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Einstein's field equations in the light of constrained hyperbolic systems

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Abstract: *Results previously known in the literature, on the hyperbolicity of Einstein's equations, are here quoted and improved. This aim is reached by applying recent techniques on constrained hyperbolic systems. The symmetric hyperbolic form is obtained, also in the four-dimensional formalism using harmonic coordinates. The case of sources due to the presence of matter is also considered, in particular from the view point of Extended Thermodynamics*

1 – Introduction

The importance of Einstein's equations is outstanding and needs no comments. The study of their hyperbolicity presents also some interesting aspects. Obviously, we don't have here the presumption to diminish previously results obtained on this subject by authoritative experts. *We want only to show how a recent general theory on hyperbolic systems, with differential and algebraic constraints, can be successfully applied also to this important problem*; indeed, the validity of the general theory is strengthened because our results are comparable with those obtained in other ways by the above mentioned experts.

Key Words and Phrases: *Einstein's field equations – Harmonic coordinates – Constrained Hyperbolic systems – Symmetric hyperbolic systems – Extended thermodynamics.*
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Let us start noticing that in [1] Strumia has shown how Einstein's equations can be reduced to a first order system of partial differential equations, but one of the hyperbolicity conditions according to FRIEDRICHS (see ref. [2], [3]) seems to fail, the one referring to the possibility, roughly speaking, to obtain the time derivatives as functions of the other quantities; see also [4] for other details. Although apparently strange, this result is just what we would expect from the covariance property; in fact, if the metric tensor $g_{\mu\nu}$ is a solution of Einstein's equations, then so is $g_{\mu'\nu'} = g_{\alpha\beta}[x^\rho(x^{\lambda'})]\partial_{\mu'}x^\alpha \partial_{\nu'}x^\beta$ i.e. the expression determined from $g_{\mu\nu}$ by a general coordinate transformation $x \to x'$. This consideration can be found in papers such as [5]-[7].

In [8] this failure of Einstein's equations to determine $g_{\mu\nu}$ uniquely is compared to the failure of Maxwell's equations to determine the vector potential uniquely. Here we propose another comparison which will be the thread of our subsequent arguments, i.e., the problem of determining the geodesic curves of a surface Σ; for the sake of simplicity, we shall consider Σ belonging to a 3-dimensional euclidean space. If $P = P(u, v)$ are the parametric equations of Σ and $P(\lambda) = P[u(\lambda), v(\lambda)]$ the equations of a geodesic curve γ, then

$$\frac{P'(\lambda)}{|P'(\lambda)|}$$

is the tangent unit vector and one has

$$\frac{d}{d\lambda}\left[\frac{P'(\lambda)}{|P'(\lambda)|}\right] = \frac{1}{\rho}\,\mathbf{n}\,\frac{ds}{d\lambda},$$

where ρ is the radius of curvature, $\mathbf{n}$ is the normal unit vector and s is its arc-length parameter; therefore, the equations of γ can be obtained from the system

$$\begin{cases} \dfrac{\partial P}{\partial u} \cdot \dfrac{d}{d\lambda}\left[\dfrac{P'(\lambda)}{|P'(\lambda)|}\right] = 0, \\[2ex] \dfrac{\partial P}{\partial v} \cdot \dfrac{d}{d\lambda}\left[\dfrac{P'(\lambda)}{|P'(\lambda)|}\right] = 0. \end{cases} \tag{1.1}$$

These equations don't determine $u(\lambda)$, $v(\lambda)$, because their linear combination, through the coefficients u' and v', is an identity; as Einstein's equations don't determine $g_{\mu\nu}$ due to the arbitrariness of the coordinates transformation, so the equations (1.1) fail to determine u and v because the parameter λ is arbitrary. One may proceed in one of the following ways:

1. Require $|P'(\lambda)| = 1$, in addition to equations (1.1); in other words, we require that λ is the arc-length parameter.
2. Require $d|P'(\lambda)|/d\lambda = 0, \quad |P'(\lambda_0)| = 1$, with λ_0 initial value of λ. Note that these conditions, together with (1.1), are equivalent to

$$\begin{cases} \dfrac{\partial P}{\partial u} \cdot P''(\lambda) = 0, \\ \dfrac{\partial P}{\partial v} \cdot P''(\lambda) = 0, \\ |P'(\lambda_0)| = 1. \end{cases}$$

   This last condition may also be omitted, being content with a λ which is a linear function of the arc-length parameter, without assuming $\lambda = s$.

In the same manner we will investigate the hyperbolicity of Einstein's equations in one of the following ways:

