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Nodal curves and Brill - Noether theory

E. BALLICO

Abstract: *Here we prove some existence theorems for special spanned line bundles on the general nodal curve of genus $g \geq 2$. We give counterexamples to similar questions for curves with seminormal singularities.*

1 – Introduction

In the first 3 sections of this paper we study the Brill - Noether theory of special divisors on the general k-gonal curve with only ordinary nodes as singularities. On an integral projective curve, Y, there are at least 4 quite different Brill - Noether theories: one can study spanned line bundles, line bundles, spanned rank 1 torsion free sheaves or rank 1 torsion free sheaves. The Brill - Noether theory of rank 1 torsion free sheaves is the only one in which the set of the solutions is always a complete scheme. Passing to the spanned subsheaf, one can reduce the Brill - Noether theory of rank 1 torsion free sheaves to the one for spanned torsion free sheaves. The Brill - Noether theory of line bundles is interesting because it concerns important closed subschemes of the non-complete scheme $\mathrm{Pic}^d(Y)$. For the relations between the last two theories for curves with only ordinary nodes or ordinary cusps as sin-

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gularities, see 2.3. The Brill - Noether theory of spanned line bundles is the more important one because it concerns the morphisms $Y \to \mathbf{P}^r$. But we have an additional problem because we are interested in k-gonal curves and their Brill - Noether theories depend very much on the singularities of the degree k pencil. For any rank 1 torsion free sheaf F on Y, set $\mathrm{Sing}(F) := \{P \in Y : F \text{ is not locally free at } P\}$. Thus $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(Y)$. We introduce the following definition.

Definition 1.1. Fix integers g, x, k and y with $k \geq 2$, $g \geq 2k + x - y + 2$, $g \geq x \geq y \geq 0$ and $x \geq 0$. Let X be a general smooth $(k-y)$-gonal curve of genus $g-x$. Call $M \in \mathrm{Pic}^{k-y}(X)$ the degree $k-y$ spanned line bundle on X and $h_M : X \to \mathbf{P}^1$ the associated morphism with $\deg(h_M) = k-y$. Take $x+y$ general points P_i, $1 \leq i \leq x-y$, A_j, $1 \leq j \leq y$, and B_j, $1 \leq j \leq y$, on Y. Fix points Q_i, $1 \leq i \leq x-y$, with $h_M(P_i) = h_M(Q_i)$ for every i. Let $\pi : X \to Y$ be the birational morphism obtained gluing together the points P_i and Q_i for $1 \leq i \leq x-y$, and the points A_j and B_j for $1 \leq j \leq y$. Hence Y is a nodal curve with $p_a(Y) = g$ and x nodes. Set $F := \pi_*(M)$. Thus F is a rank 1 torsion free sheaf on Y with $\deg(F) = k$, $\mathrm{Sing}(F) = \{\pi(A_1), \ldots, \pi(A_y)\}$ and $h^0(Y,F) = 2$. We will say that Y or the pair (Y,F) is the *general k-gonal curve of genus g with x nodes and a pencil with y singularities* or just a *general nodal k-gonal curve of genus g with type (x,y)*.

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. As a sample of our results we state here the following one which will be proved in Section 2.

Theorem 1.2. *Fix integers g, x, y, k and d with $k \geq 2+y$, $x \geq y \geq 0$, $x > 0$, $g \geq 2k+2x+1$ and $2d \geq g+2$. Let Y be the general k-gonal nodal curve of genus g with type (x,y) and F the degree k pencil with card $(\mathrm{Sing}(F)) = y$. Then there is an irreducible locally closed subset Z of $\mathrm{Pic}^d(Y)$ with $Z \neq \emptyset$, $\dim(Z) = \rho(g-x+y,d,1)-x := 2d-g+x-2-y$ such that every $R \in Z$ is spanned. If $d \leq g-x+y-1$, then we may find Z such that $h^0(Y, \mathrm{Hom}(F,R)) = 0$ for every $R \in Z$.*

The case $y = x$ is the easier one. If $y = x$ we obtain an existence result for embeddings of Y into $\mathbf{P}^r$, $r \geq 3$ (see Theorem 3.1).

In the last section we will consider seminormal curves in the sense of [17] and [9], i.e. curves with the simplest singularities compatible with their number of branches: if the singularity has r branches, then it is formally equivalent to the germ at $0 \in \mathbf{K}^r$ of the union of the r coordinate axis. We will show that for non-nodal seminormal curves the usual existence theorem for special line bundles (even non spanned ones) are not always true if one uses only the Brill - Noether number $\rho(g, r, d) := g - (r+1)(g+r-d)$ in its statement as in the case of smooth curves ([14] or [2]).

2 – Proof of 1.2

In the first part of this section we give several preliminary results needed for the proof of 1.2 and of other related results. Let Y be an integral projective curve, $\pi : X \to Y$ its normalization and F a rank 1 torsion free sheaf on Y. The sheaf $G := \pi^*(F)/\operatorname{Tors}(\pi^*(F))$ has rank 1 and no torsion. Hence $G \in \operatorname{Pic}(X)$. We claim that the natural map $\alpha : H^0(Y, F) \to H^0(X, G)$ is injective; set $x := h^0(Y, F)$ and take x general points $P_1, \ldots, P_x$ of X; there is $f \in H^0(Y, F)$ with $f(\pi(P_i)) = 0$ for $i < x$ and $f(\pi(P_x)) \neq 0$; hence $\alpha(f)(P_i) = 0$ for $i < x$ and $\alpha(f)(P_x) \neq 0$, proving the claim.We will call the integer $\delta - \deg(F) := \deg(G)$ the δ-degree of F. By [10], Lemma 1, we have $\deg(F) + p_a(X) - p_a(Y) \leq \delta - \deg(F) \leq \deg(F)$ and $\delta - \deg(F) = \deg(F)$ if and only if $F \in \operatorname{Pic}(Y)$. Furthermore, $\deg(F) - \delta - \deg(F) \geq \operatorname{card}(\operatorname{Sing}(F))$. If F is spanned, then $\pi^*(F)$ is spanned and hence G is spanned.

