Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 83-116

Stability and controllability of an abstract evolution equation of hyperbolic type and concrete applications

S. NICAISE

Abstract: *We consider the stability of an abstract evolution equation using Liu's principle based on the exponential stability of the inverse problem with a linear feedback and on an integral inequality. Russell's principle also yields some exact controllability results. Some concrete examples with new stability and controllability results illustrate the interest of our approach.*

1 – Introduction

Stability of different systems of partial differential equations of hyperbolic type with linear or nonlinear feedbacks has been recently the object of several works. Let us quote the stability of the wave equation [18], [19], [20], [23], [22], [43], [26], [10] and the references cited there, of the Petrovsky system [11], [13], [15], [1], [4], of the elastodynamic system [1], [4], [13], of Mawxell's system [3], [21], [39], [7], [36] or combination of them [17], [37]. We actually remark that the approach of recent works

Key Words and Phrases: *Stability of partial differential equations of hyperbolic type – Integral inequality – Evolution equations – Liu principle.*
A.M.S. Classification: 37D35 – 35L10

cited above has a similar structure, namely the use of Liu's principle and of some integral inequalities. Liu's principle consists in estimating the energy of the direct system by some terms related to the feedbacks using a retrograde system with final data equal to the final data of the direct system. These terms are then estimated using the exponential stability of the inverse (retrograde) problem with a linear feedback (based on Russell's principle) and an appropriated integral inequality. Therefore our goal is to present an abstract setting leading to the stability and controllability (via Russell's principle) of the abstract system, setting as large as possible to include all examples of the aforementioned papers and allowing even new applications.

More precisely we first present an abstract setting of hyperbolic type and including the above systems. General assumptions guarantee existence results as well as dissipativeness of the system. In a second step we show that the exponential decay of the energy of the solution is equivalent to the validity of a stability estimate, estimate that can be checked in some particular cases. In a third step we use the so-called Russell's principle "controllability via stability" to obtain controllability results for the abstract system. Finally using Liu's principle [28] and a new integral inequality from [7] we give sufficient conditions on a class of (quite general) feedbacks which lead to an explicit decay rate of the energy. The strength of our approach lies in the fact that the controllability and stability results (with general feedbacks) are only based on the stability estimate with a linear feedback, estimate that may be checked for an explicit problem by different techniques, like the multiplier method, microlocal analysis or any method entering in a linear framework (like nonharmonic analysis for instance). This approach was successfully initiated in [36] for Maxwell's system and is here extended to an abstract system. We further illustrate our approach by considering different examples for which new stability and controllability results are even obtained.

The schedule of the paper is the following one: the abstract setting and its well-posedness are analysed in Section 2. Section 3 is devoted to the equivalence between the exponential stability and the stability estimate. In Section 4 exact controllability results are deduced from Russell's principle. Section 5 is devoted to the stability results for a class of nonlinear feedbacks using Liu's principle. Some applications are presented in the last section.

2 – Abstract setting

In this section we describe a general abstract setting of hyperbolic type that will be used later on. It is motivated by the examples (and other ones) given in Section 6 which all enter in this setting.

Let us fix two real separable Hilbert spaces $\mathcal{H}$, $\mathcal{V}$ with respective inner products $(.,.)_{\mathcal{H}}$, $(.,.)_{\mathcal{V}}$ and such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have the standard diagram:

$$\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}' \hookrightarrow \mathcal{V}'.$$

We suppose given a bounded linear operator A_1 from $\mathcal{V}$ into $\mathcal{V}'$ and a (nonlinear) mapping B from $\mathcal{V}$ into $\mathcal{V}'$. We now define two (nonlinear) operators $\mathcal{A}^+$ and $\mathcal{A}^-$ as follows

$$D(\mathcal{A}^\pm) = \{v \in \mathcal{V} | (\pm A_1 + B)v \in \mathcal{H}\}, \tag{1}$$

$$\mathcal{A}^\pm = (\pm A_1 + B)v, \forall v \in D(\mathcal{A}^\pm). \tag{2}$$

For shortness we often drop the superscript + at $\mathcal{A}^+$.

Motivated by the examples we introduce the following assumptions:

$$\mathcal{A}^+ \text{ is maximal monotone,} \tag{3}$$

$$\mathcal{A}^- \text{ is maximal monotone,} \tag{4}$$

$$D(\mathcal{A}^+) \text{ is dense in } \mathcal{H}, \tag{5}$$

$$D(\mathcal{A}^-) \text{ is dense in } \mathcal{H}, \tag{6}$$

$$\langle A_1 u, u \rangle = 0, \forall u \in \mathcal{V}, \tag{7}$$

$$\langle Bu, u \rangle \geq 0, \forall u \in \mathcal{V}, \tag{8}$$

where hereabove and below $\langle .,. \rangle$ means the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$.

