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A multiplicity result for solutions of a nonlinear elliptic system with Neumann conditions

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RIASSUNTO: *Consideriamo un sistema ellittico con condizioni di Neumann*

$$\begin{cases} -\varepsilon\Delta u_1 + F_{x_1}(u_1,u_2) = 0 & in\ \Omega \\ -\varepsilon\Delta u_2 + F_{x_2}(u_1,u_2) = 0 & in\ \Omega \end{cases}$$

dove $\Omega \subset \mathbb{R}^n$ è un dominio aperto e limitato. Proveremo che il numero di soluzioni è strettamente legato al numero di punti critici di F usando una generalizzazione della teoria di Morse dovuta a Benci e Giannoni che si applica anche per punti critici degeneri.

ABSTRACT: *We consider an elliptic system with Neumann conditions*

$$\begin{cases} -\varepsilon\Delta u_1 + F_{x_1}(u_1,u_2) = 0 & in\ \Omega \\ -\varepsilon\Delta u_2 + F_{x_2}(u_1,u_2) = 0 & in\ \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain. We will show that the number of solutions is strictly linked with the number of critical points of F using a generalized Morse theory developed by Benci and Giannoni which works also for degenerate critical points.

1 – Introduction and statement of the result

Let us consider the problem

$$(1) \qquad \begin{cases} u_1, u_2 \in C^2(\overline{\Omega}) \\ -\varepsilon\Delta u_1 + F_{x_1}(u_1,u_2) = 0 & \text{in } \Omega \\ -\varepsilon\Delta u_2 + F_{x_2}(u_1,u_2) = 0 & \text{in } \Omega \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

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where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with sufficiently regular boundary ($n \geq 2$), $\varepsilon > 0$ is a real number and $F \in C^2(\mathbb{R}^2, \mathbb{R})$.

This problem is used as mathematical model for some phase transition problems arising from mathematical physics (see, for example, [3] and references therein).

We want to study the existence and the multiplicity of (1) under some assumptions on F and we will show that the number of solutions is strictly linked with the number of critical points of F. We observe explicitly that if $a \in \mathbb{R}^2$ is a critical point of F, then u_a, the function constantly equal to a, is a trivial solution of (1). Therefore we must be sure that each critical point found is nontrivial. In order to do this and in order to estimate from below the number of critical points, we will use Morse theory. There is a technical complication due to the fact that critical points of L_ε may be degenerate. We will overcome this difficulty by using a generalized Morse theory developed by Benci and Giannoni [2].

Similar problems are also been studied by Modica, Mortola, Passaseo and Vannella [3], [4], [5], [7]. In particular we will extend the results of Vannella [7] to the case of an elliptic system with Neumann conditions. The difficulty consists of the fact that, while in Vannella's paper there are only minimum and maximum points of F, here we have also saddle points and this requires more attention, as we will see.

Here we consider the problem (1) under these conditions:

(i.1) F is coercive, that is

$$\lim_{\|(x_1,x_2)\| \to \infty} F(x_1, x_2) = +\infty;$$

(i.2) there exist $a, b > 0$ such that

$$\forall (x_1, x_2) \in \mathbb{R}^2 \quad \forall i = 1, 2 \quad |F_{x_i}(x_1, x_2)| \leq a\|(x_1, x_2)\|^{p-1} + b$$

where $p \in]2, 2^*[$ and $2^* = \dfrac{2n}{n-2}$ if $n \geq 3$, while $p > 2$ if $n = 2$;

(i.3) F has only isolated critical points, in finite number, and in each critical point the Hessian H_F is invertible;

(i.4) there exists $r > 0$ such that, for $i = 1, 2$ and for all $(x_1, x_2) \in \mathbb{R}^2$ with $x_i > r$ we have that $F_{x_i}(x_1, x_2) > 0$, while if $x_i < -r$, then $F_{x_i}(x_1, x_2) < 0$.

The solutions of (1) are critical points of the functional $L_\varepsilon : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ defined by

$$(2) \qquad \forall u=(u_1,u_2) \qquad L_\varepsilon(u)=\frac{\varepsilon}{2}\int_\Omega \sum_{i=1}^{2}|\nabla u_i|^2 dx+\int_\Omega F(u(x))dx.$$

We can now enunciate a first version of the main result of this paper.