(2.1) Let R be the one-dimensional complete semilocal ring which is either the completion of an ordinary node or an ordinary cusp. Let $\mathbf{m}$ be the maximal ideal of R (cusp case) or the intersection of the two maximal ideals (nodal case). Let M be a torsion free finitely generated R-module with $\operatorname{rank}(M) = 1$; here we assume that if R is the completion of an ordinary node, then M has constant rank on each of the two branches of R. Since $\operatorname{char}(\mathbf{K}) = 0$, there is a complete classification of all such M: there are uniquely determined integers a, b with $a \geq 0$, $b \geq 0$, $a + b = \operatorname{rank}(M)$ such that $M \cong R^{\oplus a} \oplus \mathbf{m}^{\oplus b}$ [11]. We will need only the case $\operatorname{rank}(M) = 1$.

(2.2) Let Y be an integral projective curve with only ordinary nodes and ordinary cusps as singularities, $\pi : X \to Y$ its normalization and F a rank 1 torsion free sheaf on Y. If $P \in \operatorname{Sing}(F)$, then the completion of F at P is isomorphic either to the maximal ideal (the cusp case) or the intersection of the two maximal ideals of the completion of $\mathbf{O}_{Y,P}$ (the nodal case). Thus $\deg(F) - \delta - \deg(F) = \mathrm{card}(\mathrm{Sing}(F))$. Let Y be an integral projective curve with only ordinary nodes and only ordinary cusps as singularities. The following remark shows the relations between the Brill - Noether theory of (not necessarily spanned) line bundles on Y and the Brill - Noether theory of spanned rank 1 torsion free sheaves on Y.

Remark 2.3. Let Y be an integral projective curve and F a rank 1 torsion free sheaf such that for every $P \in \mathrm{Sing}(F)$ the curve has at P either an ordinary node or an ordinary cusp. By 2.1 for every $P \in \mathrm{Sing}(F)$ the completion of the stalk of F at P is isomorphic to the maximal ideal of the competion of the local ring $\mathbf{O}_{Y,P}$. Thus there is a unique $L \in \mathrm{Pic}(Y)$ with $F \subseteq L$, $\deg(L/F) = \mathrm{card}(\mathrm{Sing}(F))$ and $\mathrm{Supp}(L/F) = \mathrm{Sing}(F)$. We have $h^0(Y, F) \le h^0(Y, L) \le h^0(Y, F) + \mathrm{card}(\mathrm{Sing}(F))$. Furthermore, the integer $h^0(Y, L) - h^0(Y, F)$ is the number of points of $\mathrm{Sing}(F)$ at which L is spanned.

Remark 2.4. Let X be a smooth projective curve of genus q and $h : X \to \mathbf{P}^1$, $f : X \to \mathbf{P}^1$ non-constant morphisms such that the associated morphism $j := (h, f) : X \to \mathbf{P}^1 \times \mathbf{P}^1$ is birational. Set $a := \deg(h)$, $b := \deg(f)$ and assume $q < ab - a - b + 1$. By the genus formula for a divisor of type (a, b) on $\mathbf{P}^1 \times \mathbf{P}^1$ the curve $j(X)$ is singular. Assume that $j(X)$ has only nodal singularities; by [1], Proposition 2.4 and its proof, this is the case if X is a general a-gonal curve and f is general in the set of all degree b pencils on X not composed with h. Assume that the monodromy group of a generic fiber of h is the full symmetric group; since $\mathrm{char}(\mathbf{K}) = 0$ this is the case if the reduction of a fiber of X has exactly $a - 1$ elements; this condition is always satisfied if X is a general a-gonal curve and h is the associated degree a pencil. Set $z := ab - a - b + 1 - q$. By our assumptions there is a non-empty set of $2z$-ples $(P_1, Q_1, \dots, P_z, Q_z) \in X^{2z}$ with $P_i \ne Q_i$ and $j(P_i) = j(Q_i)$ for every i, i.e. $h(P_i) = h(Q_i)$ and $f(P_i) = f(Q_i)$ for every i. Take 3 general points of $\mathbf{P}^1$, say B_1, B_2 and B_3 and fix $A_i \in X$ with $j(A_i) = B_i$, $1 \le i \le 3$. Fix an integer w with $0 < w \le z$. Assume the existence of a quasi-projective integral subvariety T of the scheme $\mathrm{Hom}^b(X, \mathbf{P}^1)$ of degree b morphisms sending each A_i onto B_i, $1 \le i \le 3$, with $\dim(T) = w$, $j \in T$ and such that for every $t \in T$ the pair $j_t := (h, f_t)$ associated to the corresponding morphism $f_t : X \to \mathbf{P}^1$ satisfies the previous conditions. We claim that for a general $(P_1, \dots, P_w) \in X^w$ there is $t \in T$ and $(Q_1, \dots, Q_w) \in X^w$ such that $P_i \neq Q_i$ for every i and $j_t(P_i) = j_t(Q_i)$ for every i. Consider the following statement $T(k)$, $0 \leq i \leq w$.

Statement $T(k)$: for a general $(P_1, \dots, P_k) \in X^k$ there are a $(w-k)$-dimensional irreducible subvariety $T(P_1, \dots, P_k) \subseteq T$ and $(Q_1, \dots, Q_k) \in X^w$ with $P_i \neq Q_i$ for every i with $1 \leq i \leq k$ such that for every $t \in T(P_1, \dots, P_k)$ we have $j_t(P_i) = j_t(Q_i)$ for every i with $1 \leq i \leq k$. Furthermore, the set of all $t \in T$ satisfying this condition has codimension k in T.

The first assertion of Statement $T(w)$ is the claim we want to prove. Statement $T(0)$ is empty: just take $T(\varnothing) := T$. Assume proved $T(k)$ for some integer k with $k < w$ and take the corresponding points $Q_1, \dots, Q_j$. Set $J := \{(P, Q, t) \in X^2 \times T(P_1, \dots, P_k)$ with $P \neq Q$, $f_t(P) \notin \{f_t(P_1), \dots, f_t(P_k), B_1, B_2, B_3\}$, $f_t(P) = f_t(Q)$ and $j_t(P_i) = j_t(Q_i)$ for every $i\}$. Call $\pi_1 : J \to X$ and $\pi_3 : J \to T(P_1, \dots, P_k)$ the projections on the first and third factor. Since $w \leq z$ each fiber of π_3 is finite and non-empty. Thus every irreducible component of J has dimension $w - k > 0$. If J contains a slice $\{P\} \times X \times \{t\}$, then $f_t(X) = f_t(P)$ and hence f_t is constant; this is impossible because $\deg(j_t) = b$ by assumption. Since J is not union of slices $\{P\} \times X \times T(P_1, \dots, P_k)$, π_1 is dominant. By the assumption on the monodromy group of the generic fiber of h, for any fixed $t \in T$ and for general $P \in X$ either $h^{-1}(h(P)) \cap f_t^{-1}(f_t(P)) = \{P\}$ or $h^{-1}(h(P))$ is contained in $f_t^{-1}(f_t(P))$, i.e. $h = f_t$. We apply this observation to the general element of $T(P_1, \dots, P_k)$ to obtain the first assertion of $T(k+1)$ and to the general elements of similar codimension k irreducible component of T to obtain the last assertion of $T(k+1)$. Hence we obtain $\dim(T(P_1, \dots, P_k, P_{k+1})) < \dim(T(P_1, \dots, P_k))$ for general P_{k+1}, i.e. we obtain the last assertion of $T(k+1)$. We have $T(P_1, \dots, P_{k+1}) \neq\to$ for general P_{k+1} because of π_1 is dominant. Thus $T(k+1)$ holds. By induction we obtain $T(w)$, proving the claim.

