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The perturbation functor in the calculus of variations

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ABSTRACT: *In the framework of second order Calculus of Variations on jet bundles we show that the operator which determines the "First Variation" is a functor which we call "Perturbation Functor". This functor allows us to link the Jacobi morphism for the second variation to the first variation of a new Lagrangian. Its naturality properties are discussed. We also show that it permutes with most of the relevant cohomology functors of the Calculus of Variations and with the de Rham's one.*

0 – Introduction

In the last decades several techniques having a geometrical origin have been developed to deal with partial differential equations in general and, more particularly, for those equations which are the consequence of a variational principle (see, e.g., [1], [2], [3], [4] and references quoted therein). In all these frameworks, which are of course based on the use of the jet-prolongations (possibly of infinite order) of both the bundles and the equations involved, the tools of homological algebra have revealed themselves to be extremely powerful. As a few examples we mention: the work of ANDERSON and DUCHAMP ([5], for the introduction of cochain

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complexes in the Calculus of Variations); the work of Bryant and Griffiths ([6] and [7], where the notion of cohomological tower is extensively used); of Tulczyjew and Dedecker ([8], with the introduction of the so-called "Lagrange complex"); of Krupka ([9] and [10], with the introduction of the so-called "variational sequences"; see also [11]).

The Calculus of Variations on jet bundles is a very powerful method in Analysis, Geometry and Matematical Physics. It allows in fact a global perspective on the problems and helps, via Noether's theorem, to provide a general setting for conservation laws (see, e.g., [14]). The fundamental ingredients in this direction are contained in the notion of contact forms, of Poincaré-Cartan forms, of local and global exactness (both at the "strong" level of the bundle or at the "weak" level of the space of critical sections).

In recent investigations of ours ([15], [16], [17]) we have been considering the somehow neglected problem of second variation of a Lagrangian action from the geometrical viewpoint, together with the ensuing notion of (generalized) Jacobi equation. In particular, we have been able to show that the Euler-Lagrange equations together with the Jacobi equations are in fact the Euler-Lagrange equations of a "derived" variational principle in a larger space, governed by a "deformed Lagrangian" which is an algebraic counterpart of the first variation of the original Lagrangian (see [16] for the definition of this new Lagrangian, [15] and [18] for an application to Riemannian Geometry and [19] for a short review).

In the course of our investigations we have realized that most of the relevant constructions entering the first variation, the second variation, the Poincaré-Cartan form and the Jacobi morphism can be alltogether factorized through a functorial operation which can be given the name of "perturbation functor". The perturbation functor, denoted by $\mathcal{P}$, essentially associates to any given Lagrangian $\mathcal{L}$ its first order deformation, in such a way that all relevant quantities of the Calculus of Variations are carried over to the analogous quantities for the new Lagrangian. Such a functor $\mathcal{P}$ is not unique, owing to the well known fact that equivalent Lagrangians and equivalent Jacobi morphisms exist (see, e.g., [10], [14], [18]), although it will be possible to choose "canonical" one.

In this paper we shall develop the basic tools to construct a reasonable (and canonical) perturbation functor in the physically relevant case of Lagrangian theories of order at most two; generalizations to higher orders are of course possible and will be considered elsewhere. We shall then begin (Section 1) with a short account about the basic framework of the Calculus of Variations on jet bundles and the notion of first order deformation of a Lagrangian. Section 2 will be devoted to introduce the fundamental categories of bundles and morphisms which are needed to our purposes, as well as to define the perturbation functor $\mathcal{P}$ and discuss some of its basic features; among them, the most useful comes from a surprising aspect of the procedure which following [16] determines the deformed Lagrangian, which in turn is determined by the existence of a class of immersions (which will be investigated in this Section and which must be taken into account, not only to understand the main properties of the deformation procedure, but also to avoid mistakes which can occur in practical calculations). In Section 3 we shall briefly account on some of the many relations existing between the cohomological interpretation of our functor $\mathcal{P}$ and the existing cohomological tools of [6] and of [9]. Our comparison will be based on the introduction of suitable ideals of forms in the de Rham complex of a convenient jet-prolongation of the relevant bundle. The sub-complex we derive differs in general from the previously existing ones and, in a sense, it is intermediate between the variational complex of [9] and the whole de Rham's complex. We shall investigate how properties of $\mathcal{P}$ reflect in these three cohomological complexes, as well as in the complex introduced in [6].

Among the results of this comparison we quote the construction of a second type of "tower prolongation" (here called "Jacobi tower") obtained by iterating the action of the functor $\mathcal{P}$. This tower prolongation is in a sense the completion of the "tower prolongation" of Bryant and Griffiths and, if applied to the cohomology investigated in [6], it provides informations on the conservation laws of the higher order Jacobi fields, while, if applied to the cohomology introduced in [9], it provides informations on the "Lepagean equivalence" of higher order deformed Lagrangians. Since the notion of "Jacobi tower" applies to any "level" of the tower construction of [6], we obtain a family of cohomological groups, here called "JBG-wall" (where JBG means Jacobi, Bryant and Griffiths). An analogous construction is made for the cohomological groups of [9], since the Bryant-Griffiths tower construction applies to these groups, too. Finally, since closed ideals generate their own cohomological groups, we show that a Jacobi tower construction is possible for both ideals used by Bryant-Griffiths in [6] and by Krupka in [9]. As we said above, we also introduce a new complex in which the Euler-Lagrange form is closed and we show that even for this complex it is possible to perform the "wall construction". More detailed investigations about the interrelationship among these various cohomologies will in fact form the subject of a forthcoming paper ([20]).

Our investigation will pay a continuous attention to the "naturality" properties of the perturbation functor, especially in view of its possible applications to the problem of conservation laws. This intriguing aspect of the theory is still under investigation and will as well form the subject of a further paper ([21]). The present paper contains an appendix, which contains a few remarks about the applications to some relevant partial differential equations of parabolic type (in the sense of [6] and [7], heat equation and KdV equation included).

1 – Preliminaries and notation

In this first Section we shall recall the main framework we need in this paper.

1.1 – Basics on calculus of variations

Let us first list some basic facts about the Calculus of Variations on fibered manifolds. Notation follows closely [2] and [22], to which we refer the reader for further details.

Let $\mathcal{B} = (B, M, \pi)$ be a fibered manifold over a m-dimensional manifold M, with p-dimensional fibers. We will denote by (x^μ), $\mu \in \{1, \ldots, m\}$ a local coordinate system on M and by (x^μ, y^a), $a \in \{1, \ldots, p\}$ a fibered coordinate system on $\mathcal{B}$ over (x^μ).

The *bundle of vertical vectors of* $\mathcal{B}$ is defined as follows. We set $V\pi \equiv \mathrm{Ker}(T\pi) \subseteq TB$ and we define a bundle over B as $V\mathcal{B} = (V\pi, B, \nu_B)$, where ν_B is the appropriate restriction of the natural projection $\tau_B : TB \to B$. For notational convenience, if there is no danger of confusion, we shall write VB instead of $V\pi$. In the sequel we shall be also concerned with double fibrations $C \xrightarrow{\alpha} B \xrightarrow{\pi} M$. In this case there are two vertical bundles, namely those defined by $\mathrm{Ker}(T\alpha)$ over B and by $\mathrm{Ker}[T(\pi \circ \alpha)]$ over M, respectively; they will be respectively denoted by $V^{\mathcal{B}}C$ (or, more simply, just by VC) and by $V^M C$. Hereafter, for the sake of simplicity, "vertical" will shortly mean "vertical with respect to a given projection" whenever there is no need to specify which projection is being considered (if this is already clear from the context).

For any (regular) domain D (i.e., $D \subseteq M$ is a compact m-dimensional submanifold with sufficiently regular boundary) $\Gamma_D(\pi)$ will denote the set of (local) sections $\lambda : D \to B$. Moreover, $J^k\mathcal{B} \equiv (J^k B, B, \pi^k)$ will denote the k-th order jet-prolongation of $\mathcal{B}$, with naturally induced coordinates $(x^\mu, y^a, y^a_\mu, y^a_{\mu\nu}, \dots)$. If $\lambda \in \Gamma_D(\pi)$ is a local section, locally expressed by $(x^\mu, \lambda^a(x^\rho))$, thence its k-th order jet-prolongation $j^k\lambda$ has local expression $(x^\mu, \lambda^a(x^\rho), \partial_\nu\lambda^a(x^\rho), \partial^2_{\mu\nu}\lambda^a(x^\rho), \dots)$.

A section $\Sigma : D \to J^k B$ is said to be *holonomic* iff there exists a section $\lambda : D \to B$ such that $\Sigma = j^k\lambda$. We denote by $\Lambda M = \bigoplus_{0\leq h\leq m} \Lambda^h M$ the exterior bundle of M and by $\mathbf{\Omega}(M) = \bigoplus_{0\leq h\leq m} \mathbf{\Omega}_h(M)$ the module of its sections, i.e. of differential forms of M. We set:

$$\mathbf{ds} = dx^1 \wedge \cdots \wedge dx^m \quad , \quad \mathbf{ds}_\mu \equiv \partial_\mu \rfloor \mathbf{ds} \ , \tag{1.1}$$

where $X\rfloor$ (or, equivalently, sometimes i_X) denotes inner product with respect to a vectorfield X on M; the forms (1.1) determine a (local) basis for m-forms and $(m-1)$-forms, respectively.

A fibered morphism $\mathcal{L} : J^2 B \to \Lambda^m M$ will be called a (*second order*) *Lagrangian*. The Lagrangian $\mathcal{L}$ is locally expressed by:

$$\mathcal{L} = L(x^\mu, y^a, y^a_\mu, y^a_{\mu\nu})\mathbf{ds} \ , \tag{1.2}$$

where L is a scalar density on $J^2 B$ with respect to coordinate changes in the base manifold M. The Lagrangian $\mathcal{L}$ defines a *variational problem* (of the second order) on $\mathcal{B}$, through the *action functionals*:

$$\mathcal{A}(\lambda) = \int_D \mathcal{L} \circ (j^2\lambda) \ . \tag{1.3}$$

Critical sections are those sections $\lambda \equiv \lambda_0 \in \Gamma_D(\pi)$ such that

$$\delta\mathcal{A} \equiv \frac{\partial}{\partial\varepsilon}\mathcal{A}(\lambda_\varepsilon)_{|\varepsilon=0} = 0$$

for all homotopic 1-parameter deformations λ_ε (with $\varepsilon \in]-a, a[$, $a > 0$) which strongly fix the boundary (i.e., $j^1\lambda_\varepsilon|_{\partial D} = j^1\lambda|_{\partial D}$, for any ε).

Here and in the sequel the first variation operator δ will shortly denote the ε-derivative $\frac{\partial}{\partial \varepsilon}$ evaluated at $\varepsilon = 0$. It is well known that critical sections are those sections which satisfy the "Euler-Lagrange equations" of $\mathcal{L}$ (see below). From now on we shall consider only homotopic 1-parameter deformations which strongly fix the boundary.

1.2 – Horizontal forms and canonical momenta

For any integer k let $\mathcal{H}or(J^k\mathcal{B}) = \bigoplus_{0\leq q\leq m} \mathcal{H}or^q(J^k\mathcal{B})$ be the tensor algebra of horizontal forms of $J^k\mathcal{B}$ (i.e., those forms which vanish whenever they are evaluated on a set of vectorfields containing at least one vertical vectorfield).

Definition 1.1 (see [23]). The **horizontal differential** is the operator d_H uniquely defined on $\mathcal{H}or(J^k\mathcal{B})$ with values into $\mathcal{H}or(J^{k+1}\mathcal{B})$ and intrinsically expressed by:

$$(d_H\omega) \circ j^{k+1}\lambda = d(\omega \circ j^k\lambda) \quad \forall \lambda \in \Gamma(\pi) \ ,$$

for all $\omega \in \mathcal{H}or(J^k\mathcal{B})$, where d is the exterior differential operator of M.

Locally, d_H is determined by a family of operators d_μ acting on smooth functions, called *formal derivatives.* As an example, if $f : J^4B \to \mathbb{R}$ is a differentiable mapping, then $d_\mu f$ is the function on J^5B defined by:

$$d_\mu f = \frac{\partial f}{\partial x_\mu} + \frac{\partial f}{\partial y^a} y^a_\mu + \frac{\partial f}{\partial y^a_\nu} y^a_{\nu\mu} + \frac{\partial f}{\partial y^a_{\nu\rho}} y^a_{\nu\rho\mu} + \frac{\partial f}{\partial y^a_{\nu\rho\sigma}} y^a_{\nu\rho\sigma\mu} + \frac{\partial f}{\partial y^a_{\nu\rho\sigma\tau}} y^a_{\nu\rho\sigma\tau\mu} \ .$$

Finally, we set $d_V = d - d_H$, where d is now the exterior differential operator in $J^k\mathcal{B}$ (see [14]). It is known that $d_H^2 = 0$ and $d_V^2 = 0$, so that $d_V d_H = -d_H d_V$ because of $d^2 = 0$ (in $J^k\mathcal{B}$).

We also recall that, if $\mathcal{B} = (B, M, \pi)$ is a fibered manifold and $B_x \equiv \pi^{-1}(x)$ is its fiber over x, for any $x \in M$, then one defines the *dual vertical bundle* by setting $V^*B = \sqcup_{x\in M}(TB_x)^*$; this vector bundle $V^*\mathcal{B} = (V^*B, B, \mu_B)$ is not a sub-bundle of the cotangent bundle (T^*B, B, π_B). Let us denote by $\otimes_M$ the tensor product of vector bundles over M.