1. Require that the coordinates x^α aren't the most general ones, but the *harmonic coordinates* defined by $\Gamma_\alpha = 0$ (for the expression of Γ^α see the equation $(1.2)_6$ below). *In this way, the Einstein's equations become equations with differential and algebraic constraints.* In this framework they will be studied in Section 2, by applying the general methods outlined in ref. [4], where they have been successfully applied to the equations of relativistic fluid dynamics. See also refs. [9]-[13] for other examples of physical application, such as the relativistic magneto-fluid dynamics, the Maxwell electrodynamics, the equations of the superfluid and those of ultra relativistic gases.
   In ref. [4] it is shown also a method to eliminate the algebraic constraints, in a manner which corresponds to the following method (2).
2. Require $\partial_t \Gamma_\alpha = 0$, $(\Gamma_\alpha)_\Sigma = 0$, where $(\Gamma_\alpha)_\Sigma$ is the value of Γ_α calculated in the initial manifold Σ. This approach will be followed in Section 3. We will see that these further assumptions are equivalent

to $\partial_{(\beta}\Gamma_{\alpha)} = 0$, $(\Gamma_\alpha)_\Sigma = 0$, which have the advantage to be written in 4-dimensional notation.
Following the general methods of paper [4], we obtain the equations found in [5]-[7] in another way; in ref. [5], Fischer and Marsden have transformed these equations in the symmetric hyperbolic form, but in 3-dimensional notation. *Here we reach the same result, but in* 4-*dimensional notation.*
A third approach, which is present in literature, will be exploited in Section 4.
The case of sources due to the presence of matter will be considered in Section 5, *showing how the symmetric hyperbolic form can be obtained also in this case, and also with the equations of relativistic extended thermodynamics and similar* [14]-[16].

We conclude this section reporting the Einstein's equations.

$$G_{\mu\nu} = \chi T_{\mu\nu}, \tag{1.2}$$

with χ the einsteinian gravitational constant, $T_{\mu\nu}$ the energy tensor,

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R \quad \text{(Einstein tensor)},\\
R &= g^{\alpha\beta} R_{\alpha\beta} \quad \text{(scalar curvature)},\\
R_{\mu\nu} &= \frac{1}{2}\, g^{\alpha\beta}[-\partial^2_{\alpha\beta} g_{\mu\nu} - \partial^2_{\mu\nu} g_{\alpha\beta} + \partial^2_{\alpha\nu} g_{\mu\beta} + \partial^2_{\beta\mu} g_{\nu\alpha}] +\\
&\quad - g_{\rho\sigma}\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\alpha\beta} g^{\alpha\beta} + g^{\alpha\beta} g_{\rho\sigma}\Gamma^\rho_{\mu\alpha}\Gamma^\sigma_{\nu\beta} =\\
&= \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} \quad \text{(Ricci tensor)},\\
\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\, g^{\mu\lambda}\left(\partial_\beta g_{\lambda\alpha} + \partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta}\right) \quad \text{(Christoffel symbols)},\\
\Gamma^\mu &= \Gamma^\mu_{\alpha\beta} g^{\alpha\beta} \quad \text{(Lanczos symbols)}.
\end{aligned}$$

Obviously the Einstein's equation (1.2) can also be written in the form

$$R_{\mu\nu} = \chi\left(T_{\mu\nu} - \frac{1}{2}\, T_{\alpha\beta} g^{\alpha\beta} g_{\mu\nu}\right). \tag{1.3}$$

When sources are not present, i.e. $T_{\mu\nu} = 0$, we have the so called "*exterior case*" and equation (1.3) reduce to

$$R_{\mu\nu} = 0. \tag{1.4}$$

2 – The Einstein's equations in harmonic coordinates

The transformation equations of the contracted Christoffel symbols Γ^μ, introduced by Lanczos, are

$$(\Gamma')^\lambda = \Gamma^\rho \partial_\rho (x')^\lambda - g^{\rho\sigma} \partial^2_{\rho\sigma} (x')^\lambda .$$

Hence we can always find a coordinate system $(x')^\lambda$ where $(\Gamma')^\lambda$ vanishes; such coordinates are called *harmonic coordinates.*

From now on, in this section, we will impose to be already in harmonic coordinates, so that we have $\Gamma^\mu = 0$. The Einstein's equation (1.4) in the exterior case can be reduced to a first order system by setting $\partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}$; one obtains

$$\begin{cases} \partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}, \\[1ex] \dfrac{1}{2}\, g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}] = F_{\mu\nu}(g_{\alpha\beta}, \omega_{\alpha\beta\gamma}), \\[1ex] \partial_{[\beta}\omega_{\alpha]\mu\nu} = 0, \\[1ex] \dfrac{1}{2}\, g^{\alpha\beta}\,(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta}) = 0 \quad (\text{i.e. } \Gamma_\lambda = 0). \end{cases} \tag{2.1}$$

Equations $(2.1)_{1\text{-}3}$ *constitute a system of* 110 *equations in the* 50 *unknowns* $g_{\mu\nu}$, $\omega_{\alpha\mu\nu}$, *restricted by the four algebraic constraints* $(2.1)_4$, *so that we have only* 46 *independent variables; obviously, in the system* $(2.1)_{1\text{-}3}$ *there are also* 66 *differential constraints.*

Now a general method to study the hyperbolicity of systems with algebraic and differential constraints has been proposed in ref. [4] and already applied with success to important physical problems.

Here we find another interesting example of physical application. In a few words the method, applied to the present case, consists in multiplying the system $(2.1)_{1\text{-}3}$ on the left by a suitable matrix of rank 46 so that the resulting system is hyperbolic in the time direction defined by t_α, with $t_\alpha t^\alpha = -1$.