Lemma 2.1. *Under the assumptions* (3), (5), (7) *and* (8), *the evolution equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} + A_1 u + Bu = 0 \text{ in } \mathcal{H}, t \geq 0, \\ u(0) = u_0, \end{cases} \tag{9}$$

admits a unique (weak) solution $u \in C(\mathbb{R}_+, \mathcal{H})$ *for any* $u_0 \in \mathcal{H}$. *If moreover* $u_0 \in D(\mathcal{A})$, *the problem* (9) *admits a unique (strong) solution* $u \in W^{1,\infty}(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, D(\mathcal{A}))$ *and such that* $u(t) \in D(\mathcal{A})$, *for all* $t \geq 0$.

This system is dissipative since its energy

$$\mathcal{E}(t) = \frac{1}{2}||u(t)||^2_{\mathcal{H}},$$

is non-increasing. Moreover for $u_0 \in D(\mathcal{A})$, *we have*

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \langle Bu(t), u(t)\rangle\, dt, \forall 0 \leq S < T < \infty, \tag{11}$$

$$\frac{d}{dt}\mathcal{E}(t) = -\langle Bu(t), u(t)\rangle, \forall t \geq 0. \tag{12}$$

Under the assumptions (4), (6), (7) *and* (8), *the same results hold for* $\mathcal{A}^-$ (*with the same expression for the energy and the same identities* (11) *and* (12) *for* $u_0 \in D(\mathcal{A}^-)$).

Proof. The first assertions follow from nonlinear semigroup theory [42]. For the second assertions it suffices to show (12) since $D(\mathcal{A})$ is dense in $\mathcal{H}$. For $u_0 \in D(\mathcal{A})$, we have

$$\frac{d}{dt}\mathcal{E}(t) = \left(\frac{\partial u}{\partial t}(t), u(t)\right)_{\mathcal{H}} = -(\mathcal{A}u(t), u(t))_{\mathcal{H}},$$

by (9). From the definition of $\mathcal{A}$ and the fact that $u(t) \in \mathcal{V}$, for all $t \geq 0$, we get

$$\frac{d}{dt}\mathcal{E}(t) = -\langle A_1 u(t), u(t)\rangle - \langle Bu(t), u(t)\rangle.$$

This yields (12) owing to (7). ∎

Remark 2.2. The identity (11) remains valid for $u_0 \in \mathcal{H}$ indeed for a sequence $u_{0n} \in D(\mathcal{A})$ such that $u_{0n} \to u_0$ in $\mathcal{H}$, let u_n be the solution of (9) with initial datum u_{0n}, then they fulfill

$$\mathcal{E}_n(S) - \mathcal{E}_n(T) = \int_S^T \langle Bu_n(t), u_n(t)\rangle\, dt.$$

Since the left-hand side tends to $\mathcal{E}(S)-\mathcal{E}(T)$ (because $u_n \to u$ in $C(\mathbb{R}_+, \mathcal{H})$), the right-hand side admits also a limit that we denote by $\int_S^T \langle Bu(t), u(t)\rangle dt$. This is the so-called hidden regularity of u.

3 – Exponential stability

In this section we find a necessary and sufficient condition which guarantees the exponential stability of (9). This condition is the validity of a stabilility estimate that will be checked in some particular cases in Section 6. We closely follow the arguments of the beginning of Section 3 of [36] given in the case of Mawxell's system and that can be easily extended to our abstract setting. The proofs are nevertheless given for the sake of completeness.

In the whole section we suppose that (3), (5), (7) and (8) hold.

We start with the following definition.

Definition 3.1. We say that the pair (A_1, B) satisfies the stabilility estimate if there exist $T > 0$ and two non negative constants C_1, C_2 (which may depend on T) with $C_1 < T$ such that

$$\int_0^T \mathcal{E}(t)\, dt \leq C_1 \mathcal{E}(0) + C_2 \int_0^T \langle Bu(t), u(t)\rangle\, dt, \tag{13}$$

for all solution u of (9).

That property admits the following equivalent formulation:

Lemma 3.2. *The pair (A_1, B) satisfies the stabilility estimate if and only if there exist $T > 0$ and a positive constant C (which may depend on T) such that*

$$\mathcal{E}(T) \leq C \int_0^T \langle Bu(t), u(t)\rangle\, dt, \tag{14}$$

for all solution u of (9).

Proof.
⇒: Since $\mathcal{E}(t)$ is non-increasing, the estimate (13) implies that

$$T\mathcal{E}(T) \leq C_1\mathcal{E}(0) + C_2 \int_0^T \langle Bu(t), u(t)\rangle\, dt.$$

By Lemma 2.1 we get

$$T\mathcal{E}(T) \leq C_1\mathcal{E}(T) + (C_1 + C_2) \int_0^T \langle Bu(t), u(t)\rangle\, dt.$$

This yields (14) with $C = \frac{C_1+C_2}{T-C_1}$.
⇐: From the monotonicity of $\mathcal{E}$ we may write

$$\int_0^T \mathcal{E}(t)\, dt \leq T\mathcal{E}(0).$$

Again Lemma 2.1 yields

$$\int_0^T \mathcal{E}(t)\, dt \leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}\left(\mathcal{E}(T) + \int_0^T \langle Bu(t), u(t)\rangle\, dt\right).$$

Using the assumption (14) we obtain

$$\int_0^T \mathcal{E}(t)\, dt \leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}(1+C) \int_0^T \langle Bu(t), u(t)\rangle\, dt,$$

which is nothing else than (13). □

Examples of pairs (A_1, B) satisfying the stabilility estimate may be found in Section 6 below (see also Section 3 of [36]).

