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The Atiyah-Ward Ansätze in 3-dimensions

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Riassunto: *Sia* $\mathbb{M}$ *una varietà conforme autoduale complessa di dimensione* 4*,* Z *il suo spazio di twistori e* $\widehat{W}$ *un fibrato olomorfo di rango* 1*, su* Z*, banale sulle reti twistoriali. L'idea del* A_k *Atiyah-Ward Ansatz è di scrivere soluzioni autoduali di Yang-Mills ottenute come trasformazione di Penrose-Ward di una estensione* E

$$0 \to \mathcal{O}(-k) \otimes \widehat{W} \to E \to \mathcal{O}(k) \otimes \widehat{W}^* \to 0$$

su Z*. Siccome l'estensione* E *viene definita dal fibrato in rete* $\widehat{W}$ *e da una classe di estenzione in* $H^1\,(Z, \mathcal{O}(-2k) \otimes \widehat{W}^2)$*, il* A_k *Atiyah-Ward Ansatz dà un metodo per generare un* $SL(2,\mathbb{C})$ *campo di Yang-Mills autoduale rispetto al gruppo* $SL(2,\mathbb{C})$ *da un campo di Maxwell autoduale (che corrisponde a* $\widehat{W}$*) e da una soluzione di una equazione ausiliaria (che corrisponde all'elemento di* $H^1(Z, \mathcal{O}(-2k) \otimes \widehat{W}^2)$*. Imponendo condizioni di realtà, simili Ansätze sono stati ottenuti sulle varietà riemanniane conformi autoduali.*

In questo articolo, riduciamo il A_k *Atiyah-ward Ansatz (per* $k = 0, 1, 2$*) dalle varietà riemanniane autoduali di dimensione* 4 *alle varietà reali di Einstein-Weyl di dimensione* 3*. Otteniamo così un metodo per costruire una soluzione delle equazioni di Einstein-Weyl Bogomolny da una soluzione dell'equazione dei monopoli abeliani ed una soluzione di una equazione ausiliara. Soluzioni delle equazioni di Einstein-Weyl Bogomolny sono importanti poiché si possono usare per trovare degli esempi espliciti di varietà autoduali di dimensine* 4*.*

Abstract: *Consider* $\mathbb{M}$ *a complex self-dual conformal* 4*-manifold,* Z *its twistor space and* $\widehat{W}$ *a holomorphic line bundle over* Z *trivial on the twistor lines. The idea of the* A_k *Atiyah-Ward Ansatz is to write down self-dual Yang-Mills solutions which are the Penrose-Ward transform of an extension* E

$$0 \to \mathcal{O}(-k) \otimes \widehat{W} \to E \to \mathcal{O}(k) \otimes \widehat{W}^* \to 0$$

over Z*. Since the data which define* E *is the line bundle* $\widehat{W}$ *and the extension class* *in* $H^1(Z, \mathcal{O}(-2k) \otimes \widehat{W}^2)$*, the* A_k *Atiyah-Ward Ansatz provides a method of generating a* $SL(2,\mathbb{C})$ *self-dual Yang-Mills field from a self-dual Maxwell field (which corresponds to* $\widehat{W}$*) and a solution of an auxiliary equation (which corresponds to* $H^1(Z, \mathcal{O}(-2k) \otimes \widehat{W}^2)$*). Imposing reality conditions similar Ansätze on Riemannian self-dual conformal* 4*-manifolds have been obtained.*

In this paper we will reduce the A_k *Atiyah-Ward Ansatz (for* $k = 0, 1, 2$*) from* 4*-dimensional Riemannian self-dual manifolds to* 3*-dimensional real Einstein-Weyl manifolds. We obtain a method of constructing a solution of the Einstein-Weyl Bogomolny equations from a solution of the abelian monopole equation and a solution of an auxiliary equation. Solutions of Einstein-Weyl Bogomolny equations are important since they can be used to find explicit examples of self-dual* 4*-manifolds.*

1 – Conformal geometry

The material from this section can be found in [2], [3] and [4]. All our manifolds will be real. For a manifold M, TM and $\mathcal{E}^k_M$ will denote the tangent bundle of M and the bundle of k forms on M respectively.

Density line bundles and conformal structures. If V is a real n-dimensional vector space and w any real number, the one dimensional linear space $L^w = L^w(V)$ carrying the representation $A \to |\det A|^{w/n}$ of $\mathrm{GL}(V)$ is called the space of densities of weight w. For M an arbitrary manifold, we will use the density line bundle $L^w = L^w(TM)$ of M which is defined to be the bundle over M whose fiber at $x \in M$ is $L^w(T_xM)$. Any sub-bundle or quotient bundle of $L^k \otimes \Lambda^s(TM) \otimes \mathcal{E}^p_M$ is said to have weight $k + s - p$.

