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A kinetic approach to studying the asymptotic behaviour of convection-diffusion equations

R. CAVAZZONI

Abstract: *We present a new approach to the study of the large time behaviour of solutions to the Cauchy problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - u^{p-1}\dfrac{\partial u}{\partial x} & x \in \mathbb{R}, \qquad t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases}$$

where $p \geq 2$ and $u_0(x) \geq 0$.

1 – Introduction and main results

The aim of this paper is to show how kinetic methods can be used in the study of the long time behaviour of the solution to Cauchy problems for convection-diffusion equations having the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - u^{p-1}\dfrac{\partial u}{\partial x} & x \in \mathbb{R}, \qquad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

where $p \geq 2$ and u_0 is a nonnegative function from $L^1(\mathbb{R})$. More precisely,

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we prove that there exists a function u_∞ such that, for every $q \in [1, \infty)$,

$$\lim_{t \to +\infty} (2t+1)^{1/2(1-1/q)} \, \|u(\cdot, t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R})} = 0. \tag{2}$$

In 1950, Hopf [7] found an explicit regular solution to the viscous Burgers equation, i.e. equation (1) with $p = 2$. His proof makes use of the so called Hopf-Cole transformation, which turns the viscous Burgers equation into a linear heat equation. More recently, useful estimates on the L^r-norm of the solution to (1) have been derived (see [11] or [5]). In particular, the $L^r(\mathbb{R})$ to $L^q(\mathbb{R})$ smoothing properties of (1) have been shown to be exactly the same as the standard heat equation. In 1987, Chern and Liu [4] studied the large time behaviour of solutions of viscous conservation laws. In 1991, Escobedo and Zuazua [5] analysed the large time behaviour of solutions to the Cauchy problem for convection-diffusion equations. The main result of [5] tells us that if $p = 2$, then the general solution $u = u(x, t)$ to (1) behaves like the self-similar solution as $t \to +\infty$.

In the case where $p > 2$, it is proved in [5] that for every $r \in [1, \infty]$

$$\| u(\cdot, t) - G(\cdot, t) \|_{L^r} \longrightarrow 0 \ \text{ as } \ t \to +\infty, \tag{3}$$

where G is the heat kernel. These results have been obtained by a direct application of standard estimates for the heat kernel and by decay estimates in the integral equation associated with (1).

Related results on the long time behaviour of nonnegative solutions of nonlinear diffusion equations are contained e.g. in [6], [8], [11], [12], [13].

Our main purpose here is the use of a completely different approach to investigate the asymptotic behaviour of diffusion equations. The underlying idea of our approach is derived from the H-theorem of kinetic theory of rarefied gases [3]. In the last years the derivation of diffusion equations as a hydrodynamic limit of particles models has been a well studied subject in kinetic theory [3]. In this paper we shall look for a suitable functional to describe the evolution of the solution to problem (1), in a similar way as for the solution of the Boltzmann equation (see [3] and references cited in [3]). The kinetic approach has been recently applied by Carrillo and Toscani [2] for the N-dimensional porous medium equation: they have obtained the rate of convergence to equilibrium, by an analysis of the time evolution of the entropy production.

In the present paper we consider the asymptotic behaviour of the solution to (1) by techniques different from those of [5]. In particular, we do not make use of L^q-estimates for the derivatives of the solution. We study separately the cases $p = 2$ and $p > 2$. In the case where $p = 2$, we construct a suitable functional to prove the convergence to equilibrium by using the monotonicity in time of the functional. The result on the large time behaviour of equation (1) is proved in the following theorem.

THEOREM 1.1. *Assume that* $p = 2$ *and*

$$\begin{aligned} &u_0 \in L^1(\mathbb{R}), \qquad \int_{\mathbb{R}} u_0(x)dx = 1; \\ &u_0(x) \geq 0 \quad a.e. \quad x \in \mathbb{R}. \end{aligned} \tag{4}$$

Let u *be the solution to the Cauchy problem* (1)*. Then for every* $q \in [1, \infty)$

$$\lim_{t \to +\infty} (2t+1)^{1/2(1-1/q)} \|u(\cdot, t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R})} = 0,$$

where

$$u_\infty(x, 2t+1) = \frac{1}{(2t+1)^{1/2}} \frac{e^{-\frac{x^2}{2(2t+1)}}}{(2\pi)^{\frac{1}{2}} A - \frac{1}{2} \int_{-\infty}^{\frac{x}{(2t+1)^{1/2}}} e^{-y^2/2} dy},$$

and $A = \frac{e^{1/2}}{2(e^{1/2}-1)}$.

A variant of the techniques involved in the proof of Theorem 1.1 enables us to describe the large time behaviour of the solution to (1) when $p > 2$. Indeed, we perform the same time dependent scaling. Let us emphasize that the equation obtained after scaling has coefficients depending on t. The convex functional, which represents the physical entropy for the viscous Burgers equation, will be used to study the large time behaviour also for $p > 2$. However, in this case the functional is not monotone in time. Nevertheless, the specific form of the time derivative allows us to identify the limit as a stationary solution to the Fokker-Planck equation.

Theorem 1.2. *Assume that $p > 2$ and*

$$u_0 \in L^1(\mathbb{R}), \qquad \int_{\mathbb{R}} u_0(x)dx = 1; \qquad (5)$$
$$u_0(x) \geq 0 \quad a.e. \quad x \in \mathbb{R}.$$

Let u be the solution to the Cauchy problem (1)*. Then for every $q \in [1, \infty)$*

$$\lim_{t \to +\infty} (2t+1)^{1/2(1-1/q)} \|u(\cdot, t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R})} = 0,$$

where

$$u_\infty(x, 2t+1) = \frac{1}{(2t+1)^{1/2}} \gamma \exp\left(-\frac{x^2}{2(2t+1)}\right),$$

and $\gamma = (\int_{\mathbb{R}} \exp(-\frac{x^2}{2})dx)^{-1}$.

The paper is organized as follows. In Section 2, we recall some known results on the initial value problem (1) and we derive some bounds on the solution. Section 3 is devoted to the study of the functional to which we alluded above and to the proof of convergence results. In Section 4, we describe the asymptotic behaviour of (1) when $p \geq 2$, and conclude the proofs of Theorems 1.1 and 1.2. In the last section, we prove that our method can be applied to study the long time behaviour of the solution to a class of convection-diffusion equations in $\mathbb{R}^N$, with $N > 1$.

2 – Preliminaries

In the present section we recall some known results on the solution to the Cauchy problem (1) and we prove some technical Lemmas.