We now show that the stabilility estimate is equivalent to the exponential stability of (9).

Theorem 3.3. *The pair (A_1, B) satisfies the stabilility estimate if and only if there exist two positive constants M and ω such that*

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \tag{15}$$

for all solution u of (9).

Proof. Assume that the stabilility estimate holds, i.e., by the previous Lemma, (14) equivalently holds. The identity (11) of Lemma 2.1 then yields

$$\mathcal{E}(T) \leq C(\mathcal{E}(0) - \mathcal{E}(T)).$$

This estimate is equivalent to

$$\mathcal{E}(T) \leq \gamma \mathcal{E}(0),$$

with $\gamma = \frac{C}{1+C}$ which is < 1.

Applying this argument on $[(m-1)T, mT]$, for $m = 1, 2, \cdots$ (which is valid since our system is invariant by a translation in time), we will get

$$\mathcal{E}(mT) \leq \gamma \mathcal{E}((m-1)T) \leq \cdots \leq \gamma^m \mathcal{E}(0), m = 1, 2, \cdots$$

Therefore we have

$$\mathcal{E}(mT) \leq e^{-\omega m T} \mathcal{E}(0), m = 1, 2, \cdots$$

with $\omega = \frac{1}{T} \ln \frac{1}{\gamma} > 0$. For an arbitrary positive t, there exists $m = 1, 2, \cdots$ such that $(m-1)T < t \leq mT$ and by the nonincreasing property of $\mathcal{E}$, we conclude

$$\mathcal{E}(t) \leq \mathcal{E}((m-1)T) \leq e^{-\omega(m-1)T} \mathcal{E}(0) \leq \frac{1}{\gamma} e^{-\omega t} \mathcal{E}(0).$$

Let us now show the converse implication: from Lemma 2.1, for any $T > 0$, we may write

$$\int_0^T \langle Bu(t), u(t) \rangle \, dt = \mathcal{E}(0) - \mathcal{E}(T).$$

With the help of (15), we get

$$\int_0^T \langle Bu(t), u(t) \rangle \, dt \geq \mathcal{E}(0)(1 - M e^{-\omega T}). \tag{16}$$

The exponential decay (15) also implies

$$\int_0^T \mathcal{E}(t) dt \leq M \mathcal{E}(0) \frac{1 - e^{-\omega T}}{\omega}.$$

Consequently for all $C_1 > 0$, we may write

$$\int_0^T \mathcal{E}(t)dt \le C_1\mathcal{E}(0) + \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)\mathcal{E}(0). \tag{17}$$

Choosing T large enough so that $1-Me^{-\omega T}>0$ and $C_1<\min\{\frac{M(1-e^{-\omega T})}{\omega}, T\}$, (16) and (17) yield (13) with

$$C_2 = \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)(1-Me^{-\omega T})^{-1}.$$ □

4 – Exact controllability results

Using the results of the previous section and Russell's principle we deduce exact controllability results for the evolution equation associated with the operator $-A_1$ with controls in $L^2(]0,T[;U)$, the control space U being a given real Hilbert space such that $\mathcal{V}$ is continuously embedded into U. We then denote by I_U the embedding from $\mathcal{V}$ into U and $\mathcal{I}_U$ the mapping identifying U as a subspace of $\mathcal{V}'$, i.e.,

$$\langle \mathcal{I}_U u, v\rangle := (I_U u, I_U v)_U, \forall u, v \in \mathcal{V}.$$

The exact controllability problem may be formulated as follows: for all $u_0 \in \mathcal{H}$, we are looking for a time $T > 0$ and a control $J \in L^2(]0,T[;U)$ such that the solution u of

$$\begin{cases} \dfrac{\partial u}{\partial t} - A_1 u = J \text{ in } \mathcal{V}', t \ge 0, \\ u(0) = u_0, \end{cases} \tag{18}$$

satisfies

$$u(T) = 0. \tag{19}$$

THEOREM 4.1. *If the assumptions* (3) *to* (8) *hold for the pair* $(A_1, \mathcal{I}_U)$ *and if the pair* $(A_1, \mathcal{I}_U)$ *satisfies the stabilility estimate, then for* $T > 0$ *sufficiently large, for all* $u_0 \in \mathcal{H}$ *there exist a control* $J \in L^2(]0,T[;U)$ *such that the solution* $u \in C([0,T],\mathcal{H})$ *of* (18) *is at rest a time* T, *i.e., satisfies* (19).

