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Vector cross products and almost contact structures

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RIASSUNTO: *Si definiscono delle strutture di quasi contatto su varietà 7-dimensionali dotate di prodotto vettoriale. In particolare, si dimostra che i due prodotti vettoriali non isomorfi di* $\mathbb{R}^8$ *inducono su ogni ipersuperficie* $M \subset \mathbb{R}^8$ *due differenti strutture di quasi contatto, una delle quali coincide con l'usuale struttura definita su* M *mediante la struttura complessa di* $\mathbb{R}^8$*. Infine, si costruisce in questo modo una struttura non normale di quasi K-contatto sulla sfera* S^7*.*

ABSTRACT: *Special almost contact structures on 7-dimensional manifolds endowed with a 2-fold vector cross product have been defined. Between them, the almost contact structures induced by the two non-isomorphic 3-fold vector cross products of* $\mathbb{R}^8$ *on any orientable hypersurface* M *have been considered, proving that one of them always coincides with the structure inherited by* M *from the complex structure of* $\mathbb{R}^8$*, while the second one generally provides an unknown example of almost contact structure on* M*. In particular, in this way, a non normal almost K-contact metric structure on* S^7 *has been constructed.*

1 – Introduction

In 1969 A. Gray gave a general definition of the well-known notion of vector cross product, studying in particular vector cross products on manifolds [6]. A careful exam of vector cross products from the point of view of differential geometry was mainly suggested by their strong

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relations with almost complex structures. Moreover, vector cross products provided an approach to the study of Riemannian manifolds with holonomy group G_2 or *Spin(7)* (cf. [8]). In particular, by means of the two non-isomorphic 3-fold vector cross products of $\mathbb{R}^8$, some new almost complex structures on S^6 have been obtained in [6]. In the same way, some examples of manifolds with different almost complex structures J_1 and J_2 such that J_1 is Kählerian but J_2 is not, have been constructed.

In the light of these results on 6-dimensional case, we show that some classes of 7-dimensional manifolds admitting a 2-fold vector cross product P have an almost contact structure naturally induced by P. As special case, we consider the orientable hypersurfaces M of $\mathbb{R}^8$ which inherit two almost contact structures from the two non-isomorphic vector cross products of $\mathbb{R}^8$. The properties of the two vector cross products of $\mathbb{R}^8$ imply strong differences between these structures. In fact, one of them is just the almost contact structure naturally induced on M by the complex structure of $\mathbb{R}^8$. On the contrary, the second one provides in most cases a new example of almost contact structure on the hypersurfaces M. In particular, concerning its general properties, we prove that the conditions for the normality impose very special restrictions to the corresponding vector cross product P.

In the last section, as remarkable example, we construct the so defined almost contact structures on S^7. Besides of the well known Sasakian structure, we obtain in this way a different almost contact structure on the unitary 7-dimensional sphere. More precisely, a careful analysis by means of the results of [1], [4], shows that it can be classified as an almost K-contact, non normal structure on S^7.

2 – Preliminaries

Definition 2.1. [6] Let V be an n-dimensional vector space over the real numbers and let g be a non-degenerate bilinear form on V. An *r-fold vector cross product* on V $(1 \leq r \leq n)$ is a multilinear map P, $P : V^r \longrightarrow V$, such that

$$\|P(a_1, \cdots, a_r)\|^2 = \det(g(a_i, a_j)) = \|a_1 \wedge \cdots \wedge a_r\|^2 \quad \text{and} \tag{2.1}$$

$$g(P(a_1, \cdots, a_r), a_i) = 0,$$

for all $a_1, \cdots, a_r$ in V, with $\|a\|^2 = g(a,a)$. Let P and P' two r-fold vector cross products on V with respect to the same bilinear form g. If there exists a map $\Psi : V \to V$ such that

$$g(\Psi a, \Psi b) = g(a,b) \quad \text{and} \quad P(\Psi a_1, \cdots, \Psi a_r) = (-1)^q P(a_1, \cdots, a_r),$$

with $q \in \mathbb{N}$ even (odd), then P and P' are said *isomorphic* (*anti-isomorphic*).

In what follows we will be interested to 2-fold and 3-fold vector cross products described below (a complete classification of the vector cross products can be found in [3], [5]).

a) Let V be the orthogonal complement of the identity e of an 8-dimensional composition algebra W. Then we can define 2-fold vector cross product $P : V \times V \longrightarrow V$ on V by

$$P(a,b) = ab + g(a,b)e.$$

Two 2-fold vector cross products on the same vector space (V,g) are always isomorphic.
b) Let V be an 8-dimensional composition algebra with bilinear form g. Then the following maps $P, P' : V \times V \times V \longrightarrow V$

$$P(a,b,c) = -a(\bar{b}c) + g(a,b)c + g(b,c)a - g(c,a)b, \tag{2.2}$$

$$P'(a,b,c) = -(a\bar{b})c + g(a,b)c + g(b,c)a - g(c,a)b \tag{2.3}$$

define two non-isomorphic 3-fold vector cross products on V with respect to g. Every other 3-fold vector cross product with bilinear form g is isomorphic to either P or P'.