Remark 2.5. Let X be a smooth projective curve of genus q and $h : X \to \mathbf{P}^1$, $f : X \to \mathbf{P}^1$ non-constant morphisms such that the associated morphism $j := (h, f) : X \to \mathbf{P}^1 \times \mathbf{P}^1$ is birational. Set $a := \deg(h)$, $b := \deg(f)$ and assume $q < ab - a - b + 1$. Assume that $j(X)$ has only nodal singularities and that the monodromy group of a generic fiber of h is the full symmetric group. By our assumptions there is a non-empty set of $2z$-ples $(P_1, Q_1, \dots, P_z, Q_z) \in X^{2z}$ with $P_i \neq Q_i$ and $j(P_i) = j(Q_i)$ for every i, i.e. $h(P_i) = h(Q_i)$ and $f(P_i) = f(Q_i)$ for every i. Take 3 general points of $\mathbf{P}^1$, say B_1, B_2 and B_3 and fix $A_i \in X$ with $j(A_i) = B_i$, $1 \leq i \leq 3$. Fix an integer w with $0 < w \leq z$ and an integer $\alpha > w$. Assume the existence a quasi-projective integral subvariety T of the scheme $\mathrm{Hom}^b(X, \mathbf{P}^1)$ of degree b morphisms sending each A_i onto B_i, $1 \leq i \leq 3$, with $\dim(T) = w$, $j \in T$ and such that for every $t \in T$ the pair $j_t := (h, f_t)$ associated to the corresponding morphism $f_t : X \to \mathbf{P}^1$ satisfies the previous conditions. By Remark 2.5 for a general $(P_1, \dots, P_w) \in X^w$ there is $t \in T$ and $(Q_1, \dots, Q_w) \in X^w$ such that $P_i \neq Q_i$ for every i and $j_t(P_i) = j_t(Q_i)$ for every i. Take a general element $(P_{\alpha-w+1}, Q_{\alpha-w+1}, \dots, P_\alpha, Q_\alpha)$ of $X^{2\alpha-2w}$. Let Y be the nodal curve obtained from X gluing together each pair (P_i, Q_i), $1 \leq i \leq \alpha$. By construction Y is a nodal curve with α nodes and with a degree b pencil of type $(\alpha, \alpha - w)$.

Example 2.6. Fix an even integer $g = 2b \geq 6$ and let X be a general smooth curve of genus $g-1$. Thus X has no spanned line bundle, L, with $1 \leq \deg(L) \leq [(g-1+3)/2] = b$ and a finite set, S, of line bundles, R, with $\deg(R) = b+1$ and $h^0(X, R) = 2$. Furthermore, every $R \in S$ is spanned and $\mathrm{card}(S) = (2b)!/(b-1)!b!) \neq 0$ ([2], p. 211) . Fix P, $Q \in X$ such that for every $R \in S$ the morphism $h_R : X \to \mathbf{P}^1$ has $h_R(P) \neq h_R(Q)$. Let Y be the curve obtained from X gluing P and Q. Thus Y is a curve with $p_a(Y) = g$, a unique ordinary node as singularities and with X as normalization. Call $\pi : X \to Y$ the normalization. Thus $\pi(P) = \pi(Q)$ is the singular point. For every $R \in S$ the rank 1 torsion free sheaf $\pi_*(R)$ has degree $b+2$ and $h^0(Y, \pi_*(R)) = h^0(X, R) = 2$. The condition $h_R(P) \neq h_R(Q)$ is equivalent to the fact that R is not the pull-back of a spanned line bundle on Y. Thus the condition $h_R(P) \neq h_R(Q)$ is equivalent to the spannedness of $\pi_*(R)$. We claim that there is no $M \in \mathrm{Pic}(Y)$ with $1 \leq \deg(M) \leq b+1$ and $h^0(Y, M) \geq 2$. Assume the existence of such M. Thus $h^0(X, \pi^*(M)) \geq h^0(Y, M) \geq 2$. If $\deg(M) \leq b$ this is impossible because Y is general. Assume $\deg(M) = b+1$. Then $\pi^*(M) \in S$. We just saw that this is impossible by the choice of the pair $\{P, Q\}$. Notice that if we choose $\{P, Q\}$ general the curve Y is the general nodal curve of genus g with exactly one node. However, if we fix X general of genus $g-1 = 2b$

and take as Y' the curve obtained from X gluing together two points in the same fiber of one of the morphisms, then we obtain a nodal curve Y' with one node, normalization with general moduli and $L \in \mathrm{Pic}(Y')$ with $\deg(L) = b+1$ and L spanned, while $\rho(g, b+1, 1) = -1 < 0$. Take a rank 1 torsion free sheaf F on Y with $\deg(F) \leq b+2$ and $h^0(Y, F) \geq 2$. Since F is not locally free, we have $\deg(\pi^*(F)/\mathrm{Tors}(\pi^*(F))) < \deg(F)$ and indeed $\deg(\pi^*(F)/\mathrm{Tors}(\pi^*(F))) = \deg(F) - 1$ ([10], Lemma 1). Since $\pi^*(F)/\mathrm{Tors}(\pi^*(F)) \in \mathrm{Pic}(X)$ and $h^0(X, \pi^*(F)/\mathrm{Tors}(\pi^*(F))) \geq 2$, we obtain $\pi^*(F)/\mathrm{Tors}(\pi^*(F)) \in S$. Thus $\deg(F) = b+2$ and there is a natural bijection between S and the set of all such sheaves F.