A conformal structure on M is a positive definite bilinear form c on $L^{-1}TM$ (when tensoring with a density line bundle we generally omit the tensor product sign). If the conformal manifold (M, c) is also oriented and has dimension n, there is a Hodge star operator $*$ which maps a k form on M to an $(n-k)$ form on M with values in L^{n-2k} and which satisfies the relations

$$\begin{cases} i_X * (\alpha) = *(X \wedge \alpha) \\ *(1) = \mathrm{or}_M\,. \end{cases}$$

for any vector field X and form α on M. (Here $\mathrm{or}_M \in \Gamma(M, L^n \otimes \mathcal{E}^n_M)$ is the unit section given by the orientation. Also, for the wedge product we

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shall always use the convention $(X \wedge Y)(Z) := \langle X, Z\rangle Y$-$\langle Y, Z\rangle X$, where $\langle\cdot,\cdot\rangle$ denotes the conformal structure c.) A direct calculation shows that $*^2 = (-1)^{\frac{1}{2}n(n-1)}$.

Weyl connections. A Weyl connection on a conformal manifold is a connection on L^1. The fundamental theorem of conformal geometry (see [7]) states that on a conformal manifold (M, c) there is a bijective correspondence between Weyl connections and torsion free connections on TM which preserve the conformal structure c. The corresponding linear map sends a 1-form γ to the $co(TM)$-valued 1-form Γ defined by $\Gamma_X = \gamma(X)\,\mathrm{Id} + \gamma \wedge X$, where "Id" denotes the identity endomorphism.

The Jones and Tod construction. Let M be a 4-dimensional self-dual oriented conformal manifold and K is a non-vanishing conformal vector field. Then $B := M/K$ inherits an orientation related to the orientation of M by the formula

$$*_M(\xi \wedge \alpha) = (-1)^{k+1} *_B (\alpha)$$

which holds for any k-form α on B (where $\xi := K/|K|$), a conformal structure c_B and a Weyl derivative D^B which respect to which it is Einstein-Weyl (that is, the symmetric trace free part of the Ricci tensor of D^B vanishes). Moreover, the Einstein-Weyl space B comes with a solution (w, A) (where $A \in \mathcal{E}^1(B)$ and $w \in L^{-1}$) of the abelian monopole equation $*_B D^B(w) = dA$ from which M can be recovered: the real line bundle M is locally isomorphic to $U \times \mathbb{R}$; the conformal structure on M is given locally by the formula $\pi^*(c_B) + w^{-2}(dt + A)^2$ and the conformal vector field K is $\frac{\partial}{\partial t}$. In other words there is a correspondence between self-dual 4-spaces with symmetry and Einstein-Weyl 3-spaces with monopoles (see [3] and [5] for details about this construction).

2 – The Ansätze in 4-dimensions

The Atiyah-Ward Ansätze in 4-dimensions have been well understood (see [1] and [6]). However, our treatment of these Ansätze in the language of conformal geometry does not appear in the literature and is particularily useful for the reduction process. The proofs of the results from this section follow by direct calculations.

2.1 – The case $k = 0$

Lemma 1. *Consider ∇ a connection on a complex vector bundle W over the 4-dimensional conformal oriented manifold M.*

If $\mathbf{A} \in \Gamma(M, \mathcal{E}^1 \otimes \mathrm{End}(W^*, W))$ then the connection

$$\begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix}$$

on $W \oplus W^*$ is self-dual if and only if ∇ is a self-dual Yang-Mills field and $\mathbf{A}$ satisfies the equation $(d^{\nabla} A)_{ASD} = 0$. (Here and elsewhere the subscript "ASD" denotes the anti-self-dual part of a 2-form).

2.2 – The case $k = 1$

Notation. On a 4-dimensional conformal oriented manifold M S_- is the weight 1/2 spin bundle. It is a complex rank 2 vector bundle over M such that $\odot S_- = \Lambda^2_{ASD}(L^{-1/2}TM)_{\mathbb{C}}$ and $\Lambda^2(S_-) = L^1_{\mathbb{C}}$. (The subscript $\mathbb{C}$ denotes complexifications and $\odot$ denotes the symmetrised tensor product).