Examples of vector cross products of type a) and b) can be obtained on $\mathbb{R}^7$ and $\mathbb{R}^8$ via the non associative 8-dimensional algebra *Cay* of the *Cayley numbers*. Denoting by $\mathbb{H}$ the quaternions algebra, we can think *Cay* as the product $\mathbb{H} \times \mathbb{H}$ endowed with the multiplication

$$(z,w)(z',w') = (zz' - \bar{w}'w, w'z + w\bar{z}'), \tag{2.4}$$

where $z, w, z', w' \in \mathbb{H}$ and $\bar{q}$ denotes the conjugation in $\mathbb{H}$ of the quaternion q. Furthermore, a basis $\mathcal{B}$ and the conjugation for *Cay* are respectively given as follows:

$$\mathcal{B} = \{i_0 = (1,0), i_1 = (i,0), i_2 = (j,0), i_3 = (k,0), i_4 = (0,1), \\ i_5 = (0,i), i_6 = (0,j), i_7 = (0,k)\} \tag{2.5}$$

$$\overline{(z,w)} = (\bar{z}, -w). \tag{2.6}$$

Following [6], we say that a Riemannian manifold (M, g) has an r-fold vector cross product P if, for each $m \in M$, the corresponding tangent space $T_m M$ has an r-fold vector cross product $P_m : T_m M \times T_m M \to T_m M$, requiring that the map $m \to P_m$ be C^∞.

We recall the following remarkable theorem which will be useful for us [6]

Theorem 2.1. *Let M be an m-dimensional oriented submanifold of the n-dimensional Riemannian manifold $(\overline{M}, g)$. Suppose that the restrictions of g to M and to the normal bundle of M are nondegenerate and positive definite respectively. If $\overline{M}$ has an r-fold vector cross product $\overline{P}$ with respect to g, then $\overline{P}$ induces a k-fold vector cross product P on M with $k = r - p$, being p the codimension of M.*

A partial vice versa of the previous Theorem 2.1 has been also obtained in [6]. In fact, the author proved that the existence of an r-fold vector cross product on the unitary n-dimensional sphere S^n implies the existence of an (r+1)-fold vector cross product on $\mathbb{R}^{n+1}$. So, the only spheres admitting almost complex structures are S^2 and S^6 while S^7 surely has 2-fold vector cross products.

Other important properties of the geometry of a Riemannian manifold (M, g) admitting a vector cross product can be found in [6].

3 – Vector cross products and almost contact structures

Let M be a differentiable manifold of odd dimension $2n + 1$. An almost contact metric structure (a.c.m.s.) (φ, ξ, η, g) on M is given by a field of endomorphisms of the tangent bundle φ, a vector field ξ and a 1-form η satisfying the following relations [2]:

$$\left\{\begin{array}{lll} \varphi^2 = -I + \eta\otimes\xi, & \eta(\xi) = 1, & \\ \varphi(\xi) = 0, & \eta\circ\varphi = 0, & \text{rank } \varphi = 2n. \end{array}\right. \tag{3.7}$$

together with a Riemannian metric on M such that

$$g(X,\xi) = \eta(X) \quad , \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.8}$$

for every $X, Y \in \mathcal{X}(M)$. Then $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold.

Let now V be a 7-dimensional vector space over $\mathbb{R}$ with positive definite inner product g and let P be a 2-fold vector cross product. Given the unitary vector ξ of V, consider the (1,1)-tensor φ and the 1-form η respectively defined by the relations

$$\varphi a = P(\xi, a), \quad \eta(a) = g(\xi, a) \qquad \text{for every } a \in V. \tag{3.9}$$

Proposition 3.1. *(φ, ξ, η, g) is an almost contact metric structure on V.*

Proof. In fact, the definitions of φ and η trivially imply $\varphi\xi = 0$ and $\eta(\xi) = 1$. Moreover for $a, b, c \in V$ we have [8]

$$P(a, P(a,b)) = -\|a\|^2 b + g(a,b)a, \tag{3.10}$$

$$g(P(a,b), P(a,c)) = g(a\wedge b, a\wedge c). \tag{3.11}$$

By substituting ξ for a in (3.10) and (3.11) we respectively obtain $\varphi^2 = -I + \eta\otimes\xi$ as well as $g(\varphi b, \varphi c) = g(b,c) - \eta(b)\eta(c)$, proving the assert. □

This definition gives us the possibility to induce an almost contact metric structure on every Riemannian 7-dimensional manifold M endowed of a 2-fold vector cross product which admits a globally defined unitary vector field ξ, as the parallelizable 7-dimensional manifolds and the orientable hypersurfaces of $\mathbb{R}^8$ [6]. In the sequel we just shall focus our attention on this second class of manifolds which inherit from $\mathbb{R}^8$ further nice properties. In fact, if M is such a hypersurface of $\mathbb{R}^8$, then there exists a naturally defined global unitary vector field ξ on M given by $\xi = -JN$, where N denotes the unitary normal vector field to M and J the complex structure of $\mathbb{R}^8$. Moreover, since $\mathbb{R}^8$, conveniently identified with the Cayley algebra of octonions, possesses the two non-isomorphic 3-fold vector cross products P and P', formulas (2.2) and (2.3) define on M the following two almost contact metric structures (φ, ξ, η, g) and (φ', ξ, η, g)

$$\varphi X = -P(N, \xi, X) = N(\bar{\xi}X) - g(\xi, X)N = N(\bar{\xi}X) - \eta(X)N, \tag{3.12}$$

$$\varphi' X = -P'(N, \xi, X) = (N\bar{\xi})X - g(\xi, X)N = (N\bar{\xi})X - \eta(X)N, \tag{3.13}$$

where g is the restriction to M of the metric G of $\mathbb{R}^8$, $\xi = -JN$ and $\eta(X) = g(\xi, X)$ for every $X \in \mathcal{X}(M)$.