Theorem 1.1 (see [14]). *There exist two global bundle morphisms denoted by* $f^{\mathcal{B}}_{(1)}(\mathcal{L}) : J^3B \to \Lambda^{m-1}M \otimes_M V^*B$ *and* $f^{\mathcal{B}}_{(2)}(\mathcal{L})$:

$J^2B \to \Lambda^{m-1}M \otimes_M V^*J^1B$, and a global bundle morphism $e^{\mathcal{B}}(\mathcal{L}) : J^4B \to \Lambda^m M \otimes_M V^*B$, associated to the Lagrangian $\mathcal{L}$ and to its action (1.3), *where $V^*J^1\mathcal{B} \cong J^1V^*\mathcal{B}$ is the dual bundle of the vector bundle $VJ^1\mathcal{B} \cong J^1V\mathcal{B}$* (*these last two isomorphisms being canonical*), *which enter the following expression for the first perturbation of $\mathcal{L}$ under any homotopic variation $\lambda_\varepsilon : D \to B$ of any section $\lambda \equiv \lambda_0$:*

$$\delta(\mathcal{L} \circ j^2\lambda_\varepsilon) = e^{\mathcal{B}}(\mathcal{L}) \circ j^4\lambda + d_H[f^{\mathcal{B}}_{(1)}(\mathcal{L}) + f^{\mathcal{B}}_{(2)}(\mathcal{L})] \circ j^4\lambda\,. \tag{1.4}$$

Equation (1.4) is known as the (*global*) *first variation formula* for $\mathcal{L}$. As we said above, the critical sections of (1.3) satisfy Euler-Lagrange equations:

$$e^{\mathcal{B}}(\mathcal{L}) \circ j^4\lambda = 0\ .$$

The bundle morphisms entering (1.4) have local expressions given, respectively, by:

$$\begin{aligned} f^\mu_a &\equiv [f^{\mathcal{B}}_{(1)}(\mathcal{L})]^\mu_a = p^\mu_a - d_\nu p^{\mu\nu}_a\ , \\ f^{\mu\nu}_a &\equiv [f^{\mathcal{B}}_{(2)}(\mathcal{L})]^{\mu\nu}_a = p^{\mu\nu}_a\ , \\ e_a &\equiv [e(\mathcal{L})^{\mathcal{B}}]_a = p_a - d_\mu[f^{\mathcal{B}}_{(1)}(\mathcal{L})]^\mu_a = \\ &= p_a - d_\mu p^\mu_a + d_\nu d_\mu p^{\mu\nu}_a\ , \end{aligned} \tag{1.5}$$

having defined the *canonical momenta* $(p_a, p^\mu_a, p^{\mu\nu}_a)$ by setting

$$p_a \equiv p_a(\mathcal{L}) = \frac{\partial L}{\partial y^a}, \quad p^\mu_a \equiv p(\mathcal{L})^\mu_a = \frac{\partial L}{\partial y^a_\mu}, \quad p^{\mu\nu}_a \equiv p(\mathcal{L})^{\mu\nu}_a = \frac{\partial L}{\partial y^a_{\mu\nu}}\ . \tag{1.6}$$

The local components $(f^\mu_a, f^{\mu\nu}_a)$ of the bundle morphisms $f^{\mathcal{B}}_{(1)}(\mathcal{L})$ and $f^{\mathcal{B}}_{(2)}(\mathcal{L})$ are known as the *true momenta*, while $e^{\mathcal{B}}(\mathcal{L})$ is the *Euler-Lagrange morphism*.

Remark. Notice that the bundle morphisms above determine in turn the following tensorfields, which by an abuse of notation will be denoted by the same symbols of the corresponding morphisms:

$$\begin{aligned} f^{\mathcal{B}}_{(1)}(\mathcal{L}) &= f^\mu_a dy^a \wedge \mathbf{ds}_\mu\ , \\ f^{\mathcal{B}}_{(2)}(\mathcal{L}) &= f^{\mu\nu}_a dy^a_\mu \wedge \mathbf{ds}_\nu\ , \\ e^{\mathcal{B}}(\mathcal{L}) &= e_a dy^a \wedge \mathbf{ds}\ . \end{aligned} \tag{1.7}$$

1.3 – Contact forms and symmetries

Definition 1.2. The ideal of **contact forms** $\mathcal{K}(J^2\mathcal{B})$, is the ideal of the exterior algebra $\mathbf{\Omega}(J^3B)$ formed by those forms $\eta \in \mathbf{\Omega}(J^2B)$ which vanish along all holonomic sections $j^2\lambda$ of the bundle $J^2\mathcal{B} = (J^2B, M, \pi)$.

The ideal $\mathcal{K}(J^2\mathcal{B})$ is generated by the following family of local 1-forms:

$$\theta^a = dy^a - y^a_\sigma dx^\sigma \quad , \quad \theta^a_\mu = dy^a_\mu - y^a_{\mu\sigma} dx^\sigma \ , \tag{1.8}$$

by the ring $\mathbf{\Omega}^0(J^2B)$.

Definition 1.3. The **Poincaré-Cartan form** is the m-form along the canonical projection of J^3B onto B, having the following local expression:

$$\mathbf{\Theta} \equiv \Theta^{\mathcal{B}}(\mathcal{L}) = (f^\mu_a \theta^a + f^{\mu\nu}_a \theta^a_\nu) \wedge \mathbf{ds}_\mu + \mathcal{L} \ . \tag{1.9}$$

Finally, the form $\mathbf{\Omega} \equiv \mathbf{\Omega}^{\mathcal{B}}(\mathcal{L}) = d\mathbf{\Theta}$ is the multiplectic form of the variational problem (see [24]). This form $\mathbf{\Omega}$ determines the Euler-Lagrange equations, which can in fact be equivalently written as:

$$(j^3\sigma)^*(i_v(\mathbf{\Omega})) = 0 \ , \quad \forall v \in VJ^3B \cong J^3VB \ . \tag{1.10}$$

For more details see, e.g., [22] and [25].

Now we denote by $\mathbf{L}$ the Lie derivative operator, defined on the sections of a bundle $\mathcal{B}$ whenever the bundle is a natural bundle (see [26]) or a gauge-natural bundle (see [27], [28] and [29]).

Definition 1.4 (see [12] and [25]). An **infinitesimal symmetry** is a vectorfield $\Xi \in \mathcal{X}(J^3B)$ is said to be of $\mathcal{L}$ if:

$$\mathbf{L}_\Xi[\Theta^{\mathcal{B}}(\mathcal{L})] = 0 \ . \tag{1.11}$$

Then $E^c(\mathcal{L}, \Xi, \lambda) = (j^3\lambda)^*(\Xi \rfloor \Theta(\mathcal{L}))$ is called the **conserved Noether current associated to** Ξ.

If λ is a solution of the Euler-Lagrange equations of $\mathcal{L}$ one has $d_\mu[E^c(\mathcal{L}, \Xi, \lambda)]^\mu = 0$ (see [12] and [25]).

As we explained in the Introduction, we are here interested into investigating the naturality of the "first order perturbation" procedure, by means of a functor suitably defined on a suitable category. Obviously, the "largest" is the category on which the functor is defined, the strongest will be its naturality properties.

DEFINITION 1.5. A (local) section $\lambda : U \to B$ is said to be **admissible for** Φ if and only if the mapping $\phi_t^\lambda \equiv \phi_t \circ \lambda : U \to \phi_t^\lambda(U) = V_t$ is a local diffeomorphism.

THEOREM 1.2. *Equation* (1.11) *is meaningful even if the local* 1-*parameter group* $\Phi = \{\Phi_t\}$ *generated by* Ξ *is not a* (*local*) *group of bundle automorphisms*, *but just a group of diffeomorphisms of the total space.*

PROOF. In fact, let us set $\phi_t = \pi \circ \Phi_t : B \to M$. Then the action of Φ on λ is defined by setting

$$\lambda_t(x) \equiv (\Phi_t^* \lambda)(x) = \Phi_t \circ \lambda \circ (\phi_t^\lambda)^{-1}(x) \tag{1.12}$$

for any $x \in V_t$; the family $\{\lambda_t\}_{t\in(-\varepsilon,\varepsilon)}$, with $\varepsilon > 0$, is a homotopic variation of $\lambda \equiv \lambda_0$ and, as in [22] and [26], we have:

$$\mathbf{L}_\xi(\lambda) \equiv \left[\frac{d}{dt}\lambda_t\right]_{|t=0} = T\lambda \circ \xi_\lambda - \Xi \circ \lambda \ , \tag{1.13}$$

where $\xi_\lambda = T\pi \circ \Xi \circ \lambda$ is a vectorfield over the basis M (which depends of course on the section λ). ∎

REMARK. As a consequence, the results of [22] and [26] hold true also in this case, which is obtained by restoring the classical definition of the action of a differentiable mapping on a "field". In fact, let $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ be two fiber bundles and $F : B \to B'$ a differentiable mapping between the total spaces of the two bundles (not necessarily a bundle morphism). We set $f_F = \pi' \circ F : B \to M'$ and call it the *basic map associated to* F. We also say that a section $\lambda : M \to B$ is *admissible for* F if and only if $\tilde{f}_F = f_F \circ \lambda : M \to M'$ is a (local) diffeomorphism; in this case, of course, M and M' have to be of the same dimension. Then the classical action of F on the set of admissible sections is given by:

$$F.\lambda = F \circ \lambda \circ (f \circ \lambda)^{-1} \ , \tag{1.14}$$

for any admissible section $\lambda : M \to B$.

1.4 – Second variation of Lagrangians

We will follow [16] for the second variation.

Definition 1.6. The **first order perturbation** $\mathcal{L}_{(1)} : J^2VB \to \Lambda^m M$ of the Lagrangian $\mathcal{L} \equiv \mathcal{L}_{(0)}$ is the (unique and global) morphism with local expression given by:

$$\mathcal{L}_{(1)} \equiv L_{(1)}\mathbf{ds} = \{p_a\rho^a + p_a^\mu \rho_\mu^a + p_a^{\mu\nu}\rho_{\mu\nu}^a\}\mathbf{ds} \ , \tag{1.15}$$

where $(\rho^a, \rho_\mu^a, \rho_{\mu\nu}^a)$ are the local components of an element of VJ^2B (canonically identified with J^2VB).

The action functional associated to the Lagrangian $\mathcal{L}_{(1)}$ is given by:

$$\tilde{\mathcal{A}} = \int_D \mathcal{L}_{(1)} \circ (j^2\lambda \times j^2 v) \ , \tag{1.16}$$

for any local section $\lambda \in \Gamma_D(\pi)$ and any vertical vectorfield v which projects onto the section λ. We also set:

$$e^{\mathcal{B}}(\mathcal{L}_{(1)}) = \tilde{e}_a d\rho^a + E_a dy^a \ . \tag{1.17}$$

Theorem 1.3. *The following holds*:

$$\begin{aligned} \tilde{e}_a =& [e^{\mathcal{B}}(\mathcal{L})]_a = e_a \ , \\ E_a \equiv& [E^{\mathcal{B}}(\mathcal{L})]_a = P_a - d_\mu [F_{(1)}^{\mathcal{B}}]_a^\mu = P_a - d_\mu P_a^\mu + d_\mu d_\nu P_a^{\mu\nu} \ , \end{aligned} \tag{1.18}$$

where

$$\begin{aligned} [F_{(1)}^{\mathcal{B}}]_a^\mu \equiv& [F_{(1)}^{\mathcal{B}}(\mathcal{L})]_a^\mu = P_a^\mu - d_\nu P_a^{\nu\mu} \\ [F_{(2)}^{\mathcal{B}}]_a^{\mu\nu} \equiv& [F_{(2)}^{\mathcal{B}}(\mathcal{L})]_a^{\mu\nu} = P_a^{\mu\nu} \ , \end{aligned} \tag{1.19}$$

being

$$\begin{aligned} P_a \equiv& \frac{\partial L_{(1)}}{\partial y^a} \\ P_a^\mu \equiv& [P^{\mathcal{B}}(\mathcal{L})]_a^\mu = \frac{\partial L_{(1)}}{\partial y_\mu^a} \\ P_a^{\mu\nu} \equiv& [P^{\mathcal{B}}(\mathcal{L})]_a^{\mu\nu} = \frac{\partial L_{(1)}}{\partial y_{\mu\nu}^a} \ , \end{aligned} \tag{1.20}$$

with e_a defined by (1.5).

Proof. See [16].

Remark. With the positions of Theorem 1.3 the non-covariant part E_a represents the coefficients of the Jacobi morphism of $\mathcal{L}$ (as defined in our previous paper [15]).

Definition 1.7. The **Hessian mapping** $\mathrm{Hess}_{\mathcal{B}}(\mathcal{L})\colon J^2B\times_B J^2VB\times_B J^2VB\to\Lambda^m M$, where $\times_B$ denotes the fibered product over B, is given by:

$$\begin{aligned}\mathrm{Hess}_{\mathcal{B}}(\mathcal{L})_{(x,y)}(\xi;\rho) = [P_a]_{(x,y)}(\rho)\xi^a + [P_a^{\mu}]_{(x,y)}(\rho)\xi^a_{\mu} + \\ + [P_a^{\mu\nu}]_{(x,y)}(\rho)\xi^a_{\mu\nu}\ ,\end{aligned} \tag{1.21}$$

where $\xi = (\xi^a, \xi^a_\mu, \xi^a_{\mu\nu})$ are the local coordinates of a further point belonging to the fiber of $J^2V\mathcal{B}$ over the point of B having local coordinates (x^μ, y^a).

Equation (1.21) gives in fact the Hessian mapping of the variational problem (see [17]).

1.5 – Basic categories

We finally list the basic categories used in this paper. We shall adopt the following standard notation. If τ is any category , we shall denote by $\tau(O,O')$ the set of all morphisms in τ from O into O', being O, O' objects of τ.

i) The category $\mathcal{M}an$ having as objects the (C^∞-differentiable) manifolds and as morphisms the (C^∞-differentiable) mappings between manifolds.
ii) The category $\mathbf{B}un$ whose objects are the fiber bundles $\mathcal{B} = (B, M, \pi)$ over any manifold M (object of $\mathcal{M}an$) and whose morphisms are the usual bundle morphisms (i.e., the fiber preserving differentiable mappings between fiber bundles).
iii) By $\mathbf{VB}un$ we denote the subcategory of $\mathbf{B}un$ having as objects the vector bundles and as morphisms the linear bundle morphisms.
iv) In this last category we will make use of the subcategory $T\mathcal{M}an$ whose objects are the tangent bundles of the manifolds M of $\mathcal{M}an$ and, if M and N are two manifolds of $\mathcal{M}an$, a mapping $F : TM \to TN$ belongs to the set of morphisms $T\mathcal{M}an(TM, TN)$ in this category if and only if $F = Tf$ is the *tangent mapping of the mapping*

$f \in \mathcal{M}an(M, N)$. In the following, by an abuse of notation, we will denote simply by TM the tangent bundle (TM, M, τ_M); moreover T is the so called *tangent functor*.

v) Finally, $\mathbf{V}ec$ will denote the category of real vector spaces whose morphisms are linear mappings between pairs of real vector spaces.

The basic functor we shall need between the category $\mathcal{M}an$ and the category $\mathbf{V}ec$, namely the functor which associates to any manifold M its total de Rham cohomology group $H_{dR}(M)$, will be denoted by H_{dR}. Recently, (see, e.g., [6] and [9]), some new cohomological functors related to the Calculus of Variations and/or to partial differential equations have also been introduced in the literature.

A result which can be easily inferred by comparing [6] with [9] is that the construction needed to obtain the cohomological groups related to these functors is somehow standard. In fact, all these cohomological groups are obtained by first choosing some graded ideal $\mathcal{I}(J^kB)$ of the graded exterior algebra $\widehat{\mathbf{\Omega}}(J^kB) \equiv \mathcal{H}or(J^k(\mathcal{B}) \oplus \mathcal{K}(J^k\mathcal{B})$ ($k = \infty$ is not excluded) having the property

$$d(\mathcal{I}(J^kB)) \subseteq \mathcal{I}(J^kB) \ . \tag{1.22}$$

One then takes the quotient of $\widehat{\mathbf{\Omega}}(J^kB)$ with respect to $\mathcal{I}(J^kB)$, to obtain a cochain complex, and then considers the cohomological groups of this last complex. Notice that $\mathbf{\Omega}(J^kB) \subseteq \widehat{\mathbf{\Omega}}(J^kB)$. Finally, the ideals of $\widehat{\mathbf{\Omega}}(J^kB)$ verifying (1.22) will be called *closed ideals*.

