Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 117-129

Asymptotic behavior of convolution powers of a probability measure on harmonic extensions of *H*-type groups

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Abstract: *We give a local (central) limit theorem and a renewal theorem for radial probability measures on AN-groups.*

– Introduction

Solvable extensions of H-type groups have been objects of intensive studies in recent years, since the discovery made by E. Damek and F.Ricci [4] of a counterexample to the Lichnerowicz conjecture. Indeed, after E. Damek and F. Ricci have shown [5] that, despite the lack of symmetry, it is possible to develop on these groups a harmonic analysis similar to the one developed by Harish-Chandra for semisimple groups, several authors have investigated the possibility to extend to these groups analogous results known for rank one symmetric spaces: multipliers problems [2], Paley-Wiener theorems [6], asymptotic behavior of the heat kernel [1], to mention just a few. In this paper we follow the mainstream, but with a more probabilistic flavor. We first recall two well known results for

Key Words and Phrases: *Local limit theorem – AN-groups.*
A.M.S. Classification: 43A80 – 60BIS

the convolution powers of a nonarithmetic probability measure μ on $\mathbb{R}$. The local (central) limit theorem states that if μ has mean zero and variance σ then the sequence $n^{1/2}\mu^{*n}$ converges weakly to a multiple, which depends only on σ, of the Lebesgue measure on $\mathbb{R}$. The renewal theorem deals with the potential $U(f)(x) = \sum_{n=0}^{+\infty} \mu^{*n} * f(x)$, where f is a continuous function with compact support. We have that $\lim_{x\to-\infty} Uf(x)$ is different from zero if and only if μ has finite positive mean and the mass of f is different from zero; moreover, if this is the case, the value of the limit is the reciprocal of the mean of μ multiplied by the mass of f.

Both these results have been extended to symmetric spaces by Bougerol in [3] and in this paper we will show how Bougerol's method can be easily adapted to the setting of harmonic extensions of H-type groups. In particular we consider a radial probability measure on an AN-group with support not concentrated at the origin and we prove that the sequence $\rho^{-n}n^{3/2}\mu^{*n}$, where $0 < \rho < 1$, converges weakly to a multiple of the spherical function ϕ_0. In particular, for any compact set K of the origin $\mu^{*n}(K)$ decays exponentially. We should recall that the local asymptotic behavior of the convolution powers of a probability measure on any (amenable) connected Lie group has been determined by N. Th. Varopoulos [10]. According to Varopoulos' classification our AN-groups are in the category of NC groups and if μ is a *symmetric* (i.e. $\mu(A) = \mu(A^{-1})$ for every measurable set A) probability measure on such groups then $\mu^{*n}(K) \approx n^{-3/2}$; thus we do not have an exponential decay as for the the radial measures. To clarify the reason of this difference consider the case when the measure μ has a density f. Then μ is symmetric if and only if $f(x^{-1}) = f(x)m(x)$ a.e., where m denotes the modular function. Since radial densities are symmetric in the usual sense, i.e. $f(x^{-1}) = f(x)$, and the modular function is trivial only at the origin, we have that probability measures associated with radial densities are not symmetric.

– Preliminaries

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra endowed with a scalar product $\langle\cdot,\cdot\rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. Let $J_Z : \mathfrak{p} \to \mathfrak{z}$ the linear map defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle \quad (X, Y \in \mathfrak{p}; Z \in \mathfrak{z}).$$

Then $\mathfrak{n}$ is a Heisenberg algebra if

$$J_Z^2 = -|Z|^2 I \quad \forall\, Z \in \mathfrak{z},$$

and the corresponding simply connected group N is called of Heisenberg-type or simply H-type group. If $k = \dim \mathfrak{z}$ and $m = \dim \mathfrak{p}$ we have that m is always even so that $Q = \frac{m}{2} + k$ is a positive integer called the homogeneous dimension of N. Identifying the group with its Lie algebra via the exponential map we have that the product on N is given by

$$(X, Z)(X', Z') = \left(X + X', Z + Z' + \frac{1}{2}[X, X'] \right).$$

The semidirect product $G = N \rtimes \mathbb{R}_+$ defined by

$$(X, Z, a)(X', Z', a) = \left(X + a^{1/2} X', Z + aZ' + \frac{1}{2} a^{1/2} [X, X'], aa' \right)$$

is a solvable Lie group with Lie algebra $\mathfrak{g} = \mathfrak{p} + \mathfrak{z} + \mathbb{R}$. It is equipped with the left invariant Riemannian metric induced by the scalar product

$$\langle (X, Z, l), (X, Z', l') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + ll'$$

on $\mathfrak{g}$. The associated left Haar measure is given by

$$d^L g = dg = a^Q dX dZ \frac{da}{a}$$

while the right Haar measure is given by

$$d^R g = a dX dZ da$$

so that the group is not unimodular. We recall that if $S = KAN$ is the Iwasawa decomposition of semisimple connected Lie groups of real rank one, then the solvable group $AN = NA$ is an example of a harmonic extension of an H-type group.