Remak 3.2. It should be remarked that there is one fundamental difference between the vector cross products and the a.c.m.s. In fact, a.c.m.s. are generally defined without reference to a Riemannian metric, and if a metric exists, a compatibility condition is required. In contrast to this, a vector cross product has a unique metric associated with it (see [6], [8]).

We shall describe now some useful properties of the 2-fold vector cross products induced on M by P and P'. With this purpose, we shall use the notation $\tilde{P}$ for both P and P', as well as $\tilde{\varphi}$ for both the corresponding almost contact structures on M.

Firstly, we remark that, because of the linearity of $\tilde{P}$, applying (2.1), for every $X, Y \in \mathcal{X}(M)$, we obtain

$$\|\tilde{P}(N, X + \xi, Y)\|^2 = \|X + \xi\|^2 \|Y\|^2 - (g(X, Y) + \eta(Y))^2, \tag{3.14}$$

$$\|\tilde{P}(N, X, Y)\|^2 = \|X\|^2 \|Y\|^2 - (g(X, Y))^2 \tag{3.15}$$

and, as usual, $\|\tilde{\varphi}Y\|^2 = \|Y\|^2 - (\eta(Y))^2$. Moreover, from the equality $g(\tilde{P}(N, X + Y, Z), X + Y) = 0$ (see (2.1)), we also have

$$g(\tilde{P}(N, X, Z), Y) = -g(\tilde{P}(N, Y, Z), X) \tag{3.16}$$

for all $X, Y, Z \in \mathcal{X}(M)$. Then, taking account of the previous relations, from (3.16) we deduce the following fundamental relation

$$\tilde{P}(N, Y, \tilde{\varphi}Y) = \eta(Y)Y - \|Y\|^2\xi. \tag{3.17}$$

On the other hand, because of the enunciated properties of $\tilde{P}$ and (3.17), we can write (see also (2.7) in [8])

$$\tilde{P}(N, \tilde{\varphi}X, Y) + \tilde{\varphi}\tilde{P}(N, X, Y) = -2\eta(Y)X + \eta(X)Y + g(X, Y)\xi. \tag{3.18}$$

Thus, after some computations, by substituting in (3.18) $\tilde{\varphi}Y$ for Y, we get the new useful equation true for all $X, Y \in \mathcal{X}(M)$.

$$\tilde{P}(N, \tilde{\varphi}X, \tilde{\varphi}Y) + \tilde{P}(N, X, Y) = -\eta(X)\tilde{\varphi}Y + \eta(Y)\tilde{\varphi}X + 2g(X, \tilde{\varphi}Y)\xi\,. \tag{3.19}$$

Then, since the equality $g(\tilde{P}(X,\xi,Y),\tilde{P}(N,\xi,Y)) = -g(\tilde{P}(\tilde{\varphi}Y, \xi, Y), X)$ and (2.1) imply that $\tilde{P}(X, \xi, Y)$ is orthogonal both to $\tilde{\varphi}X$ and $\tilde{\varphi}Y$, we deduce that $\tilde{P}(Y, \xi, \tilde{\varphi}Y)$ is always parallel to N for every $Y \in \mathcal{X}(M)$. Then, we can choose the vector fields X, Y on M such that $X, Y, \tilde{\varphi}X, \tilde{\varphi}Y$ are independent to each other and give rise to the following local frame on M:

$$\{\dot{X}, \dot{Y}, \tilde{\varphi}X, \tilde{\varphi}Y, \tilde{P}_t(X, \xi, Y), \tilde{\varphi}\tilde{P}_t(X, \xi, Y), \xi\}, \tag{3.20}$$

where $\dot{X}, \dot{Y}$ and $\tilde{P}_t(X, \xi, Y)$ denote the horizontal components of X, Y and the tangent part of $\tilde{P}(X, \xi, Y)$ respectively. Finally the following theorem gives the fundamental relation which characterizes the a.c.m.s. induced on M by the vector cross products of $\mathbb{R}^8$

Theorem 3.3. *For the vector cross products $\tilde{P}$ one of the following relations holds*

$$\begin{aligned} \tilde{P}(X, N, Y) = \mp(&\tilde{P}\,(\tilde{\varphi}X, \xi, Y) - g(\tilde{\varphi}X, \tilde{\varphi}Y)N) + \\ &+ \eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X - g(\tilde{\varphi}X, Y)\xi, \end{aligned} \tag{3.21}$$

for all $X, Y \in \mathcal{X}(M)$.