In order to introduce the aforementioned functors (especially for the functor defined in [6], which is far too general with respect to the case considered here) we need some further construction. These will be given in Section 2, where we shall introduce the "perturbation functor", while the relations of our new functor with the functors of [6] and [9] will be shortly discussed in Section 3.

2 – The first order perturbation functor

Physicists make use of many "perturbation techniques", which are quite different among each other. Here we shall consider only those perturbations which were studied in an explicit way in [16], since they are the starting point from which many physical results are obtained through the Calculus of Variations; we just quote [30] and [17] (where applications to gravitational theories can be found, both in the case of General Relativity and in the case of non-linear gravitational Lagrangians) since these two papers are more closely related to our present interest.

2.1 – Definition of the functor $\mathcal{P}$

In a previous paper of ours [18] it was shown that the "complete lift" used in differential geometry (see, e.g., [31]) is nothing but a particular case of the perturbation technique recently introduced in [16]. The functor we shall be dealing with can be deduced by using the perturbation considered in the aforementioned papers and is in fact obtained by composing the tangent functor with some other suitable functors, having the same degree of naturality.

As is well known, adding any divergence to a given Lagrangian $\mathcal{L}$ does not affect Euler-Lagrange equations $e^{\mathcal{B}}(\mathcal{L})\circ j^4\lambda = 0$, but several constructions suffer changes: e.g., the Poincaré-Cartan form changes, giving then rise to different boundary terms in the action, as well as a different but dynamically equivalent version of equation (1.10) (see, e.g., [32]). Therefore, even in the class of "perturbations" considered here one can define many different "perturbations" for the same "original" set of Euler-Lagrange equations. We shall here propose a kind of a "canonical choice". We think in fact that our functor is the simplest possible one and, as an example, we shall compare it with the one which could be deduced from Taub's paper [30]. In any case, all the functors obtained in this way would be "equivalent in a suitable sense" from the viewpoint of the Calculus of Variations.

Before going on, let us first notice that there is no substantial difference between mappings and sections from the viewpoint of the Calculus of Variations. In fact, if the variational problem is defined on the set of all mappings from M into a further manifold N, one can uniquely identify any mapping $h : M \to N$ with the section $\lambda_h : M \to M \times N$ of the trivial bundle $pr_1 : M \times N \to M$, being pr_1 the natural projection on the first factor, defined by $\lambda_h(x) = (x, h(x))$. The converse is also true, as any section $\lambda : M \to B$ is nothing but a mapping which satisfies the constraint $\pi \circ \lambda = id$. An analogous remark holds also for the groups of diffeomorphisms.

In order to define our perturbation functor we need some new categories and some new functors which are easily obtained from the ones considered in Section 1. Because of our introductory remarks about conservation laws, the category $\mathbf{B}un$ does not contain enough morphisms. Hence, we denote by $\mathbf{B}$ the category having as objects all the bundles $\mathcal{B} = (B, M, \pi)$ of $\mathbf{B}un$ and as morphisms all the differentiable mappings between the total spaces of pairs of bundles. If $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ are two objects of $\mathbf{B}$, we have then $\mathbf{B}(\mathcal{B}, \mathcal{B}') = \mathcal{M}an(B, B')$. Obviously, the category $\mathbf{B}un$ is a full sub-category of this category $\mathbf{B}$.

Recall that for any pair of objects $\mathcal{B}$, $\mathcal{B}'$ of $\mathbf{B}$ and for any $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ we have set $f_F = \pi' \circ F$ and we have called it the *basic map associated to* F. It is obvious that F belongs to $\mathbf{B}un(\mathcal{B}, \mathcal{B}')$ if and only if the basic map f_F is constant along the fibers of $\mathcal{B}$. In this case f_F defines a map $f'_F : M \to M'$ which is called the "induced map" and is such that $f_F \equiv \pi' \circ F = f'_F \circ \pi$.

The second category we need is denoted by $T\mathbf{B}$. Its objects are the tangent bundles $T\mathcal{B} = (TB, TM, T\pi)$, i.e. the images under the tangent functor T of all bundles $\mathcal{B} = (B, M, \pi)$ of $\mathbf{B}$, while its morphisms are the images by T of the morphisms of $\mathbf{B}$.

A third category we shall need, denoted by $R\mathbf{B}$, is defined as follows: its objects are the fiber bundles of the trivial type $R\mathcal{B} = (\mathbb{R} \times B, \mathbb{R} \times M, id_{\mathbb{R}} \times \pi)$, where $\mathcal{B} = (B, M, \pi)$ is any object of $\mathbf{B}$, and $id_{\mathbb{R}} \times \pi : \mathbb{R} \times B \to \mathbb{R} \times M$ is defined by setting $(id_{\mathbb{R}} \times \pi)(t, y) = (t, \pi(y))$ for any $(t, y) \in \mathbb{R} \times B$. In this category a morphism $F \in R\mathbf{B}(R\mathcal{B}, R\mathcal{B}')$, being $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ objects of $\mathbf{B}$, is a pair of mappings $(id_{\mathbb{R}}, \tilde{F}) : \mathbb{R} \times B \to \mathbb{R} \times B'$. Hence we have typical morphisms $(id_{\mathbb{R}}, \tilde{F})(\varepsilon, y) = (\varepsilon, \tilde{F}(\varepsilon, y))$, where $\tilde{F} : \mathbb{R} \times B \to B'$ is the *mapping defining a homotopic variation* $F_\varepsilon \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ of $F_0 : B \to B'$, with $\varepsilon \in \mathbb{R}$; i.e., $\tilde{F}(\varepsilon, x) = F_\varepsilon(x)$, for any $\varepsilon \in \mathbb{R}$ and $x \in M$.

Remark. Since in the Calculus of Variations we are interested only into a neighborhood of $0 \in \mathbb{R}$, we can consider as homotopic variations (modulo a possible reparametrization on ε) only the families $F_\varepsilon \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$, with ε varying in the whole of $\mathbb{R}$, identifying them with the morphisms of the category $R\mathbf{B}$. Since all objects M of the category $\mathcal{M}an$ are objects of $\mathbf{B}$ via the trivial bundle structure (M, M, id_M), also the homotopic variations of local sections can be considered as morphisms in the previous category; in this last case we shall consider only homotopic variations strongly preserving the boundary (see [17]).

Theorem 2.1. *There exists a natural covariant functor from the category* $\mathbf{B}$ *into the category* $R\mathbf{B}$, *which, with an abuse of notation, will be again denoted by* R. *This functor* R *will be called the* **canonical lift**.

Proof. Immediate, by defining R as the functor which associates to any bundle $\mathcal{B} = (B, M, \pi)$ of $\mathbf{B}$ the bundle $R\mathcal{B} = (\mathbb{R} \times B, \mathbb{R} \times M, id_{\mathbb{R}} \times \pi)$ of $R\mathbf{B}$ and to any morphism $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ between the objects $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbf{B}$ the morphism $RF = (id_{\mathbb{R}} \times F) \in R\mathbf{B}(R\mathcal{B}, R\mathcal{B}')$. □

The canonical lift of F acts on a homotopic variation $\sigma : \mathbb{R} \times M \to \mathbb{R} \times \mathcal{B}$ in the following way. Let $\lambda_\varepsilon : M \to B$ be the family of mappings defining σ, i.e. $\sigma(\varepsilon, x) = \lambda_\varepsilon(x)$. We say that σ is *admissible* for RF if and only if λ_ε is admissible for F, for any $\varepsilon \in \mathbb{R}$. Then we can consider the mapping $\tau : \mathbb{R} \times M' \to B'$ defining the homotopic variation $F.\lambda_\varepsilon = F \circ \lambda_\varepsilon \circ (f_F \circ \lambda_\varepsilon)^{-1} : M' \to B'$, for any $\varepsilon \in \mathbb{R}$, being f_F the basic map associated to F. By these remarks the action of RF is defined as $(RF).\sigma = (id_{\mathbb{R}}, \tau) : \mathbb{R} \times M \to \mathbb{R} \times \mathcal{B}$.

Finally we have the further category $T^R\mathbf{B}$ whose objects are the bundles $T^R\mathcal{B} = (T\mathbb{R} \times TB, T\mathbb{R} \times TM, id_{T\mathbb{R}} \times T\pi)$, with $\mathcal{B} = (B, M, \pi)$ any object of $\mathbf{B}$, and whose morphisms are the mappings $(id_{T\mathbb{R}}, \tilde{F}) \in T^R\mathbf{B}(T^R\mathcal{B}, T^R\mathcal{B}')$, where $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ are bundles of $\mathbf{B}$ and $\tilde{F} : \mathbb{R} \times TB \to B'$ is a mapping which defines a homotopic variation $F_\varepsilon : TB \to TB'$ between linear bundle morphisms.

Definition 2.1. The **evaluation functor** $\mathcal{E}$ is the covariant functor from the category $T^R\mathbf{B}$ with values into the category $\mathbf{B}$ defined as follows: the functor $\mathcal{E}$ associates to any object $T^R\mathcal{B}$ of $T^R\mathbf{B}$ the canonical lift $RT\mathcal{B}$ of $T\mathcal{B}$ and to any morphism $(id_{T\mathbb{R}}, \tilde{F}) \in T^R\mathbf{B}(T^R\mathcal{B}, T^R\mathcal{B}')$ the morphism $(id_{\mathbb{R}}, \tilde{F}_0) \in R\mathbf{B}(RT\mathcal{B}, RT\mathcal{B}')$, via the canonical identification $T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ obtained by means of the standard chart $(\mathbb{R}, id_{\mathbb{R}})$, $\tilde{F}_0$ being defined by $\tilde{F}_0(z) = \tilde{F}(0, z)$, for all $z \in TB$.

We set now $\mathcal{E} \circ T = T_{\mathcal{E}}$

Definition 2.2. The **first order perturbation functor**, is defined on the category $\mathbf{B}$ and takes its values into the category $R\mathbf{B}$. It is the covariant functor defined by

$$\mathcal{P} = \mathcal{E} \circ T \circ R = T_{\mathcal{E}} \circ R \tag{2.1}$$

and it associates to any bundle $\mathcal{B} = (B, M, \pi)$ the bundle $\mathcal{P}\mathcal{B} = (\mathbb{R} \times TB, \mathbb{R} \times TM, id_{\mathbb{R}} \times T\pi)$.

It is now easy to see that the following holds:

PROPOSITION 2.2. *For any homotopic variation $\sigma : \mathbb{R} \times M \to \mathcal{B}$, which is assumed to be admissible for a morphism $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$, with $\mathcal{B}$ and $\mathcal{B}'$ objects in $\mathbf{B}$, one has*:

$$\mathcal{E}(T((RF).\sigma)) = T_{\mathcal{E}}((RF).\sigma) = (\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma) \ .$$

Moreover, we have:

PROPOSITION 2.3. *The functor $\mathcal{P}$ is a true perturbation functor.*

PROOF. Let us consider two bundles $\mathcal{B}=(B, M, \pi)$ and $\mathcal{B}'=(B', M', \pi')$ and a morphism $\tilde{F} \in R\mathbf{B}(R\mathcal{B}, R\mathcal{B}')$. We first notice that $T_{\mathcal{E}}\tilde{F}$ belongs to $W \equiv T^*(\mathbb{R} \times B) \otimes T(\mathbb{R} \times B') \cong (T^*\mathbb{R} \otimes T\mathbb{R}) \oplus (T^*B \otimes T\mathbb{R}) \oplus (T^*\mathbb{R} \otimes TB') \oplus (T^*B \otimes TB')$; here $\otimes$ and $\oplus$ generically denote the product bundles over the product of the bases with the natural vector bundle structures given by pairwise operations in the product of the fibers. Since the standard chart $(\mathbb{R}, id_{\mathbb{R}})$ is fixed in $\mathbb{R}$, we have the mapping $w : W \to TB'$, which acts as follows: to any element of W it associates the component belonging to $TB' \otimes T^*\mathbb{R}$, considered as forming a vector of B'. In fact, if X belongs to W, we have:

$$X = a\frac{\partial}{\partial t} \otimes dt + \omega \otimes \frac{\partial}{\partial t} + dt \otimes Y + P \ ,$$

where a is an arbitrary real number, Y a vector of B', ω a 1-form of B and P a tensor on $T^*B \otimes TB'$. Then it follows:

$$w(X) = Y \ .$$

We have thence (with an obvious meaning of the symbols used):

$$\delta\lambda = w((\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma)) \ , \tag{2.2}$$

with the obvious relation between the homotopic deformation λ_ε and the mapping σ. (We write $\delta\lambda$ for the first variation of λ_ε since it refers to the value at $\varepsilon = 0$). This proves our claim. ∎

2.2 – Coordinate expression of $\mathcal{P}$

Some of the properties which will be useful in the sequel can now be easily seen in terms of local coordinates. Hence we consider two fiber bundles $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$, a morphism $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ and a homotopic variation $\sigma : \mathbb{R} \times M \to \mathbb{R} \times B$ admissible for F. We notice that M and M' have the same dimension, since we have assumed that admissible sections exist; we denote by (x^μ, y^a) and (z^μ, y^A) natural coordinate systems in $\mathcal{B}$ and $\mathcal{B}'$, respectively, and by $z^\mu = f^\mu(x^\nu, y^a)$ the local representation of the basic map $f_F : B \to M'$. Then we have:

$$[w((\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma))]^A = \left\{ \frac{\partial F^A}{\partial y^b} - \left[\frac{\partial F^A}{\partial x^\mu} + \frac{\partial F^A}{\partial y^a} \frac{\partial \sigma^a}{\partial x^\mu} \right] C^\mu_\nu \frac{\partial f^\nu}{\partial y^b} \right\} \frac{\partial \sigma^b}{\partial \varepsilon} \ , \tag{2.3}$$

where the matrix $\|C^\mu_\nu\| \equiv \|C^\mu_\nu(j^1\sigma)\|$ is the inverse of the matrix $\|\bar{C}^\nu_\mu\|$ defined by:

$$\bar{C}^\nu_\mu \equiv \bar{C}^\nu_\mu(j^1\sigma) = \frac{\partial f^\nu}{\partial x^\mu} + \frac{\partial f^\nu}{\partial y^a} \frac{\partial \sigma^a}{\partial x^\mu} \ , \tag{2.4}$$

which has maximal rank since σ is an admissible homotopic variation. Now we notice that $T\sigma$ is a section from the basis $\mathbb{R} \times M$ into the total space $T^*(\mathbb{R} \times M) \otimes T(\mathbb{R} \times B) \cong (T^1_1\mathbb{R}) \oplus (T^*\mathbb{R} \otimes TB) \oplus (T^*M \otimes T\mathbb{R}) \oplus (T^*M \otimes TB)$. Since σ is a section of a bundle and many of its derivatives are hence constant, we can replace the previous vector bundle by the simpler vector bundle $(VB \otimes T^*\mathbb{R}) \oplus (T^*M \otimes VB)$. Finally, when the functor $T_{\mathcal{E}}$ is considered, the previous bundle simplifies further to a bundle diffeomorphic to $J^1B \times_B VB$. Hence we can define the new action of $\mathcal{P}_{\mathcal{B}}F$ by simply setting:

$$[(\mathcal{P}_{\mathcal{B}}F)_*(y, v)]^A = \left\{ \frac{\partial F^A}{\partial y^b} - (d_\mu F^A) C^\mu_\nu \frac{\partial f^\nu}{\partial y^b} \right\} v^b \ , \tag{2.5}$$

for any (y, v) not belonging to the closed subset of $J^1B \times_B VB$ having local equation:

$$\det \|\bar{C}^\nu_\mu\| = \det \|d_\mu f^\nu\| = 0 \ . \tag{2.6}$$

Equation (2.2) shows that the functor $\mathcal{P}$ defined by (2.1) is a true perturbation functor, acting through the action (2.5) and defined everywhere except a closed subset of the bundle $(J^1B \times_B VB, M, \tau)$, where τ denotes the obvious projection, determined by equation (2.6); the elements of the domain of regularity for $\mathcal{P}_\mathcal{B}F$ will be called *admissible for* $\mathcal{P}_\mathcal{B}F$. Luckily enough, the use of this complicated form of the first order perturbation functor can be avoided in most cases: we shall need it, in fact, only to study the perturbation of the Noether equation in its classical form, i.e. when the action of the 1-parameter group is defined by (1.12). In the other cases the category $\mathbf{B}un$ is enough for the study of variational problems.