Alternatively, this result may be obtained by taking suitable linear combinations of the equations $(2.1)_{1\text{-}3}$. *A possible choice is to consider the system*

$$
(2.2) \qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ h^{\mu\nu}_{\gamma\delta}\ \frac{1}{2}\ g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}] = \\ \qquad = h^{\mu\nu}_{\gamma\delta} F_{\mu\nu}(g_{\alpha\beta}, \omega_{\alpha\beta\varepsilon}), \\ t^\alpha h^{\mu\nu}_{\gamma\delta}[\partial_{[\beta}\omega_{\alpha]\mu\nu}] = 0, \\ \frac{1}{2}\ g^{\alpha\beta}(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta}) = 0, \end{cases}
$$

with

$$
(2.3) \qquad h^{\mu\nu}_{\gamma\delta} = g^{(\mu}_\gamma g^{\nu)}_\delta - \frac{1}{4} g^{\mu\nu} g_{\gamma\delta}\,.
$$

We prove now that the system $(2.2)_{1\text{-}3}$ *is hyperbolic.* Firstly, we consider the system

$$
(2.4) \qquad \begin{cases} t^\alpha t_\alpha d g_{\mu\nu} = 0, \\ h^{\mu\nu}_{\gamma\delta}\ \frac{1}{2}\ g^{\alpha\beta}[-t_\alpha d\omega_{\beta\mu\nu} - t_{(\mu} d\omega_{\nu)\alpha\beta} + t_\alpha d\omega_{\nu\mu\beta} + t_\beta d\omega_{\mu\nu\alpha}] = 0, \\ t^\alpha h^{\mu\nu}_{\gamma\delta}[t_{[\beta} d\omega_{\alpha]\mu\nu}] = 0, \\ d\left[\frac{1}{2}\ g^{\alpha\beta}\left(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta}\right)\right] = 0, \end{cases}
$$

in the unknowns $dg_{\mu\nu}$, $d\omega_{\alpha\beta\gamma}$. It is easy to see that this system has only the solution $dg_{\mu\nu} = 0$, $d\omega_{\alpha\beta\gamma} = 0$. In fact, equations $(2.4)_{1,3}$ yield $dg_{\mu\nu} = 0$,

$$
(2.5) \qquad d\omega_{\beta\mu\nu} = -t_\beta t^\delta d\omega_{\delta\mu\nu} + X_\beta g_{\mu\nu},
$$

for every X_β such that

$$
(2.6) \qquad X_\beta t^\beta = 0.
$$

After that, equation $(2.4)_4$ gives

$$X_\lambda = -t^\beta t^\delta d\omega_{\delta\beta\lambda} + \frac{1}{2} g^{\alpha\beta} t_\lambda t^\delta d\omega_{\delta\alpha\beta}.$$

By substituting in (2.5), (2.6), we obtain

$$g^{\alpha\beta} t^\delta d\omega_{\delta\alpha\beta} = -2 t^\delta t^\beta t^\lambda d\omega_{\delta\beta\lambda} \tag{2.7}$$

and

$$d\omega_{\beta\mu\nu} = -t_\beta t^\delta d\omega_{\delta\mu\nu} - g_{\mu\nu} t^\delta t^\rho d\omega_{\delta\rho\gamma}(g^\gamma_\beta + t_\beta t^\gamma).$$

At last, equation $(2.4)_2$ yields $h^{\gamma\delta}_{\mu\nu} t^\beta d\omega_{\beta\mu\nu} = 0$ from which and (2.7) the relation $d\omega_{\beta\mu\nu} = 0$ follows. This result proves that, from the system (2.2), the time derivatives can be obtained as functions of the other quantities.

To prove the hyperbolicity of the system (2.2) it suffices now to see that the following system

$$\begin{cases} t^\alpha \varphi_\alpha dg_{\mu\nu} = 0, \\ h^{\mu\nu}_{\gamma\delta} \, \frac{1}{2} \, g^{\alpha\beta}[-\varphi_\alpha d\omega_{\beta\mu\nu} - \varphi_{(\mu} d\omega_{\nu)\alpha\beta} + \varphi_\alpha d\omega_{\nu\mu\beta} + \varphi_\beta d\omega_{\mu\nu\alpha}] = 0, \\ t^\alpha h^{\mu\nu}_{\gamma\delta}[\varphi_{[\beta} d\omega_{\alpha]\mu\nu}] = 0, \\ d\left[\frac{1}{2} \, g^{\alpha\beta}\left(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta}\right)\right] = 0, \end{cases} \tag{2.8}$$

with $\varphi_\alpha = n_\alpha - \lambda t_\alpha$, has real eigenvalues λ and 46 linearly independent (l.i.) eigenvectors $dg_{\mu\nu}$, $d\omega_{\beta\mu\nu}$, for every n_α such that $n_\alpha t^\alpha = 0$, $n_\alpha n^\alpha = 1$.

Also this condition is satisfied: in fact in correspondence to the eigenvalue $\lambda = 0$, equation $(2.8)_1$ is an identity, while equation $(2.8)_3$ is equivalent to $t^\alpha h^{\mu\nu}_{\gamma\delta} d\omega_{\alpha\mu\nu} = 0$. Therefore, we have 22 equations for 50 unknowns and, consequently, 28 l.i. eigenvectors.

In correspondence to $\lambda = \pm 1$ (from which $\varphi_\beta \varphi^\beta = 0$, $\lambda \neq 0$) we obtain the eigenvectors

$$dg_{\mu\nu} = 0, \qquad d\omega_{\beta\mu\nu} = \frac{1}{\lambda}(\varphi_\beta y_{<\mu\nu>} + \varphi^\delta y_{<\delta\beta>} g_{\mu\nu}),$$

where $y_{<\mu\nu>}$ is an arbitrary symmetric traceless tensor; therefore there are 2×9 l.i. eigenvectors corresponding to $\lambda = \pm 1$. In this way, the hyperbolicity of system (2.2) has been proved.