Proof. In fact, since the scalar product $g(\tilde{P}(X,N,Y),\tilde{P}(X,\xi,Y))$ vanishes (see formulas (2.1)), considering the local vector frame (3.20) of M, we can write

$$\tilde{P}(\dot{X},N,\dot{Y}) = \alpha\tilde{\varphi}\tilde{P}_t(\dot{X},\xi,\dot{Y}) + \beta\xi, \tag{3.22}$$

where $\beta = g(\tilde{\varphi}X,Y)$. On the other hand, taking into account that $\|\tilde{P}(\dot{X},N,\dot{Y})\| = \|\tilde{P}(\dot{X},\xi,\dot{Y})\|$ and $\tilde{P}(\dot{X},\xi,\dot{Y}) = \tilde{P}_t(\dot{X},\xi,\dot{Y}) - g(\tilde{\varphi}X,Y)N$, from (3.22) we get

$$\|\tilde{P}(\dot{X},N,\dot{Y})\|^2 = \alpha^2\|\tilde{P}(\dot{X},\xi,\dot{Y})\|^2 - \alpha^2(g(\tilde{\varphi}X,Y))^2 + (g(\tilde{\varphi}X,Y))^2, \tag{3.23}$$

from which, the necessary equalities of the lengths imply $\alpha = \pm 1$. Finally, since $\tilde{\varphi}P_t(X,\xi,Y) = -\tilde{P}_t(\tilde{\varphi}X,\xi,Y)$ for every $X,Y \in \mathcal{X}(M)$, the proof of the theorem is completed by substituting in (3.23) $X - \eta(X)\xi$ for $\dot{X}$ and $Y - \eta(Y)\xi$ for $\dot{Y}$ respectively. □

Now, remembering the definition of P and P', a straightforward computation proves that the sign positive holds for P', while P obeys to (3.21) with the negative sign.

On the other hand, every orientable hypersurface $M \subset \mathbb{R}^8$ naturally inherits an almost contact metric structure $(\dot{\varphi},\xi,\eta,g)$ from the complex structure J of $\mathbb{R}^8$ defined as follows:

$$\xi = -JN, \quad \eta(X) = g(X,\xi), \qquad JX = \dot{\varphi}X + \eta(X)N \tag{3.24}$$

where $X \in \mathcal{X}(M)$ and N is the unit normal to M in $\mathbb{R}^8$.

Then, if $\overline{\nabla},\nabla$ denote the Levi-Civita connections of G on $\mathbb{R}^8$ and of g on M respectively, we have the following well-known Gauss and Weingarten equations:

$$\begin{cases} \overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \\ \overline{\nabla}_X N = -AX, \qquad X,Y,Z \in \mathcal{X}(M) \end{cases} \tag{3.25}$$

where the normal part $B(X,Y)$ of $\overline{\nabla}_X Y$, called *the second fundamental form* of M, is correlated to the symmetric Weingarten operator A by the relation $G(B(X,Y),N) = g(AX,Y)$. From (3.25) we obtain the following equations satisfied by the induced almost contact metric structure of M [11]

$$(3.26)\qquad \begin{cases} (\nabla_X \dot\varphi)Y = \eta(Y)AX - g(AX,Y)\xi, \\ (\nabla_X \eta)(Y) = \dfrac{1}{2}d\eta(X,Y) = g(\dot\varphi AX, Y), \qquad \nabla_X \xi = \dot\varphi AX. \end{cases}$$

Now, considering the identification of R^8 with the octonions algebra Cay=$\mathbb{H}\times\mathbb{H}$ and the complex structure J as the left multiplication by an imaginary unity i_k, $k = 1, \cdots, 7$, of the basis (2.5) of Cay, by a direct computation in every point $m \in M$ we get $N_m\bar\xi_m = i_k$ and $(\varphi' X)_m = (\dot\varphi X)_m$ for every vector field $X \in \mathcal{X}(M)$. In other words, the vector cross products P' gives rise just to the same almost contact metric structure induced on M by the complex structure J of $\mathbb{R}^8$.

Furthermore, directly from definition of φ and $\varphi' = \dot\varphi$ we can also prove the following theorem.

Theorem 3.4. *Let M be an orientable hypersurface of $\mathbb{R}^8$. Then the almost contact metric structures (φ, ξ, η, g) and $(\varphi' = \dot\varphi, \xi, \eta, g)$ respectively induced on M by the vector cross products P and P' of $\mathbb{R}^8$, coincide iff the horizontal fiber bundle HM is always normal to the identity vector field I_0 of the Cayley multiplication.*

Proof. As before, we consider the complex structure J of $\mathbb{R}^8$ as the left multiplication by a fixed imaginary unity i_k, $k = 1, \cdots, 7$. If we denote I_0, I_k the vector fields along M with constant component equal to i_0, i_k of $\mathcal{B}$ in every point $m \in M$, it is trivial that, when $N = \alpha I_0 + \beta I_k$ with $\alpha, \beta \in C^\infty(M)$, then the two structures coincide.