THEOREM 1. *Under hypotheses* (i.1), (i.2), (i.3) *and* (i.4), *if* L_ε *has* k *critical points and if* ε *is suitably small and it doesn't belong to a countable set, then the problem* (1) *has at least* $k-1$ *non trivial solutions.*

The outline of the paper is as follows. In Section 2 we recall some basic definitions and then we will give some properties of critical points of the functional (2). In Section 3 we will give the precise statement of Theorem 1 and we will prove it.

2 – Preliminary results

As said before, since L_ε could have degenerate critical points we cannot apply the classical Morse theory, we will use the generalized Morse theory developed by Benci and Giannoni [2]. To this end, in this section, we will see some preliminary properties of L_ε and of its critical points under assumptions (i.1), (i.2) (i.3) and (i.4).

First of all the functional L_ε is of class C^2 in $H^1(\Omega)\times H^1(\Omega)$ and, for every $u,v,w \in H^1(\Omega)\times H^1(\Omega)$,

$$dL_\varepsilon(u)[v]=\varepsilon\int_\Omega \sum_{i=1}^{2}(\nabla u_i|\nabla v_i)dx+\int_\Omega(\nabla F(u)|v)dx$$

and

$$d^2L_\varepsilon(u)[v,w]=\varepsilon\int_\Omega \sum_{i=1}^{2}(\nabla v_i|\nabla w_i)dx+\int_\Omega(H_{F(u)}v|w)dx$$

where $(\cdot|\cdot)$ is the inner product in $\mathbb{R}^2$.

We recall now some basic definitions.

Definition 2. We say that $u \in H^1(\Omega) \times H^1(\Omega)$ is a *critical point* of L_ε if

$$dL_\varepsilon(u) = 0.$$

We will denote with K_{L_ε} the set of all the critical points of L_ε.

Moreover we say that L_ε satisfies the *Palais-Smale condition* (PS) if any sequence $\{u_n\} \subset H^1(\Omega) \times H^1(\Omega)$ such that $\{L_\varepsilon(u_n)\}$ is bounded and $\lim_n dL_\varepsilon(u_n) = 0$ in the dual space of $H^1(\Omega) \times H^1(\Omega)$ has a convergent subsequence.

Definition 3. If $u \in K_{L_\varepsilon}$ and there exists $d^2 L_\varepsilon(u)$, the *Morse index* of u is the maximal dimension of a subspace of $H^1(\Omega) \times H^1(\Omega)$ on which $d^2 L_\varepsilon(u)$ is negative definite and it is denoted by $m(u)$.

The *nullity* of u is the dimension of the kernel of $d^2 L_\varepsilon(u)$.

The *large Morse index* is the sum of the Morse index and the nullity and it is denoted with $m^*(u)$.

A critical point u is called *nondegenerate* if its nullity is 0, otherwise it is called *degenerate*.

We can start with the investigation on L_ε.

Lemma 4. *There exists $r > 0$ such that, if $u = (u_1, u_2)$ is a critical point of L_ε, then $u \in [-r, r] \times [-r, r]$ and in particular $u \in L^\infty(\Omega) \times L^\infty(\Omega)$.*

Proof. By hypothesis (i.4), there exists $r > 0$ such that for all $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 > r$ we have that $F_{x_1}(x_1, x_2) > 0$.

Let $u = (u_1, u_2)$ be a critical point of L_ε. We will prove that $u_1 \in L^\infty(\Omega)$.

We take $G \in C^1(\mathbb{R}, \mathbb{R})$ bounded and such that

$$\forall t \leq r \quad G(t) = 0 \text{ and } \forall t > r \qquad 0 < G'(t) \leq M$$

with $M > 0$.

Let $v(x) = (G(u_1(x)), 0) \in H^1(\Omega) \times H^1(\Omega)$, we have

$$0 = dL_\varepsilon(u)[v] = \varepsilon \int_\Omega G'(u_1(x)) |\nabla u_1|^2 dx + \int_\Omega F_{x_1}(u(x)) G(u_1(x)) dx.$$

As the sum of these two positive quantities is zero, both of them have to vanish and observing that

$$\forall x \in \Omega \text{ with } u_1(x) > r : F_{x_1}(u(x))G(u_1(x)) > 0,$$

then

$$u_1 \leq r \text{ a.e. in } \Omega.$$

Analogously

$$u_1 \geq -r \text{ a.e. in } \Omega. \qquad \square$$

Let us now see a regularization result for solutions of (1).