Theorem 2. *Let D be a Weyl connection on a 4-dimensional oriented conformal manifold M and ∇ a connection acting on a complex line bundle W over M. Let ρ be a section of $L^{-1}W$ and suppose that W has a square root. Then the curvature of the connection $[D + \rho^{-1}(D \otimes \nabla)(\rho)] \otimes \nabla$ on $S_- \otimes W^{-1/2}$ has the form*

$$\begin{aligned} R_{X,Y} = & W^-_{X,Y} - \rho \mathcal{H}_{X,Y}(\rho^{-1}) + \\ & + \frac{1}{2}\{\rho^{-1}\Delta(\rho) \cdot X \wedge Y + F^{\nabla}(X) \wedge Y - F^{\nabla}(Y) \wedge X\}_{ASD} \end{aligned}$$

where F^{∇} is the curvature of ∇, the Hessian

$$\mathcal{H} : L^1 W^{-1} \to L^{-1}\,\mathrm{End}(TM)$$

is defined by the formula

$$\mathcal{H}(\mu) = \mathrm{sym}_0\left((D \otimes \nabla)^2(\mu) + r^D \mu\right)$$

(“sym_0” *denoting the symmetric trace free part) and the conformally invariant Laplacian*

$$\Delta : L^{-1}W \to L^{-3}W$$

is defined by

$$\Delta(\rho) = \mathrm{tr}\,(D \otimes \nabla)^2(\rho) - \frac{1}{6}\,\mathrm{scal}^D\,\rho.$$

In particular if M is a self-dual, ρ is harmonic and the connection ∇ is a self-dual Maxwell field, then the connection $[D + \rho^{-1}(D \otimes \nabla)(\rho)] \otimes \nabla$ on $S_- \otimes W^{-1/2}$ has self-dual curvature.

2.3 – The case $k = 2$

Theorem 3. *Let ∇ be a self-dual Maxwell field acting on a line bundle W over the 4-dimensional oriented conformal self-dual manifold M. Let $\rho \in \Gamma(M, \mathcal{E}^2 \otimes W)$ be non-degenerate and anti-self-dual such that $d^\nabla(\rho) = 0$, and let D be a Weyl connection on M. Define $F \in \mathcal{E}^1_M \otimes \mathrm{End}[L^{-1}\Lambda^2_{ASD}(TM) \otimes W^{-1/2}]$ by the formula $F_X(\alpha) = [\eta(\alpha) \wedge X]_{ASD}$ where $\alpha \in L^{-1}\Lambda^2_{ASD}(TM) \otimes W^{-1/2}$, $X \in TM$ and $\eta(\alpha)$ is determined by the condition $i_{\eta(\alpha)}(\rho) = \langle (D \otimes \nabla)(\rho), \alpha \rangle$. Then on the anhilator of ρ in $L^{-1}\Lambda^2_{ASD}(TM) \otimes W^{-1/2}$ the connection $D \otimes \nabla + F$ is independent of the Weyl connection D and has self-dual curvature.*

3 – The reduced equations

Assumptions. In this section we consider (B, D^B) a 3-dimensional oriented Einstein-Weyl space, (w, A) a solution of the abelian monopole equation on B and $\pi : M \to B$ the conformal submersion it generates with M an oriented 4-dimensional self-dual manifold. On B we fix a vector bundle V with a connection ∇_1 acting on it and a section $w_1 \in L^{-1} \otimes \mathrm{End}(V)$. We define a connection ∇ on $\pi^*(V)$ by the formula $\pi^*(\nabla_1) - w_1\xi$, where $\xi := w^{-1}(dt + A)$ with t the fiber coordinate of π.

One should note that (w_1, ∇_1) is a solution of the Einstein-Weyl Bogomolny equations $*_B(D^B \otimes \nabla_1)(w_1) = F^{\nabla_1}$ on B if and only if ∇ is a self-dual Yang-Mills field on M. In particular this is the correspondence between solutions of the abelian monopole equation on B and self-dual Maxwell fields on M (in the case when the rank of V is 1).

Notations. In order to simplify the notations, the tensor product connection $D^B \otimes \nabla_1$ on B applied to a section of $L^k \otimes V$ (or to a section of $L^k TB \otimes V$) will be denoted simply ∇_1. Similarly, the tensor product connection $D^B \otimes \nabla$ on M applied to a section of $L^k \otimes \pi^*(V)$ (or to a section of $L^k TM \otimes \pi^*(V)$) will be denoted simply ∇.

3.1 – The case $k = 0$.

Lemma 4. *Let $A_2 \in \Gamma(B, \mathcal{E}^1_B \otimes \mathrm{Hom}(V^*, V))$ and $w_2 \in L^{-1} \otimes \mathrm{Hom}(V^*, V)$ such that they satisfy the the relation*

$$d^{\nabla_1}(A_2) = *_B(\nabla_1(w_2) - w_1 A_2)$$

Then the 1-form $\mathbf{A} \in \Gamma(M, \mathcal{E}^1_M \otimes \mathrm{Hom}(\pi^* V^*, \pi^* V))$ defined by the formula $\mathbf{A} = \pi^*(A_2) - w_2 \xi$ satisfies the equation $d^{\nabla}(\mathbf{A})_{ASD} = 0$.