Example 2.7. Take b, g, X and S as in Example 2.6. Fix a point $A \in X$ such that for every $R \in S$ the morphism h_R is étale at A. Let $\pi' : X \to Y'$ the birational and bijective morphism with $p_a(Y') = g$, Y' with $\pi'(A)$ as unique singular point and an ordinary cusp at $\pi'(A)$. By the choice of A we may apply the proof of Example 2.6 in our situation just with notational modifications. Since as A we may take a general point of X, this description of the rank 1 torsion free sheaves of degree at most $b+2$ is the description of such sheaves for the general cuspidal curve of genus g with a unique singular point.

Examples 2.6 and 2.7 may be generalized in the following way. We omit the easy proof.

Proposition 2.8. *Let X be a smooth projective curve. Fix positive integers r and d such that for every integer $z \leq d-2$ and every $L \in \mathrm{Pic}^z(X)$ we have $h^0(X, L) \leq r$, while the set $S := \{R \in \mathrm{Pic}(X) : \deg(R) = d-1$ and $h^0(X, R) = r+1\}$ is finite. Fix P, $Q \in X$ such that for every $R \in S$ the morphism $h_R : X \to \mathbf{P}^r$ has $h_R(P) \neq h_R(Q)$. Let Y be the curve obtained from X gluing P and Q. Then there is no $M \in \mathrm{Pic}(Y)$ with $1 \leq \deg(M) \leq d$ and $h^0(Y, M) \geq r+1$. Furthermore, every* rank 1 *torsion free sheaf F on Y with $\deg(F) \leq d$ and $h^0(Y, F) \geq r+1$ has $\deg(F) = d$ and $h^0(Y, F) = r+1$ and there is $R \in S$ such that $F \cong \pi_*(R)$.*

Proposition 2.9. *Let X be a smooth projective curve. Fix positive integers r and d such that for every integer $z \leq d-2$ and every $L \in \mathrm{Pic}^z(X)$ we have $h^0(X, L) \leq r$, while the set $S := \{R \in \mathrm{Pic}(X) : \deg(R) = d-1$ and $h^0(X, R) = r+1\}$ is finite. Fix $A \in X$ such that for* *every $R \in S$ the morphism $h_R : X \to \mathbf{P}^r$ is étale at A. Let Y be the curve with $\pi : X \to Y$ as normalization map, $p_a(Y) = p_a(X)+1$ and $\pi(A)$ as an ordinary cusp. Then there is no $M \in \mathrm{Pic}(Y)$ with $1 \le \deg(M) \le d$ and $h^0(Y, M) \ge r + 1$. Furthermore, every* rank 1 *torsion free sheaf F on Y with* $\deg(F) \le d$ *and $h^0(Y, F) \ge r+1$ has* $\deg(F) = d$ *and $h^0(Y, F) = r+1$ and there is $R \in S$ such that $F \cong \pi_*(R)$.*

Remark 2.10. Fix an integer $y > 0$. Let X' be an integral projective curve and $L \in \mathrm{Pic}(X')$ with $h^0(Y, L) \le y$. Let $\alpha : X' \to Y$ be a birational morphism with Y obtained from X' creating y new nodes gluing together y general pairs of points of X'. The proof of 2.6 and 2.7 shows that there is no $R \in \mathrm{Pic}(Y)$ with $\alpha^*(R) \cong L$ and R spanned.

Remark 2.11. Let X be a smooth curve and $R \in \mathrm{Pic}(X)$ with R spanned and $h^0(X, L) = r + 1 \ge 3$. Let $h_R : X \to \mathbf{P}^r$ be the morphism induced by R. Fix P, $Q \in X$ such that $h_R(P) \ne h_R(Q)$, i.e. such that $h^0(X, R(-P-Q)) = r - 1$; this condition is satisfied for a general pair $(P, Q) \in X \times X$. Let Y be the curve obtained gluing together P and Q, i.e. let Y be the curve with $\pi : X \to Y$ as normalization map, $p_a(Y) = p_a(X) + 1$ and $\pi(P) = \pi(Q)$ as an ordinary node. Take a linear space V with $H^0(X, R(-P-Q)) \subset V \subset H^0(X, R)$, $\dim(V) = r$ and V spanning R; since $r-1 > 0$, $R(-P-Q)$ has at most finitely many base points and hence we may take as V a general linear subspace of $H^0(X, R)$ containing $H^0(X, R(-P-Q))$ and different from $H^0(X, R(-P-Q))$; in particular the set of all such linear spaces V is parametrized by an irreducible one-dimensional variety. The morphism h_V associated to V factors through π and hence there is $R_V \in \mathrm{Pic}(Y)$ with $\pi^*(R_V) = R$, $h^0(Y, R_V) = r$, R_V spanned and $\pi^*(H^0(Y, R)) = V$. Hence if $V \ne V'$, then R_V and $R_{V'}$ are not isomorphic.

Remark 2.12. Let X be a smooth curve and $R \in \mathrm{Pic}(X)$ with R spanned and $h^0(X, L) = r + 1 \ge 3$. Let $h_R : X \to \mathbf{P}^r$ be the morphism induced by R. Fix $A \in X$ such that h_R is étale at P, i.e. such that $h^0(X, R(-2P)) = r - 1$; since $\mathrm{char}(\mathbf{K}) = 0$ this condition is satisfied by a general $A \in X$. Let Y be the curve with $\pi : X \to Y$ as normalization map, $p_a(Y) = p_a(X) + 1$ and $\pi(A)$ as an ordinary cusp. Take a linear space V with $H^0(X, R(-2A)) \subset V \subset H^0(X, R)$, $\dim(V) = r$ and V spanning R; since $r - 1 > 0$, $R(-2A)$ has at most finitely many base points and hence we may take as V a general linear subspace of $H^0(X,R)$ containing $H^0(X,R(-2A))$ and different from $H^0(X,R(-2A))$; in particular the set of all such linear spaces V is parametrized by an irreducible one-dimensional variety. The morphism h_V associated to V factors through π and hence there is $R_V \in \mathrm{Pic}(Y)$ with $\pi^*(R_V) = R$, $h^0(Y,R_V) = r$, R_V spanned and $\pi^*(H^0(Y,R)) = V$. Hence if $V \neq V'$, then R_V and $R_{V'}$ are not isomorphic.