Lemma 5. *If $u = (u_1, u_2)$ is a critical point of L_ε, then its components are of class $C^2(\overline{\Omega})$ and so u is a classical solution of* (1).

Proof. We recall a regularization result holding under our assumptions (see [6]).

If $r \geq 1$ and $f \in L^r(\Omega)$, then there exists a unique solution $v \in H^{2,r}(\Omega)$ of

$$\begin{cases} -\Delta v + v = f & \text{in } \Omega \\ \gamma(\dfrac{\partial v}{\partial n}) = 0 & \text{on } \partial\Omega \end{cases}$$

where $\gamma : H^{1,r}(\Omega) \longrightarrow H^{1-\frac{1}{r},r}(\partial\Omega)$ is the trace operator.

From the Lemma 4 and the fact that F_{x_1} is continuous, we get

$$f_1(x) = -\frac{1}{\varepsilon} F_{x_1}(u(x)) + u_1(x) \in L^\infty(\Omega)$$

and so, for the previous result and the fact that $u = (u_1, u_2)$ is a critical point of L_ε, we have

$$u_1 \in H^{2,r}(\Omega) \text{ for each } r > 1.$$

By Sobolev embedding theorem we have $u_1 \in C^1(\overline{\Omega})$ and so $f_1 \in C^1(\overline{\Omega})$. Since in particular $f_1 \in L^2(\Omega)$ and $u_1 \in H^{2,2}(\Omega)$, taking $v \in C_0^\infty(\Omega) \subset H^1(\Omega)$, we have

$$0 = L_\varepsilon(u)[(v,0)] = \varepsilon \int_\Omega (\nabla u_1 | \nabla v) dx + \int_\Omega F_{x_1}(u(x)) v(x) dx =$$
$$= \int_\Omega (-\varepsilon \Delta u_1 + F_{x_1}(u(x))) v(x) dx$$

hence

$$-\varepsilon\Delta u_1 + F_{x_1}(u(x)) = 0 \text{ a.e. in } \Omega$$

and using regularity results we conclude that $u_1 \in C^2(\overline{\Omega})$. ∎

LEMMA 6. *L_ε is a coercive functional.*

PROOF. Let $(u_n) \subset H^1(\Omega) \times H^1(\Omega)$ be a sequence such that

$$\|u_n\|_{H^1(\Omega)\times H^1(\Omega)} \to \infty.$$

If $\|\nabla u_n\|_{L^2(\Omega)\times L^2(\Omega)} \to \infty$, as F is bounded from below, then $L_\varepsilon(u_n) \to \infty$.

In the other case, up to subsequence $\|u_n\|_{L^2(\Omega)\times L^2(\Omega)} \to \infty$. By Lemma 4, taken $r > 0$ sufficiently big and indicated with $R = [-r, r]^2$, without lost of generality we can suppose F such that

$$F(x_1, x_2) \geq c(x_1^2 + x_2^2) \text{ if } (x_1, x_2) \notin R.$$

Let

$$\Omega_n = \{x \in \Omega | u_n(x) \notin R.\}.$$

Since

$$\int_{\Omega\setminus\Omega_n} (|u_{n,1}(x)|^2 + |u_{n,2}(x)|^2)dx \leq 2r^2|\Omega|,$$

we get

$$\int_{\Omega_n} (|u_{n,1}(x)|^2 + |u_{n,2}(x)|^2)dx \to +\infty.$$

Finally

$$L_\varepsilon(u_n) \geq \int_\Omega F(u_n(x))dx \geq \int_{\Omega\setminus\Omega_n} F(u_n(x))dx + \int_{\Omega_n} (|u_{n,1}(x)|^2 + |u_{n,2}(x)|^2)dx,$$

and so L_ε is coercive. ∎

As seen in [7], the following lemma holds.