Proof. We first notice that

$$\begin{aligned}
d^{\nabla}(\mathbf{A}) &= d^{\pi^*(\nabla_1) - w_1 \xi}(\mathbf{A}) = \\
&= d^{\pi^*(\nabla_1)}(\mathbf{A}) - w_1 \xi \wedge \mathbf{A} = \\
&= d^{\pi^*(\nabla_1)}(\pi^*(A_2) - w_2 \xi) - w_1 \xi \wedge A_2 = \\
&= \pi^*(d^{\nabla_1}(A_2)) - \nabla_1(w_2) \wedge \xi - w_2 d^{D^B}(\xi) - w_1 \xi \wedge A_2 = \\
&= \pi^*(d^{\nabla_1}(A_2)) + \xi \wedge (\nabla_1(w_2) - w_1 A_2) - w_2 d^{D^B}(\xi)\,.
\end{aligned}$$

Then we notice that, since (w, A) satisfies the abelian monopole equation, the form $d^{D^B}(\xi)$ is self-dual, being equal to the expression

$$w^{-1}\left(\xi \wedge D^B(w) + *_M\left(\xi \wedge D^B(w)\right)\right).$$

Also, since $*_M(\alpha) = -\xi \wedge *_B(\alpha)$ for $\alpha \in \mathcal{E}^2(B)$ we obtain that the anti-self dual part of $d^{\nabla}(\mathbf{A})$ is 0 if and only if $d^{\nabla_1}(A_2) = *_B(\nabla_1(w_2) - w_1 A_2)$. The conclusion follows. ∎

3.2 – The case $k = 1$

Lemma 5. *If $s \in L_B^{-1/2} \otimes V$ is a solution of the equation*

$$\mathrm{tr}_B \nabla_1^2(s) - \frac{1}{6}\,\mathrm{scal}^{D^B}\, s + w_1^2 s = 0$$

then $w^{1/2}\pi^*(s) \in L_M^{-1} \otimes \pi^*(V)$ is a solution of the conformally invariant Laplacian coupled with the connection ∇.

Proof. Using Proposition 2.6 and Proposition 5.6 of [4] one first shows that the scalar curvatures $\mathrm{scal}_M^{D^B}$ of D^B on M and $\mathrm{scal}_B^{D^B}$ of D^B on B are related by the formula

$$\mathrm{scal}_M^{D^B} = \mathrm{scal}_B^{D^B} - \frac{9}{2}|w^{-1}D^B(w)|^2.$$

The proof then follows from a direct calculation. ∎

3.3 – The case $k = 2$

Lemma 6. *If $\alpha \in \Gamma(B, \mathcal{E}_B^1 \otimes V)$ is a solution of the system*

$$\begin{cases} d^{\nabla_1}(\alpha) = w_1 *_B(\alpha) \\ d^{\nabla_1}(*_B\alpha) = 0 \end{cases}$$

then $\pi^*(w\alpha) \in \Gamma(M, \mathcal{E}_{ASD}^2 \otimes \pi^*(V))$ satisfies

$$d^\nabla \pi^*(w\alpha) = 0.$$

Proof. First we recall that the isomorphism $\pi^*(L^{-1}\mathcal{E}_B^1) \cong \mathcal{E}_{ASD}^2$ used implicitly in the statement of the lemma associates to $\beta \in L^{-1}\mathcal{E}_B^1$ the anti-self-dual 2-form $\xi \wedge \beta - *_M(\xi \wedge \beta)$. Then we notice that

$$d^\nabla[\xi \wedge (w\alpha) - *_M(\xi \wedge (w\alpha))] = -(dt + A) \wedge \pi^*(d_1^\nabla(\alpha)) - d^{\nabla_1}(*_B\alpha)w + \\ + w_1 w\xi \wedge *_B\alpha$$

The conclusion follows by identifying the horizontal and vertical parts of the above equality. ∎

4 – The Ansätze for 3-dimensional Einstein-Weyl spaces

4.1 – The case $k = 0$

The reduced Ansatz for $k = 0$ gives an interpretation of the affine monopole equations (see [2]):

$$\begin{cases} *_B \nabla_1(w_1) = F^{\nabla_1} \\ d^{\nabla_1}(A_2) = *_B(\nabla_1(w_2) - w_1 A_2) \end{cases}$$

where $A_2 \in \Gamma(B, \mathcal{E}_B^1 \otimes \mathrm{Hom}(V^*, V))$ and $w_2 \in L^{-1} \otimes \mathrm{Hom}(V^*, V)$. The first of these equations is the abelian monopole equation. The second of these equations is obtained by reducing the auxiliary equation of the A_0 Ansatz from 4 to 3 dimensions (see Lemma 4). The following lemma holds.

LEMMA 7. *The affine monopole equations on* 3*-dimensional Einstein-Weyl manifolds are natural reductions of the A_0 Atiyah-Ward Ansatz on self-dual conformal* 4*-manifolds.*

4.2 – The case $k = 1$.

NOTATION. For B an oriented conformal 3-manifold, S is the weight 1/2-spin bundle. It is a rank two complex vector bundle on B such that $\odot^2 S = T_{\mathbb{C}} B$.