Let us consider the Cauchy problem (1). Thanks to the results by Hopf [7], we know that the viscous Burgers equation (i.e. equation (1) with $p = 2$) admits the following regular solution.

Theorem 2.1. *Let $u_0 \in L^1(\mathbb{R})$. Then*

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\, d\eta\right]\right\} dy}{\displaystyle\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\, d\eta\right]\right\} dy}, \qquad (6)$$

is a solution to the viscous Burgers equation for $t > 0$ *and satisfies the initial condition:*

$$\int_0^x u(\xi, t)\, d\xi \longrightarrow \int_0^a u_0(\xi)\, d\xi \quad \textit{as} \quad x \to a \quad \textit{and}\, t \to 0; \tag{7}$$

for every $a \in \mathbb{R}$*. If* $u_0 \in C(\mathbb{R})$*, then* $u(x,t) \longrightarrow u_0(a)$ *if* $x \to a,\ t \to 0$*. Moreover, given any* $T > 0$*, the function* u *defined by* (6) *is the unique regular solution to the viscous Burgers equation in the strip* $0 < t < T$*, satisfying* (7) *for every* $a \in \mathbb{R}$*.*

The solution to (1) given by (6) belongs to the space $C((0,\infty); L^1(\mathbb{R}))$. We shall make use of the following result of [5], on the existence of the solution together with decay rates, for the initial value problem (1).

Theorem 2.2. *Given* $u_0 \in L^1(\mathbb{R})$*, there exists a unique classical solution* $u \in C([0,\infty); L^1(\mathbb{R}))$ *to* (1)*, which satisfies the following properties:*

(i) *for every* $q \in (1,\infty)$*,* $\quad u \in C((0,\infty); W^{2,q}(\mathbb{R})) \bigcap C^1((0,\infty); L^q(\mathbb{R}))$.

(ii) *For every* $q \in [1,\infty)$*, there exists a constant* $C_q = C(q, \|u_0\|_1)$ *such that for every* $t > 0$*:*

$$\begin{cases} \|u(t)\|_q \le C_q t^{-1/2(1-1/q)}, \\ \|u(t)\|_1 \le \|u_0\|_1. \end{cases} \tag{8}$$

(iii) *Let* t_0 *be a nonnegative real number. Then there exists a positive constant* C_∞ *such that for every* $t \ge t_0$*:*

$$\|u(t)\|_\infty \le C_\infty t^{-1/2}. \tag{9}$$

If $p = 2$*, then*

$$\|u(t)\|_\infty \le C_\infty t^{-1/2},$$

for every $t > 0$*.*

One can easily verify that the unique solution in $C([0,\infty); L^1(\mathbb{R}))$ provided by Theorem 2.2 satisfies condition (6) of Theorem 2.1. Then the unique solution to the viscous Burgers equation in $C([0,\infty); L^1(\mathbb{R}))$ is the function given by (6).

Remark 2.1. Integrating equation (1) over all of $\mathbb{R}$, we obtain that the total mass of solutions is preserved for every $t > 0$:

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx. \tag{10}$$

With no loss of generality we assume that $\displaystyle\int_{\mathbb{R}} u_0(x)dx = 1$.

Lemma 2.1. *Consider equation* (1) *with* $p = 2$. *Let* u *be defined by* (6).

Then a real constant δ *exists such that*

$$u(x,t) \geq \frac{e^{-1/2}}{2t^{1/2}} \exp\left(-\frac{\delta^2}{2t}\right) \exp\left(-\frac{x^2}{2t}\right), \tag{11}$$

for every $t > 0$.

Proof. Since $\int_{\mathbb{R}} u_0(y)dy = 1$, there exists a compact interval $I \subset \mathbb{R}$ such that $\int_I u_0(y)dy \geq \frac{1}{2}$.

We have

$$\begin{aligned} &\int_{-\infty}^{+\infty} u_0(y) \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\,d\eta\right]\right\} dy \geq \\ &\geq \int_I u_0(y) \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\,d\eta\right]\right\} dy. \end{aligned} \tag{12}$$

Set $I = [-\delta, \delta]$, with $\delta \in \mathbb{R}$.

Then:

$$\begin{aligned} &\int_I u_0(y) \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\,d\eta\right]\right\} dy \geq \\ &\geq \exp\left(-\frac{1}{2}\right) \int_I u_0(y) \exp\left\{-\frac{1}{2}\frac{(x-y)^2}{2t}\right\} dy \geq \\ &\geq \frac{1}{2} \exp\left(-\frac{1}{2} - \frac{\delta^2}{2t}\right) \exp\left(-\frac{x^2}{2t}\right). \end{aligned} \tag{13}$$

Thus:

$$u(x,t) \geq \frac{A}{t^{1/2}} \exp\left(-\frac{\delta^2}{2t}\right) \exp\left(-\frac{x^2}{2t}\right); \tag{14}$$

where $A = \frac{1}{2}\exp(-\frac{1}{2})$. ∎

In the case where $p > 2$, we derive a similar estimate under an additional assumption on the initial value.

Lemma 2.2. *Let $p > 2$. Assume that $u_0(x) \geq M \exp(-\frac{|x|^2}{2})$ for some positive constant M and for a.e. $x \in \mathbb{R}$. Then there exist positive constants B and C such that*

$$u(x,t) \geq \frac{Be^{-C}}{t^{1/2}} \exp\left(\frac{C}{t^{\frac{p-2}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right), \tag{15}$$

for every $t > 1$.

Proof. Let us define $f : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ as

$$f(x,t) = \frac{M}{(2(t+1))^{1/2}} \exp(-\alpha t) \exp\left(-\frac{|x|^2}{2(t+1)}\right),$$

for $(x,t) \in \mathbb{R} \times [0, +\infty)$, where α is a positive constant to be chosen later. We prove that the function f is a subsolution to the equation (1) in $\mathbb{R} \times (0,1)$. We have

$$f(x,0) \leq u_0(x),$$

for every $x \in \mathbb{R}$. It is not difficult to see that

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - f^{p-1}\frac{\partial f}{\partial x} \leq \left(\frac{1}{2(t+1)} - \alpha + \beta e^{-(p-1)\alpha t}\frac{M^{p-1}}{(t+1)^{p/2}}\right) f; \tag{16}$$

where $\beta = 2^{\frac{1}{2}} \max_{z\in\mathbb{R}}[-\sqrt{2}ze^{-z^2(p-1)}]$. Then, if $\alpha > \frac{1}{2} + \beta M^{p-1}$,

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - f^{p-1}\frac{\partial f}{\partial x} \leq 0,$$

for $(x,t) \in \mathbb{R} \times (0,1)$. As a consequence of the comparison principle proved in [10], we obtain that for every $x \in \mathbb{R}$, $u(x,1) \geq B\exp(-\frac{|x|^2}{4})$, with $B = \frac{M}{2}e^{-\alpha}$.