If $g = an, a \in \mathbb{R}_+, n \in N$ we denote by $a(g)$ the element a and by $r(g) = d(g, e)$ the geodesic distance of g from the identity e. Furthermore we denote by $S_r = \{ g \in G : d(g, e) = r \}$ the geodesic sphere of radius r.

A function is said to be radial if it depends only on the geodesic distance or equivalently if it is constant on every geodesic sphere. The space of the continuous (resp. smooth) radial functions with compact support is denoted by $C_c(G)^{\#}$ (resp. $C_c^{+\infty}(G)^{\#}$). A (probability) measure is said to be radial if $\chi_r * \mu = \mu * \chi_r = \mu$,where, for every $r > 0$, χ_r denotes the normalized surface measure induced on S_r by the Haar measure dg. Obviously if μ has a density f we have that μ is radial if and only if f is a radial function. The spherical functions are the radial eigenfunctions of the Laplace-Beltrami operator Δ on G, normalized at the origin. They are real analytic, since Δ is elliptic, and have the following properties [5]:

- all the spherical functions are of the form
$$\phi_z(g) = \phi_z(r(g)) = \int_{S_r} a(y)^{Q/2-z} d\chi_r(y), \quad z \in \mathbb{C};$$
- $\phi_z(r) = \phi_{-z}(r)$;
- $\phi_z(r)$ are holomorphic function of z uniformly bounded in z and r for $-\frac{Q}{2} \leq \Re(z) \leq \frac{Q}{2}$.

The Fourier transform of a radial measure μ is defined as

$$\mathcal{F}\mu(z) = \int_G \phi_z(g) d\mu(g)$$

and obviously the Fourier transform of a radial function is defined as the Fourier transform of the associated measure. If $f \in C_c^{+\infty}(G)^{\#}$ then its Fourier transform $\mathcal{F}f$ is a symmetric entire function that decays exponentially on every vertical line. Moreover the following inversion formula holds true [9], [1]:

$$f(r) = \frac{2^{k-3}\Gamma\left(\frac{m+k+1}{2}\right)}{\pi^{(m+k+3)/2}} \int_{\mathbb{R}} \mathcal{F}f(is)\phi_{is}(r)|c(is)|^{-2} ds$$

where c denotes the Harish-Chandra function i.e.

$$c(z) = \frac{2^{Q-2z}\Gamma(2z)\Gamma\left(\frac{m+k+1}{2}\right)}{\Gamma\left(\frac{Q+2z}{2}\right)\Gamma\left(\frac{m+4z+2}{4}\right)}, \quad z \in \mathbb{C}.$$

In the following we will denote by C_1 the constant in front of the integral in the inversion formula.

– The Fourier transform of a measure

LEMMA 1. *Let μ be a nonsingular radial probability measure on G. Then the Fourier Transform $\mathcal{F}\mu$ has the following properties*:

1. $\mathcal{F}\mu(t+is)$ *is continuous in the strip* $S=\{t+is\in\mathbb{C}: s\in\mathbb{R}, -\frac{Q}{2}\leq t\leq\frac{Q}{2}\}$ *and holomorphic in its interior;*
2. $|\mathcal{F}\mu(t+is)|<\mathcal{F}\mu(t),\ \ s\neq 0, -\frac{Q}{2}\leq t\leq\frac{Q}{2}$ *and* $\mathcal{F}\mu(t)<\mathcal{F}\mu(\frac{Q}{2})=1,\ \ -\frac{Q}{2}<t<\frac{Q}{2}$;
3. $\limsup\limits_{s\to\infty}|\mathcal{F}\mu(t+is)|<\mathcal{F}\mu(t),\quad -\frac{Q}{2}\leq t\leq\frac{Q}{2}$.

PROOF.