In ref. [4] we find also a method to get rid of the algebraic constraints; its application to our case leads, as a first step, to the system

$$
(2.9)\qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ h^{\mu\nu}_{\gamma\delta}\,\dfrac{1}{2}\,g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}]+ \\ \quad +g^{\mu\nu} g_{\gamma\delta} t^\alpha \partial_\alpha \omega_{\mu\nu} = h^{\mu\nu}_{\gamma\delta} F_{\mu\nu}, \\ t^\alpha h^{\mu\nu}_{\gamma\delta}(\partial_{[\beta}\omega_{\alpha]\mu\nu}) + t^\alpha h^{\mu\nu}_{\gamma\delta} t_\beta \partial_\alpha \omega_{\mu\nu} + g_{\gamma\delta} t^\alpha \partial_\alpha \psi_\beta = 0, \end{cases}
$$

for the determination of the variables $g_{\mu\nu}$, $\omega_{\beta\mu\nu}$, $\omega_{\mu\nu} = \omega_{\nu\mu}$, ψ_β, constrained by $(2.1)_4$. This system is also hyperbolic and has the advantage to have an equal number of equations and of independent variables; when $\omega_{\mu\nu} = 0$, $\psi_\beta = 0$ it reduces to the system (2.2) and, moreover, if $\omega_{\mu\nu} = 0$, $\psi_\beta = 0$ on an initial hypersurface Σ, then $\omega_{\mu\nu} = 0$, $\psi_\beta = 0$ will propagate also off Σ.

The system (2.9) has been obtained from (2.2) by considering more equations and more independent variables, an idea somehow similar to that conceived in Extended Thermodynamics.

The second, and last, step leads to the system

$$
(2.10)\qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ h^{\mu\nu}_{\gamma\delta}\,\dfrac{1}{2}\,g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}]+ \\ \quad +g^{\mu\nu} g_{\gamma\delta} t^\alpha \partial_\alpha \omega_{\mu\nu} = h^{\mu\nu}_{\gamma\delta} F_{\mu\nu}, \\ t^\alpha h^{\mu\nu}_{\gamma\delta}(\partial_{[\beta}\omega_{\alpha]\mu\nu}) + t^\alpha h^{\mu\nu}_{\gamma\delta} t_\beta \partial_\alpha \omega_{\mu\nu} + \\ \quad + g_{\gamma\delta} t^\alpha \partial_\alpha \left[\dfrac{1}{2}\,g^{\mu\nu}(2\omega_{\mu\nu\beta} - \omega_{\beta\mu\nu})\right] = 0, \end{cases}
$$

in the independent variables $g_{\mu\nu}$, $\omega_{\beta\mu\nu}$, $\omega_{\mu\nu}$; *in this way all the constraints, both differential and algebraic, have been eliminated still maintaining the property to be hyperbolic and to have an equal number of equations and of independent variables.* Obviously, by setting $\omega_{\mu\nu} = 0$ in equations (2.10), the differential constraints arising are only identity and one obtains a system in the "old variables" but without algebraic constraints; this system is hyperbolic and, moreover, if $(2.1)_4$ holds on an initial hypersurface Σ, then it will be satisfied also off Σ.

Another method to obtain this result is exposed in the next section.

3 – The Einstein's equations with the further condition $\partial_{(\alpha}\Gamma_{\beta)} = 0$

We can easily see that the following relations hold

$$(3.1) \qquad \Gamma_\beta = 0 \qquad \Leftrightarrow \qquad \begin{cases} \partial_0 \Gamma_\beta = 0 \\ (\Gamma_\beta)_\Sigma = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \partial_{(\alpha}\Gamma_{\beta)} = 0 \\ (\Gamma_\beta)_\Sigma = 0, \end{cases}$$

where $(\Gamma_\beta)_\Sigma$ is the value of Γ_β on an initial space-like hypersurface Σ. The first equivalence in (3.1) is trivial; the second one is based on the fact that $\partial_0\Gamma_\beta = 0, (\Gamma_\beta)_\Sigma = 0$ implies $\partial_0(\partial_i\Gamma_\beta) = 0$, $(\partial_i\Gamma_\beta)_\Sigma = 0$, from which $\partial_i\Gamma_\beta = 0$ follows. Vice versa, if the equations in the right hand side of (3.1) hold, then we have

$$\begin{cases} \partial_0\Gamma_0 = 0 \quad \Rightarrow \quad \partial_0\left(\partial_i\Gamma_0\right) = 0, \\ (\partial_i\Gamma_0)_\Sigma = 0, \end{cases}$$

and, consequently, $\partial_i\Gamma_0 = 0$; this result allows to obtain, from $\partial_{(\alpha}\Gamma_{\beta)} = 0$, for $\alpha, \beta = 0, \ldots, i$, that $\partial_0\Gamma_i = 0$. In this way the second equivalence in (3.1) has been proved.