PROPOSITION 2.4. *In the category* $\mathbf{B}un$, *the first order perturbation functor restricted to a simpler functor* $\widehat{\mathcal{P}}$ *which does not depend any longer on* $j^1\sigma$, *but only on the* ε*-derivative of* σ.

PROOF. In fact, in this case (2.5) becomes

$$[w((\mathcal{P}_\mathcal{B}F).(T_\mathcal{E}\sigma))] = \frac{\partial F^A}{\partial y^a}(\delta\lambda)^a \frac{\partial}{\partial y^A} \tag{2.7}$$

and all the sections become admissible. Hence $(\mathcal{P}_\mathcal{B}F)$ can be considered as a fiberpreserving linear mapping defined on VB taking its values into VB'. As a consequence, we can replace $\mathcal{P}$ with a new functor $\widehat{\mathcal{P}}$, which associates to any bundle $\mathcal{B}$ over M the bundle $V\mathcal{B}$ over M endowed with the obvious projection and which transforms morphisms according to (2.7). In other words, $\widehat{\mathcal{P}}$ associates to any mapping $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$ the mapping $\widehat{\mathcal{P}}_\mathcal{B}F \in \mathbf{B}un(V\mathcal{B}, V\mathcal{B}')$ defined by:

$$(\widehat{\mathcal{P}}_\mathcal{B}F)_{(y,v)} = \frac{\partial F^A}{\partial y^a} v^a \frac{\partial}{\partial y^A} \ , \tag{2.8}$$

for any vertical vector v over $y \in B$, having local components v^a. This ends our proof. □

Definition 2.3. The functor $\widehat{\mathcal{P}}$ is called **reduced first-order perturbation functor**.

Remarks.

1). Equation (2.8) shows that in this case $\widehat{\mathcal{P}}_{\mathcal{B}}F : VB \to VB'$ is a bundle morphism also with respect to the bundle structures $VB \to B$ and $VB' \to B'$ and moreover $F : B \to B'$ is the map induced by $\widehat{\mathcal{P}}_{\mathcal{B}}F$. Because of this, in the sequel we shall omit to write the induced maps and diagrams, for the sake of brevity.

2). We remark that in this case $\widehat{\mathcal{P}}$ can be alternatively defined as the unique functor which associates to any object $\mathcal{B}$ in the category $\mathbf{B}un$ its vertical bundle $V\mathcal{B}$ and to any bundle morphism $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$, with $\mathcal{B}$ and $\mathcal{B}'$ objects of $\mathbf{B}un$, the unique bundle morphism $\widehat{\mathcal{P}}(F) : VB \to VB'$ defined by setting:

$$\delta(F \circ \lambda) = \widehat{\mathcal{P}}_{\mathcal{B}}(F)(\delta\lambda) \tag{2.9}$$

for all mappings $\sigma : \mathbb{R} \times M \to B$ which define a homotopic variation of a section $\lambda : M \to B$.

Theorem 2.5. *When the functor $\widehat{\mathcal{P}}$ is restricted to curves, as in the case of Riemannian Geometry, it essentially coincides with the tangent functor T.*

Proof. This follows easily from (2.4) and (2.8). ∎

Many of the consequences of Theorem (2.5) existence are well known, even if they were never explicitely introduced as a consequence of variational principles (this aspect of Riemannian Geometry includes more properties than what people generally think; as an example of this fact we just quote [17], where the curvature of general variational problems of "harmonic type" is discussed in some detail). The results related to the existence of the perturbation functor for curves are in fact known as consequences of the *complete lift* (see [31]) and are related to our functor in the following way. The fiber bundle $\mathbb{R} \times M \to \mathbb{R}$ can be associated to the Riemannian manifold (M, g) and curves can be thought as sections of this bundle in an obvious way. Since we have $V(\mathbb{R} \times M) = \mathbb{R} \times TM$, to any differentiable mapping $f : M \to M'$ one can associate the mapping $F_f : \mathbb{R} \times M \to \mathbb{R} \times M'$ defined by $F_f(\varepsilon, x) = (\varepsilon, f(x))$. Then $\widehat{\mathcal{P}}_{\mathbb{R}\times M}(F_f) = (id, Tf) : \mathbb{R} \times TM \to \mathbb{R} \times TM'$ and the total differential Tf of f is nothing but the complete lift of f. Most of the constructions related to the variational aspects of Riemannian Geometry, e.g. those discussed in [31], will then coincide with our results (see also [18] for more details).

2.3 – Some Lagrangian properties of the reduced first order perturbation functor $\widehat{\mathcal{P}}$

Let us now consider the tensor bundle $\tau M = (\mathbf{T}M, M, \chi_M)$ with total space $\mathbf{T}^*M = \bigoplus_{(r,s)\in\mathbf{N}^2} T^r_s M$, where $T^r_s M$ is the bundle of tensors of type (r,s), for any $(r, s) \in \mathbb{N}^2$ and $(r, s) \neq (0, 0)$, while $T^0_0 M = M \times \mathbb{R}$. We set $T^0_r M \equiv T_r M$, for any $r \in \mathbb{N}$. We stress that, if $\mathcal{B}$, $\mathcal{B}'$ are two objects in $\mathbf{B}un$ and $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$, then the reduced first order perturbation functor $\widehat{\mathcal{P}}$ determines the map $\widehat{\mathcal{P}}_{\mathcal{B}}(F)$ which associates to any vertical vector of the total space of the bundle $V\mathcal{B}$ a contravariant vector of the total space of $\mathcal{B}'$. Hence, at a first sight, this functor seems to have nothing to do with Lagrangians which are instead determined by mappings from J^2B into $\Lambda^m M$. However, this is not the case, since it easy to see that each vertical bundle VT_rM splits as follows with a natural projection:

$$pr_1 : VT_rM \cong (T_rM) \oplus_M (T_rM) \to T_rM\,, \quad \forall r \in \mathbf{N}\,. \tag{2.10}$$

Proposition 2.6. *Let $\mathcal{L} : B \to \Lambda^m M$ be a Lagrangian. The following holds*:

$$\widehat{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L} = (\mathcal{L}, \mathcal{L}_{(1)}) : VJ^2B \cong J^2VB \longrightarrow V\Lambda^m M \cong \Lambda^m M \oplus_M \Lambda^m M\,, \tag{2.11}$$

where $\mathcal{L}_{(1)}$ is the first order perturbation of the Lagrangian $\mathcal{L}$.

Proof. It is a straightforward consequence of results of [16] together equation (1.15) of Section 1, since both the reduced first order perturbation functor and the identification (2.10) preserve symmetries. □

Remarks. A virtual application of a strictly analogous functor is due to Taub, who explicitly introduced a Lagrangian previously used in an implicit way to study the stability of relativistic gaseous masses (see [30] and the papers quoted therein). The perturbed Lagrangian used by Taub is the following:

$$\tilde{\mathcal{L}}_y(v) = \{e^{\mathcal{B}}(\mathcal{L})\}_y(v) + \{d_H[f^{\mathcal{B}}_{(1)}(\mathcal{L}) + f^{\mathcal{B}}_{(2)}(\mathcal{L})]\}_y(v) \; , \tag{2.12}$$

for any $y \in J^2B$ and $v \in VB$, both projecting onto the same point of B (see (1.4) of [30]). As a consequence, between Taub's and our perturbation there is only a "difference in simplicity", since (2.12) is obtained from (2.11) by means of a formal integration by parts, i.e. by adding to the Lagrangian a divergence which does not affect the variational problem (see [32]). This difference might however have some relevance, not only because of the different complication in the calculations; in fact we know that divergences determine those physical quantities which are pushed to the boundary of the region considered and enter the conservation laws through Stokes' theorem, so that they cannot be arbitrarily changed. This is true not only in classical physics, but also in General Relativity (see [32] for an example related to the Komar superpotential).

We conclude this Section by noticing that the morphism $\widehat{\mathcal{P}}(\mathcal{L})$ contains the first order "deformed" Lagrangian $\mathcal{L}_{(1)}$ of $\mathcal{L} \equiv \mathcal{L}_{(0)}$ in the sense of [16] and hence it contains informations on the Jacobi equations of the variational problems.

2.4 – The reduced functor $\overline{\mathcal{P}}$

In order to consider all the other geometric objects related to the Calculus of Variations, we need a more sophisticated construction than (2.10). For this purpose we first recall some results of [31]. Let us denote by $\mathcal{T}^r_s(M)$ the module of tensorfields of type (r,s) on M, being $\mathcal{T}^0_0(M) \equiv \mathbf{\Omega}_0(M) \equiv \mathcal{F}(M)$ the ring of smooth functions, and we set $\mathcal{T}(M) \equiv \bigoplus_{(r,s)\in \mathbb{N}^2} \mathcal{T}^r_s(M)$. We also denote by (x^μ, v^ν) the local coordinates induced on the tangent bundle TM by a local coordinate system (U, x^μ) on M.

Proposition 2.7 (see, e.g., [31] for a proof). *There exists an $\mathcal{F}(M)$-linear isomorphism from $\mathcal{T}(M)$ into $\mathcal{T}(TM)$, denoted by v and called* **vertical lift**, *defined by*:

$$(S\otimes T)^v = S^v \otimes T^v, \quad \forall\, S \in \mathcal{T}^r_s(M), \forall\, T \in \mathcal{T}^h_k(M), \forall\, r,s,h,k \in \mathbb{N} \tag{2.13a}$$

and

$$\left(\frac{\partial}{\partial x^\mu}\right)^v = \frac{\partial}{\partial v^\mu} \quad , \quad (dx^\mu)^v = dx^\mu \ , \tag{2.13b}$$

for any $\mu \in \{1, \ldots, m\}$).

We need a further definition:

Definition 2.4. The **complete lift** is the $\mathbb{R}$-linear map $c : \mathcal{T}(M) \to \mathcal{T}(TM)$ defined by:

$$f^c \equiv df : TM \to \mathbb{R} \ , \quad \forall f \in \mathcal{F}(M) \ , \tag{2.14a}$$

$$\begin{gathered}(S \otimes T)^c = S^c \otimes T^v + S^v \otimes T^c \, , \\ \forall S \in \mathcal{T}^r_s(M) \ , \ \forall T \in \mathcal{T}^h_k(M) \ , \ \forall r, s, h, k \in \mathbb{N}\end{gathered} \tag{2.14b}$$

and

$$\left(\frac{\partial}{\partial x^\mu}\right)^c = \frac{\partial}{\partial x^\mu} \quad , \quad (dx^\mu)^c = dv^\mu \quad , \ \forall \mu \in \{1, \ldots, m\} \ . \tag{2.14c}$$

Notice that if X and S are a vectorfield and a tensorfield defined on M, respectively, then the following relation between Lie derivatives exists:

$$(\mathbf{L}_X(S))^c = \mathbf{L}_{X^c}(S^c) \ .$$

Proposition 2.8. *Let us fix* $(r, s) \in \mathbb{N}^2$, *with* $r + s \geq 1$, *and let* $S \in \mathcal{T}^r_s(M)$ *be a tensorfield. Consider the total differential* $TS : TM \to T(T^r_s M)$ *and the complete lift* $S^c : TM \to T^r_s(TM)$. *Then there exists an immersion* $\xi \equiv \xi^r_s(M) : T^r_s(TM) \to T(T^r_s M)$ *such that* $\phi^r_s(M) \circ TS = S^c$, *for any* $S \in \mathcal{T}^r_s(M)$.

PROOF. In fact, a tensor $S \in T^r_s M$ belonging to the fiber over $x \in M$ has local expression:

$$S = S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} \ . \tag{2.15}$$

Denoting by $(x^\mu, S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})$ the local coordinate system of $T^r_s M$ induced by the local coordinate system (U, x^μ) of M, we can write the local expression of a vector $X \in T_S(T^r_s M)$, being $S \in T^r_s M$ a tensor over a point $x \in U$, as follows:

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}} \ . \tag{2.16}$$

We can always find a tensorfield $\widetilde{S}$ defined on U such that:

$$(X^\mu \partial_\mu \widetilde{S}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})_x = X^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \quad , \quad (\widetilde{S}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})_x = S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \ , \tag{2.17}$$

where $Y = X^\mu \frac{\partial}{\partial x^\mu} \in T_x M$ and $S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}$ are the local components of S. We stress that (2.17) is equivalent to $(T\widetilde{S})_Y = X$. Then we set:

$$\xi^r_s(X) \equiv (\widetilde{S}^c)_Y \ , \tag{2.18}$$

since the tensor on the right hand side does not depend on the local coordinate system nor it depends on the tensorfield $\widetilde{S}$. By using (2.13) and (2.14) one can see that:

$$\begin{aligned} \xi^r_s(X) = {} & X^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial v^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial v^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} + \\ & + \sum_{h=1}^{r} S^{\mu_1 \dots \mu_h \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial v^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_h}} \otimes \dots \otimes \frac{\partial}{\partial v^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} + \\ & + \sum_{h=1}^{s} S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_h \dots \nu_s} \frac{\partial}{\partial v^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial v^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dv^{\nu_h} \otimes \dots \otimes dx^{\nu_s} \ . \end{aligned} \tag{2.19}$$

This proves our claim. □

REMARK. We stress that, if $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ are two fiber bundles and $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$ is a bundle morphism, then equation (2.8) can be equivalently written as:

$$\widehat{\mathcal{P}}_{\mathcal{B}}(F) = [(TF)_{|VB}]^\perp \ , \tag{2.20}$$

where $[\dots]^{\perp}$ denotes the "vertical part" obtained by projection through the natural projection of TB onto $V^M B$.