LEMMA 7. *L_ε verifies (PS) and $\exists\Psi$ of class C^1 with Ψ' completely continuous such that $L_\varepsilon(u) = \frac{\varepsilon}{2}\langle u, u\rangle_{H^1(\Omega)\times H^1(\Omega)} + \Psi(u)$.*

LEMMA 8. *The set K_{L_ε} of critical points of L_ε is compact.*

PROOF. Since L_ε satisfies (PS), it is sufficient to prove that K_{L_ε} is bounded in $H^1(\Omega) \times H^1(\Omega)$.

Let $u \in K_{L_\varepsilon}$, we have:

$$dL_\varepsilon(u)[u] = \varepsilon \int_\Omega \sum_{i=1}^{2} |\nabla u_i|^2 dx + \int_\Omega (\nabla F(u)|u) dx = 0.$$

By Lemma 4 the claim immediately follows. ∎

Following propositions deal with the Morse index of trivial critical points of L_ε.

In the following, we will denote with $0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_i \le \cdots$ the eigenvalues of $-\Delta$ on Ω with Neumann boundary conditions, while if $a = (a_1, a_2) \in \mathbb{R}^2$, u_a will denote a function constantly equal to a in Ω.

Let s be a saddle point of F and let μ^1, μ^2 be the eigenvalues of $H_{F(s)}$ with $\mu^1 < 0 < \mu^2$.

PROPOSITION 9. *If $\varepsilon \ne \frac{-\mu^1}{\lambda_j}$ for all $j \ge 1$, then u_s is nondegenerate. Besides let $\bar{\jmath}$ be the natural number such that $\varepsilon \in]\frac{-\mu^1}{\lambda_{\bar{\jmath}}}, \frac{-\mu^1}{\lambda_{\bar{\jmath}-1}}[$, or otherwise $\bar{\jmath} = 1$ if $\varepsilon \in]\frac{-\mu^1}{\lambda_1}, +\infty[$, then:*

$$m(u_s) = \bar{\jmath}.$$

PROOF. Let us observe that u_s is degenerate if and only if there exists $v \in H^1(\Omega) \times H^1(\Omega)$ such that for all $w \in H^1(\Omega) \times H^1(\Omega)$

$$0 = d^2 L_\varepsilon(u_s)[v, w] = \varepsilon \int_\Omega \sum_{i=1}^{2} (\nabla v_i|\nabla w_i) dx + \int_\Omega (H_{F(u_s)} v|w) dx$$

and so if and only if, indicated with $-\Delta v$ the vector $(-\Delta v_1, -\Delta v_2)$, the equation

$$-\varepsilon \Delta v + H_{F(u_s)} v = 0 \tag{3}$$

has a nonzero solution in $H^1(\Omega) \times H^1(\Omega)$.

Since $H_{F(u_s)}$ is a symmetric matrix, there exists an invertible matrix B such that

$$BH_{F(u_s)}B^{-1} = D \quad \text{and} \quad H_{F(u_s)} = B^{-1}DB$$

with $D = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$. It is easy to verify that for all matrices A we have $-\varepsilon\Delta(Av) = A(-\varepsilon\Delta v)$, therefore v is a solution of (3) if and only if

$$0 = B^{-1}B(-\varepsilon\Delta v + H_{F(u_s)}B^{-1}Bv) = B^{-1}(-\varepsilon\Delta(Bv) + BH_{F(u_s)}B^{-1}Bv)$$

and so Bv is a nonzero solution of

$$-\varepsilon\Delta w + Dw = 0. \tag{4}$$

Conversely, we can prove that if w satisfies the (4), then $B^{-1}w$ is a solution of (3). The degeneration of u_s is equivalent to the existence of a nonzero solution of (4), that is $w = (w_1, w_2)$ such that

$$\begin{cases} -\varepsilon\Delta w_1 + \mu^1 w_1 = 0 \\ -\varepsilon\Delta w_2 + \mu^2 w_2 = 0 \end{cases}$$

and this happens if and only if $\frac{-\mu^1}{\varepsilon}$ or $\frac{-\mu^2}{\varepsilon}$ is an eigenvalue of $-\Delta$. Since $\mu^1 < 0 < \mu^2$, u_s is nondegenerate if and only if $\varepsilon \neq \dfrac{-\mu^1}{\lambda_j}$ for all $j \geq 1$.