Let us define $g : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ as

$$g(x,t) = \frac{Be^{-C}}{t^{1/2}} \exp\left(\frac{C}{t^{\frac{p-2}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right),$$

where the positive constant C will be fixed later and $(x,t) \in \mathbb{R} \times [1, \infty)$. We have that for every $x \in \mathbb{R}$, $u(x,1) \geq g(x,1)$. Moreover,

$$\begin{aligned} &\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} - g^{p-1}\frac{\partial g}{\partial x} = \\ &= \left(-\frac{C(p-2)}{2t^{\frac{p-2}{2}+1}} - \frac{x}{2t}\left(\frac{Be^{-C}}{t^{1/2}} \exp\left(\frac{C}{t^{\frac{p-2}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right)\right)^{p-1}\right) g \leq \\ &\leq \left(-\frac{C(p-2)}{2t^{\frac{p}{2}}} + \frac{B^{p-1}\bar{\beta}}{t^{p/2}}\right) g; \end{aligned} \tag{17}$$

where $\bar{\beta} = 2^{\frac{p-2}{2}}\beta$.

If we choose C sufficiently large, then we obtain the last expression is negative.

Therefore, due to the comparison principle, inequality (15) follows. ∎

3 – Convergence results

We now look for solutions to (1) having the form:

$$u(x,t) = \frac{1}{R(t)} v\left(\frac{x}{R(t)}, L(t)\right) = \frac{1}{R(t)} v(y, \tau); \tag{18}$$

where $R(t), L(t)$ are unknown functions. Let us impose that the previous function satisfies (1) and determine what functions $R(t), L(t)$ are admissible, in such a way that the initial values of u and v are the same.

The time-dependent scaling and the use of a suitable functional are the main novelty of our approach. Instead of working directly with equation (1), we analyse the asymptotic behaviour of the solution v to the problem

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial v}{\partial x} + yv - e^{-(p-2)\tau} v^p\right] & y \in \mathbb{R}, \quad \tau > 0, \\ v(y,0) = v_0(y) \quad y \in \mathbb{R}. \end{cases} \tag{19}$$

Indeed, it is easily seen that the solutions u and v to problems (1) and (19), respectively, are related by

$$\begin{aligned} u(x,t) &= \frac{1}{(2t+1)^{1/2}} v\left(\frac{x}{(2t+1)^{1/2}}, \log(2t+1)^{1/2}\right), \\ v(y,\tau) &= e^{\tau} u\left(ye^{\tau}, \frac{e^{2\tau}-1}{2}\right). \end{aligned} \tag{20}$$

Let us notice that $u_0 = v_0$.

We shall prove below convergence results on studying separately the following two cases: $p = 2$ and $p > 2$.

In Section 5, we shall extend the results for a class of convection-diffusion equations in $\mathbb{R}^N$.

3.1 – Case $p = 2$

We introduce a suitable functional for the Fokker-Planck type equation (19). We shall prove the time monotonicity of the functional and its decay to zero as $\tau \to +\infty$, in order to study the asymptotic decay to a fixed equilibrium state v_∞ of the solution to (19).

We first derive the equilibrium state v_∞, by looking for a stationary solution to (19):

$$\frac{\partial v_\infty}{\partial y} + y v_\infty - v_\infty^2 = 0, \tag{21}$$

i.e.:

$$v_\infty(y) = \frac{e^{-y^2/2}}{(2\pi)^{\frac{1}{2}} \dfrac{e^{1/2}}{2\left(e^{1/2}-1\right)} - \dfrac{1}{2}\displaystyle\int_{-\infty}^{y} e^{-s^2/2} ds}, \tag{22}$$

where v_∞ is positive and $\displaystyle\int_{\mathbb{R}} v_\infty(y)dy = 1$.

Notice that $v_\infty = v_1$, where, in accordance with the result proved in [1], v_1 is the unique self-similar solution to (1) with a smooth profile verifying

$$\frac{\partial}{\partial x'}\left[\frac{\partial v_1}{\partial x'} + x' v_1 - v_1^2\right] = 0. \tag{23}$$

Since $u \in C\left((0,\infty); W^{2,q}(\mathbb{R})\right) \bigcap C^1\left((0,\infty); L^q(\mathbb{R})\right)$ for every $q \in (1,\infty)$, by Theorem 2.2, the solution v to the Cauchy problem (19), belongs to the same class.

Moreover, $\|v(\cdot,\tau)\|_\infty \leq C_\infty$ for every $\tau > 0$. Thanks to Lemma 2.1, we have, as $\tau > 0$,

$$\tag{24} \begin{cases} \text{(P1)} \quad \displaystyle\int_{\mathbb{R}} v(y,\tau)dy = 1; \\ \text{(P2)} \quad v(y,\tau) \geq \dfrac{2^{\frac{1}{2}} e^{\tau} e^{-1/2}}{2\left(e^{2\tau}-1\right)^{\frac{1}{2}}} \exp\left(-\dfrac{2\delta^2}{e^{2\tau}-1}\right) \exp\left(-\dfrac{y^2 e^{2\tau}}{e^{2\tau}-1}\right). \end{cases}$$

Let us prove the following preliminary result.

Lemma 3.1. *Let v be the solution to* (19). *Then*

$$\tag{25} \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{v_\infty(y)} - 1 - \log \frac{v(y,\tau)}{v_\infty(y)}\right) e^{-y^2/2} dy < +\infty,$$

for every $\tau > 0$.

Proof. Thanks to (P2), fixed any $\delta > 0$, we have:

$$\tag{26} \frac{v(y,\tau)}{v_\infty(y)} \geq C(\delta) \exp\left[\left(-\frac{e^{2\tau}}{e^{2\tau}-1} + \frac{1}{2}\right) y^2\right] = C(\delta)\exp(-\gamma y^2),$$

for every $\tau > \delta > 0$, for some positive constant γ. Moreover,

$$\tag{27} -\log \frac{v(y,\tau)}{v_\infty(y)} < -\log\left(C(\delta)\exp(-\gamma y^2)\right) < \gamma y^2 - \log C(\delta).$$

Thus,

$$\tag{28} \begin{aligned} 0 &\leq \left(\frac{v(y,\tau)}{v_\infty(y)} - 1 - \log \frac{v(y,\tau)}{v_\infty(y)}\right) e^{-y^2/2} < \\ &< \bar{C} v(y,\tau) + \left(-1 + \gamma y^2 - \log C(\delta)\right) e^{-y^2/2}, \end{aligned}$$

where $\bar{C}$ is a positive constant. Thus (25) follows. $\square$

Let us introduce now the following functional, which represents the physical relative entropy for the viscous Burgers equation.