This suggest to consider the equations

$$(3.2) \qquad \partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}, \qquad R_{\mu\nu} = 0, \qquad \partial_{[\beta}\omega_{\alpha]\mu\nu} = 0, \qquad \partial_{[\mu}\Gamma_{\nu]} = 0.$$

This system has more differential constraints than the system (2.1), but has no algebraic constraints because $(2.1)_4$ has to be imposed only on the initial manifold. The method in ref. [4] already applied in Section 2 to equations (2.1), can now be applied to the system (3.2). One obtains

$$(3.3) \qquad \begin{aligned} &t^\alpha\partial_\alpha g_{\mu\nu} = t^\alpha\omega_{\alpha\mu\nu}, \qquad R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta}g^\gamma_{(\mu}g^\delta_{\nu)}\partial_{[\beta}\omega_{\gamma]\delta\alpha} = 0, \\ &t^\alpha\left(\partial_{[\beta}\omega_{\alpha]\mu\nu}\right) = 0, \end{aligned}$$

which is the new counterpart of system (2.2).

We note that the equations $(3.3)_2$ substantially coincide with those proposed by Fourès-Bruhat, Fischer and Marsden in refs. [5], [7], i.e.,

$$R_{\mu\nu} - g_{\alpha(\mu}\partial_{\nu)}\Gamma^\alpha - 2g^{\alpha\beta}g^\gamma_{(\mu}g^\delta_{\nu)}\partial_{[\beta}\omega_{\gamma]\delta\alpha} = 0$$

or, equivalently,

$$R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta} g^{\gamma}_{(\mu} g^{\delta}_{\nu)} \partial_{[\beta}\omega_{\gamma]\delta\alpha} + \Gamma^{\alpha}\omega_{(\nu\mu)\alpha} = 0;$$

the only difference is in the last term which doesn't involve the derivatives of the variables and, therefore, does not affect the study of hyperbolicity.

We retain very interesting to see how a general method such that of ref. [4] *leads to equations obtained in other ways in literature.* These equations $(3.3)_2$ writes explicitly

$$-\frac{1}{2}\, g^{\alpha\beta}\partial_{\alpha}\omega_{\beta\mu\nu} = F_{\mu\nu} - \frac{1}{2}\, g^{\gamma}_{(\mu} g^{\delta}_{\nu)} \omega^{\alpha\beta}_{\gamma}(2\omega_{\alpha\beta\delta} - \omega_{\delta\alpha\beta}), \tag{3.4}$$

where we have used the relation $\partial_{\gamma} g^{\psi\eta} = -\omega_{\gamma}{}^{\psi\eta}$ which comes from $g^{\psi\theta}$ contracted with the derivative with respect to x_{γ} of the relation

$$g_{\theta\delta} g^{\delta\eta} = \delta^{\eta}_{\theta}.$$

To prove the hyperbolicity of the system (3.3) is now an easy task because the system

$$-dg_{\mu\nu} = 0, \qquad -\frac{1}{2}\, t^{\beta} d\omega_{\beta\mu\nu} = 0, \qquad t^{\alpha} t_{[\beta} d\omega_{\alpha]\mu\nu} = 0,$$

imply $dg_{\mu\nu} = 0$, $d\omega_{\beta\mu\nu} = 0$, while the system

$$\lambda dg_{\mu\nu} = 0, \qquad -\frac{1}{2}\, \varphi^{\beta} d\omega_{\beta\mu\nu} = 0, \qquad t^{\alpha} \varphi_{[\beta} d\omega_{\alpha]\mu\nu} = 0,$$

has 50 l.i. eigenvectors, i.e.

- the 30 l.i. solutions of $t^{\beta} d\omega_{\beta\mu\nu} = 0$, $n^{\beta} d\omega_{\beta\mu\nu} = 0$, corresponding to the real eigenvalue $\lambda = 0$,
- the 20 l.i. solutions of $dg_{\mu\nu} = 0$, $d\omega_{\beta\mu\nu} = x_{\mu\nu}\varphi_{\beta}$, (with $x_{\mu\nu}$ an arbitrary symmetric tensor) corresponding to the real eigenvalues $\lambda = \pm 1$ (i.e., $\varphi_{\beta}\varphi^{\beta} = 0$).

But a more interesting aspect is that the system (3.3) *can be put in the symmetric form*; this result has been obtained by Fischer and Marsden for their system of equations, but in 3-dimensional formalism. Here we obtain in 4-dimensional notation the same result for our system (3.3). In fact, it is equivalent to

$$\left\{\begin{aligned} & t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ & t_\tau \left(-\frac{1}{2}\, g^{\alpha\beta} \partial_\alpha \omega_{\beta\mu\nu}\right) - t^\alpha \partial_{[\tau} \omega_{\alpha]\mu\nu} = t_\tau K_{\mu\nu}, \end{aligned}\right. \tag{3.5}$$

where we have used the expression (3.4) for the equations $(3.3)_2$ and we have called $K_{\mu\nu}$ the second member of (3.4).(1)

The system (3.5) is symmetric; in fact, if we take a linear combination of its left-hand sides through the coefficients $\lambda^{\mu\nu}$, $\lambda^{\tau\mu\nu}$ and substitute ∂_α with δ, we obtain(2)

$$\begin{aligned} & t^\alpha \left(\lambda^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2}\, \lambda^{\tau\mu\nu} \delta \omega_{\tau\mu\nu}\right) + \\ & - \frac{1}{2} \left(t^\beta g^{\alpha\beta'} \lambda_{\beta'\mu'\nu'} \delta\omega_{\beta\mu\nu} + t^\tau \lambda_{\tau\mu'\nu'} g^{\alpha\beta} \delta\omega_{\beta\mu\nu}\right) g^{\mu'\mu} g^{\nu'\nu}. \end{aligned}$$

This expression doesn't change if we exchange $\lambda_{\mu\nu}$ with $\delta g_{\mu\nu}$ and $\lambda_{\tau\mu\nu}$ with $\delta\omega_{\tau\mu\nu}$, thus proving the symmetric form of (3.5).