To prove the vice versa, suppose $(N\bar\xi)X = N(\bar\xi X)$ for every $X \in \mathcal{X}(M)$, with $N = \alpha I_0 + \beta I_k + \gamma Z$, for differentiable functions α, β, γ on M such that $\alpha^2+\beta^2+\gamma^2 = 1$ and some unitary vector field Z orthogonal to I_0, I_k. In the sequel, to simplify our proof, without loosing the generality, we shall suppose $Z = I_j$, for some $j \neq k$. Then, if W is the vector field defined by $W = I_k Z$, W is also an imaginary unity satisfying the equation $ZW = I_k$. Furthermore, for every point $m \in M$, the vectors i_0, i_k, Z_m, W_m are four orthonormal elements in $\mathbb{R}^8$; as a consequence, we obtain the decomposition $\mathbb{R}^8 = span[i_0, i_k, Z_m, W_m] \oplus \mathbb{R}^4$. Let now X

a unitary vector field given by $X = (0, y)$, with $y_m \in \mathbb{R}^4$ and $\|y\| = 1$. Obviously, $X \in \mathcal{X}(M)$ so that we must have $N(\bar{\xi}X) = (N\bar{\xi})X$. We already know that $(N\bar{\xi})X = I_k(0, y)$. We shall compute now the left part considering that $\xi = \beta I_0 - \alpha I_k - \gamma W$, and consequently $\bar{\xi} = \beta I_0 + \alpha I_k + \gamma W$. Then, taking into account the definition and the properties of Z and W, we have

$$\begin{aligned} N(\bar{\xi}X) &= (\alpha I_0 + \beta I_k + \gamma Z)((\beta I_0 + \alpha I_k + \gamma W)(0, y)) = \\ &= (\alpha I_0 + \beta I_k + \gamma Z)(\beta I_0(0, y) + \alpha I_k(0, y) + \gamma W(0, y)) = \\ &= (\alpha^2 + \beta^2)I_k(0, y) + 2\alpha\gamma W(0, y) + 2\beta\gamma Z(0, y) - \gamma^2 I_k(0, y). \end{aligned} \tag{3.27}$$

Because of the equality of the structures, the previous equation (3.27) implies

$$(\alpha^2 + \beta^2 - \gamma^2)I_k + 2\alpha\gamma W + 2\beta\gamma Z = I_k, \tag{3.28}$$

from which, since $\|N\|^2 = \alpha^2 + \beta^2 + \gamma^2 = 1$, we finally get

$$\alpha\gamma W + \beta\gamma Z = \gamma^2 I_k \tag{3.29}$$

and then $\gamma = 0$. Consequently we obtain $N = \alpha I_0 + \beta I_k$, completing the proof of the theorem. □

On the other hand, by using (3.26), we can also see that the structure (φ, ξ, η, g) and the old structure $(\dot{\varphi}, \xi, \eta, g)$ induced on M by J are strictly related. In particular, we have the fundamental relation

$$\begin{aligned} (\nabla_X \varphi)Y &= -(\nabla_X P)(\xi, Y) - P(\nabla_X \xi, Y) = \\ &= -(\nabla_X P)(\xi, Y) - P(\dot{\varphi}AX, Y); \end{aligned} \tag{3.30}$$

here, abusing a little of notations, we still denote by P the 2-fold vector cross product induced on M by the 3-fold vector cross product P of $\mathbb{R}^8$, that is $P(X, Y) = P(N, X, Y)$, for all $X, Y \in \mathcal{X}(M)$. The equation (3.30) appears more significant in the light of the results of [6], [8], where the authors studied the covariant derivative of vector cross products on manifolds.

Definition 3.1 [6]. Let (M, g) be a Riemannian manifold endowed of an r-fold differentiable vector cross product P on M associated to g.

Let ∇ and δ denote the Riemannian connection and the coderivative of M relative to g respectively, and let Π be the (r+1)-fold differential form determined by P: $\Pi(X_1, \cdots, X_{r+1}) = g(P(X_1, \cdots, X_r), X_{r+1})$ for all $X_1, \cdots, X_{r+1} \in \mathcal{X}(M)$. Then
(a) P is parallel if $\nabla P = 0$;
(b) P is nearly parallel if $\nabla_{X_1}(P)(X_1, X_2, \cdots, X_r) = 0$ for all vector fieds $X_1, X_2, \cdots, X_r$ on M;
(c) P is almost parallel if $d\Pi = 0$;
(d) P is semiparallel if $\delta\Pi = 0$.

Leaving from the definition above, in [8] it has been showed the following theorem

THEOREM 3.5. *Let M be an orientable hypersurface of $\mathbb{R}^8$ with unit norm N and let P denote one of the 2-fold vector cross products determined on M by the ordinary vector cross products of $\mathbb{R}^8$. Then:*
(i) *P is parallel if and only if M is totally geodesic (i.e. M is a part of a hyperplane);*
(ii) *P is nearly-parallel if and only if M is totally umbilical (i.e. M is a part of a sphere);*
(iii) *P is semiparallel if and only if M is minimal.*

The previous results and relations obviously give rise to several consequences for the induced a.c.m.s. too. For example, the conditions for (φ, ξ, η, g) to be normal result to be very strong and imply some restrictive relations for the vector cross product P of M. As it is well known, an a.c.m.s. (φ, ξ, η, g) on a manifold M is said to be normal when the (1,2)-tensor $N = N_\varphi + d\eta \otimes \xi$, where N_φ denotes the Nijenhuis tensor of φ, identically vanishes on M. The normality is one of the most remarkable properties for an a.c.m.s.; it insures the integrability of a naturally defined almost complex structure on $N \times \mathbb{R}$ (see [2] for more details and several examples).