Definition 3.1. For every solution v to (19), let $L : \mathbb{R}^+ \to \mathbb{R}$ be defined as

$$L(\tau) = \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{v_\infty(y)} - 1 - \log \frac{v(y,\tau)}{v_\infty(y)} \right) e^{-y^2/2} dy. \tag{29}$$

The main property of L in connection with our discussion is that L is a monotone non increasing function of τ when v is the solution to (19).

Lemma 3.2. *Let v be the solution to* (19)*. Then for every $\tau > 0$,*

$$\lim_{|y|\to+\infty} \left(\frac{1}{\tilde{v}_\infty} - \frac{1}{\tilde{v}} \right) \left(\frac{\partial v}{\partial y} + yv - v^2 \right) = 0, \tag{30}$$

where $\tilde{v}_\infty(y) = e^{y^2/2} v_\infty(y)$ and $\tilde{v}(y,\tau) = e^{y^2/2} v(y,\tau)$.

Proof.
Let us divide the proof in two steps.
1) We prove that

$$\lim_{|y|\to+\infty} \frac{1}{\tilde{v}_\infty} \left(\frac{\partial v}{\partial y} + yv - v^2 \right) = 0. \tag{31}$$

Since $v(\cdot,\tau) \in W^{2,q}(\mathbb{R})$, for every $\tau > 0$ and for every q, with $1 < q < \infty$, then:

$$\lim_{|y|\to+\infty} v(y,\tau) = \lim_{|y|\to+\infty} \frac{\partial v}{\partial y}(y,\tau) = 0.$$

Moreover, $\tilde{v}_\infty^{-1}$ is bounded. Thanks to formula

$$u(t) = G(t) * u_0 - \int_0^t \nabla G(t-s) * u^2(s) ds, \tag{32}$$

we have

$$\lim_{|x|\to+\infty} xu(x,t) = 0,$$

for $t > 0$.

Hence,

$$\lim_{|y|\to+\infty} yv(y,\tau) = 0.$$

2) Let us prove now that

$$\lim_{|y|\to+\infty} \frac{1}{\tilde{v}} \left(\frac{\partial v}{\partial y} + yv - v^2 \right) = 0. \tag{33}$$

Making use of the previous integral formula (32) for the solution to (1) yields $\frac{\partial u}{\partial x}$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} -\frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} u_0(y)dy + \\ &+ \int_0^t \frac{1}{4\pi(t-s)} \int_{\mathbb{R}} \frac{1}{2(t-s)} e^{-\frac{(x-y)^2}{4(t-s)}} u^2(y,s)ds + \\ &- \int_0^t \frac{1}{4\pi(t-s)} \int_{\mathbb{R}} \frac{(x-y)^2}{4(t-s)^2} e^{-\frac{(x-y)^2}{4(t-s)}} u^2(y,s)ds. \end{aligned} \tag{34}$$

Owing to the lower bound of Lemma 2.1, we have that

$$\frac{1}{u} \left| \frac{\partial u}{\partial x} \right| \exp\left(-\frac{x^2}{2(2t+1)} \right) \le \left| \frac{\partial u}{\partial x} \right| t^{1/2} e^{1/t} \exp\left(\frac{x^2}{4t(2t+1)} \right). \tag{35}$$

Therefore,

$$\lim_{|x|\to+\infty} \frac{1}{u} \left| \frac{\partial u}{\partial x} \right| \exp\left(-\frac{x^2}{2(2t+1)} \right) = 0. \tag{36}$$

Thus, on performing the time dependent scaling, we get the conclusion. □

Let us now prove the time monotonicity of the relative entropy L.

LEMMA 3.3. *Let v be the solution to* (19) *and let L be defined by* (29). *Then*

$$\frac{d}{d\tau} L(\tau) \le 0,$$

for $\tau > 1$.

Proof. It is easily seen from formula (32) that $\frac{\partial v}{\partial \tau} \in L^1(\mathbb{R})$.
Integration by parts yields:

$$
\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}}\left(\frac{v}{v_\infty}-1-\log\frac{v}{v_\infty}\right)e^{-y^2/2}dy =\\
&=\int_{\mathbb{R}}\left(\frac{1}{v_\infty}-\frac{1}{v}\right)\frac{\partial v}{\partial \tau}e^{-y^2/2}dy =\\
&=\int_{\mathbb{R}}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\frac{\partial \tilde{v}}{\partial \tau}e^{-y^2/2}dy =\\
&=\int_{\mathbb{R}}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\frac{\partial}{\partial y}\left[\tilde{v}^2e^{-y^2/2}\frac{\partial}{\partial y}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\right]dy =\\
&=\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\left(\frac{\partial v}{\partial y}+yv-v^2\right)\Big|_{-\infty}^{+\infty}+\\
&\quad-\int_{\mathbb{R}}\tilde{v}^2e^{-y^2/2}\left(\frac{\partial}{\partial y}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\right)^2dy.
\end{aligned}
\tag{37}
$$

Thus, thanks to Lemma 3.2, we have

$$
\frac{dL}{d\tau}=-\int_{\mathbb{R}}\tilde{v}^2e^{-y^2/2}\left(\frac{\partial}{\partial y}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\right)^2dy\le 0,
\tag{38}
$$

for $\tau > 1$. ∎

Let I be the function from $\mathbb{R}^+$ into $\mathbb{R}$ given by

$$
I(\tau)=-\frac{d}{d\tau}L(\tau).
\tag{39}
$$

Remark 3.1. Thanks to Lemma 3.3,

$$
\int_\delta^{+\infty} I(s)ds = L(\delta)-L(\infty) < +\infty,
$$

for any $\delta > 1$. Since $I(\tau) \geq 0$ for $\tau > 1$, then there exists a sequence $\tau_k \to +\infty$ such that $I(\tau_k) \to 0$ as $k \to +\infty$.