The first part of the thesis is proved, let us show the last one.

Let $(e_i)_{i\in\mathbb{N}}$ be the orthonormal basis of $L^2(\Omega)$ such that e_i is the eigenfunction relative to λ_i:

$$\begin{cases} e_i \in H^1(\Omega) \cap C^\infty(\Omega) & \\ -\Delta e_i = \lambda_i e_i & \text{in } \Omega \\ \dfrac{\partial e_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $p^1 = (p^1_1, p^1_2)$ and $p^2 = (p^2_1, p^2_2)$ be the orthonormal eigenfunctions of $H_{F(u_s)}$ relative respectively to μ^1 and μ^2. For $l = 1, 2$ and $j \in \mathbb{N}$, denoted with

$$v^l_j = p^l e_j = (p^l_1 e_j, p^l_2 e_j),$$

$\{v_j^l\}_{j\in\mathbb{N}}^{l=1,2}$ is a basis of $H^1(\Omega)\times H^1(\Omega)$.
For the calculus of

$$d^2L_\varepsilon(u_s)[v_j^l,v_j^l]=\varepsilon\int_\Omega\sum_{i=1}^2|\nabla p_i^l e_j|^2dx+\int_\Omega(H_{F(u_s)}v_j^l|v_j^l)dx \tag{5}$$

we observe that for all i

$$\int_\Omega|\nabla p_i^l e_j|^2dx=(p_i^l)^2\int_\Omega|\nabla e_j|^2dx=(p_i^l)^2\lambda_j$$

and so the first term of (5) becomes

$$\varepsilon\int_\Omega\sum_{i=1}^2|\nabla p_i^l e_j|^2dx=\varepsilon\sum_{i=1}^2(p_i^l)^2\lambda_j=\varepsilon\lambda_j$$

while for the second we have

$$\begin{aligned}\int_\Omega(H_{F(u_s)}v_j^l|v_j^l)dx&=\int_\Omega(H_{F(u_s)}p^le_j|p^le_j)dx=(H_{F(u_s)}p^l|p^l)\int_\Omega(e_j)^2dx=\\&=(H_{F(u_s)}p^l|p^l)=(\mu^lp^l|p^l)=\mu^l\|p^l\|^2=\mu^l.\end{aligned}$$

Therefore

$$d^2L_\varepsilon(u_s)[v_j^l,v_j^l]=\varepsilon\lambda_j+\mu^l.$$

Now we observe that for all $j\in\mathbb{N}$

$$d^2L_\varepsilon(u_s)[v_j^2,v_j^2]>0. \tag{6}$$

At the contrary

$$d^2L_\varepsilon(u_s)[v_0^1,v_0^1]=\mu^1<0 \tag{7}$$

and for all $j\geq 1$

$$d^2L_\varepsilon(u_s)[v_j^1,v_j^1]<0\iff\varepsilon\lambda_j+\mu^1<0\iff\varepsilon<\frac{-\mu^1}{\lambda_j}. \tag{8}$$

Let us prove now that

$$\text{if } (l_1,j_1)\neq(l_2,j_2) \text{ then } d^2L_\varepsilon(u_s)[v_{j_1}^{l_1},v_{j_2}^{l_2}]=0.$$

In fact

$$d^2L_\varepsilon(u_s)[v^{l_1}_{j_1}, v^{l_2}_{j_2}] = \varepsilon\int_\Omega \sum_{i=1}^{2}(\nabla p^{l_1}_i e_{j_1}|\nabla p^{l_2}_i e_{j_2})dx + \int_\Omega (H_{F(u_s)} v^{l_1}_{j_1}|v^{l_2}_{j_2})dx =$$
$$= \varepsilon(p^{l_1}|p^{l_2})\int_\Omega (\nabla e_{j_1}|\nabla e_{j_2})dx + \mu^{l_1}(p^{l_1}|p^{l_2})\int_\Omega e_{j_1}e_{j_2}dx = 0.$$

Therefore in our hypotheses and by (6), (7) and (8) $d^2L_\varepsilon(u_s)$ is negative definite on

$$\bigoplus_{0\le j\le \bar{\jmath}-1} \mathbb{R}v^1_j$$

and positive on

$$\overline{\bigoplus_{j\ge \bar{\jmath}} \mathbb{R}v^1_j \oplus \bigoplus_{j\in\mathbb{N}} \mathbb{R}v^2_j}$$

and so $m(u_s) = \bar{\jmath}$. □

The next proposition is obvious.