Consider firstly that for a normal a.c.m.s. (φ, ξ, η, g) the equation $\mathcal{L}_\xi \varphi = 0$ is always satisfied and ξ is a Killing vector field [2]. Then, taking account of (3.25), from the equation $g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$ we get the relation $\dot{\varphi} A = A \dot{\varphi}$ which assures that $(\dot{\varphi}, \xi, \eta, g)$ is normal [11]. In other words, the normality of $(\dot{\varphi}, \xi, \eta, g)$ is a first necessary condition for the normality of (φ, ξ, η, g). On the other hand, from $\mathcal{L}_\xi \varphi = 0$, for every vector field X on M we also get

$$
\begin{aligned}
0 &= [\xi, \varphi X] - \varphi[\xi, X] = P(\xi, [\xi, X]) - [\xi, P(\xi, X)] = \\
&= P(\xi, \nabla_\xi X) + P(\xi, \nabla_X \xi) = \\
&= (\nabla_\xi P)(\xi, X) - \dot{\varphi} A \varphi X + \varphi \dot{\varphi} A X.
\end{aligned}
\tag{3.31}
$$

Then, when (φ, ξ, η, g) is normal, P obeys to the equation

$$
(\nabla_\xi P)(\xi, X) = \dot{\varphi} A \varphi X - \varphi \dot{\varphi} A X \tag{3.32}
$$

for every $X \in \mathcal{X}(M)$. We remark that, taking account of Theorem 3.5, in the case of a nearly parallel vector cross product, (3.32) implies the commutativity of φ and $\dot{\varphi}$.

A further condition for the derivative of P follows from the equation: $(\mathcal{L}_{\varphi X} \eta)(Y) - (\mathcal{L}_{\varphi Y} \eta)(X) = 0$, always satisfied on a normal almost contact manifold [2]. In fact, after some computations this equation becomes

$$
\begin{aligned}
0 &= (\nabla_{\varphi X} \eta)(Y) + \eta(\nabla_Y \varphi X) - (\nabla_{\varphi Y} \eta)(X) - \eta(\nabla_X \varphi Y) = \\
&= g(\dot{\varphi} A \varphi X, Y) + \eta((\nabla_Y \varphi) X) - g(\dot{\varphi} A \varphi Y, X) - \eta((\nabla_X \varphi) Y),
\end{aligned}
\tag{3.33}
$$

from which, since A and $\dot{\varphi}$ commute, we also obtain

$$
\eta((\nabla_X P)(\xi, Y)) = \eta((\nabla_Y P)(\xi, X)), \tag{3.34}
$$

for every $X, Y \in \mathcal{X}(M)$ supposing (φ, ξ, η, g) normal.

4 – The induced structure (φ, ξ, η, g) of S^7

It is known that the existence of an a.c.m.s. on a differentiable manifold M is equivalent to the existence of a reduction of the structural group $\mathcal{O}(2n+1)$ to $\mathcal{U}(n) \times 1$. If we denote by Φ the fundamental 2-form of $(M, \varphi, \xi, \eta, g)$ defined by $\Phi(X, Y) = g(X, \varphi Y)$ and by ∇ the Riemannian connection of g, the covariant derivative $\nabla \Phi$ is a covariant tensor of degree 3 which has various symmetry properties.

For every odd dimensional real vector space V endowed with an a.c.m.s. (φ, ξ, η, g), let $\mathcal{C}(V)$ be the vector space of 3-forms on V having the same symmetries of $\nabla\Phi$, i.e.

$$\mathcal{C}(V) = \{\alpha \in \otimes_3^0 V \mid \alpha(a,b,c) = -\alpha(a,c,b) = \\ = -\alpha(a,\varphi b,\varphi c) + \eta(b)\alpha(a,\xi,c) + \eta(c)\alpha(a,b,\xi)\},$$

for all $a, b, c \in V$.

A decomposition of $\mathcal{C}(V)$ into twelve components $\mathcal{C}_i(V)$ mutually orthogonal, irreducible and invariant under the action of $\mathcal{U}(n) \times 1$ has been obtained in [1] and [4]. Applying this algebraic decomposition to the geometry of the a.c.m.s., for each invariant subspace we obtain a different class of almost contact metric manifolds; more precisely, we shall say M of class $\mathcal{C}_k$, $k = 1, \ldots, 12$, if, for every $m \in M$, the 3-form $(\nabla\Phi)_m$ of the vector space $(T_mM, \varphi_m, \xi_m, \eta_m, g_m)$ belongs to $\mathcal{C}_k(T_mM)$.
Our aim now is to study the almost contact structures induced in the unitary 7-dimensional sphere S^7 of $\mathbb{R}^8$ in the light of the cited decompositions. Considering the identification $\mathbb{R}^8 \equiv \mathbb{H} \times \mathbb{H}$, we have

$$S^7 = \Big\{m = (x,y) \in \mathbb{H} \times \mathbb{H} \equiv \mathbb{R}^8; |x|^2 + |y|^2 = 1\Big\}$$

and the normal vector field on S^7 is $N = (x, y)$. Then, a vector field $X = (u, v)$ is tangent to S^7 at $m = (x, y)$ if and only if it satisfies: $g(N, X) = < x, u > + < y, v > = 0$, where we denoted by $<,>$ the usual scalar product in $\mathbb{R}^4 \equiv \mathbb{H}$. Finally, the tangent vector field $\xi = -JN$ at $m = (x, y)$ is given as usual by $\xi_m = -i_k(x, y)$.