From Remarks 2.11 and 2.12 and the existence part of Brill - Noether theory on smooth curves we obtaing at once the following result.

Corollary 2.13. *Let Y be an integral projective curve with only ordinary nodes and only ordinary cusps as singularities. Set $g := p_a(X)$ and $x := \mathrm{card}(\mathrm{Sing}(Y))$. Fix integers r, d with $r \geq 2$ and $\rho(g-x, r+x, d) \geq 0$. Then there exists an integer $b \leq d$ and $L \in \mathrm{Pic}(Y)$ with $\deg(L) = b$, $h^0(Y,L) \geq r+1$ and L spanned.*

Proof of Theorem 1.2. Let X be a general smooth $(k-y)$-gonal curve of genus $g-x$. Call $M \in \mathrm{Pic}^{k-y}(X)$ the degree $k-y$ pencil. First assume $d \leq g-x+y-1$. We apply [8], part (2) of Cor. 1 of Section 1, to X with respect to the following data: $g' := g-x$, $k' := k-y$, $r = f = 1$, $d = \deg(E) = y$, $\gamma = g'+1$. Since $(g-x+y+2)/2 \leq d \leq g-x+y-1$, we obtain the existence of a spanned $T \in W^1_d(X)$ with $h^0(X, T \otimes M^*) = 0$. Alternatively, we could quote here [6], Theorem 2.2.2. Thus there is an irreducible component W of $W^1_d(X)$ with $W \neq \emptyset$, $\dim(W) \geq \rho(g-x,1,d)$ and such that a general $N \in W$ is spanned and with $h^0(X, N \otimes M^*) = 0$. By our numerical assumptions we have $\rho(g-x,1,d) \geq x$. We claim that for a general ordered set of $x+y$ points $(P_1, \dots, P_{x-y}, A_1, \dots, B_y)$ there is $(Q_1, \dots, Q_{x-y}) \in X^{x-y}$ with $Q_i \neq P_i$ for every i and a locally closed irreducible subset Z of W with $Z \neq \emptyset$, $\dim(Z) = \dim(W) - x$ and such that for every $R \in Z$ we have $h_R(P_i) = h_R(Q_i)$ for every $i \leq x-y$. The claim and 1.2 in this range follow from Remark 2.10, the proof of Remark 2.4 (see in particular Statement $T(k)$ and Remark 2.5. Now assume $g-x+y \leq d \leq g-x+y+k-3$. We apply [8], part (2) of Cor. 1 of Section 1, $k-4$ times with respect to the integers $g' := g-x+y$, $r = f$ with $2 \leq f \leq k-3$, $\gamma = g'+r = g-x+y+f$, $d = \deg(E)$ and conclude in the same way. Now assume $d > g-x$. By assumption we have $x-y < g/2$ and hence $\dim(\mathrm{Pic}^d(X)) \geq x-y$. For a general

$R \in \mathrm{Pic}^d(X)$ we have $h^1(X, R) = 0$ and R is spanned. We apply the previous proof taking as Z a non-empty open subset of $\mathrm{Pic}^d(X)$.

3 – Embeddings in $\mathbf{P}^r$ and the Lüroth semigroup

By [5], Section 1, 2 and 3, for all integers d, g and r with $r \geq 3$ and either $d \geq g+r$ or $d-r < g \leq d-r+[(d-r-2)/(r-2)]$ there is an irreducible component $W(d,g,r)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^r)$ of degree d curves of $\mathbf{P}^r$ with arithmetic genus g such that a general $C \in W(d,g,r)$ is smooth, connected and non-degenerate and with $h^1(C, N_{C,r}) = 0$, where $N_{C,r}$ is the normal bundle of C in $\mathbf{P}^r$. In particular $W(d,g,r)$ is generically smooth and of dimension $h^0(C, N_{C,r}) = (r+1)d - (r-3)(g-1)$. If $\rho(g,r,d) \geq 0$, then $W(d,g,r)$ contains smooth curves with general moduli ([5], Proposition 3.1). If $d \geq g+r$ for a general $C \in W(d,g,r)$ we have $h^1(C, \mathbf{O}_C(1)) = 0$ and hence $h^0(C, \mathbf{O}_C(1)) = d+1-g$. If $d \leq g+r$ for a general $C \in W(d,g,r)$ we have $h^0(C, \mathbf{O}_C(1)) = r+1$.

Theorem 3.1. *Fix integers g, k, x, r with $x > 0$, $k \geq 2+x$ and $r \geq 3$; assume either $d \geq g+r$ or the existence of an integer $t > 0$ and an integer $e \geq 3x$ such that $d = r+2+e+t(r-2)$ and $g = r+2+e-3x+t(r-1)$. Let Y be a general k-gonal nodal curve of genus g and type (x,x). Then there exists a very ample $L \in \mathrm{Pic}^d(Y)$ with $h^0(Y, L) \geq r+1$ and such that for a general embedding $j : Y \to \mathbf{P}^r$ associated to L we have $j(Y) \in W(d,g,r)$.*

Proof. The (omitted) case $x = 0$ is [4], part (a) of Theorem 0.1. The case $d \geq g+r$ is trival, taking non special embeddings. Hence from now on we will assume $d < g+r$. We will modify the proof of [4], Theorem 0.1, to obtain 3.1. For all integers d', g', x' with $0 \leq x' \leq k-2$ and $d' = r+2+e'+t'(r-2)$, $g' = e'-3x'+t'(r-1)$ (as in the statement of 3.1) call $A(d', g', x')$ the following assertion:

Assertion $A(d', g', x')$: there is a pair (C, T) with the following properties:

(i) $C \in W(d', g', r)$ and C satisfies the thesis of 3.1 for the parameters r, k, d', g', x' and $h^1(C, N_{C,r} \otimes \mathbf{I}_Z) = 0$, where Z is the first infinitesimal neighborhood of $\mathrm{Sing}(C)$ in C;

(ii) T is a subset of C_{reg} contained in a positive divisor, D, of the degree $k-x'$ pencil of C with $\text{card}(T) = r+2$ and such that T is in linearly general position, i.e. such that every proper subset T' of T spans a linear subspace $\langle T' \rangle$ of $\mathbf{P}^r$ with $\dim(\langle T' \rangle) = \text{card}(T') - 1$.