The result of this section has been achieved at the cost of dealing with modified Einstein's equations, i.e., $(3.3)_2$. A more elegant result will be obtained in the next section by introducing suitable equations for Γ^α.

4 – The unmodified Einstein's equations

In ref. [7], Fischer et al. obtain equations involving Γ^α, drawing it from a consequence of Bianchi identities, i.e., $\nabla_\alpha G^{\alpha\beta} = 0$ (where ∇ is the operator of covariant derivation) or, in other words, from

$$\partial_\mu G^{\mu\nu} + G^{\rho\nu} \Gamma^\mu_{\rho\mu} + G^{\mu\rho} \Gamma^\nu_{\rho\mu} = 0. \tag{4.1}$$

It seems strange that an equation may be obtained from an identity!

(1) Note that $(3.5)_2$ contracted with t^τ gives $(3.3)_2$ and, after that, it remains $(3.3)_3$.
(2) Note that
$$t^\alpha \lambda^{\tau\mu\nu} \partial_\tau \omega_{\alpha\mu\nu} = t^\tau \lambda^{\alpha\mu\nu} \partial_\alpha \omega_{\tau\mu\nu} \quad \Rightarrow \quad t^\tau \lambda^{\alpha\mu\nu} \delta\omega_{\tau\mu\nu}.$$

The reason of this apparent paradox is that (4.1) is an identity when applied to the whole Einstein tensor $G^{\mu\nu}$, while Fischer et al. apply it to

$$G^{\mu\nu} = \left(g^{\mu(\beta} g^{\gamma)\nu} - \frac{1}{2}\, g^{\mu\nu} g^{\beta\gamma} \right) g_{\alpha\beta} \partial_\gamma \Gamma^\alpha ,$$

i.e. the expression of $G^{\mu\nu}$ calculated on a solution of $R_{\mu\nu} - g_{\alpha(\mu}\partial_{\nu)}\Gamma^\alpha = 0$. In this way they obtain a system of the form

$$\frac{1}{2}\, g^{\beta\nu} \partial^2_{\beta\nu} \Gamma^\mu + A^{\beta\mu}_\alpha (g_{\gamma\delta}, \partial_\lambda g_{\gamma\delta}) \partial_\beta \Gamma^\alpha = 0. \tag{4.2}$$

Therefore (4.2) is not a consequence of Einstein's equations, but of their "modified" expressions which are equivalent to them only under the further assumption of harmonic coordinates; in this case, it is obvious that also (4.2) is an identity! On the other hand, one may consider (4.2) as further equations to consider jointly with Einstein's ones, disregarding their origin; assuming the validity of (4.2) is less restrictive than assuming harmonic coordinates.

So, let us consider the system

$$\begin{cases} R_{\mu\nu} = 0 \quad \text{(Einstein's equations)}, \\ \dfrac{1}{2}\, g^{\beta\nu} \partial^2_{\beta\nu} \Gamma^\mu = -A^{\beta\mu}_\alpha (g_{\gamma\delta}, \partial_\lambda g_{\gamma\delta}) \partial_\beta \Gamma^\alpha \quad \text{(Fischer's equations)}. \end{cases} \tag{4.3}$$

It is expressed in terms of $g_{\mu\nu}$ and of its first, second and third derivatives; obviously, it can be reduced to a first order system considering $g_{\mu\nu}$, $\partial_\alpha g_{\mu\nu}$, $\partial^2_{\alpha\beta} g_{\mu\nu}$ as independent variables.

But the third derivatives of $g_{\mu\nu}$ intervene only through the second derivatives of Γ^μ; therefore, one can consider $g_{\mu\nu}$, $\partial_\alpha g_{\mu\nu}$, Γ^μ, $\partial_\alpha \Gamma^\mu$ as independent variables except for the algebraic constraints

$$\Gamma^\mu = \frac{1}{2}\, g^{\mu\lambda} g^{\alpha\beta} \left(2\partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta} \right). \tag{4.4}$$

The system (4.3) can now be reduced to a first order one by defining $\omega_{\alpha\beta\gamma} = \partial_\alpha g_{\beta\gamma}$, $S^\mu_\beta = \partial_\beta \Gamma^\mu$, i.e.