Following our general results, the almost contact metric structure (φ', ξ, η, g) of S^7 is just the canonical Sasakian structure $(\dot{\varphi}, \xi, \eta, g)$ induced by the complex structure J of $\mathbb{R}^8$ and, comparing with [4], it belongs to $\mathcal{C}_6$. In what follows we classify the a.c.m.s. (φ, ξ, η, g) that the unitary 7-dimensional sphere inherits from the vector cross product P (see (3.12)) following [4].

Then, since S^7 is a totally umbilical hypersurface of $\mathbb{R}^8$ with $A = -I$, Theorem 3.5 assures that P is nearly parallel so that

$$(\nabla_X P)(Y, X) = 0 \tag{4.35}$$

holds for every $X, Y \in \mathfrak{X}(S^7)$.

Furthermore, comparing the Gauss and Weingarten equations (3.25) with (3.26), we obtain

$$\nabla_X \xi = -JX + \eta(X)N = -\dot{\varphi}X. \tag{4.36}$$

And finally (3.30) and (4.35) for $Y = \xi$ imply

$$(\nabla_X \varphi)X = -P(\nabla_X \xi, X) = P(\dot{\varphi}X, X), \tag{4.37}$$

or, equivalently,

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = P(\dot{\varphi}X, Y) + P(\dot{\varphi}Y, X), \tag{4.38}$$

for every X, Y vector fields on S^7.

Before giving the complete decomposition of a.c.m.s. (φ, ξ, η, g) of S^7, we prove the following

Proposition 4.1. *Let Φ be the fundamental* 2*-form of $(S^7, \varphi, \xi, \eta, g)$ and let ∇ be the Riemannian connection of g. Then, for the covariant derivative $\nabla\Phi$ of Φ the following equation holds for all $X, Y \in \mathcal{X}(S^7)$*

$$(\nabla_X \Phi)(Y, \xi) - (\nabla_{\varphi X} \Phi)(\varphi Y, \xi) = g(\varphi X, \dot{\varphi}Y) - g(\varphi Y, \dot{\varphi}X). \tag{4.39}$$

Proof. The proof of the proposition follows from (3.30). In fact, taking into account that S^7 is a totally umbilical hypersurface with $A = -I$, for all $X, Y \in \mathcal{X}(S^7)$ we get

$$\begin{aligned}(\nabla_X \Phi)(Y, \xi) &= -g(Y, (\nabla_X P)(\xi, \xi)) + g(Y, P(\dot{\varphi}X, \xi)) = \\ &= g(\varphi Y, \dot{\varphi}X).\end{aligned} \tag{4.40}$$

Developing in the same way $(\nabla_{\varphi X} \Phi)(\varphi Y, \xi)$ we obtain (4.39). ∎

The above proposition has a very important meaning. Since $(\nabla_X\Phi)(Y,\xi)-(\nabla_{\varphi X}\Phi)(\varphi Y,\xi)$ is generally different from zero, we deduce from (4.39) that the endomorphisms φ and $\dot\varphi$ don't commute each other. Then, taking account of the results concerning the normality of (φ,ξ,η,g), we can already state that the structure is non normal.

The following theorem concludes the exam of (φ,ξ,η,g). For an extensive and detailed description of the twelve classes of $\mathcal{C}$ we shall refer to [4].

THEOREM 4.2. *(S^7,φ,ξ,η,g) is of class $\mathcal{D}_1\oplus\mathcal{C}_6\oplus\mathcal{C}_7\oplus\mathcal{C}_{10}$. In particular, (φ,ξ,η,g) is a non normal almost K-contact on S^7.*

PROOF. Following [4], we split the space $\mathcal{C}(T_mS^7)$, $m\in S^7$, into the direct sum

$$\mathcal{C}(T_mS^7)=\mathcal{D}_1\oplus\mathcal{D}_2\oplus\mathcal{D}_3, \tag{4.41}$$

where

$$\begin{cases} \mathcal{D}_1=\mathcal{C}_1\oplus\cdots\oplus\mathcal{C}_4= \\ \quad =\{\alpha\in\mathcal{C}(V)\mid\alpha(\xi,x,y)=\alpha(x,\xi,y)=0\} \\ \mathcal{D}_2=\mathcal{C}_5\oplus\cdots\oplus\mathcal{C}_{11}= \\ \quad =\{\alpha\in\mathcal{C}(V)\mid\alpha(x,y,z)= \\ \quad =\eta(x)\alpha(\xi,y,z)+\eta(y)\alpha(x,\xi,z)+\eta(z)\alpha(x,y,\xi)\} \\ \mathcal{D}_3=\mathcal{C}_{12}=\{\alpha\in\mathcal{C}(V)\mid\alpha(x,y,z)= \\ \quad =\eta(x)\eta(y)\alpha(\xi,\xi,z)+\eta(x)\eta(z)\alpha(\xi,y,\xi)\}. \end{cases} \tag{4.42}$$