When applied to vectorfields, equation (2.20) gives then rise to vectorfields which determine local 1-parameter groups having a trivial action on the Lagrangians obtained by (2.11), since the the "horizontal" components of the original vectorfields are lost. The existence of the family $\xi^r_s \equiv \xi^r_s(B) : T^r_s(TB) \to T(T^r_s B)$ and equation (2.19) allow us to associate to the functor $\widehat{\mathcal{P}}$ a new functor $\overline{\mathcal{P}}$ in all the cases in which $\widehat{\mathcal{P}}$ acts on tensorfields on the manifold B, considered as obvious bundle morphisms. In fact, a tensorfield $S \in \mathcal{T}^r_s(B)$ can be considered as a morphism $S : B \to T^r_s B$, with respect to the bundle structure of $\mathcal{B}$ and the obvious bundle structure $T^r_s B \to M$. Then, by using (2.17), we can set

$$\overline{\mathcal{P}}_{\mathcal{B}}(S) = \xi^r_s(\widehat{\mathcal{P}}_{\mathcal{B}}(S)) \ . \tag{2.21}$$

The local expression of $\overline{\mathcal{P}}_{\mathcal{B}}(S)$ can be easily calculated by using the local expression of ξ^r_s given by (2.16) for any tensorfield S of type (r,s) on B. This gives quite complicated formulae in the general case, since several terms are involved. We shall thence limit ourselves to write these formulae only for vectorfields and 1-forms, because they will be needed below. Hence, we set:

$$\begin{aligned} &\left(\frac{\partial}{\partial x^\mu}\right)^v = \frac{\partial}{\partial v^\mu} \ , \quad \left(\frac{\partial}{\partial y^a}\right)^v = \frac{\partial}{\partial v^a} \ , \\ &\left(\frac{\partial}{\partial x^\mu}\right)^c = \frac{\partial}{\partial x^\mu} \ , \quad \left(\frac{\partial}{\partial y^a}\right)^c = \frac{\partial}{\partial y^a} \ , \\ &(dx^\mu)^v = dx^\mu \ , \qquad (dy^a)^v = dy^a \ , \\ &(dx^\mu)^c = dv^\mu \ , \qquad (dy^a)^v = dv^a \ . \end{aligned} \tag{2.22}$$

Let $X = X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial y^a}$ be the local expression of a vector field X and $\omega = \omega_\mu dx^\mu + \omega_a dy^a$ be the local expression of a 1-form, defined on B. Then, for any vertical vector $v = v^a \frac{\partial}{\partial y^a}$, we have:

$$\overline{\mathcal{P}}_{\mathcal{B}}(X)_v = X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial y^a} + v^b \frac{\partial X^\mu}{\partial y^b} \frac{\partial}{\partial v^\mu} + v^b \frac{\partial X^a}{\partial y^b} \frac{\partial}{\partial v^a} \tag{2.23a}$$

and

$$(2.23b) \qquad [\overline{\mathcal{P}}_{\mathcal{B}}(\omega)]_v = \omega_\mu dx^\mu + \omega_a dy^a + v^b \frac{\partial \omega_\mu}{\partial y^b} dv^\mu + v^b \frac{\partial \omega_a}{\partial y^b} dv^a \ .$$

2.5 – The variational component of $\widetilde{\mathcal{P}}$

In order to determine the last functor into which we are interested we need some further construction. Let $(x^{\mu'}, y^{a'})$ $(\mu, \mu' \in \{1, \ldots, m\})$ be a further local bundle coordinate system whose domain intersects the domain of the coordinate system (x^μ, y^a). We will denote by $\varphi^{\mu'}(x^\mu)$ and $\psi^{a'}(x^\mu, y^a)$ the transition functions, together with their inverses ϕ^μ and ψ^a. Let us consider the tangent bundle TB and let us recall that in the charts induced on this manifold the following transformation laws hold:

$$(2.24) \qquad \begin{aligned} &\text{i)} \quad x^{\mu'} = \varphi^{\mu'}(x^\mu) \ , \\ &\text{ii)} \quad y^{a'} = \psi^{a'}(x^\mu, y^a) \ , \\ &\text{iii)} \quad v^{\mu'} = v^\mu \varphi^{\mu'}_\mu \\ &\text{iv)} \quad v^{a'} = v^\mu \psi^{a'}_\mu + v_a \psi^a_{a'} \ , \end{aligned}$$

for any $v = v^\mu \frac{\partial}{\partial x^\mu} + v^a \frac{\partial}{\partial y^a} \in T_y B$ in a point $y \in B$ belonging to the intersection domain. Here and in the sequel we set $\varphi^\mu_{\mu'} = \partial_{\mu'} \varphi^\mu$, $\psi^a_{\mu'} = \partial_{\mu'} \psi^a$ and so on. Now, we consider the subbundle $\pi^{VB} : (\pi_{TB})^{-1}(VB) = \tau^* VB \to VB$ of the cotangent bundle $(T^*(TB), TB, \pi_{TB})$ and a 1-form $\alpha = \alpha_\mu dx^\mu + \alpha_a dy^a + \beta_\mu dv^\mu + \beta_a dv^a \in \tau^* VB$. Then, the transformation laws (2.24) induce the following transformations on the local components of α:

$$(2.25) \qquad \begin{aligned} &\text{i)} \quad \alpha'_{\mu'} = \alpha_\mu \varphi^\mu_{\mu'} + \alpha_a \psi^a_{\mu'} + \beta_a \psi^a_{a'\mu'} \psi^{a'}_b v^b \ , \\ &\text{ii)} \quad \alpha_{a'} = \alpha_a \psi^a_{a'} + \beta_a \psi^a_{a'b'} \psi^{b'}_b v^b \ , \\ &\text{iii)} \quad \beta_{\mu'} = \beta_{\mu'} \varphi^\mu_{\mu'} + \beta_a \psi^a_{\mu'} \ , \\ &\text{iv)} \quad \beta_{a'} = \beta_a \psi^a_{a'} \ . \end{aligned}$$

On the other hand, one obtains from (2.4) the transition functions on the bundle VB by simply setting $v^\mu = 0$. The corresponding transformation laws of the local components of a 1-form $r = \rho_\mu dx^\mu + \rho_a dy^a + \sigma_a dv^a$

defined on VB are then given by:

$$
\text{(2.26)} \qquad
\begin{aligned}
&\text{i)} \quad \rho'_{\mu'} = \rho_\mu \varphi^\mu_{\mu'} + \rho_a \psi^a_{\mu'} + \sigma_a \psi^a_{a'\mu'} \psi^{a'}_b v^b \ , \\
&\text{ii)} \quad \rho_{a'} = \rho_a \psi^a_{a'} + \sigma_a \psi^a_{a'b'} \psi^{b'}_b v^b \ , \\
&\text{iii)} \quad \beta_{a'} = \beta_a \psi^a_{a'} \ .
\end{aligned}
$$

THEOREM 2.9. *Let us consider the vector bundle $T^*B \oplus_B T^*VB \to VB$, in which the fiber over a vector $v \in V_yB$, with $y \in B$, is given by $T^*_y \oplus T^*_vVB$, with the obvious structure of real vector space. There exists a bundle isomorphism $\eta_* : \tau^*VB \to T^*B \oplus_B T^*VB$ which associates to any covariant vector $\alpha = \alpha_\mu dx^\mu + \alpha_a dy^a + \beta_\mu dv^\mu + \beta_a dv^a$ of τ^*VB over the vector v of VB the ordered pair (ω, ρ) of $T^*B \oplus_B T^*VB$, being $\omega = \beta_\mu dx^\mu + \beta_a dy^a$ and $\rho = \alpha_\mu dx^\mu + \alpha_a dy^a + \beta_a dv^a$, with the covariant vector ρ belonging to the fiber of T^*VB over v.*

PROOF. Immediate by comparing (2.25), (2.26) together with the transformation laws of T^*B.

REMARK. The bundle $T^*B \oplus_B T^*VB \to VB$ possesses a naturally induced structure of vector bundle. Moreover, the bundle over VB of covariant tensors of order r determined by the vector bundle structure on $T^*B \oplus_B T^*VB$ turns out to be isomorphic to $T_rB \oplus_B T_rVB$, for any $r > 0$. Hence, if ν_rVB denotes the restriction of the bundle of covariant tensors T_rTB to VB, we can consider the power $(\eta_*)_r : \nu_rVB \to T_rB \oplus_B T_rVB$.

DEFINITION 2.5. We set $\phi_r = \xi^0_r \circ (\eta_*)_r$, for any $r \geq 0$ and $\Phi \equiv (\phi_r)_{r \geq 1}$. Then we have the following covariant functor:

$$
\text{(2.27)} \qquad \widetilde{\mathcal{P}}_{\mathcal{B}} \equiv pr_2 \circ \phi_r \circ \mathcal{P}_{\mathcal{B}} \equiv pr_2 \circ \xi^0_r \circ \overline{\mathcal{P}}_{\mathcal{B}} : \mathcal{T}_rB \to \mathcal{T}_rVB \ ,
$$

where $pr_2 : T_rB \oplus_B T_rVB \to T_rVB$ is the canonical projection. The functor $\widetilde{\mathcal{P}}$ acts on the appropriate categories which can be easily determined and it is called the **variational component of the reduced first order perturbation functor**.

In order to determine the action of the functor $\widetilde{\mathcal{P}}$ on the local components of covariant tensorfields we need some more pieces of notation. Let us denote by $\mathcal{A}_r(h)$ the set of multiple indices

$$
A_r(h) = (\mu_1, \dots, \mu_h, a_1, \dots, a_{r-h}) \ ,
$$

with $h \in \{1, \dots, r\}$. We make the convention that the multiple indices in which the μ's do not appear are of the type $A_r(0)$ and the multiple indices not having the a's are of the type $A_r(r)$. We shall call the previous multiple indices *basic multiple indices*. The standard action of the permutation group G_r on the basic multiple indices determines all the multiple indices needed to study the tensors of B. We consider now the set of local covariant tensors of B defined by:

$$dz^{\sigma(A_r(h))} = dz^{\sigma(\mu_1)} \otimes \cdots \otimes dz^{\sigma(\mu_h)} \otimes dz^{\sigma(a_1)} \otimes \cdots \otimes dz^{\sigma(a_{r-h})} \ , \tag{2.28}$$

for any $A_r(h) \in \mathcal{A}_r$ and $\sigma \in G_r$, having set $dz^\mu = dx^\mu$ and $dz^a = dy^a$, for any $\mu \in \{1, \dots, r\}$ and any $a \in \{1, \dots, p\}$, respectively. Then the family $(dz^{\sigma(A_r(h))})$, obtained when σ spans G_r and $A_r(h)$ spans $\mathcal{A}_r$, is a local system of generators of T_rB, which is obtained from the standard local basis of T_rB by repeating exactly $h!(r-h)!$-times each element $dz^{\sigma(A_r(h))}$, for any $\sigma \in G_r$ and $A_r(h) \in \mathcal{A}_r$, for any $r > 0$. Moreover, if $S \in \mathcal{T}_rB$, we have:

$$S = \sum_{h=0}^{r} \sum_{\sigma \in G_r} \frac{1}{h!(r-h)!} S_{\sigma(A_r(h))} dz^{\sigma(A_r(h))} \ , \tag{2.29}$$

where $S_{\sigma(A_r(h))}$ are the standard local components of S and the Einstein convention on the multiple indices $A_r(h)$ is used without any danger of confusion.

Then, for all sections $\omega : B \to T_rB$, having local expression:

$$\omega = \sum_{h=0}^{r} \sum_{\sigma \in G_r} \frac{1}{h!(r-h)!} \omega_{\sigma(A_r(h))} dz^{\sigma(A_r(h))} \ , \tag{2.30}$$

we have:

$$\begin{aligned} \widetilde{\mathcal{P}}_{\mathcal{B}}(\omega) = & \sum_{h=0}^{r} \sum_{\sigma \in G_r} \frac{1}{h!(r-h)!} v^a \partial_a [\omega_{\sigma(A_r(h))}] dz^{\sigma(A_r(h)} + \\ & + \sum_{h=0}^{r-1} \sum_{\sigma \in G_r} \frac{1}{h!(r-1-h)!} \omega_{\sigma(A_r(h)\hat{a})} dz^{\sigma(A_r^1(h)\hat{a})} \ , \end{aligned} \tag{2.31}$$

having set

$$A_r^1(h)\hat{a} \equiv (\mu_1, \dots, \mu_h, a_1, \dots, a_{r-h-1}, \hat{a})$$

and

$$dz^{\sigma(A^1_r(h)\hat{a})} = dz^{\sigma(\mu_1)} \otimes \cdots \otimes dz^{\sigma(\mu_h)} \otimes dz^{\sigma(a_1)} \otimes \cdots \otimes dz^{\sigma(a_{r-h-1})} \otimes dz^{\sigma(\hat{a})} ,$$

being, in this case, $dz^{\hat{a}} = dv^{\hat{a}}$, for any $\hat{a} \in \{1, \ldots, p\}$. Obviously, the functor $\mathcal{P}$ can be easily obtained from $\widetilde{\mathcal{P}}$ in these cases.