Notice that Z in condition (i) of $A(d', g', r)$ is an effective Weil divisor of degree $3(\text{card}(\text{Sing}(C))$. Assume $A(d', g', x')$ and take C, D, T satisfying it. Fix an integer t with $0 \le t \le r+1$. We want to prove $A(d'+r, g'+t, x')$. Since any $r+3$ points of $\mathbf{P}^r$ in lineraly general position are contained in a unique rational normal curve, it is easy to check the existence of a rational normal curve D with D intersecting quasi-transversally C, $D \cap C \subset T$ and $\text{card}(D \cap C) = t+1$. Set $W := C \cup E$. By [16], proof of Theorem 5.2, (or [12] and a dimensional count, or [5]) and [5], 2.3 and 3.1, we have $h^1(W, N_{W,r}) = 0$, $W \in W(d'+r, g'+t, r)_{\text{reg}}$ and the nodal curve W is smoothable. If $t \ge 2$ the nodal curve W is stable, while if $0 \le t \le 1$ it is only semistable. Fix a subset A of E with $A \cap C = \emptyset$ and $\text{card}(A) = t+1$. Let V be the pencil of divisors on E generated by $D \cap C$ and A. Using Knudsen - Harris - Mumford theory of admissible coverings ([13], Section 4) we get that the stable reduction of W in the moduli scheme $M^{-}_{g'+t}$ of stable curves of genus $g'+t$ of the variety of smooth $(k-x')$-gonal curves. We may even assume for general C that W has no non-trivial automorphism, i.e. we may even assume that $M^{-}_{g'+t}$ is smooth at the point corresponding to the stable reduction of W. By [5], Theorem 3.1, or the proof of [16], 5.2, the rational map τ from $\text{Hilb}(\mathbf{P}^r)$ to $M^{-}_{g'+t}$ is dominant. A dimensional count shows that near W the fiber of τ over $\tau(W)$ has the smallest a priori possible dimension. Thus τ is flat at W and hence open at W. The proof of [5], Lemma 1.2, gives also $h^1(W, N_{W,r} \otimes \mathbf{I}_Z) = 0$ and this means that we may do the previous limit without smoothing the nodes in C, i.e. that W is a flat limit inside $\text{Hilb}(\mathbf{P}^r)$ of a family of nodal gonal curves of type (x', x'). Taking A general, we see how to obtain the last condition of $A(d'+r, g'+t, x')$. Hence we may continue and cover all triples (d, g, x') claimed by 3.1 if we may start the induction with some k-gonal curve of type (x', x'). However, at the beginning we only know the case $x' = 0$ (for instance from part (a) of [4], Theorem 0.1). To start this procedure for the first x steps we will increase by one the integer x', i.e. we will pass from x' to $x'+1$. This is possible without modifying the proof of [5], Lemma 1.2, only if

$t \leq r-2$. For simplicity we will use it and hence in the first x steps will loose $3x$ in the upper bound of the genus with respect to the degree. This explains the the term "$-3x$" in the expression of g in the statement of 3.1. We fix $P \in C \cap E$ and call B the union of Z and the first infinitesimal neighborhood of P in W. Thus $\deg(B) = 3 + \deg(Z) = 3 + 3x'$. As in [5], Lemma 2.1, using a Mayer - Vietoris exact sequence and the description of $N_{W,r}$ we obtain $h^1(W, N_{W,r} \otimes \mathbf{I}_B) = 0$. Then we may apply a partial smoothing in which we may preserve $x'+1$ nodes (the ones of $\mathrm{Sing}(C) \cup \{P\}$), obtaing the case $x'+1$ needed.

REMARK 3.2. In the proof of 3.1 if $d \leq g+r$, then we found L with $h^0(Y, L) = r+1$.

PROPOSITION 3.3. *Fix integers g, x, k and d with $k \geq 2+x$, $x > 0$, $g \geq 2k+2x+1$ and $2d \geq 2g+6$. There is a nodal k-gonal curve Y of genus g and type (x,x) with as normalization a general $(k-x)$-gonal curve, X, of genus $g-x$ and with the following property. There is $R \in \mathrm{Pic}^d(Y)$ with $h^0(Y,R) = 3$, R spanned and such that the associated morphism $h_R : Y \to \mathbf{P}^2$ is étale at every point of $\mathrm{Sing}(Y)$, it is birational and the curve $h(Y)$ has only ordinary nodes as singularities except one point, P; P is an ordinary point of multiplicity $\deg(R)-k$, $h_R^{-1}(P) \cap \mathrm{Sing}(Y) = \emptyset$ and the degree $k-x$ pencil on X is induced by the pencil of lines in $\mathbf{P}^2$ passing through P.*

PROOF. Let X be a general smooth k-gonal curve of genus $g-x$. Call $M \in \mathrm{Pic}^{k-x}(X)$ the degree $k-x$ pencil. By [15] (or see [8], theorem in part 2 of the introduction, or, for its statement, the introduction of [1] or [7], 2.2) there is an irreducible component W of $W^2_d(X)$ with $W \neq \emptyset$, $\dim(W) = \rho(g-x,d,2)$ such that a general $N \in W$ is spanned, $h^0(X, N \otimes M^*) = 1$, the corresponding morphism is birational, and its image, C, with only ordinary nodes except one point, P, which is an ordinary point of multiplicity $d-k+x$. Furthermore, M is induced by the pencil of lines through P. Fix x of the singular points, say $B_1, \dots, B_x$, of C and let Y be the partial normalization of C in which we normalize all nodes except the ones corresponding to the points $B_1, \dots, B_x$. Y solves our problem.

DEFINITION 3.4 Let Y be an integral projective curve. Set $LS(Y) := \{d \in \mathbf{Z}$: there is a spanned line bundle L on Y with $\deg(L) = d\}$ and

$LS(Y)' := \{d \in \mathbf{Z}$: there is a spanned rank 1 torsion free sheaf F on Y with $\deg(F) = d\}$. $LS(Y)$ will be called the *Lüroth semigroup* of Y. $LS(Y)$ is a semigroup of the set, $\mathbf{N}$, of non-negative integers. $LS(Y)'$ will be called the *singular Lüroth set* of Y. It is very easy to find a nodal curve Y such that $LS(Y)'$ is not a semigroup (see the proof of Example 2.6 and 1.2 for $x = y = 1 \ll k \ll g$).