Let us define $v_k : \mathbb{R} \to \mathbb{R}$ as

$$v_k(y) = v(y, \tau_k), \quad k \in N.$$

PROPOSITION 3.1. *A real constant c exists such that if w_∞ is the function defined by $\frac{1}{w_\infty} = \frac{1}{v_\infty} + c$, then the sequence of functions $(v_k)_{k\in N}$ converges a.e. in $\mathbb{R}$ to w_∞ as $k \to +\infty$.*

PROOF. Thanks to the previous remark and to (P2)

$$\lim_{k\to+\infty} \int_{\mathbb{R}} e^{-2y^2} \left(\frac{\partial}{\partial y} \left(\frac{1}{\tilde{v}_\infty} - \frac{1}{\tilde{v}_k} \right) \right)^2 dy = 0.$$

Consequently,

$$\frac{\partial}{\partial y} \left(\frac{1}{\tilde{v}_k} - \frac{1}{\tilde{v}_\infty} \right) \longrightarrow 0, \text{ strongly in } L^2_{\mathrm{loc}}(\mathbb{R}).$$

Since $\frac{1}{\tilde{v}_k}$ is a bounded sequence of functions in $L^2_{\mathrm{loc}}(\mathbb{R})$, then there exists a function w_∞ such that

$$\frac{1}{\tilde{v}_k} \longrightarrow \frac{1}{\tilde{w}_\infty}, \text{ strongly in } W^{1,2}_{\mathrm{loc}}(\mathbb{R});$$

as $k \to +\infty$ and

$$\frac{\partial}{\partial y} \left(\frac{1}{\tilde{w}_\infty} \right) = \frac{\partial}{\partial y} \left(\frac{1}{\tilde{v}_\infty} \right), \ a.e.\, y \in \mathbb{R}.$$

Hence,

$$\frac{1}{\tilde{w}_\infty} = \frac{1}{\tilde{v}_\infty} + c;$$

for some constant $c \in \mathbb{R}$; therefore

$$w_\infty(y) = \frac{e^{-\frac{y^2}{2}}}{c + (2\pi)^{\frac{1}{2}} \dfrac{e^{1/2}}{2\,(e^{1/2} - 1)} - \dfrac{1}{2} \displaystyle\int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds}. \tag{40}$$

Since $v_k > 0$, we have $w_\infty > 0$. ∎

3.2 – Case $p > 2$

After performing the time-dependent scaling (20) in equation (1) as $p > 2$, we study the large time behaviour of the solution to (1).

As a consequence of the results in Theorem 2.2, we have that $\|v(\cdot,\tau)\|_{L^\infty} \leq C_\infty$, for every $\tau \geq \tau_0$ and thanks to Lemma 2.2,

$$(41)\quad \begin{cases} (\mathrm{P1}') & \displaystyle\int_{\mathbb{R}} v(y,\tau)dy = 1; \\ (\mathrm{P2}') & v(y,\tau) \geq \dfrac{2^{\frac{1}{2}} e^{\tau} B e^{-C}}{(e^{2\tau}-1)^{\frac{1}{2}}} \exp\left(\dfrac{2^{(p-2)/2}C}{(e^{2\tau}-1)^{(p-2)/2}}\right) \exp\left(-\dfrac{|y|^2 e^{2\tau}}{e^{2\tau}-1}\right); \end{cases}$$

where the constants B, C have been defined in Lemma 2.2.

We shall prove in this case that the large time behaviour of the solution v is determined by the following equation:

$$(42)\qquad \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial y}(yv).$$

We refer to (42) as the Fokker-Planck equation associated to (1) in the case where $p > 2$.

A stationary solution to (42) is given by the function:

$$(43)\qquad \bar{v}_\infty(y) = \gamma \exp\left(-\frac{y^2}{2}\right),$$

where γ is a constant. Let us choose γ in such a way

$$(44)\qquad \int_{\mathbb{R}} \gamma \exp\left(-\frac{y^2}{2}\right) dy = 1.$$

In the present section we prove that the large time behaviour of (1) is given by the function $\bar{v}_\infty$.

Similarly as the case of the viscous Burgers equation, we define the following convex nonnegative functional.

Definition 3.2. Let v be the solution to (19). Let L be the function from $\mathbb{R}^+$ into $\mathbb{R}$ given by

$$(45)\qquad L(\tau) = \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{\bar{v}_\infty(y)} - 1 - \log\frac{v(y,\tau)}{\bar{v}_\infty(y)}\right) e^{-|y|^2/2} dy.$$

Lemma 3.4. *If v is a solution to* (19), *then*

$$L(\tau) < +\infty.$$

Moreover

Lemma 3.5. *Let v be the solution to* (19). *Then*

$$\lim_{|y|\to+\infty}\left(\frac{1}{\bar v_\infty}-\frac{1}{v}\right)\left(\frac{\partial v}{\partial y}+yv-e^{-(p-2)\tau}v^p\right)e^{-|y|^2/2}=0, \tag{46}$$

for every $\tau > 0$.

The proofs of Lemmas 3.4 and 3.5 follow the same lines as those of Lemmas 3.1 and 3.2, respectively and will be omitted for brevity.

Lemma 3.6. *Let L be defined as in* (45). *Then*

$$\begin{aligned}&\frac{dL}{d\tau}=\\ &=-\int_{\mathbb{R}}\left(v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)^2-v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)e^{-(p-2)\tau}v^p\right)dy,\end{aligned}\tag{47}$$

for $\tau > 0$.

Proof. Thanks to formula

$$u(t)=G(t)*u_0-\frac{1}{p}\int_0^t\nabla G(t-s)*u^p(s)ds, \tag{48}$$

we have on integrating by parts,

$$\begin{aligned}&\frac{d}{d\tau}\int_{\mathbb{R}}\left(\frac{v}{\bar v_\infty}-1-\log\frac{v}{\bar v_\infty}\right)e^{-|y|^2/2}dy=\\ &=\int_{\mathbb{R}}\left(\frac{1}{\bar v_\infty}-\frac{1}{v}\right)\frac{\partial v}{\partial\tau}e^{-|y|^2/2}dy=\\ &=e^{-|y|^2/2}\left(\frac{1}{\bar v_\infty}-\frac{1}{v}\right)\left(\frac{\partial v}{\partial y}+yv-e^{-(p-2)\tau}v^p\right)\Big|_{-\infty}^{+\infty}+\\ &\quad-\int_{\mathbb{R}}\left(v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)^2+v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)e^{-(p-2)\tau}v^p\right)dy.\end{aligned}\tag{49}$$

$\square$

LEMMA 3.7. *Set $v(y,\tau_j) = v_j(y)$, where $j \in \mathbb{N}$. Let v be the solution to* (19). *Then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ such that:*

$$\lim_{k\to+\infty} \int_{\mathbb{R}} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y} + y v_k\right)^2 dy = 0. \tag{50}$$