Remark. We conclude this part by noticing that, if Ξ is a vectorfield and ω is a covariant tensorfield both defined on B, then from (2.14d) and the definition of $\widetilde{\mathcal{P}}$ we easily obtain:

$$\mathbf{L}_{\overline{\mathcal{P}}_{\mathcal{B}}(\Xi)}(\widetilde{\mathcal{P}}_{\mathcal{B}}(\omega)) = \widetilde{\mathcal{P}}_{\mathcal{B}}(\mathbf{L}_{\Xi}(\omega)) \ . \tag{2.32}$$

3 – Relations of the functor $\mathcal{P}$ with the calculus of variations and with some cohomological functors

3.1 – Action on forms of the perturbation functors

Let $\mathcal{B} = (B, M, \pi)$ be a fiber bundle. We shall denote by $\mathcal{T}^*(B) = \bigoplus_{r\in\mathbb{N}} \mathcal{T}_r(B)$ the direct sum of the modules $\mathcal{T}_r(B)$ of tensorfields of type (0,r), i.e. the sections of the bundle $\tau^* B$. We also recall that, if $\mathcal{B}$, $\mathcal{B}'$ are objects in $\mathbf{B}un$, then the functor $\widehat{\mathcal{P}}$ defines a map $\widehat{\mathcal{P}}_{\mathcal{B},\mathcal{B}'} : \mathbf{B}un(\mathcal{B}, \mathcal{B}') \to \mathbf{B}un(\widehat{\mathcal{P}}(\mathcal{B}), \widehat{\mathcal{P}}(\mathcal{B}')) = \mathbf{B}un(V\mathcal{B}, V\mathcal{B}')$, which transforms a morphism $f \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ into the morphism $\widehat{\mathcal{P}}(f) \in \mathbf{B}(V\mathcal{B}, V\mathcal{B}')$, given by (2.8). This holds also for the functor $\widetilde{\mathcal{P}}$.

A number of results holds becuase of (2.14):

Proposition 3.1. *The variational component $\widetilde{\mathcal{P}}$ of the reduced first order perturbation functor $\widehat{\mathcal{P}}$ acts as a derivative on $\mathcal{T}^*(B)$, considered as a $\mathcal{T}^*(M)$-algebra, via the natural identification induced by pull-back along $\pi : B \to M$.*

As a consequence, by using the simplified notation introduced in Section 2, we have:

$$\widetilde{\mathcal{P}}_{\mathcal{B}}(\omega \otimes \omega') = \widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\omega) \otimes \omega' + \omega \otimes \widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\omega') \tag{3.1}$$

and

$$\widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\alpha \otimes \omega) = \alpha \otimes \widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\omega) \ , \tag{3.1'}$$

for all $\omega \in \mathcal{T}_r(B)$, $\omega' \in \mathcal{T}_s(B)$, $\alpha \in \mathcal{T}_h(M)$ and $r, s, h \in \mathbb{N}$; i.e., in this section we shall consider the vertical lift as an identification morphism.

PROPOSITION 3.2. *The variational component* $\widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}$ *of the reduced first order perturbation functor preserves the symmetries of tensors.*

Hence, when $\widetilde{\mathcal{P}}_{\mathcal{B}}$ is reduced to the exterior algebra of B, we can replace in (3.1) the tensor product with the exterior product, so that:

$$\widetilde{\mathcal{P}}_{\mathcal{B}}(\mathbf{\Omega}(B)) \subseteq \mathbf{\Omega}(VB) \; . \tag{3.2}$$

PROPOSITION 3.3. *The functor* $\widetilde{\mathcal{P}}_{\mathcal{B}}$ *is localizable*; *i.e., if* N *is an open submanifold of* M *and if* $\pi : B' \to N$ *defines a sub-bundle of* $\mathcal{B}$, *then:*

$$[\widetilde{\mathcal{P}}_{\mathcal{B}}(\omega)]_{|\mathcal{B}'} = \widetilde{\mathcal{P}}_{\mathcal{B}'}(\omega_{|\mathcal{B}'}) \; , \tag{3.3}$$

for any $\omega \in \mathbf{\Omega}(B)$.

Now we are ready to prove one of the main results of this paper.

THEOREM 3.4. *There exists a morphism* $\widetilde{\mathcal{P}}^*_{\mathcal{B},\Lambda B} : H_{dR}B \to H_{dR}VB$, *being* H_{dR} *the de Rham* ($\mathbb{R}$*-valued*) *cohomology functor.*

PROOF. We first recall that a function $f : B \to \mathbb{R}$ can be identified with a section $f : B \to B \times \mathbb{R}$ of the bundle $pr_1 : B \times \mathbb{R} \to B$. Since we have the identification $pr_1 : V^M(B \times \mathbb{R}) \cong (VB) \times \mathbb{R} \to VB$, the mapping $\widetilde{\mathcal{P}}_{\mathcal{B}}(f)$ is a section of this bundle, and hence a function. For this function we have locally:

$$\begin{aligned} \widetilde{\mathcal{P}}_{\mathcal{B}}\left(\frac{\partial f}{\partial x^\mu}\right) &= \frac{\partial(\widetilde{\mathcal{P}}_{\mathcal{B}} f)}{\partial x^\mu} \, , \\ \widetilde{\mathcal{P}}_{\mathcal{B}}\left(\frac{\partial f}{\partial y^a}\right) &= \frac{\partial(\widetilde{\mathcal{P}}_{\mathcal{B}} f)}{\partial y^a} \, , \\ \frac{\partial(\widetilde{\mathcal{P}}_{\mathcal{B}} f)}{\partial v^a} &= \frac{\partial f}{\partial y^a} \; ; \end{aligned} \tag{3.4}$$

the above identities hold since, in virtue of (2.14) we have:

$$[\widetilde{\mathcal{P}}_{\mathcal{B}}(f)](v) = \left(\frac{\partial f}{\partial y^a}(y)\right) v^a \ , \tag{3.5}$$

for any $v \equiv v^a \frac{\partial}{\partial y^a} \in VB$ projecting onto $y \in B$. From (2.14) and (3.4), it also follows:

$$\widetilde{\mathcal{P}}_{\mathcal{B}}(d\omega) = d(\widetilde{\mathcal{P}}_{\mathcal{B}}\omega) \ , \tag{3.6}$$

for all $\omega \in \mathbf{\Omega}_r(B)$ and any integer $r \in \{1, \ldots, m+p\}$. Hence, the map $\widetilde{\mathcal{P}}_{\mathcal{B},\Lambda B}$ is a cochain morphism from $\mathbf{\Omega}(B)$ into $\mathbf{\Omega}(VB)$. As a consequence, it defines a morphism $\widetilde{\mathcal{P}}^*_{\mathcal{B},\Lambda B} : H_{dR}D \to H_{dR}(VB)$, as we planed. ∎

3.2 – Fundamental properties of $\widetilde{\mathcal{P}}$

Now, we consider the bundle $J^2V\mathcal{B}$ together with its natural bundle structure $J^2VB \to VB$ and the local basis for the contact 1-forms, given by:

$$\widetilde{\theta}^a = dv^a - v^a_\sigma dx^\sigma \ , \quad \widetilde{\theta}^a_\mu = dv^a_\mu - v^a_{\mu\sigma} dx^\sigma \ . \tag{3.7}$$

The family of 1-forms defined by combining (1.8) together with (3.7) determines a local basis for the contact 1-forms with respect to the bundle structure $J^2VB \to B$. Moreover, from (2.14) we have:

$$\widetilde{\mathcal{P}}_{J^2\mathcal{B}}(\theta^a) = \widetilde{\theta}^a \ , \quad \widetilde{\mathcal{P}}_{J^2\mathcal{B}}(\theta^a_\mu) = \widetilde{\theta}^a_\mu \ . \tag{3.8}$$

We need two technical lemmae:

LEMMA 3.5. *Let $f : J^2B \to \mathrm{I\!R}$ be a function, which induces the mapping $d_V f : J^3B \to T^*M$ and the perturbation $\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f : J^2VB \to \mathrm{I\!R}$, where the obvious identifications with sections are used. Considering the induced morphisms $d_V(\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f) : J^3VB \to T^*J^3VB$ and $\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_V f) : J^3VB \to T^*J^3VB$, the following hold*

$$d_V(\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_V f) \ , \tag{3.9a}$$

$$d_H(\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_H f) \ . \tag{3.9b}$$

PROOF. The lemma follows easily from (2.14), (3.4) and (3.8). Equation (3.9b) holds because of (3.6) and (3.9a), being $d_H = d - d_V$. ∎

Analogous calculations give the following lemma:

LEMMA 3.6. *Let $\omega \in \mathbf{\Omega}(J^2B)$ be any form the following hold*:

$$d_V(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\omega) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_V\omega) \ , \tag{3.10a}$$

$$d_H(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\omega) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_H\omega) \ . \tag{3.10b}$$

Using the previous lemmae, by simple calculations one obtains also the following fundamental result:

THEOREM 3.7. *The variational component of the reduced first order perturbation functor $\widetilde{\mathcal{P}}$ satisfies the following "naturality properties":*

$$\begin{aligned}
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(f^{\mathcal{B}}_{(1)}(\mathcal{L})) &= f^{V\mathcal{B}}_{(1)}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(f^{\mathcal{B}}_{(2)}(\mathcal{L})) &= f^{V\mathcal{B}}_{(2)}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^4\mathcal{B}}(e^{\mathcal{B}}(\mathcal{L})) &= e^{V\mathcal{B}}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(\Theta^{\mathcal{B}}(\mathcal{L})) &= \Theta^{V\mathcal{B}}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(\mathbf{\Omega}^{\mathcal{B}}(\mathcal{L})) &= \mathbf{\Omega}^{V\mathcal{B}}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ .
\end{aligned} \tag{3.11}$$

Finally, from (1.17), (2.9), (2.11) and (3.11) we deduce that:

THEOREM 3.8. *The morphism $\widetilde{\mathcal{P}}_{J^4\mathcal{B}}(e^{\mathcal{B}}(\mathcal{L}))$ is the Jacobi morphism of $\mathcal{L}$ and the following holds*:

$$\delta^2(\mathcal{L}\circ j^2\lambda_\varepsilon) = \widetilde{\mathcal{P}}_{J^4\mathcal{B}}(e^{\mathcal{B}}(\mathcal{L}))\circ j^4 v + \delta[(d_H f^{\mathcal{B}}_{(1)}(\mathcal{L}) + d_H f^{\mathcal{B}}_{(2)}(\mathcal{L}))\circ j^4\lambda_\varepsilon] \tag{3.12}$$

REMARK. Equation (3.12) gives the second variation of the Lagragian $\mathcal{L}$ expressed by the variational component $\widetilde{\mathcal{P}}$ of the first order perturbation functor $\widehat{\mathcal{P}}$.

3.3 – The comparison between cohomologies

Now, we determine some relations among the variational component of the first order perturbation functor $\widetilde{\mathcal{P}}$ and some of the functors defined by other authors in various cohomological theories related to problems involving partial differential equations. Obviously, these relations can be considered as a further measure of the naturality of the functors $\mathcal{P}$, $\widehat{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ introduced here. To this purpose we consider two different versions of the cohomological theory introduced by ANDERSON and DUCHAMP (see [5]) and developed by many authors, among which we recall VINOGRADOV ([33]; see [6] and [9] for a more detailed bibliography). We shall also introduce a third version of Vinogradov's cohomological theory, which better exploits the naturality of the functors introduceded here and puts forward some problems which apparently were not considered in the previous literature known to us.

The cohomology considered in [6] is not extremely well suited to include the global versions of the Euler-Lagrange equations. In fact, the only case known to us in which this cohomological theory works well for variational problems is the case obtained by taking $B = M \times \mathbb{R}^p$ (with p any integer) and $\pi = pr_1 : B \to M$ (see [34]). Moreover, the "tower construction" of [7] does not seem to be suited to include the differential equations ensuing from variational problems, as we shall shortly see below. We shall thence suggest a "naive" solution for both problems. We recall once again that the construction considered here has the unique purpose of testing the naturality of the variational component of the first order perturbation functor. Accordingly, "better for our purposes" will not in general mean "better" (especially when one considers the important results of [7] and [10]), even if we believe that it could be useful to compare some of the possible constructions together with their applications. Finally, we stress that the variational methods involve many more types of partial differential equations than people generally think, as it will be pointed out by the examples of the Appendix (related to "parabolic" systems of partial differential equations in the sense of [6], heat equations and KdV equations included). This remark can be especially useful for the cohomological groups considered here, since the problems coming from the degeneracy of the Lagrangian and from the signature of its associated Hessian do not seem to play an important role, at least for the moment.

Let again $\mathcal{B} = (B, M, \pi)$ be a bundle. Let us denote by $\pi^h_k : J^h B \to J^k B$ the canonical projections, for any $h, k \in \mathbf{N}$, with $h > k$ and let us set $J^0\mathcal{B} = \mathcal{B}$. Then we have canonical inclusions $(\pi^h_k)^* : \mathbf{T}^* J^k B \to \mathbf{T}^* J^h B$, for any $h, k \in \mathbf{N}$, with $h > k$; we shall use $(\pi^h_k)^*$ as identification morphisms. Then, more or less clearly, the specific construction of [9] suggests to overcome the use of the bundle $J^\infty\mathcal{B}$ of infinite jet prolongations of sections of $\mathcal{B}$ which has better "flatness" properties but has a complicated topology (see, e.g., [34]), by just considering and suitably working on jet bundles of order $k+1$, being k the highest order on which the r-forms used depend. Since in our hypotheses $de^{\mathcal{B}}(\mathcal{L}) = d_V e^{\mathcal{B}}(\mathcal{L})$ depends on the elements of $J^5 B$, for any Lagrangian $\mathcal{L}$ on $J^2 B$, we shall consider $\mathbf{\Omega}(J^k B) \subseteq \mathbf{\Omega}(J^6 B)$, for any $k \le 5$. We shall also consider $\mathbf{\Omega}(M) \subseteq \mathbf{\Omega}(B) \subseteq \mathbf{\Omega}(J^6 B)$, via the identification morphism $\pi^* : \mathbf{\Omega}(M) \to \mathbf{\Omega}(J^0 B) = \mathbf{\Omega}(B)$. The previous identifications allow us to consider the ring of smooth functions $\mathbf{\Omega}_0(J^k B)$ as a sub-ring of the ring of smooth functions $\mathbf{\Omega}_0(J^6 B)$ which are constant along the fibers of the bundle $\pi^6_k : J^6 B \to J^k B$, with $k < 6$. We shall denote by $\widetilde{\mathbf{\Omega}}^h_r(J^k B)$ the $\mathbf{\Omega}_0(J^k B)$-module of r-forms along the canonical projection $\pi^h_k : J^h B \to J^k B$, for any $h, k \le 5$, with $h > k$. Finally, we denote by $\widetilde{\mathbf{\Omega}}_r(M)$ the $\mathbf{\Omega}_0(J^k B)$-submodule of r-forms along the canonical projection $\bar{\pi}^k : \pi \circ \pi^k_0 : J^k B \to M$, for any $k \le 5$; also this module will be considered as a sub-module of $\mathbf{\Omega}_r(J^6 B)$, for all $r \in \{1, \ldots, m\}$.