Proposition 3.5. *Fix integers g, k and x with $x > 0$, $k \geq 2 + x$ and $g \geq 2k - x + 3$. Let Y be a general k-gonal nodal curve of genus g with type (x, x). Then the singular Lüroth set $LS(Y)'$ of Y contains the integers $t(k - x) + x$ for $1 \leq t \leq \min\{x, [(g - x)/(k - x)]\}$, $t(k - x)$ for $\min\{x + 1, [(g - x)/(k - x)]\} < t \leq [(g - x)/(k - x)]$ and all integers $\beta \geq [(g - x + 3)/2] + x$.*

Proof. Let $\pi : X \to Y$ be the normalization map. Thus X is a general smooth $(k-x)$-gonal curve of genus $g-x$. Let $M \in \mathrm{Pic}^{k-x}(X)$ be the degree $k - x$ pencil. By [3] and [6], Theorem 2.2 (see the discussion in [6], 0.2), the Lüroth semigroup $LS(X)$ of X contains the integers $t(k-x)$ (induced by $M^{\otimes t}$) and all the integers α with $[(g-x+3)/2] \leq \alpha \leq g - x$. If $A \in \mathrm{Pic}(X)$, then $\deg(\pi_*(A)) = \deg(A) + x$ and $h^0(Y, \pi_*(A)) = h^0(X, A)$. Hence to show that $t(k-x)+x \in LS(Y)'$ it is sufficient to show that $\pi_*(M^{\otimes t})$ is spanned, while to show that $[(g - x + 3)/2] + x + e \in LS(Y)'$ it is sufficient to find $A \in \mathrm{Pic}^{[(g-x+3)/2]+e}(X)$, A spanned with $\pi_*(A)$ spanned, i.e. with $\pi_*(A)$ spanned at each point of $\mathrm{Sing}(Y)$. Move Y keeping fixed X, i.e. move the $2x$ points $\pi^{-1}(\mathrm{Sing}(Y))$. Since any symmetric product of X is irreducible and the type is (x, x), for a general Y we obtain that either $\pi_*(M^{\otimes t})$ is spanned or it is not spanned at each point of $\mathrm{Sing}(Y)$. Assume that the second possibility occurs and call B the subsheaf of $\pi_*(M^{\otimes t})$ spanned by $H^0(Y, \pi_*(M^{\otimes t}))$. We may even assume that t is the first integer for which this possibility occurs. Again, by the irreducibility of the symmetric product we obtain that $\pi_*(M^{\otimes t})/B$ has the same length, v, at each point of $\mathrm{Sing}(Y)$. By [3] we have $h^0(X, M^{\otimes t}) = t + 1$ for $t \leq [(g - x)/(k - x)]$. First assume $t < x$. X and hence M are fixed. By Remark 2.10 for general Y there is no spanned $R \in \mathrm{Pic}(Y)$ with $M^{\otimes t} \cong \pi^*(R)$. If $t \geq x + 1$ we have $h^0(X, M^{\otimes t}) \geq x + 2$ and hence applying x times Proposition 2.7 we obtain the existence of a spanned $R \in \mathrm{Pic}(Y)$ with $M^{\otimes t} \cong \pi^*(R)$. Now take $A \in \mathrm{Pic}(X)$

computing one of the integers α of $LS(X)$ with $[(g-x+3)/2] \le \alpha \le g-x$ and A general. By [6], Theorem 2.2, and the generality of A we have $h^0(X,A) = 2$. Thus using Remark 2.10 we obtain easily that for general Y and general A the corresponding sheaf $\pi_*(A)$ has no subsheaf B with length $(\pi_*(A)/B) \ge x$ and $h^0(Y,B) = 2$, i.e. we obtain the spannedness of $\pi_*(A)$. Every integer $u \ge g+1$ may be realized as an element of $LS(Y)$ and hence of $LS(Y)'$ just taking a general non-special $R \in \mathrm{Pic}^u(Y)$.

Proposition 3.6. *Fix integers g, k, x, y and t with $x > 0$, $x \ge y \ge 0$, $k \ge 2+x$ and $g \ge 2k-x+3$ and $t < (g-x)/2+1$. Let Y be a general k-gonal nodal curve of genus g with type (x,y). If $t \ne a(k-y)$ for all integers a, then $t \notin LS(Y)$. If $t = a(k-y)$ for some integer a and $y > a$, then $t \notin LS(Y)$.*

Proof. Let $\pi : X \to Y$ be the normalization map. Thus X is a general smooth $(k-y)$-gonal curve of genus $g-x$. Let $M \in \mathrm{Pic}^{k-y}(X)$ be the degree $k-y$ pencil. By [1], Theorem 0.1, there is no $L \in \mathrm{Pic}^t(X)$ with L spanned, unless $t = a(k-y)$ for some integer a and in this case we have $L \cong M^{\otimes a}$. Hence $t \notin LS(Y)$ if $t \ne a(k-y)$ for every integer a. Assume $t = a(k-y)$. By [3] we have $h^0(X, M^{\otimes a}) = a+1$. Apply Remark 2.10 and the assuption $y > a$.

Lemma 3.7. *Fix integers g, k and x with $x \ge 0$ and $g \ge 2k+x+3$. Let Y be a general k-gonal nodal curve of type $(x,0)$ and $M \in \mathrm{Pic}^k(Y)$ the degree k spanned line bundle on Y. Then for all integers t with $0 \le t \le [g/(k-1)]$ we have $h^0(Y, M^{\otimes t}) = t+1T$.*

Proof. Let $\pi : X \to Y$ be the normalization. Since $h^0(Y,M) \ge 2$ the value for $h^0(Y, M^{\otimes t})$ is the minimal a priori possible and hence we may use semicontinuity. By definition of general nodal curve of type $(x,0)$, X is a general smooth k-gonal curve of type $(x,0)$. By [3] we have $h^0(X, \pi^*(M^{\otimes t})) = t+1$ for $t \le [(g-x)/(k-1)]$. Thus we may assume $[(g-x)/(k-1)] < t \le [g/(k-1)]$. We modify the proof of [3]. We need to find an integral nodal curve, T, of type (k,a) (some a) on $\mathbf{P}^1 \times \mathbf{P}^1$ with normalization of genus $g-x$, at least x nodes and such that a subset, S, of $\mathrm{Sing}(T)$ with $\mathrm{card}(S) = \mathrm{card}(\mathrm{Sing}(T)) - x$ satisfies a certain cohomological condition (say $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{I}_S(k-2,b)) = 0$ for a suitable b). The existence of an integral nodal curve in $\mathbf{P}^1 \times \mathbf{P}^1$ with that numerical invariants follows from [1], Proposition 3.7 and Proposition 4.1. By semicontinuity we may even choose as x "omitted" nodes for the cohomological condition is any subset of $\mathrm{Sing}(T)$ we prefer and hence it is sufficient to have $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{I}_{\mathrm{Sing}(T)}(k-2,b)) \leq x$. This is even easier than in [3] (case $x = 0$ with cardinality of the singular set $\mathrm{card}(\mathrm{Sing}(T)) - x$).