Proposition 10. *If c is a minimum point of F, then $m(u_c) = m^*(u_c) = 0$.*

Let d be a maximum point of F and let η^1, η^2 be the eigenvalues of $H_{F(d)}$ with $\eta^1 \le \eta^2 < 0$.

Proposition 11. *If $\varepsilon \ne \frac{-\eta^1}{\lambda_j}$ and $\varepsilon \ne \frac{-\eta^2}{\lambda_j}$ for all $j \ge 1$, then u_d is nondegenerate. Besides for $l = 1, 2$ let $\bar{\jmath}_l$ be the natural number such that $\varepsilon \in]\frac{-\eta^l}{\lambda_{\bar{\jmath}_l}}, \frac{-\eta^l}{\lambda_{\bar{\jmath}_l-1}}[$, or otherwise $\bar{\jmath}_l = 1$ if $\varepsilon \in]\frac{-\eta^l}{\lambda_1}, +\infty[$, then:*

$$m(u_d) = \bar{\jmath}_1 + \bar{\jmath}_2$$

in particular $m(u_d) \ge 2$.

PROOF. In a similar way of Proposition 9 we can prove that u_d is nondegenerate.

Let $q^1 = (q_1^1, q_2^1)$ and $q^2 = (q_1^2, q_2^2)$ the orthonormal eigenfunctions of $H_{F(u_d)}$ relative respectively to η^1 and η^2. For $l = 1, 2$ and $j \in \mathbb{N}$, indicated with

$$w_j^l = q^l e_j,$$

$\{w_j^l\}_{j\in\mathbb{N}}^{l=1,2}$ is a basis of $H^1(\Omega) \times H^1(\Omega)$.
We have

$$d^2 L_\varepsilon(u_d)[w_j^l, w_j^l] = \varepsilon\lambda_j + \eta^l.$$

Now we observe that for $l = 1, 2$ and for all $j \in \mathbb{N}$

$$d^2 L_\varepsilon(u_d)[w_0^l, w_0^l] = \eta^l < 0 \tag{9}$$

while for all $j \geq 1$

$$d^2 L_\varepsilon(u_d)[w_j^l, w_j^l] < 0 \iff \varepsilon\lambda_j + \eta^l < 0 \iff \varepsilon < \frac{-\eta^l}{\lambda_j}. \tag{10}$$

We can prove that

$$\text{if } (l_1, j_1) \neq (l_2, j_2) \text{ then } d^2 L_\varepsilon(u_d)[w_{j_1}^{l_1}, w_{j_2}^{l_2}] = 0.$$

So in our hypotheses and by (9) and (10) $d^2 L_\varepsilon(u_d)$ is negative definite on

$$\bigoplus_{0\leq j\leq \bar{\jmath}_l - 1}^{l=1,2} \mathbb{R} w_j^l$$

and positive on

$$\overline{\bigoplus_{j\geq \bar{\jmath}_l}^{l=1,2} \mathbb{R} w_j^l}$$

hence $m(u_d) = \bar{\jmath}_1 + \bar{\jmath}_2$. ∎

Remark 12. As said before we will use the generalized Morse theory developed by Benci and Giannoni [2] which works for a big class $\mathcal{F}$ of functionals on $H^1(\Omega)\times H^1(\Omega)$. Since in Example 5.2 of [2] we can replace the hypothesis that f is bounded with the one that K_f is bounded and since Lemma 7 holds, L_ε belongs to $\mathcal{F}$. Therefore as L_ε is bounded from below, by Theorem 5.9 of [2], we have a generalized Morse equality

$$i_\lambda(K_{L_\varepsilon}) = 1 + (1+\lambda)Q_\lambda \tag{11}$$

where $i_\lambda(K_{L_\varepsilon})$ is the Morse index of K_{L_ε}, that is a formal series in one variable λ with coefficients in $\mathbb{N}\cup\{+\infty\}$ that, in the classical theory, is nothing else that the Morse polynomial. In fact, if u is a nondegenerate critical point of L_ε, we have

$$i_\lambda(\{u\}) = \lambda^{m(u)}.$$

3 – Main result

In this section we will present the main result of the paper giving the precise statement of Theorem 1.