PROOF. Let us suppose by contradiction it does not exist a sequence $(\tau_k)_k$ in such a way that

$$\lim_{k\to+\infty} \int_{\mathbb{R}} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y} + y v_k\right)^2 dy = 0.$$

Let I be the function $I : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\begin{aligned} I(\tau) = &\int_{\mathbb{R}} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y} + y v\right)^2 dy + \\ &+ \int_{\mathbb{R}} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y} + y v\right) e^{-(p-2)\tau} v^p dy. \end{aligned} \tag{51}$$

On integrating by parts and making use of the L^∞-norm estimates for the function v, we deduce that the second integral in (51) tends to 0 as $\tau \to +\infty$. Therefore, there exists $T > 0$ such that for every $\tau > T$

$$I(\tau) > 0.$$

Moreover, $\frac{dL}{d\tau} \le 0$ as $\tau > T$; then

$$\int_T^\infty I ds = L(T) - L(\infty) < \infty.$$

Thus we can find a sequence $(\tau_j)_{j\in\mathbb{N}}$ such that $I(\tau_j) \to 0$ as $j \to +\infty$ and we have a contradiction. □

The proof of the following result follows the same lines as the proof of Proposition 3.1.

PROPOSITION 3.2. *A real constant C exists in such a way that if $\bar{w}_\infty$is the function defined by $\bar{w}_\infty = C\bar{v}_\infty$, then the sequence of functions $(v_k)_{k\in N}$ converges a.e. in $\mathbb{R}$ to $\bar{w}_\infty$.*

4 – Proofs of Theorems 1.1 and 1.2.

Let us begin with the proof of the following inequality, which allows us to obtain the decay of the function v towards the equilibrium v_∞ as $\tau \to +\infty$.

LEMMA 4.1. *Let v be the solution to* (19) *and w_∞ be a positive function. Assume there exist positive constants θ_1, θ_2 such that $\theta_1 e^{-y^2/2} < w_\infty(y) < \theta_2 e^{-y^2/2}$, a.e. $y \in \mathbb{R}$. Then for every $\tau > 0$*

$$\|v(\cdot,\tau)-w_\infty(\cdot)\|^2_{L^1(\mathbb{R})} \le \tilde{B} \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{w_\infty(y)} - 1 - \log \frac{v(y,\tau)}{w_\infty(y)}\right) e^{-|y|^2/2} dx, \tag{52}$$

where $\tilde{B} = \tilde{B}(\|v\|_1, \theta_1, \theta_2)$ is a suitable positive constant.

PROOF. Let $\alpha \in \mathbb{R}$, $\alpha > 2e^2$. Let us fix $\tau > 0$ and define the following set:

$$A_\tau = \left\{ x \in \mathbb{R} : \frac{v(y,\tau)}{w_\infty(y)} > \frac{\alpha}{2} \right\}.$$

Let us denote by f the function:

$$f : [1, +\infty) \to \mathbb{R}, \ : f(z) = z - 1 - 2\log(z).$$

We have that $f(z) > 0$ for every $z > \frac{\alpha}{2}$.

Thus,

$$\begin{aligned} \int_{A_\tau} (v - w_\infty)\, dy &= \int_{A_\tau} \left(\frac{v}{w_\infty} - 1\right) w_\infty dy \le \\ &\le \int_{A_\tau} \left(\frac{v}{w_\infty} - 1\right) \theta_2 e^{-y^2/2} dy \le \\ &\le \int_{A_\tau} 2\theta_2 \left(\frac{v}{w_\infty} - 1 - \log \frac{v}{w_\infty}\right) e^{-y^2/2} dy\,. \end{aligned} \tag{53}$$

Since

$$\begin{aligned} 0 &\le \int_{A_\tau} \left(\frac{v}{w_\infty} - 1 - \log \frac{v}{w_\infty}\right) e^{-y^2/2} dy \le \\ &\le \int_{A_\tau} \frac{v}{w_\infty} e^{-y^2/2} dy \le \frac{1}{\theta_1}, \end{aligned} \tag{54}$$

we have that:

$$\left(\int_{A_\tau}(v-w_\infty)\,dy\right)^2 \le \frac{2\theta_2}{\theta_1}\int_{A_\tau}\left(\frac{v}{w_\infty}-1-\log\frac{v}{w_\infty}\right)e^{-y^2/2}dy. \tag{55}$$

Set $B_\tau = \mathbb{R}\backslash A_\tau$ and define the function

$$g:\mathbb{R}^+\to\mathbb{R},\quad g(z)=(z-1)^2-\alpha(z-1-\log z).$$

One can verify that $g(z)\le 0$ if $0<z\le\frac{\alpha}{2}$. Hence, for any given $\tau>0$,

$$\left(\frac{v}{w_\infty}-1\right)^2 e^{-y^2/2}\le\alpha\left(\frac{v}{w_\infty}-1-\log\frac{v}{w_\infty}\right)e^{-y^2/2}, \tag{56}$$

for $y\in B_\tau$, and

$$\begin{aligned}\left(\int_{B_\tau}(v-w_\infty)\,dy\right)^2 &\le \int_{B_\tau}\theta_2^2 e^{-y^2/2}dy\int_{B_\tau}\left(\frac{v}{w_\infty}-1\right)^2 e^{-y^2/2}dy\le\\ &\le\theta_2^2\,(2\pi)^{1/2}\int_{B_\tau}\left(\frac{v}{w_\infty}-1\right)^2 e^{-y^2/2}dy.\end{aligned} \tag{57}$$

Then

$$\begin{aligned}&\|v(\cdot,\tau)-w_\infty(\cdot)\|^2_{L^1(\mathbb{R})}\le\\ &\le\left(\theta_2^2\,(2\pi)^{1/2}\,\alpha+\frac{2\theta_2}{\theta_1}\right)\int_{\mathbb{R}}\left(\frac{v}{w_\infty}-1-\log\frac{v}{w_\infty}\right)e^{-y^2/2}dy,\end{aligned} \tag{58}$$

if $\tau>0$. □

Let us prove now Theorem 1.1 to state the large time behaviour of the solution to (1) in the case where $p=2$.