1) The spherical functions $\phi_z(g)$ are holomorphic functions of $z\in\mathbb{C}$ that are uniformly bounded in the strip S. This on the one hand implies that the Fourier transform of μ is continuous on S and on the other hand, by Cauchy's formula, that also the derivatives of $\phi_z(g)$ are uniformly bounded in any substrip $-\frac{Q}{2}+\epsilon\leq\Re(z)\leq\frac{Q}{2}-\epsilon$. Thus the integral $\int_G|\frac{d^l}{dz}\phi_z(g)|d\mu(g)$ is convergent and this guarantees that the function $\mathcal{F}\mu(z)$ is smooth in the interior of S and that $\frac{d^l}{dz}\mathcal{F}\mu(z)=\int_G\frac{d^l}{dz}\phi_z(g)d\mu$.
2) We will first show that analogous inequalities hold for the spherical functions. This has been proved in [5], but for us it is essential to check that the inequalities are strict. If $s\in\mathbb{R}\backslash\{0\},\ -\frac{Q}{2}<t<\frac{Q}{2}$ and $|g|=r\neq 0$,

$$\begin{aligned}|\phi_{t+is}(g)|&=\left|\int_{S_r}a(y)^{\frac{Q}{2}-t-is}d_{\chi_r}(y)\right|<\\&<\int_{S_r}a(y)^{\frac{Q}{2}-t}d_{\chi_r}(y)<\\&<\left(\int_{S_r}a(y)^{Q}d_{\chi_r}(y)\right)^{\frac{Q/2-t}{Q}}=\\&=\phi_{-\frac{Q}{2}}(g)^{\frac{Q/2-t}{Q}}\ =\ \phi_{\frac{Q}{2}}(g)^{\frac{Q/2-t}{Q}}\ =\ 1,\end{aligned}$$

where the first inequality follows from the passage of the absolute value under the integral and the second one by Jensen's inequality. They are strict because the function $a(y)$ is not constant on S_r, for $r > 0$ and the first inequality also holds for $t = \pm\frac{Q}{2}$. Then 2) follows from the fact that the support of μ is not concentrated at the origin.

3) This is an immediate consequence of the analogous property of the spherical functions which, in turn, follows from the classical Riemann-Lebesgue lemma. $\square$

LEMMA 2. *Let μ be as in the previous lemma and consider the function of real variable $h(s) = \mathcal{F}\mu(is)$. Then*

1. *The first derivative of h vanishes at zero*;
2. *The second derivative of h at zero is strictly negative.*

PROOF. The first statement follows from the symmetry of the spherical functions, namely $\phi_{is}(g) = \phi_{-is}(g)$. By the proof of the previous lemma we have

$$\frac{d^2}{ds}h(0) = \int_G \frac{d^2}{ds}a(y)^{\frac{Q}{2}-is}\bigg|_{s=0} d\mu(y) = -\int_G \ln(a(y))^2 a(y)^{\frac{Q}{2}} d\mu(y)$$

which is clearly nonpositive. It is equal to zero if and only if $a(y) = 1$ μ a.e. and this is not the case since the support of μ is not concentrated at the identity. $\square$

LEMMA 3. *The Harish-Chandra function has the following properties*:

1. *For $s \in \mathbb{R}$ we have*

$$\lim_{n\to+\infty} n|c(is/\sqrt{n})|^{-2} = 4s^2 \left| \frac{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{m+2}{4}\right)}{\Gamma\left(\frac{m+k+1}{2}\right)2^Q} \right|^2 = 4s^2 C_2;$$

2. *$c(z)^{-1}$ is holomorphic in the region $S_Q = \{z \in \mathbb{C} : \Re(z) > -\frac{Q}{2}\}$ and there exists k such that*

$$|c(z)|^{-1} \leq k(1+|z|)^k \quad z \in S_Q.$$

Proof. Both statements are immediate consequence of the definition of c and well known properties of the Gamma function. For instance 1 follows from the fact the $\Gamma(z)$ is holomorphic and different from zero for $\Re(z) > 0$ and that $\frac{1}{\Gamma(2is)} \approx 2is$ for s in a neighborhood of the origin. ∎

– The local limit theorem

Theorem 1. *Let μ be a nonsingular radial probability measure on G and f a continuous function with compact support. Then*

$$\lim_{n\to+\infty} n^{3/2}\mathcal{F}\mu(0)^{-n}\int_G f(g)d\mu^{*n}(g) = C\int_G f(g)\phi_0(g)dg$$

where $C = C_1C_2\int_{\mathbb{R}} s^2\exp(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)})ds$.