THEOREM 8. *Let (∇_1, w_1) be a solution of the abelian monopole equation defined on the line bundle V over the* 3*-dimensional oriented Einstein-Weyl space B. Suppose that V has a square root. If $s \in L^{-1/2}V$ is a solution of the equation*

$$\mathrm{tr}\, \nabla_1^2(s) - \frac{1}{6} \mathrm{scal}^{D^B}\, s + w_1^2 s = 0$$

then the Higgs field

$$-\frac{1}{2} s^{-1} \nabla_1(s) \in L_B^{-1} \otimes \mathrm{End}(L_B^{-1/4} S \otimes V^{-1/2})$$

and the connection $(D^B+\Gamma)\otimes\nabla_1$ *on* $L_B^{-1/4}S\otimes V^{-1/2}$ *with*

$$\Gamma_X=\frac{1}{2}s^{-1}(\nabla_1)_X(s)\cdot\mathrm{Id}-\frac{1}{2}w_1X+\frac{1}{2}*_B\left(s^{-1}\nabla_1(s)\wedge X\right)$$

(*where* $X\in TB$) *satisfy the Einstein-Weyl Bogomolny equations on* B.

PROOF. Consider (w,A) a solution of the abelian monopole equation on B and $\pi:M\to B$ the conformal submersion it generates with M a 4-dimensional self-dual oriented conformal manifold. From Theorem 2 and Lemma 5 we obtain a self-dual Yang-Mills field on $S_-\otimes\pi^*(V^{-1/2})$ defined by the formula $(D^B+\rho^{-1}\nabla(\rho))\otimes\nabla$, where S_- is the spin bundle over M such that $\odot^2S_-=L^{-1}\Lambda^2_{ASD}(TM)$, $\rho:=w^{1/2}s\in L^{-1}\otimes\pi^*(V)$ and the connection ∇ on $\pi^*(V^{-1/2})$ is induced by the connection $\nabla:=\pi^*(\nabla_1)-w_1\xi$ on $\pi^*(V)$. The proof has three steps.

1. STEP 1. We show that on S_- the connection $D^B+\rho^{-1}\nabla(\rho)$ is the connection

$$\pi^*(D^B)+\frac{1}{2}\left(\frac{1}{2}w^{-1}D^B(w)+s^{-1}\nabla_1(s)-w_1\xi\right)\cdot\mathrm{Id}+F$$

where $F\in T^*M\otimes\mathrm{End}\left(S_-\otimes\pi^*(V^{-1/2})\right)$ is

$$F_X=[(s^{-1}\nabla_1(s)-w_1\xi)\wedge X]_{ASD}$$

For this we first notice that

$$\nabla(\rho)=\frac{1}{2}w^{-1/2}D^B(w)s+w^{1/2}(\nabla_1(s)-w_1\xi\cdot s)$$

and

$$\rho^{-1}\nabla(\rho)=\frac{1}{2}w^{-1}D^B(w)+s^{-1}\nabla_1(s)-w_1\xi$$

It follows that on L^1 the connection $D^B+\rho^{-1}\nabla(\rho)$ is equal to

$$D^B+\frac{1}{2}w^{-1}D^B(w)+s^{-1}\nabla_1(s)-w_1\xi.$$

On L^1 we define the Weyl connection D^{sd} to be $D^B+\frac{1}{2}w^{-1}D^B(w)$ which has the property that on $L^{-1/2}S_-$ it is the pull-back of D^B on

$L^{-1/2}S$ (see [2]). Since on L^1 the connection $D^B+\rho^{-1}\nabla(\rho)$ is equal to $D^{sd}+s^{-1}\nabla_1(s)-w_1\xi$, it follows that on $L^{-1/2}S_-$ (which is of weight 0) it is equal to $\pi^*(D^B)+F$, with $F\in\Gamma(M,\mathcal{E}^1_M\otimes \mathrm{End}[L^{-1/2}S_-])$ defined by

$$F_X=[(s^{-1}\nabla_1(s)-w_1\xi)\wedge X]_{ASD}$$

(for every $X\in TM$). Then on $L^{1/2}\otimes L^{-1/2}S_-$ the connection $D^B+\rho^{-1}\nabla(\rho)$ is the tensor product $[D^{sd}+\frac{1}{2}(s^{-1}\nabla_1(s)-w_1\xi)\cdot \mathrm{Id}]\otimes[\pi^*(D^B)+F]$, which is equal to $[D^B+\frac{1}{2}(\frac{1}{2}\frac{D^B(w)}{w}+s^{-1}\nabla_1(s)-w_1\xi)\cdot \mathrm{Id}]\otimes[\pi^*(D^B)+F]$, because $D^{sd}=D^B+\frac{1}{2}w^{-1}D^B(w)$ on L^1. The claim follows.