We shall use the following known results (see [9]):

Proposition 3.9. *The following contact forms*:

$$\theta^a_{\mu\nu\rho} = dy^a_{\mu\nu\rho} - y^a_{\mu\nu\rho\sigma} dx^\sigma, \qquad \theta^a_{\mu\nu\rho\sigma} = dy^a_{\mu\nu\rho\sigma} - y^a_{\mu\nu\rho\sigma\tau} dx^\tau \ , \tag{3.13}$$
$$\theta^a_{\mu\nu\rho\sigma\varepsilon} = dy^a_{\mu\nu\rho\sigma\varepsilon} - y^a_{\mu\nu\rho\sigma\varepsilon\tau} dx^\tau \ ,$$

together with the contact forms defined by (1.8), *the forms* dx^μ *and the forms* $dy^a_{\mu\nu\rho\sigma\varepsilon\eta}$, *determine a local basis* C *of the* $\mathbf{\Omega}_0(J^6 B)$*-module* $\mathbf{\Omega}_1(J^6 B)$ *and hence generate* $\mathbf{\Omega}(J^6 B)$. *Moreover, the subset* C' *obtained from* C *by removing only all the forms* dx^μ *and* $dy^a_{\mu\nu\rho\varepsilon\tau}$ *generates an ideal of* $\mathbf{\Omega}(J^6 B)$, *known as the ideal of* contact forms.

One of the most substantial differences beetwen the viewpoint of [6] and the viewpoint of [9] is the definition of "solution of a system of differential equations". In fact, let $\mathcal{I}$ be an ideal of $\mathbf{\Omega}(J^6B)$ and $\sigma \in \Gamma_D(\pi)$ be a local section, being D a domain in M. In [6] the section σ is said to be a solution of the system of partial differential equations defined by $\mathcal{I}$ if and only if $(j^6\sigma)^*(\mathcal{I}_{|J^6E}) = 0$, being $E = \pi^{-1}(D)$ the total space of the bundle over D naturally induced by the bundle structure of $\mathcal{B}$ and $(j^6\sigma)^* : \mathbf{\Omega}(J^6E) \to \mathbf{\Omega}(M)$ the total differential of $j^6\sigma : D \to J^6B$. In the Calculus of Variations a section σ is instead a solution of the system of partial differential equations defined by $\mathcal{I}$ if and only if $\mathcal{I} \circ \sigma \equiv \{\omega \circ j^6\sigma / \omega \in \mathcal{I}\} = 0$. This alternative definition of solution can be easily inferred from the general theory, since if $\omega_i \circ j^6\sigma = 0$ for a family $(\omega_i)_{i\in I}$ where $I \neq \emptyset$ is any set of indices, then $\omega \circ\ j^6\sigma = 0$ for all ω belonging to the ideal $\mathcal{I}$ generated by the family $(\omega_i)_{i\in I}$.

The definition of solution used in [6] cannot be applied immediately to the Euler-Lagrange equations, since they are globally defined by an (m+1)-form which is locally of the type $e_a\theta^a \wedge \mathbf{ds}$, while $(j^6\sigma)^*(\theta^a) = 0$ holds for all $a \in \{1, \dots, p\}$ because of the very definition of the structure forms θ^a. We stress moreover that the solution suggested in [34] for variational problems defined on the trivial bundle $\mathcal{B}$ given by $pr_1 : M \times \mathbb{R}^p \to M$ is however viable, only due to the fact that one can avoid the use of the contact forms θ^a by fixing on $\mathbb{R}^p$ the standard atlas containing the unique chart $(\mathbb{R}^p, id_{\mathbb{R}^p})$. This obstacle can be overcame by first noticing that all general constructions of [6] continue to hold if one replaces the closed ideal $\mathcal{I}_{\mathrm{var}}$ of $\mathbf{\Omega}(J^\infty B)$, used in [6], with any family of closed ideals $\mathcal{I}_{i\in I}$ $(I \neq \emptyset)$. Then we make the following "naive" suggestion: instead of considering the ideal generated by means of the Euler-Lagrange form $e^{\mathcal{B}}(\mathcal{L})$, we consider the family of ideals generated by the family of m-forms $(i_{\mathbf{v}}(\mathbf{\Omega}))_{\mathbf{v}\in\mathcal{V}(J^3B)}$ together with the family of contact forms already considered in [6], where $\mathcal{V}(J^3B)$ is the module of vertical vectorfields defined on $J^3\mathcal{B}$. The elements of $\mathcal{V}(J^3B)$ must be here considered as a *mere parameters*; for this reason we shall use boldface letters to denote them. This construction allows us to use the results of [6] also in the variational case, since (1.10) holds as an equivalent of the Euler-Lagrange equations. This suggestion could be useful out of the context of this paper, since equation (A.5) of the Appendix shows that $i_{\mathbf{v}}(\mathbf{\Omega})$ belongs to the closed ideal $\mathcal{I}_{\mathrm{var}}$ generated by the contact forms of the adapted basis and by the m-form $\mathbf{ds}$, for any $\mathbf{v} \in \mathcal{V}(J^3B)$. Hence, the previous ideal can be replaced by this last one, obtaining cohomological groups which do not depend on the Lagrangian $\mathcal{L}$.

Remark. By using a variant of the construction presented in [9], one can avoid to introduce the notation needed when using the infinite jet bundle $J^\infty\mathcal{B}$ by just noticing that the set $C' \cup \{\mathbf{ds}\}$ generate a closed ideal $\mathcal{I}'_{\mathrm{var}}(J^6B)$ of $\mathbf{\Omega}(J^6B)$. This fact allows us to consider the cohomological groups $H'_{\mathrm{var}}(J^6B)$ of the quotient cochain complex $\widehat{\mathbf{\Omega}}(J^6B)/\mathcal{I}'_{\mathrm{var}}(J^6B)$ (recall that $\mathbf{\Omega}(J^kB) \subseteq \widehat{\mathbf{\Omega}}(J^kB) = \mathcal{H}or(J^k\mathcal{B}) \oplus \mathcal{K}(J^k\mathcal{B})$), having the total differential modulo $\mathcal{I}'_{\mathrm{var}}(J^6B)$ as a coboundary operator. Then the cohomological group H_{var} is obtained by considering the projective limit of $H'_{\mathrm{var}}(J^6B)$, in the obvious way.

Also the "tower construction" of [6] (in the following it will be called BG-*tower construction*, because the variational component of the first order perturbation functor will determine a further tower which will be called here the *Jacobi tower*) is not well suited to include the Euler-Lagrange equations of variational problems. In fact, the (m+1)-form $d_H(i_{\mathbf{v}}(\mathbf{\Omega}))$ vanishes, when $\mathbf{v} \in \mathcal{V}(J^3B)$ is considered as a mere parameter, as in (1.10), and the BG-tower construction coincides essentially with the horizontal derivative. In order to overcome this problem we shall assume for simplicity that M is orientable and that a global volume form $\mathbf{vol}$ is fixed on M. Then, there exists a unique 1-form $\widetilde{\mathbf{\Omega}}$ on J^6B such that:

$$(3.14) \qquad i_{\mathbf{v}}(\mathbf{\Omega}) = i_{\mathbf{v}}(\widetilde{\mathbf{\Omega}} \wedge \mathbf{vol}) \quad , \quad \forall \mathbf{v} \in \mathcal{V}(J^3B) \ .$$

Following an idea first developed in [17] one can now consider the family of Lagrangians $\mathcal{L}^1_{\mathbf{v}} = i_{\mathbf{v}}(d_H\widetilde{\mathbf{\Omega}})\mathbf{vol} : J^4B \times_M TM \to \Lambda^m M$, locally defined by:

$$(3.15) \qquad \mathcal{L}^1_{\mathbf{v}}(j^4\lambda, X) = (j^6\lambda)^*(d_\mu(\widetilde{\mathbf{\Omega}}_a)v^a X^\mu)\mathbf{vol} \ ,$$

being $\widetilde{\mathbf{\Omega}}_a$, v^a and X^μ the local components of $\widetilde{\mathbf{\Omega}}$, $\mathbf{v}$ and X, respectively, where λ is any section, X is a vectorfield defined on M and $\mathbf{v}$ an element of $\mathcal{V}(J^3B)$. The first variation of the family of Lagrangians (3.14) splits into:

$$(3.16a) \qquad (j^6\lambda)^* d_\mu(\widetilde{\mathbf{\Omega}}) = 0 \ ,$$

$$(3.16b) \qquad [\delta(j^6\lambda)^*(d_\mu(\widetilde{\mathbf{\Omega}}_a))]v^a X^\mu = 0 \ .$$

As is well known, the inverse problem of the Calculus of Variations (i.e., the problem of finding a variational principle whose Euler-Lagrange equation is fixed a priori) is not at all trivial, while the same problem becomes in a sense trivial if one allows the possibility of considering new variables which are not a priori restricted to satisfy any further condition (even if, in some cases, one can consider to be more important the advantages coming from the introduction of variational methods than the problems coming from the "triviality" of the new variables; see, e.g., [37]). Replacing the original Euler-Lagrange equations with the new equations (3.16) goes in fact in this "trivial" direction. However, the usefulness of this alternative variational principle is garanteed by the results of [6] and [7], which ensure that the solutions of the equation (3.16a) are of practical importance, while the second equation (3.16b) does not eliminate any solution of the first equation (3.16a), since it is always verified by the vectorfield $X = 0$ and hence it preserves at least a copy of any solution of the first equation. A second question is whether the relation between $\mathbf{\Omega}$ and $\widetilde{\mathbf{\Omega}}$ preserves or not the informations on the variational problem contained in the first (m+1)-form. A positive answer can be obtained by remarking that being M orientable there exists an atlas of M in which the local expression of $\widetilde{\mathbf{\Omega}}$ coincides with the local expression of $\mathbf{\Omega}$. In any case, most of the relations needed between $\mathbf{\Omega}$ and $\widetilde{\mathbf{\Omega}}$ can be easily deduced from the results of [16].

Proposition 3.10. *The following inlcusion holds*:

$$\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\mathcal{I}'_{\text{var}}(J^6B)) \subseteq \mathcal{I}'_{\text{var}}(J^6VB) \tag{3.17}$$

and hence the corresponding homological construction can be easily iterated.

Proof. This can be easily seen by using (2.7) together with the appropriate extension of (3.8) to all the involved contact forms. ∎

Remark. In particular equations (3.6) and (3.17) entail that $\widetilde{\mathcal{P}}$ induces a morphism between the corresponding cohomological groups, which will be denoted by the same letter (with an abuse of notation).

The new tower construction obtained as in [6] by iterating the application of $\widetilde{\mathcal{P}}$ to the equivalent Euler-Lagrange equations (1.10) of the

Lagrangians which are obtained by iterating the action of $\widetilde{\mathcal{P}}$ on the original Lagrangian $\mathcal{L}$ has a sure meaning, since it determines the "higher order Jacobi fields" (the reasons for which those fields must be considered are strictly analogous to those well explained in [6], whereby they refer to conservation laws rather than to higher order variations, as here).

Let us now notice that according to the method developed in [6] and to (3.15) the other "levels" of the relevant BG-tower also determine variational problems (in a sense "associated" to the original one we are considering). Accordingly, the previous construction can also be iterated for each level. For the other "levels" of the BG-tower construction the problem of their usefulness comes from the "triviality" of the variational principle (3.15); again, because of (3.16), this problem is related to the usefulness of "Jacobi fields" for generic systems of partial differential equations, which does not seem to be clear to us, nor it has been considered in the existing literature. We limit ourselves to remark that, as in the case of variational problems, also for generic differential equations "Jacobi fields" determine the directions in which a homotopic variation of a solution is still determined by means of solutions. This suggests us to give the following definition:

Definition 3.1. We will call **Jacobi tower** the set of cohomological groups so obtained by iterating the action of $\widetilde{\mathcal{P}}$ on $\mathcal{L}$ while the k^{th}-**Jacobi tower** will be the set of cohomological groups obtained by iterating the action of $\widetilde{\mathcal{P}}$ on the Lagrangians constructed by iterating (3.15) till the k-th term of the corresponding BG-tower. We shall call JBG-**wall** the complete set of cohomological groups obtained in this way.

Let us now turn to consider the approach of "variational sequences". Differently from [6], the construction of [9] is explicitly worked out for variational problems, hence it does not present the problems coming from the definition of solutions of a differential partial equation we discussed before. A further observation of [9] is that one does not need the structure of graded exterior algebra on a quotient cochain complex of $\mathbf{\Omega}(J^6B)$ in order to define its cohomological groups, but simply an Abelian group structure. Finally, a last observation can be obtained from the comparison of [6] and [9]. In fact, if $\mathcal{I}$ is a graded complex of closed modules and $(\mathcal{I}^r)_{1\leq r\leq N}$ is its gradation, then $\mathcal{I}^r$ can be obtained by setting $\mathcal{I}^r = \widetilde{\mathcal{I}}^r + d\widetilde{\mathcal{I}}^{r-1}$, for $1 < r \leq N$ and $\mathcal{I}^1 = \widetilde{\mathcal{I}}^1$, where each of the modules $\widetilde{\mathcal{I}}^r$ can be chosen by using different criteria for each $1 \leq r \leq N$. As an example, in [9] the graded module $\mathcal{I}_K$, with gradation $(\mathcal{I}_K^r)$, is obtained by taking the graded family of modules which are the kernels of a suitable family of $\mathbf{\Omega}^0(J^6B)$-linear applications. This family can be easily described in the following way. Let $\mathcal{I}_K^1(J^6U) = \widetilde{\mathcal{I}}_K^1(J^6U)$ be the submodule of $\mathbf{\Omega}^1(J^6U)$, generated by the set C', of all contact forms on J^6U, having set $J^6U \equiv (\pi \circ \pi_0^6)^{-1}(U)$, for any U open set of M which is the domain of a local coordinate system. Let us also set $\widetilde{\mathcal{I}}_K^r(J^6U) = \mathcal{I}_K^1(J^6U) \wedge \mathbf{\Omega}^{r-1}(J^6U)$, for any $r \in \{2, \dots m\}$, for any open set U of M, on which a local coordinate system is defined. Finally, we set $\widetilde{\mathcal{I}}_K^r(J^6U) = (\mathcal{I}_K^1(J^6U))^{r-m+1} \wedge \mathbf{\Omega}^{m-1}(J^6U)$, where $(\mathcal{I}_K^1(J^6U))^p$ denotes the p-th power with respect to the wedge product, for any $r \in \{m+1, \dots N\}$, where N is the dimension of J^6B. Then, $\widetilde{\mathcal{I}}_K^r$ is the submodule of $\mathbf{\Omega}(J^6B)$ of r-forms whose restrictions belong to $\widetilde{\mathcal{I}}_K^r(U)$, for any $r \in \{2, \dots N\}$ and any open subset U of M which is the domain of a local coordinate system and $\mathcal{I}_K^r = d\widetilde{\mathcal{I}}_K^{r-1} + \widetilde{\mathcal{I}}_K^r$, for any $r \in \{2, \dots N\}$. Again, $\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\mathcal{I}_K)$ is contained into the module obtained with the same criteria starting from the variational problem $\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\mathcal{L})$. As a consequence the suitable Jacobi tower can be constructed and analogous remarks hold, as in the previous cohomological groups.