Proof. Let $\mathcal{R}$ be the operator of radialization [4], then $\int_G f(g)d\mu^{*n} = \int_G \mathcal{R}f(g)d\mu^{*n}(g)$, so we can suppose that f is radial. Since $C_c^\infty(G)^\#$ is dense in $C_c(G)^\#$ [6], we can suppose that f is also smooth. Then, by the Paley-Wiener theorem, the function f has an integrable Fourier transform and thus, by the Fourier inversion formula, we have

$$\begin{aligned}\mathcal{F}\mu(0)^{-n}n^{3/2}\int_G f(g)d\mu^{*n}(g) =&\mathcal{F}\mu(0)^{-n}n^{3/2}f*\mu^{*n}(e) =\\ &=C_1\mathcal{F}\mu(0)^{-n}n^{3/2}\int_{\mathbb{R}}\mathcal{F}f(is)\mathcal{F}\mu(is)^n|c(is)|^{-2}ds.\end{aligned}$$

Notice that for any positive η

$$\lim_{n\to+\infty} C_1n^{3/2}\int_{|s|>\eta}\left(\frac{\mathcal{F}\mu(is)}{\mathcal{F}\mu(0)}\right)^n\mathcal{F}f(is)|c(is)|^{-2}ds = 0$$

since, by Lemma 1, there exists $0<\epsilon<1$ such that $|\frac{\mathcal{F}\mu(is)}{\mathcal{F}\mu(0)}|^n \le \epsilon^n$, for $|s|\ge\eta$ and the integrand is uniformly dominated by the integrable function $C|\mathcal{F}f(is)(1+|s|^k)|$. Lemma 2 and Taylor's formula give

$$\mathcal{F}\mu(is) = \mathcal{F}\mu(0) + \frac{1}{2}\frac{d^2}{ds^2}\mathcal{F}\mu(0)s^2 + o(s^2), \quad |s|\le\eta,$$

which implies that for $|s| \leq \eta\sqrt{n}$, $|\mathcal{F}\mu(is/\sqrt{n})| \leq \exp(-cs^2/n)$ and

$$\lim_{n\to+\infty}\left(\frac{\mathcal{F}\mu(is/\sqrt{n})}{\mathcal{F}\mu(0)}\right)^n = \lim_{n\to+\infty}\left(1+\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)n}\right)^n = \exp\left(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)}\right).$$

Performing a change of variable $s \to s/\sqrt{n}$ and taking in account Lemma 3 we have

$$\lim_{n\to+\infty} C_1 n^{3/2}\int_{|s|\leq\eta}\left(\frac{\mathcal{F}\mu(is)}{\mathcal{F}\mu(0)}\right)^n \mathcal{F}f(is)|c(is)|^{-2}ds =$$

$$= \lim_{n\to+\infty} C_1\int_{|s|\leq\eta\sqrt{n}}\left(\frac{\mathcal{F}\mu\left(\frac{is}{\sqrt{n}}\right)}{\mathcal{F}\mu(0)}\right)^n \mathcal{F}f\left(\frac{is}{\sqrt{n}}\right)n\left|c\left(\frac{is}{\sqrt{n}}\right)\right|^{-2}ds =$$

$$= C_1C_2\int_{\mathbb{R}}\exp\left(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)}\right)s^2\mathcal{F}f(0)ds =$$

$$= C_1C_2\int_{\mathbb{R}}\exp\left(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)}\right)s^2ds\int_G f(g)\phi_0(g)dg. \qquad \square$$

– **The renewal theorem**

We first recall the classical renewal theorem for the potential of a probability measure on $\mathbb{R}$. Let μ be a nonarithmetic noncentered probability measure on $\mathbb{R}$, define the potential measure $\gamma = \sum_{n=0}^{\infty}\mu^{*n}$ and set $U(f)(x) = \gamma * f(x)$ for $f \in C_c(\mathbb{R})$. Notice that if $\hat{\mu}$ denotes the Euclidean Fourier transform, and f is also smooth, we have, using the Fourier inversion formula, that

$$U(f)(x) = \sum\mu^{*n} * f(x) = \frac{1}{2\pi}\lim_{b\uparrow 1}\int_{\mathbb{R}}\frac{1}{1-b\hat{\mu}(\xi)}\hat{f}(\xi)e^{-i\xi x}d\xi. \tag{1}$$