Proof of Theorem 1.1. As a consequence of Proposition 3.1, we obtain by Lebesgue theorem that as $k\to+\infty$,

$$\int_{\mathbb{R}}\left(\frac{v(y,\tau_k)}{w_\infty(y)}-1-\log\frac{v(y,\tau_k)}{w_\infty(y)}\right)e^{-y^2/2}dy\to 0. \tag{59}$$

Thanks to the result of Lemma 4.1 and Proposition 3.1, we have that $\|v(\cdot,\tau_k)-w_\infty(\cdot)\|_{L^1(\mathbb{R})}\longrightarrow 0$, as $k\to+\infty$. It follows that $\|w_\infty(\cdot)\|_{L^1(\mathbb{R})}=1$

and $w_\infty = v_\infty$. Since for every $\tau > 0$, $\|v(\cdot,\tau)\|_\infty \leq C_\infty$, we have by interpolation that

$$\lim_{\tau\to+\infty} \|v - v_\infty\|_{L^p(\mathbb{R})} = 0;$$

as $1 < p < \infty$.

After performing the time-dependent scaling, we get the conclusion. □

We prove now the main result on the large time behaviour of the solution to (1) as $p > 2$.

In contrast to the case where $p = 2$, we are not able to define a suitable functional L, which is monotone non increasing in time. Nevertheless, one can prove that $\int_{\mathbb{R}}(\frac{v(y,\tau)}{\bar{v}_\infty(y)} - 1 - \log \frac{v(y,\tau)}{\bar{v}_\infty(y)})e^{-y^2/2}dy$ converges to zero as $\tau \to +\infty$.

Proof of Theorem 1.2. Thanks to the inequality of Lemma 4.1 and to Proposition 3.2, we deduce that $\bar{w}_\infty = \bar{v}_\infty$. Let us prove now that the function $L(\tau)$ converges to zero as $\tau \to +\infty$.

Step 1. The function I is continuous in τ, thanks to the results of Theorem 2.1. We have to consider the following three cases:

1) there exists $T > 0$ such that $I(\tau) \geq 0$ for every $\tau \geq T$; then L is a Lyapunov functional and the conclusion follows as for the case where $p = 2$.
2) There exists $T > 0$ such that $I(\tau) \leq 0$ for every $\tau > T$. It follows that:

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} v^{-2}e^{-y^2/2}\left(\frac{\partial v}{\partial y} + yv\right)^2 dy \leq \\ &\leq -\int_{\mathbb{R}} v^{-2}e^{-y^2/2}\left(\frac{\partial v}{\partial y} + yv\right)\frac{1}{e^{(p-2)\tau}}v^p dy. \end{aligned} \tag{60}$$

Thus, on integrating by parts in the last integral of (60), we have

$$\lim_{\tau\to+\infty}\int_{\mathbb{R}} v^{-2}e^{-y^2/2}\left(\frac{\partial v}{\partial y} + yv\right)^2 dy = 0.$$

3) The function $I(\tau)$ changes the sign as $\tau \in [0,\infty)$. Let $(\tau_i)_{i\in I} \in [0,\infty)$ such that $I(\tau_i) = 0$. Suppose $I(\tau) > 0$ as $\tau \in [\tau_{i-1}, \tau_i)$ and $I(\tau) \leq 0$

as $\tau \in [\tau_i, \tau_{i+1}]$. Then $L(\tau_{i-1}) \geq L(\tau) \geq 0$ for $\tau \in [\tau_{i-1}, \tau_i)$ and $L(\tau_{i+1}) \geq L(\tau) \geq 0$ as $\tau \in [\tau_i, \tau_{i+1}]$.

As a consequence of $I(\tau_i) = 0$, we have that $L(\tau_i) \to 0$ as $\tau_i \to +\infty$. Thanks to the previous inequalities, $L(\tau) \to 0$ as $\tau \to +\infty$.

Step 2. Thanks to the inequality proved in Lemma 4.1, we obtain the result of Theorem 1.2 in a similar way as in the case where $p = 2$ if the intial value u_0 satisfies the assumption of Lemma 2.2.

By density argument the result can be proved in the general case. The solution to the Cauchy problem (1) satisfies indeed the following $L^1(\mathbb{R})$-contraction property proved in [5]:

$$\|u(\cdot, t) - \bar{u}(\cdot, t)\|_1 \leq \|u_0 - \bar{u}_0\|_1,$$

for every $t \geq 0$. Consider now a nonnegative initial value $u_0 \in L^1(\mathbb{R})$ and approximate u_0 in $L^1(\mathbb{R})$ by a sequence of functions $(u_{0,n})_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ such that $u_{0,n}(x) \geq M_n \exp(-\frac{x^2}{2})$, a.e. $x \in \mathbb{R}$, where M_n are positive constants. Let u_n be the solution to (1) with initial value $u_{0,n}$. Thanks to the result on the long time behaviour of u_n and the $L^1(\mathbb{R})$-contraction property, we get the conclusion. $\square$

5 – Concluding remarks

Consider the following class of convection-diffusion equations in $\mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle u - a \cdot \nabla(u^p) & x \in \mathbb{R}^N, \qquad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{61}$$

where $p > 1 + \frac{1}{N}$, $N > 1$; $a \in \mathbb{R}^N$ and u_0 is a nonnegative function from $L^1(\mathbb{R}^N)$.

In the present section we will prove that our procedure can be used in the study of the long time behaviour of the solution to Cauchy problem (61).

Given $u_0 \in L^1(\mathbb{R}^N)$, there exists a unique classical solution $u \in C([0, \infty);\ L^1(\mathbb{R}^N))$ to (61), which satisfies the following properties (see [5]):

(i) for every $q \in (1,\infty)$, $u \in C((0,\infty); W^{2,q}(\mathbb{R}^N)) \bigcap C^1((0,\infty); L^q(\mathbb{R}^N))$;

(ii) for every $q \in [1,\infty)$, there exists a constant $C_q = C(q, \|u_0\|_1)$ such that for every $t > 0$:

$$\begin{cases} \|u(t)\|_q \leq C_q t^{-N/2(1-1/q)}, \\ \|u(t)\|_1 \leq \|u_0\|_1. \end{cases} \tag{62}$$

(iii) Let t_0 be a nonnegative real number. Then there exists a positive constant C_∞ such that for every $t \geq t_0$:

$$\|u(t)\|_\infty \leq C_\infty t^{-N/2}. \tag{63}$$

By studying the problem in a similar way as the case where $p > 2$ and $N = 1$, we can prove the following result.