Proof. For concrete problems the proof is quite standard. We adapt it to our abstract setting as follows. For further purposes we prefer to solve the inverse problem (so that the asumption "$(A_1, \mathcal{I}_U)$ satisfies the stabilility estimate" is replaced by "$(-A_1, \mathcal{I}_U)$ satisfies the stabilility estimate"): Given $p_0 \in \mathcal{H}$, we are looking for $K \in L^2(]0,T[;U)$ such that the solution $p \in C([0,T],\mathcal{H})$ of

$$\left\{ \begin{array}{l} \dfrac{\partial p}{\partial t} + A_1 p = K \quad \text{in} \quad \mathcal{V}', t \geq 0, \\ p(T) = p_0, \end{array} \right. \tag{20}$$

satisfies

$$p(0) = 0. \tag{21}$$

Indeed if the above problem has a solution the conclusion follows by setting

$$u(t) = -p(T-t).$$

We solve problem (20) and (21), using a backward and an inward system with linear boundary feedbacks $\mathcal{I}_U$: First given f_0 in $\mathcal{H}$, we consider $f \in C([0,T],\mathcal{H})$ the unique solution of

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial t} + A_1 f - \mathcal{I}_U f = 0 \text{ in } \mathcal{H}, t \geq 0, \\ f(T) = f_0. \end{array} \right. \tag{22}$$

Its existence following from Lemma 2.1 by setting $\tilde{u}(t) = f(T-t)$. Moreover applying Theorem 3.3 to $\tilde{u}(t)$ we get

$$\mathcal{E}(f(t)) \leq M e^{-\omega(T-t)} \mathcal{E}(f_0). \tag{23}$$

Second we consider $g \in C([0,T],\mathcal{H})$ the unique solution of (whose existence and uniqueness still follow from Lemma 2.1)

$$\left\{ \begin{array}{l} \dfrac{\partial g}{\partial t} + A_1 g + \mathcal{I}_U g = 0 \text{ in } \mathcal{H}, t \geq 0, \\ g(0) = f(0). \end{array} \right. \tag{24}$$

We now take $p = g - f$. From (22) and (24), p satisfies (20) with

$$K = -\mathcal{I}_U g - \mathcal{I}_U f. \tag{25}$$

Let us further consider the mapping Λ from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\Lambda(f_0) = g(T).$$

We show that for $T > 0$ such that $d := Me^{-\omega T} < 1$, the mapping $\Lambda - I$ is invertible by proving that $\|\Lambda\|_{\updownarrow L(\mathcal{H},\mathcal{H})} = \sqrt{d}$. Indeed using successively the definition of Λ, Lemma 2.1, the initial condition of problem (24) and the estimate (23) we have

$$\begin{aligned} \|\Lambda f_0\|_{\mathcal{H}}^2 = 2\mathcal{E}(g(T)) \leq 2\mathcal{E}(g(0)) \leq \\ \leq 2\mathcal{E}(f(0)) \leq 2Me^{-\omega T}\mathcal{E}(f_0) = d\|f_0\|_{\mathcal{H}}^2. \end{aligned}$$

Since $\Lambda - I$ is invertible for any $p_0 \in \mathcal{H}$, there exists a unique $f_0 \in \mathcal{H}$ such that

$$p_0 = p(T) = g(T) - f(T) = (\Lambda - I) f_0. \tag{26}$$

The proof will be complete if we can show that $K \in L^2(]0, T[; U)$. For that purpose, we remark that Lemma 2.1 (identity (11) applied to $\tilde{u}$ and g which has a meaning thanks to the hidden regularity) yields

$$\begin{aligned} \mathcal{E}(f(T)) - \mathcal{E}(f(0)) &= \int_0^T \|I_U f(t)\|_U^2 \, dt, \\ \mathcal{E}(g(0)) - \mathcal{E}(g(T)) &= \int_0^T \|I_U g(t)\|_U^2 \, dt. \end{aligned}$$

Summing these two identities and using the initial condition of problem (24), the final condition of (22) and the definition of Λ, we obtain

$$\int_0^T (\|I_U f(t)\|_U^2 + \|I_U g(t)\|_U^2) \, dt = \mathcal{E}(f(T)) - \mathcal{E}(g(T)) \leq \frac{1}{2}\|f_0\|_{\mathcal{H}}^2.$$

Using the identity (26) and the boundedness of $(I - \Lambda)^{-1}$ we finally arrive at the estimate

$$\int_0^T (\|I_U f(t)\|_U^2 + \|I_U g(t)\|_U^2) dt \leq \frac{1}{2}\|(I - \Lambda)^{-1} p_0\|_{\mathcal{H}}^2 \leq \frac{1}{2(1-\sqrt{d})^2}\|p_0\|_{\mathcal{H}}^2. \tag{27}$$

This proves that K given by (25) belongs to $L^2(]0, T[; U)$. ∎

Remark 4.2. Thanks to the assumptions (5) and (6) the (weak) solution $p \in C([0,T];\mathcal{H})$ of (20) and (21) can be approximated (in$C([0,T];\mathcal{H})$) by a sequence $p_\epsilon \in W^{1,\infty}(\mathbb{R}_+,\mathcal{H}) \cap L^\infty(\mathbb{R}_+,\mathcal{V})$, $\epsilon > 0$, of (strong) solution of (20) with $K_\epsilon \in L^2(]0,T[;U)$ and $p_{0\epsilon} \in \mathcal{V}$ such that

$$K_\epsilon \to K \text{ in } L^2(]0,T[;U) \text{ as } \epsilon \to 0, \tag{28}$$

$$I_U p_\epsilon \to I_U p \text{ in } L^2(]0,T[;U) \text{ as } \epsilon \to 0. \tag{29}$$