The asymptotic behavior of the function $U(f)(x)$ is well known [8]:

- If μ does not have first moment, then $\lim_{x\to\pm\infty} Uf(x) = 0$;
- if μ has first moment and its mean m is positive, then $\lim_{x\to-\infty} Uf(x) = \int f(y)dy/m$ and $\lim_{x\to+\infty} Uf(x) = 0$;

- if μ has first moment and its mean m is negative, then $\lim_{x\to+\infty} Uf(x) = \int f(y)dy/m$ and $\lim_{x\to-\infty} Uf(x) = 0$.

We consider the map $\pi_2 : G \to \mathbb{R}$ that sends $g = na(g)$ to $\ln(a(g))$. This induces a map on the space of measures by setting $\pi_2(\nu)(B) = \nu(\pi_2^{-1}(B))$ for any Borel set in $\mathbb{R}$. In the following we will denote by μ_A the real measure $\pi_2(\mu)$. Notice that $\mathcal{F}(\mu)(\frac{Q}{2} + is) = \hat{\mu}_A(-s)$. In particular if μ is a nonsingular probability measure on G we have, by Lemma 1, that μ_A is a nonarithmetic probability measure on $\mathbb{R}$. We say that μ has first moment if $\int_G |\ln(a(g))|d\mu(g) < +\infty$ and, if this is the case, we call $m = \int_G \ln(a(g))d\mu(g) > 0$ the mean of μ. Obviously μ has first moment if and only if μ_A has (classical) first moment and if this is the case the mean of μ coincides with the (classical) mean of μ_A.

Theorem 2. *Let μ be a nonsingular radial probability measure on G. Then if μ has mean m*

$$\lim_{r\to+\infty} e^{Qr} \sum_{n=0}^{+\infty} \mu^{*n} * f(r) = \frac{4\pi C_1 \int_G f dg}{c\left(\frac{Q}{2}\right) m} \qquad \forall\, f \in C_c(G)^{\#},$$

while the above limit is zero if μ does not have first moment.

Proof. By the density of $C_c^\infty(G)^{\#}$ in $C_c(G)^{\#}$ we can suppose f smooth. Then the Fourier inversion formula gives

$$e^{Qr}\sum_{n=0}^{+\infty}\mu^{*n} * f(r) = C_1 e^{Qr} \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1 - \mathcal{F}\mu(is)} \phi_{is}(r)|c(is)|^{-2} ds \tag{2}$$

Recalling the asymptotic expansion of the spherical functions [2]

$$\phi_{is}(r) = c(is) \sum_{l=0}^{+\infty} \Gamma_l(is) e^{(is-l-\frac{Q}{2})r} + c(-is) \sum_{l=0}^{+\infty} \Gamma_l(-is) e^{(-is-l-\frac{Q}{2})r}$$

where $\Gamma_0 \equiv 1$ and $\Gamma_l(\cdot)$ are holomorphic functions on $\{z \in \mathbb{C} : \Re(z) < \frac{1}{2}\}$ that satisfy the estimates [2]

$$\sup_{\Re(z)\le 0} |\Gamma_l(z)| \le d(1+l)^d \tag{3}$$

for some constant d independent of z. Then, since $|c(is)|^2 = c(is)c(-is)$, (2) equals

$$C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(-is)^{-1} \sum_{l=0}^{+\infty} \Gamma_l(is) e^{(is-l+\frac{Q}{2})r} ds+$$

$$+ C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(is)^{-1} \sum_{l=0}^{+\infty} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds.$$

The change of variable $is \to -is$ in the first integral allows us to write the above sum as

$$2C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(is)^{-1} \sum_{l=0}^{+\infty} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds.$$

Note that, by (3), $\lim_{r\to+\infty} \sum_{l=\frac{Q}{2}+1}^{+\infty} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} = 0$ for all $s \in \mathbb{R}$ and thus, applying the Lebesgue dominated convergence theorem, we are left to estimate

$$\begin{aligned} &\lim_{r\to+\infty} 2C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(is)^{-1} \sum_{l=0}^{\frac{Q}{2}} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds = \\ &= 2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-b\mathcal{F}\mu(is)} c(is)^{-1} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds \end{aligned} \tag{4}$$