PROPOSITION 3.8. *Fix integers g, k and x with $x \geq 0$ and $g \geq 2k + x + 3$. Let Y be a general k-gonal nodal curve of type $(x, 0)$ and $M \in \mathrm{Pic}^k(Y)$ the degree k spanned line bundle on Y. For any integer z with $1 \leq z < (g - x + 3)/2$ the following conditions are equivalent:*

(i) *$z = tk$ for some integer t;*
(ii) *$z \in LS(Y)$;*

Furthermore, if $z = tk < (g - x + 3)/2$ the only rank 1 *spanned line bundle, L, with* $\deg(L) = z$ *is $M^{\otimes t}$.*

PROOF. Since M is spanned, $tk \in LS(Y)$ for every integer t. Thus it is sufficient to show that every spanned line bundle L with $\deg(L) \leq (g - x + 3)/2$ is of the form $M^{\otimes t}$. Let $\pi : X \to Y$ be the normalization. Since $\pi^*(M)$ is a spanned line bundle on the general k-gonal curve X, this is [1], Theorem 2.6.

4 – Seminormal singularities

In this section we will consider seminormal curves in the sense of [17] and [9], i.e. curves with the simplest singularities compatible with their number of branches: if the singularity has r branches, then it is formally equivalent to the germ at $0 \in \mathbf{K}^r$ of the union of the r coordinate axis. A seminormal curve singularity is Gorenstein if and only if it is an ordinary double point. The conductor of a seminormal one-dimensional local ring R is the maximal ideal of R.

DEFINITION 4.1. Let Y be a projective seminormal curve and $\pi : X \to Y$ its normalization. Set $g := p_a(Y)$ and $q := p_a(X)$. For every $P \in \mathrm{Sing}(Y)$, set $s(P) := \mathrm{card}(\pi^{-1}(P))$. We may order the integers $s(P)$, $P \in \mathrm{Sing}(Y)$ in non-decreasing order, allowing repetitions. If $\mathbf{K} = \mathbf{C}$ the topological type of $Y(\mathbf{C})$ is unique determined by the integers

$(g, q; \mathrm{card}(\mathrm{Sing}(Y); s(P)_{P\in \mathrm{Sing}(Y)})$. Notice that $g = q + \sum_{P\in \mathrm{Sing}(Y)} s(P) - \mathrm{card}(\mathrm{Sing}(Y))$. We call this set the *numerical data*. The *weight* weight(τ) of the numerical data τ or of the curve Y is the maximum of the integers $s(P)$, $P \in \mathrm{Sing}(Y)$. We will say that Y is *general* or that it is general for a prescribed numerical data if X is a general smooth curve of genus q and the set $\pi^{-1}(\mathrm{Sing}(Y))$ is general in X. We will say that Y is *general for the fixed normalization* X if $\pi^{-1}(\mathrm{Sing}(Y))$ is general in X.

Remark 4.2. Let Y be a seminormal curve and $\pi : X \to Y$ its normalization. Fix $L \in \mathrm{Pic}(Y)$. For every $f \in H^0(Y, L)$ and $P \in \mathrm{Sing}(Y)$ with f vanishing at P the section $\pi^*(f)$ of $\pi^*(L)$ vanishes at each point of $\pi^{-1}(\mathrm{Sing}(P))$. Fix $h \in H^0(X, \pi^*(L))$ and assume that for every $P \in \mathrm{Sing}(Y)$ h has the same value for a fixed trivialization of L near P and hence of $\pi^*(L)$ around $\pi^{-1}(P)$ at each point of $\pi^{-1}(P)$. Then h is of the form $\pi^*(f)$ for some $f \in H^0(Y, L)$ because conductor of a seminormal one-dimensional local ring R is the maximal ideal of R.

Remark 4.3. Let Z be an integral projective curve, $L \in \mathrm{Pic}(Z)$, $V \subseteq H^0(Z, L)$ a linear subspace with $\dim(V) \geq 2$. Then for every $P \in Z$ there is subspace $V(P)$ of V with $\dim(V(P)) \geq \dim(V) - 1$ and such that every $f \in V(P)$ vanishes at P.

Remarks 4.2 and 4.3 and the definition of general seminormal curve with fixed normalization give at once the following result.

Lemma 4.4. *Let X be a smooth projective curve of genus $q \geq 0$. Fix an integer d and let x be the maximal dimension of an irreducible component of $G^r_d(X)$. Fix a type τ for seminormal curves with normalization of genus q and* weight$(\tau) > x$. *Let Y be the general seminormal curve of type τ with X as normalization. Then for every $L \in \mathrm{Pic}(Y)$ with* $\deg(L) \leq d$ *we have $h^0(Y, L) \leq r$.*

Remark 4.5. Use the notation of Lemma 4.4. Notice that for a fixed q and any genus q curve we may find a type τ with weight$(\tau) > d$ but $d < g$. In this sense there is no hope just using the Brill - Noether numbers $\rho(g, r, d)$ to have on general seminormal curves the usual Brill - Noether theory using line bundles, even if we do not require that the line bundles considered are spanned.

The proof of Remark 2.4 and Remark 2.5 give the following result.

PROPOSITION 4.6. *Fix integers g, q, d and e with $0 < e \leq g - q$. Let X be a smooth curve of genus q and assume the existence of an irreducible component, T, of $G^1_d(X)$ with $\dim(T) \geq g - q + e$ and such that for a general pair $(R, V) \in T$ the line bundle R is spanned by V. Fix a type τ for seminormal curves with genus g, normalization of genus q and e singular points. Let Y be a general seminormal curve with Y as normalization and type τ. Then there is a spanned line bundle of degree d on Y.*

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Indirizzo dell'Autore:

E. Ballico – Dept. of Mathematics – Università di Trento – 38050 Povo (TN) (Italia)
E-mail: ballico@science.unitn.it

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