$$(4.5) \qquad \begin{cases} \partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}, \\ -\dfrac{1}{2}\, g^{\alpha\beta}\partial_\alpha \omega_{\beta\mu\nu} = F_{\mu\nu}(g_{\gamma\delta}, \omega_{\lambda\gamma\delta}) + \\ \quad -\dfrac{1}{2}\, g^\gamma_{(\mu} g^\delta_{\nu)} \omega_\gamma{}^{\alpha\beta}\,(2\omega_{\alpha\beta\delta} - \omega_{\delta\alpha\beta}) - S_{\mu\nu}, \\ \partial_{[\beta}\omega_{\alpha]\mu\nu} = 0, \\ \partial_\alpha \Gamma^\mu = S^\mu_\alpha, \\ \dfrac{1}{2}\, g^{\alpha\beta}\partial_\alpha S^\mu_\beta = -A^{\beta\mu}_\alpha (g_{\gamma\nu}, \omega_{\lambda\gamma\nu}) S^\alpha_\beta, \\ \partial_{[\beta} S^\mu_{\alpha]} = 0, \end{cases}$$

where we have used the identity

$$R_{\mu\nu} = R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta} g^\gamma_{(\mu} g^\delta_{\nu)} \partial_{[\beta}\omega_{\gamma]\delta\alpha} + S_{\mu\nu} + 2g^{\alpha\beta} g^\gamma_{(\mu} g^\delta_{\nu)} \partial_{[\beta}\omega_{\gamma]\delta\alpha},$$

transformed by the expression (3.4) for $R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta} g^\gamma_{(\mu} g^\delta_{\nu)} \partial_{[\beta}\omega_{\gamma]\delta\alpha}$ and by the (4.5) for $\partial_{[\beta}\omega_{\gamma]\delta\alpha}$. Moreover $(4.5)_6$ is the integrability condition on $(4.5)_4$ such as $(4.5)_3$ is the integrability condition on $(4.5)_1$.

Now it can be easily seen that the system (4.5) is hyperbolic, without considering the constraints (4.4); therefore, it is sufficient to impose (4.4) only on the initial manifold Σ and then it will propagate off Σ. Also (4.5) can be written in a symmetric form, i.e.,

$$(4.6) \qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ t_\tau \left(-\dfrac{1}{2}\, g^{\alpha\beta}\partial_\alpha \omega_{\beta\mu\nu} \right) - t^\alpha \partial_{[\tau}\omega_{\alpha]\mu\nu} = \\ \quad = t_\tau \left[F_{\mu\nu} - \dfrac{1}{2}\, g^\gamma_{(\mu} g^\delta_{\nu)} \omega_\gamma{}^{\alpha\beta}\,(2\omega_{\alpha\beta\delta} - \omega_{\delta\alpha\beta}) - S_{\mu\nu} \right], \\ t^\alpha \partial_\alpha \Gamma^\mu = t^\alpha S^\mu_\alpha, \\ t_\tau \left(\dfrac{1}{2}\, g^{\alpha\beta}\partial_\alpha S^\mu_\beta \right) + t^\alpha \partial_{[\tau} S^\mu_{\alpha]} = -t_\tau A^{\beta\mu}_\alpha S^\alpha_\beta. \end{cases}$$

Obviously, the system

$$- dg_{\mu\nu} = 0, \qquad - t_\tau \, \frac{1}{2} \, t^\beta d\omega_{\beta\mu\nu} - t^\alpha t_{[\tau} d\omega_{\alpha]\mu\nu} = 0,$$

$$- d\Gamma^\mu = 0, \qquad t_\tau \, \frac{1}{2} \, t^\beta dS^\mu_\beta + t^\alpha t_{[\tau} dS^\mu_{\alpha]} = 0,$$

has only the solution $dg_{\mu\nu} = 0$, $d\omega_{\alpha\mu\nu} = 0$, $d\Gamma^\mu = 0$, $dS^\mu_\alpha = 0$. Moreover, the eigenvectors are the solutions of the system

$$\lambda dg_{\mu\nu} = 0, \qquad - t_\tau \, \frac{1}{2} \, \varphi^\beta d\omega_{\beta\mu\nu} - t^\alpha \varphi_{[\tau} d\omega_{\alpha]\mu\nu} = 0,$$

$$\lambda d\Gamma^\mu = 0, \qquad t_\tau \, \frac{1}{2} \, \varphi^\beta dS^\mu_\beta + t^\alpha \varphi_{[\tau} dS^\mu_{\alpha]} = 0;$$

one obtains the eigenvalues

- $\lambda = 0$, to which correspond, as eigenvectors, the 42 l.i. solutions of $t^\beta d\omega_{\beta\mu\nu} = 0$, $n^\beta d\omega_{\beta\mu\nu} = 0$, $t^\beta dS^\mu_\beta = 0$, $n^\beta dS^\mu_\beta = 0$;
- $\lambda = \pm 1$, and the corresponding 28 l.i. eigenvectors $dg_{\mu\nu} = 0$, $d\omega_{\beta\mu\nu} = x_{\mu\nu}\varphi_\beta$, $d\Gamma^\mu = 0$, $dS^\mu_\alpha = X^\mu \varphi_\alpha$, with $x_{\mu\nu}$ an arbitrary symmetric tensor, and X^μ an arbitrary 4-vector.

In the next section will be considered the case where we have sources due to the presence of matter.