2. Step 2. We change the weight of the bundle $S_-\otimes\pi^*(V^{-1/2})$ in order to get a self-dual Yang-Mills field with horizontal part independent of w.
   The change of weight is realized by the diffeomorphism
   $$w^{1/4}:S_-\otimes\pi^*(V^{-1/2})\to L_M^{-1/4}S_-\otimes\pi^*(V^{-1/2})$$
   The self-dual Yang-Mills field $[D^B+\rho^{-1}\nabla(\rho)]\otimes\nabla$ on $S_-\otimes\pi^*(V^{-1/2})$ induces a self-dual Yang Mills field
   $$[\pi^*(D^B)+\frac{1}{2}(s^{-1}\nabla_1(s)-w_1\xi)\cdot \mathrm{Id}+F]\otimes\nabla$$
   on $L_M^{-1/4}S_-\otimes\pi^*(V^{-1/2})$, with F having the same formal expression as in Step 1.

3. Step 3. We determine the horizontal and the vertical parts of the self-dual Yang-Mills field on $L_M^{-1/4}S_-\otimes\pi^*(V^{-1/2})$. They will provide the connection and the Higgs field which form a solution of the Einstein-Weyl Bogomolny equations on B.
   For this, we first notice that on $\pi^*(V^{-1/2})$ the connection ∇ is $\pi^*(\nabla_1)+\frac{1}{2}w_1\xi$, and that the Yang-Mills field on $L_M^{-1/4}S_-\otimes\pi^*(V^{-1/2})$ becomes $\pi^*(D^B\otimes\nabla_1)+\frac{1}{2}s^{-1}\nabla_1(s)\cdot\mathrm{Id}+F$, with F a 1-form with values in $\mathrm{End}\left(L_M^{-1/4}S_-\otimes\pi^*(V^{-1/2})\right)$ defined by
   $$F_X=[(s^{-1}\nabla_1(s)-w_1\xi)\wedge X]_{ASD}$$

$(X \in TM)$. It follows that the vertical part of the self-dual Yang-Mills field is $F_\xi = -\frac{1}{2}s^{-1}\nabla_1(s)$ (since $(D^B \otimes \nabla_1)_\xi(s)$ is 0, the connection ∇_1 being a pull-back connection on $L^{-1/2} \otimes \pi^*(V)$, ξ being vertical and s being the pull-back of a section on the base). The horizontal part of the self-dual Yang-Mills field is $D^B \otimes \nabla_1 + \Gamma$, where

$$\Gamma_X = \frac{1}{2}s^{-1}(\nabla_1)_X(s) \cdot \mathrm{Id} + i_\xi(F_X)$$

$(X \in TB)$. A simple calculation shows that

$$i_\xi(F_X) = -\frac{1}{2}w_1 X + \frac{1}{2} *_B (s^{-1}\nabla_1(s) \wedge X)$$

and the conclusion follows. $\square$

4.3 – The case $k = 2$

Theorem 9. *Let (w_1, ∇_1) be a solution of the abelian monopole equation defined on the line bundle V over the 3-dimensional oriented Einstein-Weyl space B. Suppose that V has a square root. If $\alpha \in \Gamma(B, \mathcal{E}^1_B \otimes V)$ satisfies*

$$\begin{cases} d^{\nabla_1}(\alpha) = w_1 *_B (\alpha) \\ d^{\nabla_1}(*_B\alpha) = 0 \end{cases}$$

then the Higgs field $H \in L^{-1} \otimes \mathrm{End}(L^{-1/2}TB \otimes V^{-1/2})$ defined by

$$H(\chi) = |\alpha|^{-2} *_B (\alpha \wedge \langle \nabla_1(\alpha), \chi \rangle) + \frac{1}{2}w_1\chi$$

and the connection $D^B \otimes \nabla_1 + G$ with $G \in T^*B \otimes \mathrm{End}(L^{-1/2}TB \otimes V^{-1/2})$ given by

$$G(X)(\chi) = |\alpha|^{-2}[\langle (\nabla_1)_\alpha(\alpha), \chi \rangle X + \langle (\nabla_1)_X(\alpha), \chi \rangle \alpha - \langle \nabla_1(\alpha), \chi \rangle \alpha(X)]$$

(where $X \in TB$) induces a solution the Einstein-Weyl Bogomolny equations on B defined on the orthogonal complement of α in the bundle $L^{-1/2}TB \otimes V^{-1/2}$.