Let $c_1, \cdots, c_{k_1}$ be the minimum points of F, $d_1, \cdots, d_{k_2}$ the maximum points and $s_1, \cdots, s_{k_3}$ the saddle points. For all $1 \le h_2 \le k_2$ we indicate with $\eta^1_{h_2}, \eta^2_{h_2}$ the eigenvalues of $H_{F(d_{h_2})}$ and for all $1 \le h_3 \le k_3$ we indicate with $\mu^1_{h_3}$ the only negative eigenvalue of $H_{F(s_{h_3})}$.

Theorem 13. *Under hypotheses* (i.1), (i.2) (i.3) *and* (i.4), *if* ε *is sufficiently small and such that* $\varepsilon \neq \frac{-\mu^1_{h_3}}{\lambda_j}, \varepsilon \neq \frac{-\eta^1_{h_2}}{\lambda_j}$ *and* $\varepsilon \neq \frac{-\eta^2_{h_2}}{\lambda_j}$ *for all* $j \ge 1$, *for all* $1 \le h_2 \le k_2$ *and for all* $1 \le h_3 \le k_3$, *then the problem* (1) *has at least* $k_1 + k_2 + k_3 - 1$ *nontrivial solutions.*

Proof. Let K denote the set of nontrivial critical points of L_ε. By Propositions 9, 10 and 11 each trivial critical point of L_ε is nondegenerate and so

$$K_{L_\varepsilon} = K \cup \{u_{c_1}\} \cup \cdots \cup \{u_{c_{k_1}}\} \cup \{u_{d_1}\} \cup \cdots \cup \{u_{d_{k_2}}\} \cup \{u_{s_1}\} \cup \cdots \cup \{u_{s_{k_3}}\}$$

where each subset is an isolated critical set of K_{L_ε}.

So, by (11), Theorem 5.8 of [2] and Remark 12, we have

$$
\begin{aligned}
(12)\qquad i_\lambda(K) + \lambda^{m(u_{c_1})} + \cdots + \lambda^{m(u_{c_{k_1}})} + \lambda^{m(u_{d_1})} + \\
+ \cdots + \lambda^{m(u_{d_{k_2}})} + \lambda^{m(u_{s_1})} + \\
+ \cdots + \lambda^{m(u_{s_{k_3}})} = 1 + (1+\lambda)Q_\lambda.
\end{aligned}
$$

Writing

$$
i_\lambda(K) = \sum_{h\in\mathbb{N}} a_h\lambda^h \quad\text{and}\quad Q_\lambda = \sum_{l\in\mathbb{N}} b_l\lambda^l
$$

from Proposition 10 we get $m(u_{c_1}) = \cdots = m(u_{c_{k_1}}) = 0$ and so the (12) becomes

$$
\begin{aligned}
(13)\qquad \sum_{h\in\mathbb{N}} a_h\lambda^h + k_1 + \lambda^{m(u_{d_1})} + \cdots + \lambda^{m(u_{d_{k_2}})} + \lambda^{m(u_{s_1})} + \cdots + \lambda^{m(u_{s_{k_3}})} = \\
= 1 + (1+\lambda)\sum_{l\in\mathbb{N}} b_l\lambda^l
\end{aligned}
$$

Suppose that $\varepsilon < \min_{1\le h_3\le k_3} \dfrac{-\mu^1_{h_3}}{\lambda_1}$, then, by Proposition 9, for all $1 \le h_3 \le k_3$ we have $m(u_{s_{h_3}}) \ge 2$, while, by Proposition 11, for all $1 \le h_2 \le k_2$ we have $m(u_{d_{h_2}}) \ge 2$.

Comparing the coefficients of the same degree of (13), we get

$$
a_1 = b_0 + b_1 \ge b_0 = k_1 - 1 + a_0 \ge k_1 - 1
$$

so there exist at least $k_1 - 1$ critical points, counted with their multiplicity, with Morse index 1.