5 – The case of interaction with matter

Let us consider now the expression (1.3) with $\chi \neq 0$, for Einstein's equations. Thanks to the identity $\nabla_\alpha G^{\alpha\beta} = 0$ and to $(1.2)_1$, it yields

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{5.1}$$

Usually, this equation doesn't suffice to include the contribution of matter and we have more equations; they can be written in the form

$$\nabla_\alpha T^{\alpha A} = P^A, \quad \text{for } A = 1, \dots, N. \tag{5.2}$$

Obviously, for some values of A the equation (5.2) coincide with (5.1); $T^{\alpha A}$ and P^A are functions of the independent variables. In particular, in Extended Thermodynamics (see for example, refs. [14]-[16]), the equations (5.2) assume the symmetric hyperbolic form by taking suitable independent variables λ_A which define the so called "*mean field*"; more clearly, the equations (5.2) become

$$\frac{\partial T^{\alpha A}}{\partial \lambda_B} \nabla_\alpha \lambda_\beta = P^A, \tag{5.3}$$

with $\frac{\partial T^{\alpha A}}{\partial \lambda_B} = \frac{\partial T^{\alpha B}}{\partial \lambda_A}$, $\frac{\partial T^{\alpha A}}{\partial \lambda_B} u_\alpha$ being a convex functions of λ_B.

But this result is achieved by considering constant the metric tensor $g_{\mu\nu}$; if we avoid this assumption, let us see how the equations (5.3) modify. The equations (5.2) by taking λ_B and $g_{\mu\nu}$ as independent variables, become

$$\frac{\partial T^{\alpha A}}{\partial \lambda_B} \nabla_\alpha \lambda_\beta = P^A - \frac{\partial T^{\alpha A}}{\partial g_{\mu\nu}} \omega_{\alpha\mu\nu}. \tag{5.4}$$

Therefore, the only difference is in the second members which don't involve the derivatives of the field. We can now consider the system constituted by (3.5) with $t^\alpha = u^\alpha$ and by (5.4)(or, alternatively, by (4.6) with $t^\alpha = u^\alpha$ and by (5.4)) and see that it is symmetric hyperbolic in the time direction u^α; moreover, the characteristic velocities don't exceed the speed of light and therefore, for Strumia's Lemma [1], they are hyperbolic in any other time direction.

For the sake of simplicity, let us consider only the example given by the equations of fluid dynamics

$$\nabla_\alpha(\rho u^\alpha) = 0, \qquad \nabla_\alpha[(e+p)u^\alpha u^\beta + p g^{\alpha\beta}] = 0. \tag{5.5}$$

Here ρ, e, p can be considered functions of the entropy density s and of the temperature T and they satisfy the Gibbs relation

$$T\,ds = \frac{1}{\rho}\,de + (e+p)\,d\left(\frac{1}{\rho}\right). \tag{5.6}$$

If we take $\lambda = -s + \frac{e+p}{\rho T}$, $\lambda^\alpha = \frac{u^\alpha}{T}$ as independent variables, the Gibbs relation gives $d\lambda$ from which one obtains

$$\frac{\partial p}{\partial \lambda} = \rho T, \qquad \frac{\partial p}{\partial T} = \frac{e+p}{T}, \qquad \frac{\partial(\rho T)}{\partial T} = \frac{\partial}{\partial \lambda}\left(\frac{e+p}{T}\right), \tag{5.7}$$

the last of which is the integrability condition on $(5.7)_{1,2}$.

Moreover, the above mentioned functions ρ, e, p satisfy the physical conditions

$$
\frac{\partial\rho}{\partial\lambda} > 0, \qquad \begin{vmatrix} \dfrac{\partial\rho}{\partial\lambda} & T\,\dfrac{\partial\rho}{\partial T} \\ T\,\dfrac{\partial\rho}{\partial T} & T\,\dfrac{\partial}{\partial T}\left(\dfrac{e+p}{T}\right) \end{vmatrix} > 0,
$$

$$
\begin{vmatrix} \dfrac{\partial\rho}{\partial\lambda} & T\,\dfrac{\partial\rho}{\partial T} & \rho \\ T\,\dfrac{\partial\rho}{\partial T} & T\,\dfrac{\partial}{\partial T}\left(\dfrac{e+p}{T}\right) & \dfrac{e+p}{T} \\ \rho & \dfrac{e+p}{T} & \dfrac{e+p}{T} \end{vmatrix} \geq 0. \tag{5.8}
$$

The system (5.4), *for this case, reads*

$$
\begin{aligned}
&\frac{\partial\rho}{\partial\lambda}\,\lambda^\alpha\partial_\alpha\lambda + \left(\rho g^{\alpha\delta} + T^2\frac{\partial(\rho T)}{\partial T}\,\lambda^\alpha\lambda^\delta\right)\partial_\alpha\lambda_\delta = \\
&= P(\lambda, \lambda^\gamma, g_{\mu\nu}, \omega_{\delta\mu\nu}), \left(\rho g^{\alpha\beta} + T^2\frac{\partial(\rho T)}{\partial T}\,\lambda^\alpha\lambda^\beta\right)\partial_\alpha\lambda + \\
&\quad + \left[3(e+p)T\lambda^{(\alpha}g^{\beta\delta)} + T^2\frac{\partial[(e+p)T^2]}{\partial T}\lambda^\alpha\lambda^\beta\lambda^\delta\right]\partial_\alpha\lambda_\delta = \\
&= P^\beta(\lambda, \lambda^\gamma, g_{\mu\nu}, \omega_{\delta\mu\nu}),
\end{aligned} \tag{5.9}
$$

which is manifestly symmetric.

Coupling it with (3.5) *or with* (4.6), *one obtains the whole system of equations, which is also symmetric and hyperbolic.*

Obviously, many other situations may be considered, but here we are satisfied with this one.

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