PROOF. Consider (w, A) a solution of the abelian monopole equation on B and $\pi : M \to B$ the conformal submersion it generates with M a 4-dimensional self-dual oriented conformal manifold. Let $\rho := w(\xi \wedge \alpha - *_B \alpha)$ and ∇ the connection on $\pi^*(V)$ defined by $\nabla := \pi^*(\nabla_1) - w_1 \xi$. Preserving the notations from Theorem 3, we obtain the self-dual Yang-Mills field $D \otimes \nabla + F$ acting on the anhilator of ρ in $L^{-1}\Lambda^2_{ASD}(TM) \otimes \pi^*(V^{-1/2})$. Since $L^{-1}\Lambda^2_{ASD}(TM) \cong \pi^*(TB)$, the self-dual Yang-Mills field can be considered to act on the anhilator of $\pi^*(\alpha)$ in $\pi^*(TB \otimes V^{-1/2})$. Also, the self-dual Yang-Mills field is independent of the choice of the Weyl connection D. We shall choose the Weyl connection D to be D^{sd} (see the proof of Theorem 8).

The proof has three steps.

1. STEP 1. We show that the self-dual Yang-Mills field acting on $\pi^*(TB \otimes V^{-1/2})$ is

$$\pi^*(D^B \otimes \nabla_1) + \frac{1}{2}\left(w^{-1} D^B(w) + w_1 \xi\right) \cdot \mathrm{Id} + F$$

where F is a 1-form with values in $\mathrm{End}\left(\pi^*(TB \otimes V^{-1/2})\right)$ such that when X is basic

$$\begin{aligned} F_X(\chi) = -|\alpha|^{-2}\langle(\nabla_1)_\alpha(\chi), \alpha\rangle X - |\alpha|^{-2}\alpha(X)\langle(\nabla_1)(\alpha), \chi\rangle + \\ + |\alpha|^{-2}\langle(\nabla_1)_X(\alpha), \chi\rangle\alpha \end{aligned}$$

and

$$F_\xi(\chi) = |\alpha|^{-2} *_B (\alpha \wedge \langle \nabla_1(\alpha), \chi\rangle)$$

for $\chi \in TB \otimes V^{-1/2}$.

For this, we first note that $D^{sd} = \pi^*(D^B) + \frac{1}{2} w^{-1} D^B(w) \cdot \mathrm{Id}$ on $\pi^*(TB)$, and that the product connection $D^{sd} \otimes \nabla$ on $\pi^*(TB \otimes V^{-1/2})$ is the connection $\pi^*(D^B \otimes \nabla_1) + \frac{1}{2}\left(w^{-1} D^B(w) + w_1\xi\right) \cdot \mathrm{Id}$. Next we determine F as a 1-form with values in $\mathrm{End}\left(\pi^*(TB \otimes V^{-1/2})\right)$. For this let $\chi \in TB \otimes V^{-1/2}$ orthogonal to α and $\beta := \xi \wedge \chi - *_M(\xi \wedge \chi) \in L^{-1}\Lambda^2_{ASD}(TM) \otimes \pi^*(V^{-1/2})$. Then (see also the definition of F from Theorem 3) the 1-form F with values in $\mathrm{End}\left(\pi^*(TB \otimes V^{-1/2})\right)$ is defined by

$$F_X(\chi) = i_\xi\left([\eta(\beta) \wedge X]_{ASD}\right)$$

where $X \in TM$ and $\eta(\beta) \in L^{-1}TM \otimes \pi^*(V^{-1/2})$ is determined by the equality

$$\rho(\eta(\beta), Y) = -\langle \rho, (D^{sd} \otimes \nabla)_Y(\beta) \rangle$$

which holds for every $Y \in TM$. Our next aim is to determine $\eta(\beta)$ explicitly. Let $\eta(\beta) = h(\beta) + v(\beta)$ with $h(\beta)$ horizontal and $v(\beta) = \lambda(\beta) \cdot \xi$ is vertical. The connections $D^{sd} \otimes \nabla$ and $\pi^*(D^B \otimes \nabla_1)$ differ by a multiple of the identity on $\pi^*(TB \otimes V^{-1/2})$. This multiple will be ignored since applied to β it is always going to be killed by the inner product with ρ. Also, recall that in our convention of notations the tensor product connection $D^B \otimes \nabla_1$ (or its pull-back) applied to a section of $TB \otimes V^{-1/2}$ (or to a section of its pull-back $\pi^*(TB \otimes V^{-1/2})$) will be simply denoted ∇_1 and $\pi^*(\nabla_1)$ respectively.
Let $Y = \xi$ in the relation $\rho(\eta(\beta), Y) = -\langle \rho, (D^{sd} \otimes \nabla)_Y(\beta) \rangle$. Using the fact that $\pi^*(\nabla_1)_\xi(\beta) = 0$ we get $\rho(\eta(\beta), \xi) = 0$ or equivalently $\langle \alpha, h(\beta) \rangle = 0$. Now let Y be basic. A simple calculation shows that

$$\rho(\eta(\beta), Y) = w\lambda(\beta)\alpha(Y) + w{*_B}(h(\beta) \wedge Y \wedge \alpha).$$