Indeed as $f_0 = (\Lambda - I)^{-1} p_0$, by (5), there exists $f_{0\epsilon} \in D(\mathcal{A})$ such that

$$\|f_0 - f_{0\epsilon}\|_{\mathcal{H}} \leq \epsilon. \tag{30}$$

Consider f_ϵ the strong solution of (22) with final datum $f_{0\epsilon}$. By the dissipativeness of the energy, we get

$$\|f(t) - f_\epsilon(t)\|_{\mathcal{H}} \leq \|f_0 - f_{0\epsilon}\|_{\mathcal{H}} \leq \epsilon, \forall t \in [0,T]. \tag{31}$$

Similarly since $f_\epsilon(0)$ belongs to $\mathcal{H}$, by (6), there exists $g_{0\epsilon} \in D(\mathcal{A}^-)$ such that

$$\|g_{0\epsilon} - f_\epsilon(0)\|_{\mathcal{H}} \leq \epsilon. \tag{32}$$

We then consider g_ϵ the strong solution of (24) with initial datum $g_{0\epsilon}$. The dissipativeness of the energy yields

$$\begin{aligned}\|g(t) - g_\epsilon(t)\|_{\mathcal{H}} &\leq \|g(0) - g_{0\epsilon}\|_{\mathcal{H}} \leq \\ &\leq \|f(0) - f_\epsilon(0)\|_{\mathcal{H}} + \|f_\epsilon(0) - g_{0\epsilon}\|_{\mathcal{H}} \leq 2\epsilon, \forall t \in [0,T],\end{aligned}$$

by (31) and (32).

The estimates (31) and (33) show that $p_\epsilon := g_\epsilon - f_\epsilon$ tends to $p = g - f$ in $C([0,T];\mathcal{H})$ as ϵ goes to 0. Finally by Lemma 2.1 we may write

$$\int_0^T \|I_U(f(t) - f_\epsilon(t))\|_U^2 \, dt \leq 2\|f_0 - f_{0\epsilon}\|_{\mathcal{H}}^2$$
$$\int_0^T \|I_U(g(t) - g_\epsilon(t))\|_U^2 \, dt \leq 2\|g(0) - g_\epsilon(0)\|_{\mathcal{H}}^2.$$

These two estimates, the estimates (30), (33) and the definitions of $K_\epsilon := -\mathcal{I}_U g_\epsilon - \mathcal{I}_U f_\epsilon$, of p_ϵ, K and p lead to the properties (28) and (29). ∎

5 – Stability in the nonlinear case

Here we use Liu's principle [28] and an integral inequality from [7] to deduce decay rates of the energy using appropriate nonlinear feedbacks. In view of the examples below we assume that the control space U is of the form

$$U = \prod_{j=1}^{J} U_j, \tag{34}$$

where for all $j = 1, \cdots, J \in \mathbb{N}^\star := \mathbb{N} \setminus \{0\}$, U_j is a closed subspace of $L^2(X_j, \mu_j)^{N_j}$, when $(X_j, \updownarrow A_j, \mu_j)$ is a measure space such that $\mu_j(X_j) < \infty$ and $N_j \in \mathbb{N}^\star$. For all $j = 1, \cdots, J$, we suppose given a mapping $g_j : \mathbb{R}^{N_j} \to \mathbb{R}^{N_j}$ such that

$$(g_j(x) - g_j(y)) \cdot (x - y) \geq 0, \forall x, y \in \mathbb{R}^{N_j} \text{ (monotonicity)}, \tag{35}$$

$$g_j(0) = 0, \tag{36}$$

$$|g_j(x)| \leq M(1 + |x|), \forall x \in \mathbb{R}^3, \tag{37}$$

for some positive constant M. We finally suppose that B is given by

$$\langle Bu, v\rangle = \sum_{j=1}^{J} \int_{X_j} g_j((I_U u)_j(x_j)) \cdot (I_U v)_j(x_j)\, d\mu_j(x_j), \tag{38}$$

where we recall that I_U is the embeding from $\mathcal{V}$ to U and therefore $(I_U u)_j$ is the j^{th} component of $I_U u$.

Remark that the conditions (35) and (36) guarantee the assumption (8) on B, while (37) guarantees that B is well defined. In most examples these conditions guarantee that the assumptions (3) and (4) hold (see Section 6 for some illustrations). We further remark that these conditions always hold for $g_j(x) = x$, corresponding to linear controls, i.e., $B = \mathcal{I}_U$.

We now recall the integral inequality obtained in [7] (compare with Theorem 9.1 of [22] or its extension by P. Martinez [31], [32]).

