 2u_0\delta^{\alpha}_{\beta}\eta + 2(u_\gamma\delta^{\alpha}_{\beta} + u_\beta\delta^{\alpha}_{\gamma})\eta^{\gamma} - 2u^{\alpha}\eta_\beta + \\ &\quad - i(u_\beta{}^{\alpha} + 2\delta^{\alpha}_{\beta}u_\gamma u^{\gamma} + 2u^{\alpha}u_\beta)\eta\end{aligned} \tag{114}$$

Also $\hat{Q}^{\overline{\alpha}}_{\beta\gamma} = Q^{\overline{\alpha}}_{\beta\gamma}$, *i.e.* $Q^{\overline{\alpha}}_{\beta\gamma}$ *is a gauge invariant. Here* $u_\alpha{}^{\rho} = u_{\alpha\overline{\beta}}\,g^{\rho\overline{\beta}}$ *and* u_{AB} *denote second order covariant derivatives* (*with respect to* ∇^*).

Proof. Note that $\eta(\zeta) = 0$, $\zeta(\lambda) = 0$ and $\hat{g}_{\alpha\overline{\beta}} = \lambda\,g_{\alpha\overline{\beta}}$. Therefore

$$\hat{\xi} = e^{-2u}\{\xi + iu^{\alpha}\xi_\alpha - iu^{\overline{\alpha}}\xi_{\overline{\alpha}}\}\,,$$

where $u^{\alpha} = g^{\alpha\overline{\beta}}u_{\overline{\beta}}$. A straightforward calculation (based on (12)) shows that (107)-(108) yield (110)-(111). To prove (112) one establishes first the identity

$$\begin{aligned}e^{2u}[\hat{\xi},\xi_{\overline{\beta}}] &= [\xi,\xi_{\overline{\beta}}] + \Big\{2iu_{\overline{\beta}}u^{\alpha} - iu^{\alpha}{}_{\overline{\beta}} + \frac{1}{2}(Q^{\alpha}_{\overline{\beta}\,\overline{\gamma}} - Q^{\alpha}_{\overline{\gamma}\,\overline{\beta}})u^{\overline{\gamma}}\Big\}\xi_\alpha + \\ &\quad + \{iu^{\overline{\alpha}}{}_{\overline{\beta}} - 2iu_{\overline{\beta}}u^{\overline{\alpha}} + i(\Gamma^{\overline{\alpha}}_{\gamma\overline{\beta}}u^{\gamma} - \Gamma^{\overline{\alpha}}_{\overline{\gamma}\,\overline{\beta}}u^{\overline{\gamma}})\}\xi_{\overline{\alpha}}\end{aligned} \tag{115}$$

[and then (109) is shown to imply (112)]. Also (113) follows from

$$A_{\alpha\beta} = -g([\xi,\xi_\alpha]\,,\,\xi_\beta)$$

and (115). To prove (114) one substitutes from (110)-(112) into

$$\hat{\omega}^{\alpha}_{\beta} = \hat{\Gamma}^{\alpha}_{\gamma\beta}\,\hat{\eta}^{\gamma} + \hat{\Gamma}^{\alpha}_{\overline{\gamma}\beta}\,\hat{\eta}^{\overline{\gamma}} + \hat{\Gamma}^{\alpha}_{\hat{0}\beta}\,\hat{\eta}\,,$$

where $\{\hat{\eta}^\gamma\}$ is the admissible coframe (corresponding to $\{\xi_\alpha, \hat{\xi}\}$) given by

$$\hat{\eta}^\alpha = \eta^\alpha - i\, u^\alpha\, \eta\,.$$

Finally, from

$$Q^{\overline{\alpha}}_{\beta\gamma}\, g_{\overline{\alpha}\mu} = 2i\, g([\xi_\mu, \xi_\beta]\,,\, \xi_\gamma)$$

it follows that $\hat{Q}^{\overline{\alpha}}_{\beta\gamma} = Q^{\overline{\alpha}}_{\beta\gamma}$. ∎

We shall prove the following

Theorem 11. *Let (M,η,g) be a contact Riemannian manifold and G_η its Fefferman metric. Then, under a gauge transformation* (12), *G_η changes conformally, i.e. $G_{\hat{\eta}} = (\lambda \circ \pi)\, G_\eta$. Consequently the restricted conformal class $[G_\eta] = \{(\lambda \circ \pi)\, G_\eta \,:\, \lambda \in C^\infty(M),\ \lambda > 0\}$ is a gauge invariant.*

Theorem 11 is the contact Riemannian analogue of the main result in [28]. S. Tanno considered (cf. [39], p. 363) the second order differential operator Δ_H [coinciding with the sublaplacian Δ_b when $T_{1,0}(M)$ is integrable] given by

$$\Delta_H u = \Delta u - \xi(\xi(u)), \quad u \in C^\infty(M)\,,$$

where Δ is the ordinary Laplacian of the Riemannian manifold (M, g). To prove Theorem 11 we contract the indices α, β in (114) and use

$$\Delta_H u = 2\, {u_\alpha}^\alpha + 2\, i\, n\, u_0$$

[a consequence of definitions and of (29)] to yield

$$\hat{\omega}^\alpha_\alpha = \omega^\alpha_\alpha + n\, du + (n+2)(u_\alpha \eta^\alpha - u^\alpha \eta_\alpha) - i\Big\{2(n+1) u_\alpha u^\alpha + \frac{1}{2}\Delta_H u\Big\}\eta\,. \tag{116}$$