In the following u_j will indicate $u_{s_{h_3}}$ or $u_{d_{h_2}}$ for all $1 \le j \le k_2 + k_3$. Let us prove that for all $1 \le j \le k_2 + k_3$ there exists a nontrivial critical point of L_ε with index $m(u_j) + 1$ or $m(u_j) - 1$ and they are all different each other.

Suppose in a first moment that all the elements of the set $\{-\eta^1_{h_2} | 1 \le h_2 \le k_2\} \cup \{-\mu^1_{h_3} | 1 \le h_3 \le k_3\}$ are different each other. Let us order them in a strictly increasing order $\rho_1 < \rho_2 < \cdots < \rho_{k_2+k_3}$. We want to prove that

(14) for all $1 \le j \le k_2 + k_3$ we have $m(u_{j+1}) \ge m(u_j) + 2$.

By [1] (Theorem 14.6) there exists ε sufficiently small such that for all $1 \leq j \leq k_2 + k_3$ the interval $]\frac{\rho_j}{\varepsilon}, \frac{\rho_{j+1}}{\varepsilon}[$ contains at least two of eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$.

There are different cases. We will study the most complex one, the others follow immediately.

Suppose that there exist h and h' such that $-\eta_h^1 < -\eta_{h'}^1$, then we can choose ε sufficiently small such that the interval $]\frac{-\eta_h^1}{\varepsilon}, \frac{-\eta_{h'}^1}{\varepsilon}[$ contains so many eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ that $m(u_{d_{h'}}) \geq m(u_{d_h}) + 2$.

In this way we have that the difference of the Morse indexes of any two elements of $\{u_{d_{h_2}} | 1 \leq h_2 \leq k_2\} \cup \{u_{s_{h_3}} | 1 \leq h_3 \leq k_3\}$ is at least two and so we have shown the (14).

For $j = 1$, from (13) we know that $b_{m(u_1)} \neq 0$ or $b_{m(u_1)-1} \neq 0$, and so respectively $a_{m(u_1)+1} \neq 0$ or $a_{m(u_1)-1} \neq 0$, therefore there exists a critical point with index $m(u_1) + 1$ or $m(u_1) - 1$ which is different from all the $u_{c_{h_1}}$ and from all the critical points found previously.

Now assuming the assert true for j, let us prove that it is also true for $j + 1$.

In fact $m(u_{j+1}) \neq 0$ yields $b_{m(u_{j+1})} \neq 0$ or $b_{m(u_{j+1})-1} \neq 0$ and so there exists a critical point of L_ε with index $m(u_{j+1}) + 1$ or $m(u_{j+1}) - 1$ which is nontrivial and, by the (14), different from all the critical points already found.

If, at the contrary, $-\mu_{h_3}^1$ and $-\eta_{h_2}^1$ are not all different each other, there are various cases. If ε is chosen sufficiently small, we have

- $-\mu_{h_3}^1 = -\eta_{h_2}^1 \Longrightarrow m(u_{d_{h_2}}) \geq m(u_{s_{h_3}}) + 2$;
- $-\eta_h^1 = -\eta_{h'}^1$ and $-\eta_h^2 < -\eta_{h'}^2 \Longrightarrow m(u_{d_{h'}}) \geq m(u_{d_h}) + 2$.

The last two cases are $-\mu_h^1 = -\mu_{h'}^1$ and, simultaneously, $-\eta_h^1 = -\eta_{h'}^1$ and $-\eta_h^2 = -\eta_{h'}^2$. We may have

$$\exists \bar{\jmath}, l \text{ such that } \rho_{\bar{\jmath}-1} < \rho_{\bar{\jmath}} = \cdots = \rho_{\bar{\jmath}+l} < \rho_{\bar{\jmath}+l+1}$$

where we have excluded the cases already discussed. Reasoning as before, $b_{m(u_{\bar{\jmath}})} + b_{m(u_{\bar{\jmath}})-1} \geq l + 1$ and there exist at least $l + 1$ critical points, counted with their multiplicity, whose Morse index is equal to $m(u_{\bar{\jmath}}) - 1$ or to $m(u_{\bar{\jmath}}) + 1$. ∎

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