Theorem 5.1. *Let $\mathcal{E} : [0, +\infty) \to [0, +\infty)$ be a non-increasing mapping satisfying*

$$\int_S^\infty \phi(\mathcal{E}(t))\, dt \leq T\mathcal{E}(S), \forall S \geq 0, \tag{39}$$

for some $T > 0$ and some strictly increasing convex mapping ϕ from $[0, +\infty)$ to $[0, +\infty)$ such that $\phi(0) = 0$. Then there exist $t_1 > 0$ and c_1 depending on T and $\mathcal{E}(0)$ such that

$$\mathcal{E}(t) \leq \phi^{-1}\left(\frac{\psi^{-1}(c_1 t)}{c_1 T t}\right), \forall t \geq t_1, \tag{40}$$

where ψ is defined by

$$\psi(t) = \int_t^1 \frac{1}{\phi(s)}\, ds, \forall t > 0. \tag{41}$$

Remark 5.2. Theorem 5.1 yields exactly the same decay rate as in Theorem 9.1 of [22] when $\phi(t) = t^{1+\alpha}$ for some $\alpha > 0$ (case leading to polynomial decay). Note furthermore that the integral inequality of P. Martinez [31], [32] is different from our integral inequality but gives similar asymptotic behaviour for the energy. □

We now give the consequence of this result to our system (9).

Theorem 5.3. *Assume that the assumptions* (3) *to* (8) *hold for the pairs (A_1, B) and $(A_1, \mathcal{I}_U)$. Let g_j, $j = 1, \cdots, J$ satisfy* (35) *to* (37) *as well as*

$$g_j(x) \cdot x \geq m|x|^2, \forall x \in \mathbb{R}^{N_j} : |x| \geq 1, \tag{42}$$

$$|x|^2 + |g_j(x)|^2 \leq G(g_j(x) \cdot x), \forall x \in \mathbb{R}^{N_j} : |x| \leq 1, \tag{43}$$

for some positive constant m and a concave strictly increasing function $G : [0, \infty) \to [0, \infty)$ such that $G(0) = 0$. If the pair $(-A_1, \mathcal{I}_U)$ satisfies the stabilility estimate, then there exist $c_2, c_3 > 0$ and $T_1 > 0$ (depending on T, $\mathcal{E}(0)$, $\mu_j(X_j)$, $j = 1, \cdots, J$) such that

$$\mathcal{E}(t) \leq c_3 G\left(\frac{\psi^{-1}(c_2 t)}{c_2 T t}\right), \forall t \geq T_1, \tag{44}$$

for all solution u of (9), *where ψ is given by* (41) *for ϕ defined by*

$$\phi(s) = T\mu G^{-1}\left(\frac{s}{c_3}\right), \tag{45}$$

where $\mu = \min_{j=1,\cdots,J} \mu_j(X_j)$.

PROOF. By the density of $D(\mathcal{A})$ into $\mathcal{H}$, it suffices to prove (44) for data in $D(\mathcal{A})$. In that case let u be the (strong) solution of (9) and consider p the solution of problem (20) and (21) with $p_0 = u(T) \in D(\mathcal{A})$ with $T > 0$ sufficiently large (whose existence was established in Theorem 4.1). Consider further a sequence p_ϵ of strong solution of (20) with final data $p_{0\epsilon}$ tending to p in $C([0,T],\mathcal{H})$ as ϵ goes to zero and satisfying (28) and (29) (see Remark 4.2).

By (9) and (20) we may write

$$\langle \partial_t u + A_1 u + Bu, p_\epsilon \rangle_{\mathcal{V}',\mathcal{V}} + \langle \partial_t p_\epsilon + A_1 p_\epsilon - K_\epsilon, u \rangle_{\mathcal{V}',\mathcal{V}} = 0.$$

This may be written equivalently

$$(\partial_t u, p_\epsilon)_{\mathcal{H}} + (\partial_t p_\epsilon, u)_{\mathcal{H}} + \langle A_1 u, p_\epsilon \rangle_{\mathcal{V}',\mathcal{V}} + \langle A_1 p_\epsilon, u \rangle_{\mathcal{V}',\mathcal{V}} + \\ + \langle Bu, p_\epsilon \rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, u \rangle_{\mathcal{V}',\mathcal{V}} = 0$$

As the assumption (7) yields

$$\langle A_1 u, p_\epsilon \rangle_{\mathcal{V}',\mathcal{V}} + \langle A_1 p_\epsilon, u \rangle_{\mathcal{V}',\mathcal{V}} = 0,$$

the above identity reduces to

$$(\partial_t u, p_\epsilon)_{\mathcal{H}} + (\partial_t p_\epsilon, u)_{\mathcal{H}} + \langle Bu, p_\epsilon \rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, u \rangle_{\mathcal{V}',\mathcal{V}} = 0$$

Integrating this identity for $t \in (0,T)$, we get

$$(u(T), p_\epsilon(T))_{\mathcal{H}} - (u(0), p_\epsilon(0))_{\mathcal{H}} + \int_0^T (\langle Bu, p_\epsilon \rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, u \rangle_{\mathcal{V}',\mathcal{V}})\, dt = 0.$$