Let us now remark that the papers [9] and [10] were published before [6] and [7], so that they present problem analogous to those we already mentioned for the tower construction of [6]. In fact, the Euler-Lagrange morphism $e^{\mathcal{B}}(\mathcal{L})$ of a Lagrangian $\mathcal{L}$ is such that $d_H e^{\mathcal{B}}(\mathcal{L}) = 0$ holds. We overcome this problem by assuming that M is orientable and that a volume form **vol** is fixed on M. Then, there exists a unique 1-form $\tilde{e}^{\mathcal{B}}(\mathcal{L})$ on $J^6(\mathcal{B})$ such that

$$e^{\mathcal{B}}(\mathcal{L}) = \tilde{e}^{\mathcal{B}}(\mathcal{L}) \wedge \mathbf{vol} \,. \tag{3.18}$$

Again, we have the family of Lagrangians $\mathcal{L}^{1\mathbf{v}} = (d_H \tilde{e}^{\mathcal{B}})_{\mathbf{v}} \otimes \mathbf{vol} : J^4B \times_M TM \to \Lambda M$, $\mathbf{v} \in J^6VB$, defined by:

$$\mathcal{L}^{1\mathbf{v}}(j^4\sigma, X) = ((d_\mu e_a)v^a X^\mu)\mathbf{vol} \,. \tag{3.19}$$

Also in this case all the considerations already made for the BG-tower construction will follow, so that at the end we have a second "wall construction" for the Euler-Lagrange differential equation (1.10) which differs from the standard BG-wall and contains other informations on the same class of problems. These informations are obviously related to the "Lepagean (equivalent) forms", i.e. to the m-forms of J^6B which are suitably obtained from $\mathcal{L}$ to determine the same Euler-Lagrange equation (see [10]).

Remarks. Let us finally make a couple of remarks, which in a sense point towards suggestions which could be in contrast with each other. If one chooses the family of modules $\widetilde{\mathcal{I}}$ in such a way that it has a maximum number of null spaces (as we shall suggest below), then the properties of the cohomological groups obtained will be of course "closer" to the properties of the full de Rham groups of the bundle. On the other hand, when $\mathcal{B}$ coincides with the trivial bundle $pr_1 : [0,1] \times M \to [0,1]$, the family $(\theta^a \wedge dt)$, where θ^a is now given by $\theta^a = dy^a - \dot{y}^a dt$, generates a closed ideal of differential forms which determines cohomological groups isomorphic to the de Rham cohomology of M, while the cohomological groups of [6] and [9] considered here are necessarily trivial.

Example. As an example of a way to obtain cohomological groups which are "close" to the de Rham ones, we consider the submodule $\mathcal{I}_1^{m+2}$ of $\mathbf{\Omega}^{m+2}(J^6B)$ locally generated by the (m+2)-forms $\theta^a_{\mu_1\dots\mu_h} \wedge \theta^b_{\nu_1\dots\nu_k} \wedge \mathbf{ds}$ $(h, k \leq 6)$. Then, by taking $\widetilde{\mathcal{I}}_1^r = 0$, for any $r \neq m+2$, one obtains a closed graded module and hence a cohomological graded group. In this complex $(\mathcal{I}_1^r)_{1\leq r\leq N}$, the Euler-Lagrange form $e^{\mathcal{B}}(\mathcal{L})$ is a cochain in $\mathcal{I}_1^{m+1}$ (see (A.5)) and hence it determines a non-trivial cohomology class. What is important here is that the combined action of the k-jet extensions and of the variational components of the first order perturbation functor allows us to construct the corresponding JBG-wall: this possibilty is a further sign of the naturality of the functors considered here. Finally, let us denote by $\widehat{\mathcal{I}}$ anyone of the graded modules $\mathcal{I}_{BG}$, $\mathcal{I}_K$ and $\mathcal{I}_1$. Then, the restriction of the total differential to $\widehat{\mathcal{I}}$ determines a structure of cochain complex, which in turn determines cohomological groups. Again, the "Jacobi tower construction" can be performed for those groups since $\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\widehat{\mathcal{I}}) \subseteq \widehat{\mathcal{I}}$. These cohomological groups could be useful, as the case of the trivial bundle $[0,1] \times M$ (which is related to the variational aspects of Riemannian Geometry) shows. We conclude this part by remarking that one could try to find the "best" closed submodule $\mathcal{I}$, if it exists, which would contain most of the informations encoded into the cohomological groups considered here. These problems will be considered in [20].

Conclusive Remark. We conclude our paper by stressing that in the general case the first order perturbation functor is compatible with (1.11), via (2.18), so that the conserved Noether currents of the original Lagrangian $\mathcal{L}$ are transformed by $\overline{\mathcal{P}}_{\mathcal{B}}$ into the Noether currents of the deformed Lagrangian $\widetilde{\mathcal{P}}(\mathcal{L}) \equiv \mathcal{L}_{(1)}$. More details will be given in [21].

– Appendix

A.1 – Augmented variational principles and examples

The basic justification for the introduction of the BG-tower comes from the KDV equation (see [6] and [7]), hence it seems important to indicate methods which allow one to write this equation as the Euler-Lagrange equation of a non-trivial variational principle. This problem has an importance of its own for other reasons, which are well explained in the Introduction to Chapter 2 of the book [36]. As a consequence, many methods have been developed to solve the inverse problem of the Calculus of Variations, even in those cases in which it is clear from the beginning that Lagrangians which determine the system of partial differential equations considered do not exist (e.g., the case of heat equations and KdV equations).

Unfortunately, people interested into this "generalized aspect of the inverse problem" have paid more attention to the systems of partial differential equations coming from technical applications rather than from Mathematics and Physics. In this Appendix, instead of applying one of the existing methods to the KdV equation we prefer to suggest a new one, because this choice will require simple calculations and will suggest that, if one does not find the existing Lagrangians to be satisfactory, one can always try to find new ones. The method considered here belongs to a larger class of methods in which the basic tool is the addition of new variables to the original variables of the given system of partial differential equations. The first example we mention is the method known as "method of mirror variables", explicitely introduced by Glansdorff and Prigogine (see [38]), following an earlier example of a hydrodynamical principle stated by Bateman (see [39]) and elaborated by Morse and Feshbach (see [40]) with some contributions (see, e.g., the papers quoted therein and in [37]). This method consists in adding to the variables of the problem, which will be subjected to variations, an identical number of "mirror variables", which are considered as mere parameters. Obviously, this addition implies in many cases that some new solutions are added to the solutions of the system of partial differential equations one started from.

The method proposed here consists instead in adding just one new dependent variable to the original dependent variables of the problem by requiring that a Lagrangian exists so that: (i) among its Euler-Lagrange equations the equation for the new variable has a simple and possibly "canonical" solution; (ii) in corrispondence with this solution, the remaining Euler-Lagrange equations reduce to the original system of the original variables or, at least, have the same set of solutions. In this way, it is easy to control the relations between the geometric objects related to the "associated Lagrangian" with those related to the original system of partial differential equations (e.g., one might require that the group of gauge transformations of the system coincides with the sub-group of gauge transformations of the associated Lagrangian which preserves the chosen solution for the extra variable; and so on). For the heat equation and the KdV equation the obvious choice for the new variable is what we call *admissible time measure*.

Example A.1 - The heat equation: For the case of the heat equation, let us consider the trivial bundle $\mathcal{B} = (\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^m, pr)$, where pr is the canonical projection of $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^2$ onto $\mathbb{R} \times \mathbb{R}^m$ and let us look for a first order Lagrangian $\mathcal{L} = L(t, x^\mu, \tau, q, \dots)\mathbf{ds}$, where t, x^μ, τ and q the time coordinate, the spatial coordinate and the admissible time measure, while $\mathbf{ds}$ is the standard volume form on $\mathbb{R} \times \mathbb{R}^m$ and "dots" replace the remaining variables, i.e. the partial space-time derivatives of τ and q. Moreover, we require that one of the two Euler-Lagrange equations of $\mathcal{L}$ is satisfied by the solution $\tau = t$ and that in correspondence of this solution the remaining equation coincides with the heat equation. There exists a large class of functions L determining Lagrangians with this property. The whole class can be determined by using a procedure analogous to the one used in [37] to show that the heat equation cannot be determined by a variational principle in the classical sense (see [37] paragraph 2.6, pp. 65-66). The following function seems to be the simplest function belonging to this class:

$$L = q\left(1 - \frac{\partial\tau}{\partial t}\right) + \delta^{ij}\frac{\partial\tau}{\partial x^i}\frac{\partial q}{\partial x^j}\ . \tag{A.1}$$

The Euler-Lagrange equations of the "associated" Lagrangian are in fact:

$$\frac{\partial q}{\partial t} - \delta^{ij}\frac{\partial^2 q}{\partial x^i \partial x^j} = 0 \tag{A.2a}$$

and

$$1 - \frac{\partial\tau}{\partial t} - \delta^{ij}\frac{\partial^2 \tau}{\partial x^i \partial x^j} = 0\ , \tag{A.2b}$$

with the obvious meaning of the symbols used. One sees immediately that $\tau = t$ makes (A.2b) satisfied, so that (A.2a) reduces to nothing but the heat equation $\frac{\partial q}{\partial t} - \Delta q = 0$ in flat space $\mathbb{R}^m$.

EXAMPLE A.2 - THE KDV EQUATION: In the case of KdV equation we take $m = 1$, hence $\mathcal{L} = L(t, x, \tau, u \dots)\mathbf{ds}$, with the obvious meaning of the symbols used. Even in this case, the set of all functions L whose Euler-Lagrange equations allow the solution $\tau = t$ so that in correspondence of this solution the remaining equation becomes the KdV equation, is large. The following function seems to be the simplest one belonging to this class:

$$L = 6u^2\frac{\partial\tau}{\partial x} + u\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\frac{\partial^2 u}{\partial x^2} - u\ . \tag{A.3}$$

This is a second order Lagrangian, whose Euler-Lagrange equations are:

$$12u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{A.4a}$$

and

$$6u\frac{\partial\tau}{\partial x} + \frac{\partial\tau}{\partial t} - 1 + \frac{\partial^3\tau}{\partial x^3} = 0\ . \tag{A.4b}$$

It is immediate to see that equation (A.2b) is satisfied by $\tau = t$ so that (A.4a) reduces to the standard KdV equation required.

A.2 – Some technical formulae

We list in this Appendix some formulae used in this papers. Let us consider the $\mathbb{R}$-linear mapping $\Theta^{\mathcal{B}} : \widetilde{\mathbf{\Omega}}_2^m(M) \to \widetilde{\mathbf{\Omega}}_3^m(J^1\mathcal{B})$, where $\widetilde{\mathbf{\Omega}}_k^h$ are the module introduced in Section 3, which associates to any second order Lagrangian $\mathcal{L}$ over M its Poincaré-Cartan m-form. Then we can express the Euler-Lagrange (m+1)-form $e^{\mathcal{B}}(\mathcal{L})$ of any Lagrangian $\mathcal{L}$ on M by means of the multiplectic form $\mathbf{\Omega}^{\mathcal{B}}(\mathcal{L}) = d\Theta^{\mathcal{B}}(\mathcal{L})$. In fact, a simple calculation shows that:

$$
\begin{aligned}
e^{\mathcal{B}}(\mathcal{L}) = \mathbf{\Omega}^{\mathcal{B}}(\mathcal{L}) &+ (d_\nu \frac{\partial^2 L}{\partial y^b \partial y^a_{\mu\nu}} - \frac{\partial^2 L}{\partial y^b \partial y^a_\mu})\theta^b \wedge \theta^a \wedge \mathbf{ds}_\mu + \\
&+ \left(d_\nu \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\mu\nu}} + \frac{\partial^2 L}{\partial y^b \partial y^a_{\rho\mu}} + \frac{\partial^2 L}{\partial y^a \partial y^b_{\rho\mu}} - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_\mu} \right) \theta^b_\rho \wedge \theta^a \wedge \mathbf{ds}_\mu + \\
&+ \left(d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} + \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\sigma\mu}} - \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_\mu} \right) \theta^b_{\rho\sigma} \wedge \theta^a \wedge \mathbf{ds}_\mu + \\
&+ \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\tau}} \theta^b_{\rho\sigma\tau} \wedge \theta^a \wedge \mathbf{ds}_\mu - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\mu\nu}} \theta^b_\rho \wedge \theta^a_\nu \wedge \mathbf{ds}_\mu + \\
&- \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} \theta^b_{\rho\sigma} \wedge \theta^a_\nu \wedge \mathbf{ds}_\mu \ ,
\end{aligned} \tag{A.5}
$$

where $\mathcal{L} = L\mathbf{ds}$ holds locally. From the previous equation, by standard calculations we get:

$$
d[e^{\mathcal{B}}(\mathcal{L})] = \{\alpha_{ab}\theta^b + \alpha^\rho_{ab}\theta^b_\rho + \alpha^{\rho\sigma}_{ab}\theta^b_{\rho\sigma} + \alpha^{\rho\sigma\mu}_{ab}\theta^b_{\rho\sigma\mu} + \alpha^{\rho\sigma\mu\tau}_{ab}\theta^b_{\rho\sigma\mu\tau}\} \wedge \theta^a \wedge \mathbf{ds} \,, \tag{A.6}
$$

being

$$
\alpha_{ab} = d_\mu \left(d_\nu \frac{\partial^2 L}{\partial y^b \partial y^a_{\mu\nu}} - \frac{\partial^2 L}{\partial y^b \partial y^a_\mu} \right) \ , \tag{A.7}
$$

$$
\alpha^\rho_{ab} = 2\frac{\partial^2 L}{\partial y^{[a} \partial y^{b]}_\rho} + 2d_\mu \left(\frac{\partial^2 L}{\partial y^b \partial y^a_{\rho\mu}} - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_\mu} + d_\nu \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\mu\nu}} \right) \ , \tag{A.8}
$$

$$
\begin{aligned}
\alpha^{\rho\sigma}_{ab} = d_\mu d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} &+ 2\frac{\partial^2 L}{\partial y^{(b} \partial y^{a)}_{\rho\sigma}} + 2d_\nu \frac{\partial^2 L}{\partial y^b_r \partial y^a_{\sigma\nu}} + \\
&- d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_\nu} - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_\sigma} \,,
\end{aligned} \tag{A.9}
$$

$$\alpha_{ab}^{\rho\sigma\mu} = 2d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} + \frac{\partial^2 L}{\partial y^b_{\rho} \partial y^a_{\sigma\mu}} - \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu}} \tag{A.10}$$

and

$$\alpha_{ab}^{\rho\sigma\mu\tau} = \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\tau}} \ . \tag{A.11}$$

The coefficients (A.7)-(A.11) are the relevant coefficients which enter the Jacobi form of the given Lagrangian $\mathcal{L}$; see [16], [17] for details.

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