Differentiating we obtain

$$\begin{aligned} d\,\hat{\omega}^\alpha_\alpha \equiv d\,\omega^\alpha_\alpha &+ (n+2) d(u_\alpha \eta^\alpha - u^\alpha \eta_\alpha) + \\ &- i\Big\{2(n+1) u_\alpha u^\alpha + \frac{1}{2}\Delta_H u\Big\} d\eta\,, \quad \operatorname{mod} \eta\,. \end{aligned}$$

On the other hand (by (30))

$$d(u_\alpha \eta^\alpha - u^\alpha \eta_\alpha) \equiv -(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha})\, \eta^\alpha \wedge \eta^{\overline{\beta}}, \quad \mathrm{mod}\ \eta^\alpha \wedge \eta^\beta,\ \eta^{\overline{\alpha}} \wedge \eta^{\overline{\beta}},\ \eta\,,$$

hence

$$d\,\hat{\omega}_\alpha^\alpha \equiv d\,\omega_\alpha^\alpha - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha})\, \eta^\alpha \wedge \eta^{\overline{\beta}} - i\Big\{2(n+1)u_\alpha u^\alpha + \frac{1}{2}\Delta_H u\Big\}\, d\eta\,.$$

Let us multiply by i, take traces in the resulting identity, use the calculations in Section 4.2, and observe the cancellation of the terms involving the Tanno tensor (by the gauge invariance in Lemma 10). We obtain

$$e^{2u}\,\hat{\rho}^* = \rho^* - 2(n+1)\Delta_H u - 4n(n+1)\, u_\alpha u^\alpha\,. \tag{117}$$

Using (116)-(117) and the gauge invariance of the fibre coordinate $\hat{\gamma} = \gamma$, one may derive the transformation law for the real 1-form σ in Section 4.2 [under a gauge transformation (12)]

$$\hat{\sigma} = \sigma + \pi^*\{i(u_\alpha \eta^\alpha - u^\alpha \eta_\alpha) + u_\alpha u^\alpha\, \eta\}\,.$$

Next, note that $L_\eta = -2\eta^\alpha \odot \eta_\alpha$ hence

$$L_{\hat{\eta}} = e^{2u}\{L_\eta - 2i(u_\alpha \eta^\alpha - u^\alpha \eta_\alpha) \odot \eta - 2u_\alpha u^\alpha\, \eta \odot \eta\}$$

and then $G_{\hat{\eta}} = e^{2u\circ\pi}\, G_\eta$. □

Remark 5. If $\hat{\eta} = v^{2/n}\eta$, that is $\lambda = e^{2u} = v^{2/n}$, then (117) becomes

$$-\frac{2(n+1)}{n}\Delta_H v + \rho^* v = \hat{\rho}^* v^{(n+2)/n}$$

which has been obtained by S. Tanno, [39], by a different technique. As stated in [39], this is the contact Riemannian analogue of the *CR Yamabe problem* in [26]. Whether the Yamabe equation for the (generalized) Fefferman metric G_η projects on M to give (117) is unclear as yet. Indeed the wave operator $\square$ [i.e. the Laplace-Beltrami operator of $(F(M), G_\eta)$] pushes forward to an operator $\pi_*\square$ which turns out to be precisely $\frac{1}{2}\Delta_H$ (cf. [3], p. 30) yet it is unknown whether the scalar curvature of G_η

projects on (a multiple of) ρ^* [the proof in [28] employs the Chern-Moser *normal form*, which is unavailable yet for a contact Riemannian manifold (with nonintegrable almost CR structure)].

As another consequence of Lemma 10 one obtains

THEOREM 12. *Let (M, η, g) be a contact Riemannian manifold. Let $(E, H) \to M$ be a Hermitian vector bundle and $\overline{\partial}_E$ a pre-$\overline{\partial}$-operator on E. Let D be the unique connection in E extending $\overline{\partial}_E$, parallelizing H and of zero curvature trace. Then D is a gauge invariant (and $\Lambda_g R^D = 0$ a gauge invariant condition).*

PROOF. Note that $\hat{D}_X s = D_X s$ for any $X \in H(M)$ and any $s \in \Gamma^\infty(E)$. Let $\{\xi_\alpha\}$ be a local orthonormal frame of $T_{1,0}(M)$. Then (by the very definitions)

$$\widehat{B}(\xi_\alpha, \xi_{\overline{\alpha}})s = B(\xi_\alpha, \xi_{\overline{\alpha}})s + 2(u^{\overline{\beta}} D_{\xi_{\overline{\beta}}} s - u^\beta D_{\xi_\beta} s)\,,$$

hence

$$e^{2u}(\Lambda_{\hat{g}} \hat{B})s = (\Lambda_g B)s + 2in(u^\alpha D_{\xi_\alpha} s - u^{\overline{\alpha}} D_{\xi_{\overline{\alpha}}})\,.$$

Finally, a calculation based on (104) with $P = 0$ leads to $\hat{D}_\xi s = D_\xi s$. $\square$

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