By the definitions of K_ϵ and B we arrive at

$$(u(T), p_\epsilon(T))_{\mathcal{H}} - (u(0), p_\epsilon(0))_{\mathcal{H}} = \int_0^T \Bigg((K_\epsilon, I_U u)_U + \\ - \sum_{j=1}^J \int_{X_j} g_j((I_U u)_j(x_j)) \cdot (I_U p_\epsilon)_j(x_j)\, d\mu_j(x_j) \Bigg)\, dt$$

Passing to the limit in ϵ and using the initial and final conditions on p, we have obtained

$$2\mathcal{E}(T)=\int_0^T\left((K,I_Uu)_U-\sum_{j=1}^J\int_{X_j} g_j((I_Uu)_j(x_j))\cdot(I_Up)_j(x_j)\,d\mu_j(x_j)\right)dt$$

Cauchy-Schwarz's inequality leads finally to

$$\begin{aligned}2\mathcal{E}(T) &\leq \|K\|_{L^2(0,T;U)}\|I_Uu\|_{L^2(0,T;U)}+\\ &\quad+\|I_Up\|_{L^2(0,T;U)}\left(\sum_{j=1}^J\int_0^T\int_{X_j}|g_j((I_Uu)_j(x_j))|^2\,d\mu_j(x_j)dt\right)^{1/2}.\end{aligned} \tag{46}$$

Let us remark that the estimate (27) and the final conditions on p yield

$$\int_0^T(\|I_Uf(t)\|_U^2+\|I_Ug(t)\|_U^2)\,dt\leq\frac{1}{(1-\sqrt{d})^2}\mathcal{E}(T).$$

This estimate, the definition of K and $p=g-f$ lead to

$$\int_0^T\|K(t)\|_U^2\,dt\leq\frac{2}{(1-\sqrt{d})^2}\mathcal{E}(T)$$
$$\int_0^T\|I_Up(t)\|_U^2\,dt\leq\frac{2}{(1-\sqrt{d})^2}\mathcal{E}(T).$$

Inserting these estimates in (46) we arrive at

$$\begin{aligned}\mathcal{E}(T) &\leq\frac{1}{(1-\sqrt{d})^2}\times\\ &\quad\times\left(\sum_{j=1}^J\int_0^T\int_{X_j}\{|(I_Uu)_j(x_j)|^2+|g_j((I_Uu)_j(x_j))|^2\}\,d\mu_j(x_j)dt\right).\end{aligned} \tag{47}$$

We now estimate the right-hand side of (47) as follows: For all $j=1,\cdots,J$ introduce

$$\Sigma_j^+=\{(x,t)\in X_j\times(0,T)||(I_Uu)_j(x,t)|>1\},$$
$$\Sigma_j^-=\{(x,t)\in X_j\times(0,T)||(I_Uu)_j(x,t)|\leq 1\}.$$

Let us split up

$$\int_0^T \int_{X_j} \{|(I_U u)_j(x_j)|^2 + |g_j((I_U u)_j(x_j))|^2\} \, d\mu_j(x_j) \, dt = I_j^+ + I_j^-,$$

where

$$I_j^+ := \int_{\Sigma_j^+} \{|(I_U u)_j(x_j)|^2 + |g_j((I_U u)_j(x_j))|^2\} \, d\mu_j(x_j) \, dt,$$
$$I_j^- := \int_{\Sigma_j^-} \{|(I_U u)_j(x_j)|^2 + |g_j((I_U u)_j(x_j))|^2\} \, d\mu_j(x_j) \, dt.$$

The assumptions (42) and (37) lead to

$$I_j^+ \leq c_4 \int_{\Sigma_j^+} (I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j)) \, d\mu_j(x_j) dt,$$

for some positive constant c_4 (depending on m and M). By (11) and the property

$$g_j(x) \cdot x \geq 0, \forall x \in \mathbb{R}^{N_j}, \tag{48}$$

following from (35) and (36) we arrive at

$$I_j^+ \leq c_4(\mathcal{E}(0) - \mathcal{E}(T)). \tag{49}$$

Similarly by the assumption (43) and the monotonicity of G we have

$$I_j^- \leq \int_{\Sigma_j^-} G((I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j))) \, d\mu_j(x_j) dt \leq$$
$$\leq \int_0^T \int_{X_j} G((I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j))) \, d\mu_j(x_j) dt.$$

Jensen's inequality then yields

$$I_j^- \leq T\mu_j(X_j) G\left(\frac{1}{T\mu_j(X_j)} \int_0^T \int_{X_j} (I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j)) d\mu_j(x_j) dt\right).$$

By (11), we arrive at

$$I_j^- \le T\mu_j(X_j) G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu_j(X_j)}\right). \tag{50}$$

The estimates (49) and (50) into the estimate (47) and the monotonicity of G give

$$\mathcal{E}(T) \le c_5 \left\{\mathcal{E}(0)-\mathcal{E}(T) + G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right)\right\},$$

for some positive constant c_5 (depending on T and $\max_j \mu_j(X_j)$), where we recall that $\mu = \min_j \mu_j(X_j)$. This finally leads to

$$\mathcal{E}(0)=\mathcal{E}(0)-\mathcal{E}(T)+\mathcal{E}(T)\le\max\{1,c_5\}\left\{(\mathcal{E}(0)-\mathcal{E}(T))+G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right)\right\}.$$