On the other hand we have

$$\begin{aligned}\langle \rho, (D^{sd} \otimes \nabla)_Y(\beta) \rangle &= \langle \rho, (\pi^* \nabla_1)_Y(\beta) \rangle = \\ &= \langle \rho, \xi \wedge (\nabla_1)_Y(\chi) - {*_M}(\xi \wedge (\nabla_1)_Y(\chi)) \rangle = \\ &= 2w \langle (\nabla_1)_Y(\chi), \alpha \rangle\end{aligned}$$

and we obtain

$$\lambda(\beta)\alpha(Y) + {*_B}(h(\beta) \wedge Y \wedge \alpha) = -2\langle (\nabla_1)_Y(\chi), \alpha \rangle.$$

Now this relation determines the vertical as well as the horizontal part of $\eta(\beta)$: to determine the vertical part $v(\beta)$ we take $Y := \alpha$ to get

$$\lambda(\beta) = -2|\alpha|^{-2} \langle (\nabla_1)_\alpha(\chi), \alpha \rangle.$$

To determine the horizontal part $h(\beta)$ we take Y orthogonal to α to get

$${*_B}(h(\beta) \wedge Y \wedge \alpha) = -2\langle (\nabla_1)_Y(\chi), \alpha \rangle$$

Since the horizontal part of $\eta(\beta)$ is orthogonal to α we can write it down explicitly from the above relation:

$$h(\beta) = -2|\alpha|^{-2} *_B (\alpha \wedge \langle \nabla_1(\alpha), \chi \rangle) .$$

The claim now follows.

2. STEP 2. We change the weight of the bundle $\pi^*(TB \otimes V^{-1/2})$ in order to get a self-dual Yang-Mills field with horizontal part independent of w. The change of weight is realized by the diffeomorphism $w^{1/2}$: $\pi^*(TB \otimes V^{-1/2}) \to \pi^*(L^{-1/2}TB \otimes V^{-1/2})$. On the anhilator of α in $\pi^*(L^{-1/2}TB \otimes V^{-1/2})$ we obtain the self-dual Yang-Mills field

$$\pi^*(D^B \otimes \nabla_1) + \frac{1}{2} w_1 \xi \cdot \mathrm{Id} + \tilde{F}$$

the 1-form $\tilde{F}$ with values in $\mathrm{End}\left(\pi^*(L^{-1/2}TB \otimes V^{-1/2})\right)$ having the same formal expression as F.

3. STEP 3. We identify the horizontal and vertical parts of the self-dual Yang-Mills field on $\pi^*(L^{-1/2}TB \otimes V^{-1/2})$ in order to get the connection and the Higgs field which form a solution of the Einstein-Weyl Bogomolny equations on B. This follows from a simple calculation, and we obtain the statement of the theorem. □

5 – Examples

To any solution $u \in C^\infty(\mathbb{R}^3)$ of the Toda equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0$$

there is an associated Einstein-Weyl space defined in a gauge (called LeBrun-Ward gauge) $\mu \in L^1$ by

$$\begin{cases} g = e^u(dx^2 + dy^2) + dz^2 \\ \omega = -u_z dz \end{cases}$$

Here g is the metric $\mu^{-2}c$ and ω is the connection 1-form of the Einstein-Weyl connection on L^1 relative to the gauge μ.

A direct calculation shows that $\mu^{-\frac{1}{2}}$ is a solution of the equation of Lemma 5 and dz is a solution of the equation of Lemma 6 (with V trivial of rank one).

Theorem 10. *Let (B, D^B) be a Toda Einstein Weyl space. Then the couples $(D^B + \Gamma, -\frac{1}{4}u_z dz)$ and $(D^B + G, H)$ defined by*

$$\Gamma_X = \frac{1}{4}u_z X(z) \cdot \mathrm{Id} + \frac{1}{4} * (u_z dz \wedge X)$$

(for $X \in TB$) and

$$\begin{aligned} H(\chi) &= -\frac{1}{2}u_z * (dz \wedge \chi) \\ G_X(\chi) &= u_z \chi(z) X + \left(\frac{3}{2}u_z \chi(z)\langle X, \frac{\partial}{\partial z}\rangle - \frac{1}{2}u_z \langle X, \chi\rangle\right) dz + \\ &\quad - X(z)\left(\frac{3}{2}u_z \chi(z)\frac{\partial}{\partial z} - \frac{1}{2}u_z \chi\right) \end{aligned}$$

(for $X \in TB$ and $\chi \in L^{-1}TB$, $\chi(z) = 0$) are solutions of the Einstein-Weyl Bogomolny equations.

The main importance of the reduced Ansätze comes from the fact that solutions of the Einstein-Weyl Bogomolny equations on a 3-dimensional Einstein-Weyl space generate self-dual 4-manifolds (see [2]). Explicit examples of self-dual 4-manifolds using the reduced Ansätze still need to be developed.

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