If $0 < b < 1$ the function $F_l(z) = \frac{\mathcal{F}f(z)}{1-\mathcal{F}\mu(z)} c(z)^{-1} \Gamma_l(-z) e^{(-z-l+\frac{Q}{2})r}$ is holomorphic in $\{z \in \mathbb{C} : 0 < \Re(z) < \frac{Q}{2}\}$ and continuous on its closure. Since $F_l(t+is)$ is rapidly decreasing for $s \to \infty$, we can use the Cauchy integral formula to shift the contour of integration obtaining that (4) is equal to

$$2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{\mathbb{R}} \frac{\mathcal{F}f\left(is+\frac{Q}{2}\right)}{1-b\mathcal{F}\mu\left(is+\frac{Q}{2}\right)} c\left(is+\frac{Q}{2}\right)^{-1} \Gamma_l\left(-is-\frac{Q}{2}\right) e^{(-is-l)r} ds.$$

If $|s| > \eta$ for a fixed positive number η the quantity $|1 - b\mathcal{F}\mu(is+\frac{Q}{2})| = |1 - b\hat{\mu_A}(-s)|$ is bounded from below uniformly in b and thus, by the classical Riemann Lebesgue lemma, we are reduced to estimate

$$
\begin{aligned}
(5)\qquad & 2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{|s|\le\eta} \frac{\mathcal{F}f\left(is+\dfrac{Q}{2}\right)}{1-b\hat{\mu_A}(-s)} c\left(is+\frac{Q}{2}\right)^{-1} \times \\
& \times \Gamma_l\left(-is-\frac{Q}{2}\right) e^{(-is-l)r} ds
\end{aligned}
$$

The function $g_l(s) = \mathcal{F}f(is+\frac{Q}{2})c(is+\frac{Q}{2})^{-1}\Gamma_l(-is-\frac{Q}{2})$, for η sufficiently small, can be written, by Taylor's formula, as

$$
(6)\qquad g_l(s) = g_l(0) + \frac{d}{ds} g_l(0)s + s^2 M_l(s), \quad |s| \le \eta
$$

where $M_l(s)$ are bounded. On the other hand $|1 - b\hat{\mu_A}(is)| \ge cb|s|^2$, $\forall |s| \le \eta$ and thus, using again the Riemann Lebesgue lemma, we have

$$
(7)\qquad \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{|s|\le\eta} \frac{s^2 M(s)}{1-b\hat{\mu_A}(is)} e^{(-is-l)r} ds = 0.
$$

We can find smooth functions h_l with compact support whose Fourier transforms satisfy

$$
\hat{h_l}(-s) = g_l(0) + \frac{d}{ds} g_l(0)s + o(s), \quad |s| \le \eta,
$$

so that, taking in account (6) and (7), we have that (5) is equal to

$$
\begin{aligned}
(8)\qquad & 2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{|s|\le\eta} \frac{\hat{h_l}(-s)}{1-b\hat{\mu_A}(-s)} e^{(-is-l)r} ds = \\
& = 4\pi C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} e^{-lr} \lim_{b\uparrow 1} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{h_l}(s)}{1-b\hat{\mu_A}(s)} e^{isr} ds.
\end{aligned}
$$

By (1) the limit in b is nothing but the potential $Uh_l(-r)$ associated with the measure μ_A. By the classical renewal theorem we have that if μ and thus μ_A, does not have first moment then

$$
\lim_{r\to+\infty} e^{-rl} U h_l(-r) = 0, \quad \forall\, l = 0, 1, \dots, \frac{Q}{2}.
$$

If the measure μ and thus μ_A has mean m then the above limit is zero for all $l \neq 0$ while

$$\lim_{r\to+\infty} Uh_0(-r) = \frac{\int_{\mathbb{R}} h_0(x)dx}{m} = \frac{\hat{h}_0(0)}{m} =$$

$$= \frac{g_0(0)}{m} = \frac{\mathcal{F}\mu\left(\frac{Q}{2}\right)\Gamma_0\left(-\frac{Q}{2}\right)c\left(\frac{Q}{2}\right)^{-1}}{m} = \frac{\int_G f(g)dg}{c\left(\frac{Q}{2}\right)m}$$

which, in virtue of (8), concludes the proof of the theorem. ∎

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Lavoro pervenuto alla redazione il 4 settembre 2002
ed accettato per la pubblicazione il 9 giugno 2003.
Bozze licenziate il 4 settembre 2003

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