Because of (4.42), we can consider the covariant derivative $(\nabla\Phi)_m$, $m\in S^7$, of the fundamental 2-form Φ of (φ,ξ,η,g), as the sum of three components $\alpha_k\in\mathcal{D}_k,\ k=1,2,3$:

$$(\nabla\Phi)_m=\alpha_1+\alpha_2+\alpha_3. \tag{4.43}$$

At first we remark that, since P is nearly parallel on S^7, the components α_k have very simple expressions. In fact, (3.30) and (4.35) imply

$$\begin{aligned}(\nabla_\xi\Phi)(X,Y)&=g(X,(\nabla_\xi\varphi)Y)= \\ &=-g(X,P(\nabla_\xi\xi,Y))=g(X,P(\dot\varphi\xi,Y))=0\end{aligned} \tag{4.44}$$

for all $X, Y \in \mathcal{X}(S^7)$. From the previous relation we deduce that $\alpha_3 = 0$ which means that $\nabla\Phi$ has not component in $\mathcal{D}_3$ and then in $\mathcal{C}_{12}$. Moreover, $\nabla_\xi\Phi = 0$ assures that the structure is almost K-contact (see [4]).

In order to compute the complete decomposition, let us consider that the equation $\nabla_X\xi = -\dot{\varphi}X$ implies that $\alpha_2(X, \varphi X, \xi) = g(\varphi X, (\nabla_X\varphi)\xi) = g(\varphi X, \varphi\dot{\varphi}X) = 0$, which, following [4], simply says that $\nabla\Phi$ doesn't have component in $\mathcal{C}_5$.

Now, making the necessary computations, we find

$$\sum_l \alpha_2(e_l, e_l, \xi) = \sum_l g(e_l, (\nabla_{e_l}\varphi)\xi) = -\sum_l g(\varphi e_l, \dot{\varphi}e_l), \tag{4.45}$$

with $\{e_l\}$ an orthonormal basis for T_mS^7. If, in particular, we consider the basis $\mathcal{B}$ (2.5) for the octonions algebra *Cay*, we can choose on T_mS^7 the basis $\{e_l = i_lN\}$, $l = 1, \cdots, 7$, obtaining that $\sum\limits_l \alpha_2(e_l, e_l, \xi) = \frac{1}{3}(1 - \|i_kN + Ni_k\|^2)$. From this last equation, we get that the component β_6 of $\nabla\Phi$ in $\mathcal{C}_6$ is given by [4]: $\beta_6(X, Y, Z) = \mu(g(X, Y)\eta(Z) - g(X, Z)\eta(Y))$ with $\mu = 1 - \|i_kN + Ni_k\|^2$.

To find the other components of the structure in $\mathcal{D}_2$, let us denote by θ the remaining part α_2. If we write $\theta = \theta_+ \oplus \theta_-$, with $\theta_\pm(X, Y, \xi) = \frac{1}{2}(\theta(X, Y, \xi) \pm \theta(\varphi X, \varphi Y, \xi))$ for all $X, Y \in \mathcal{X}(S^7)$, a direct computation proves that

$$\theta(X, Y, \xi) + \theta(Y, X, \xi) = \theta(\varphi X, \varphi Y, \xi) + \theta(\varphi Y, \varphi X, \xi) \tag{4.46}$$

getting also

$$\theta_-(X, Y, \xi) + \theta_-(Y, X, \xi) = 0, \tag{4.47}$$

which yields the vanishing of the component in $\mathcal{C}_9$.

On the other hand, θ_+ expresses just the sum of components in $\mathcal{C}_7$ and $\mathcal{C}_8$ [4]. More precisely we have

$$\beta_{7,8}(X, Y, \xi) = \frac{1}{2}(\theta_+(X, Y, \xi) \pm \theta_+(Y, X, \xi)). \tag{4.48}$$

Because of this relation, due to the symmetry in X and Y of θ_+, the component in $\mathcal{C}_8$ vanishes identically. Finally, concerning the component

β_{11} in $\mathcal{C}_{11}$, since for definition $\beta_{11}(X,Y,Z)=\eta(X)\beta_{11}(\xi,Y,Z)$ the shown equality $\nabla_\xi\Phi=0$ gives $\beta_{11}=0$ too.

Then the only other components of $\nabla\Phi$ in $\mathcal{D}_2$ are β_7,β_{10} which are respectively given by

$$\beta_7(X,Y,\xi)=-\frac{1}{2}(g(\dot\varphi X,\varphi Y)+g(\dot\varphi Y,\varphi X))-\mu g(X,Y),\quad X,Y\in\mathcal{X}(S^7), \tag{4.49}$$

$$\beta_{10}(X,Y,\xi)=-\frac{1}{2}(g(\dot\varphi X,\varphi Y)-g(\dot\varphi Y,\varphi X)),\quad X,Y\in\mathcal{X}(S^7),X,Y\perp\xi. \tag{4.50}$$

Now, a laborious direct check of the belonging conditions for the twelve classes of almost hermitian structures given in [7], shows that the restriction of the structure to the horizontal subbundle HS^7 gives a generic almost hermitian structure.

Then, finally, we obtain $\nabla\Phi=\alpha_1+\beta_6+\beta_7+\beta_{10}$, proving the theorem. ∎

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