THEOREM 5.1. *Assume that $p > 1 + \frac{1}{N}$ and*

$$\begin{aligned} &u_0 \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} u_0(x)dx = 1; \\ &u_0(x) \geq 0 \ a.e.\ x \in \mathbb{R}^N. \end{aligned} \tag{64}$$

Let u be the solution to the Cauchy problem (61)*. Then for every $q \in [1,\infty)$*

$$\lim_{t\to+\infty} (2t+1)^{N/2(1-1/q)} \|u(\cdot,t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R}^N)} = 0,$$

where

$$u_\infty(x, 2t+1) = \frac{1}{(2t+1)^{N/2}} \gamma \exp\left(-\frac{x^2}{2(2t+1)}\right),$$

and $\gamma = (\int_{\mathbb{R}^N} \exp(-\frac{x^2}{2})dx)^{-1}$.

We just give an outline of the proof.

PROOF. Let us divide the proof in the following four steps.

STEP 1. Following the same procedure of the proof of Lemma 2.2, we prove that if $u_0(x) \geq M \exp(-\frac{|x|^2}{2})$ for some positive constant M and for a.e. $x \in \mathbb{R}^N$, then there exist positive constants B and C such that

$$u(x,t) \geq \frac{Be^{-C}}{t^{N/2}} \exp\left(\frac{C}{t^{\frac{N(p-1)-1}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right), \tag{65}$$

for every $t > 1$.

STEP 2. After performing a time-dependent scaling, we study the long time behaviour of the solution to the following problem:

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = \nabla \cdot [\nabla v + yv - ae^{-(pN-N-1)\tau} v^p] & y \in \mathbb{R}^N, \quad \tau > 0, \\ v(y,0) = v_0(y) & y \in \mathbb{R}^N. \end{cases} \tag{66}$$

Notice that the solutions u and v to problems (61) and (66) respectively, are related by

$$\begin{aligned} u(x,t) &= \frac{1}{(2t+1)^{N/2}} v\left(\frac{x}{(2t+1)^{1/2}}, \log{(2t+1)^{1/2}}\right), \\ v(y,\tau) &= e^{N\tau} u\left(ye^{\tau}, \frac{e^{2\tau}-1}{2}\right). \end{aligned} \tag{67}$$

The large time behaviour of the solution to (66) is determined by the following equation:

$$\frac{\partial v}{\partial \tau} = \triangle v + \nabla \cdot (yv). \tag{68}$$

A stationary solution is given by the function:

$$\bar{v}_\infty(y) = \gamma \exp\left(-\frac{|y|^2}{2}\right), \tag{69}$$

where γ is a constant. We fix γ in such a way that

$$\int_{\mathbb{R}^N} \gamma \exp\left(-\frac{|y|^2}{2}\right) dy = 1. \tag{70}$$

Step 3. Similarly as the case of the viscous Burgers equation, we define a nonnegative functional. Let v be the solution to (66). Let L be the function from $\mathbb{R}^+$ into $\mathbb{R}$ given by

$$L(\tau) = \int_{\mathbb{R}^N} \left(\frac{v(y,\tau)}{\bar{v}_\infty(y)} - 1 - \log \frac{v(y,\tau)}{\bar{v}_\infty(y)} \right) e^{-|y|^2/2} dy. \tag{71}$$

One can prove that if v is a solution to (66), then $L(\tau) < +\infty$.

Moreover, if v is the solution to (66), then

$$\lim_{|y_i| \to +\infty} \left(\frac{1}{\bar{v}_\infty} - \frac{1}{v} \right) \left(\frac{\partial v}{\partial y_i} + y_i v - a_i e^{-(Np-N-1)\tau} v^p \right) e^{-|y|^2/2} = 0, \tag{72}$$

for every $\tau > 0$ and $i = 1, ..., N$.

Consider L defined above. On integrating by parts, we prove that

$$\begin{aligned} \frac{dL}{d\tau} = & - \int_{\mathbb{R}^N} \sum_{i=1}^{N} \left(v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right)^2 + \right. \\ & \left. - v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right) a_i e^{-(Np-N-1)\tau} v^p \right) dy, \end{aligned} \tag{73}$$

for $\tau > 0$.

Set $v(y, \tau_j) = v_j(y)$, where $j \in \mathbb{N}$. Let v be the solution to (66). Then there exists a sequence $(v_k)_{k \in \mathbb{N}}$ such that:

$$\lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\mathbb{R}^N} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y_i} + y_i v_k \right)^2 dy = 0. \tag{74}$$

Let us suppose by contradiction that it does not exist a sequence $(\tau_k)_k$ in such a way that

$$\lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\mathbb{R}^N} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y_i} + y_i v_k \right)^2 dy = 0.$$

Let I be the function $I : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\begin{aligned} I(\tau) = & \sum_{i=1}^{N} \int_{\mathbb{R}^N} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right)^2 dy + \\ & + \sum_{i=1}^{N} \int_{\mathbb{R}^N} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right) a_i e^{-(Np-N-1)\tau} v^p dy. \end{aligned} \tag{75}$$

On integrating by parts and making use of the L^∞-norm estimates for the function v, we deduce that the second integral in (75) tends to 0 as $\tau \to +\infty$. Therefore, there exists $T > 0$ such that for every $\tau > T$

$$I\,(\tau) > 0.$$

Moreover, $\dfrac{dL}{d\tau} \leq 0$ as $\tau > T$; then

$$\int_T^\infty I ds = L(T) - L(\infty) < \infty.$$

Thus we can find a sequence (τ_j) such that $I(\tau_j) \to 0$ as $j \to +\infty$ and we get a contradiction.

Similarly as the cases studied in Section 3, we can prove that there exists a real constant C in such a way that if $\bar{w}_\infty$ is the function defined by $\bar{w}_\infty = C\bar{v}_\infty$, then the sequence of functions $(v_k)_{k\in N}$ converges a.e. in $\mathbb{R}^N$ to $\bar{w}_\infty$.

Step 4. The inequality proved in Lemma 4.1 holds true even in the case where the functions v and w_∞ are defined in $\mathbb{R}^N$. Moreover, the conclusion of the proof of Theorem 5.1 is achieved by following the same procedure as for the proof of Theorem 1.2 in Section 4. ∎

We have tried to apply the method to study the long time behaviour of the solution to (61) in the case where $p = 1 + \dfrac{1}{N}$ and $N > 1$. Unfortunately we are not able to conclude the proof because of technical difficulties due to the lack of informations about the qualitative properties of the self-similar solution to (61) (see [1]).

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Indirizzo dell'Autore:

R. Cavazzoni – Dipartimento di Matematica e Appl. Architettura – Università di Firenze – Piazza Ghiberti, 27 – 50122 Firenze (Italia)
E-mail: cavazzon@interfree.it