As $\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu} \le \frac{\mathcal{E}(0)}{T\mu}$, the concavity of G yields a constant c_6 (depending continuously on T, $\mathcal{E}(0)$ and μ) such that

$$\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu} \le c_6 G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right).$$

These two estimates lead to

$$\mathcal{E}(0) \le c_3 G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right),$$

for some $c_3 > 0$ (depending on T, $\mathcal{E}(0)$, $\max_j \mu_j(X_j)$, and $\min_j \mu_j(X_j)$).

Using this argument in $[t, t+T]$ instead of $[0, T]$ we have shown that

$$\mathcal{E}(t) \le c_3 G\left(\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{T\mu}\right) = \phi^{-1}(\mathcal{E}(t)-\mathcal{E}(t+T)), \forall t \ge 0, \tag{51}$$

when we recall that ϕ was defined by (45).

We conclude by Theorem 5.1 since Lemma 5.1 of [7] shows that the estimate (51) guarantees that $\mathcal{E}$ actually satisfies (39). ∎

The assumption (42) forbids the use of bounded functions g_j which could be a drawback for some applications. Our next purpose is to obtain a variant of the above result when some mappings g_j do not satisfy (42) adapting the arguments of Theorem 9.10 of [22]. The price to pay is to assume some regularity results for elements of $D(\mathcal{A})$.

THEOREM 5.4. *Assume that the assumptions* (3) *to* (8) *hold for the pairs* (A_1, B) *and* $(A_1, \mathcal{I}_U)$. *Let* g_j, $j = 1, \cdots, J$ *satisfy* (35) *to* (37) *as well as* (43) *for some concave strictly increasing function* $G : [0, \infty) \to [0, \infty)$ *such that* $G(0) = 0$. *Assume further that* $J = J_1 \cup J_2$ *with* $J_1 \cap J_2 = \emptyset$, *that for all* $j \in J_1$, g_j *satisfies* (42) *and there exists* $c_7 > 0$ *and* $\alpha > 2$ *such that for all* $j \in J_2$ *and all* $u \in D(\mathcal{A})$, $(I_U u)_j$ *belongs to* $L^\alpha(X_j, \mu_j)$ *with the estimate*

$$(52)\qquad \left(\int_{X_j} |(I_U u)_j(x_j)|^\alpha \, d\mu_j(x_j)\right)^{1/\alpha} \leq c_7 \|u\|_{D(\mathcal{A})},$$

where we recall that $\|u\|_{D(\mathcal{A})} = \|\mathcal{A}u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}$. *If the pair* $(-A_1, \mathcal{I}_U)$ *satisfies the stabilility estimate, then for every* $u_0 \in D(\mathcal{A})$, *the solution* u *of* (9) *satisfies*

$$(53)\qquad \mathcal{E}(t) \leq c_3 G_1\left(\frac{\psi_1^{-1}(c_2 t)}{c_2 T t}\right), \forall t \geq T_1,$$

for some $c_2, c_3 > 0$ *and* $T_1 > 0$ *(depending on* T, $\mathcal{E}(0)$, $\mu_j(X_j)$, $j = 1, \cdots, J$, α *and* $\|u_0\|_{D(\mathcal{A})}$*), where* ψ_1 *is given by* (41) *for* ϕ_1 *defined by* (45) *with* G_1 *instead of* G, *the function* G_1 *being defined by*

$$G_1(x) = G(x) + x^s, \forall x \geq 0,$$

with $s = \frac{\alpha-2}{\alpha-1} \in (0, 1)$.

PROOF. We repeat the proof of Theorem 5.3 except for the estimation of I_j^+ when $j \in J_2$, where we now obtain the following estimation: First by (37) we remark that

$$(54)\qquad I_j^+ \leq (1 + 4M^2) J_j^+.$$

where

$$J_j^+ := \int_{\Sigma_j^+} |(I_U u)_j(x_j)|^2 \, d\mu_j(x_j).$$

So it remains to estimate J_j^+. For that estimation we remark that the assumption (43) yields

$$g_j(x) \cdot x \geq m_j|x|, \forall x \in \mathbb{R}^{N_j} : |x| \geq 1, \tag{55}$$

for some positive constant m_j. Indeed we notice that (43) and the property $G(0) = 0$ directly imply that

$$g_j(\xi) \cdot \xi > 0, \forall |\xi| = 1.$$

Denoting by $m_j = \min_{|\xi|=1}(g_j(\xi) \cdot \xi)$ we have already proved (55) for $|x| = 1$. For $|x| > 1$ let $\xi = x/|x|$, then by the monotonicity of g_j we have

$$(g_j(x) - g_j(\xi)) \cdot (|x| - 1)\xi \geq 0,$$

which implies

$$g_j(x) \cdot \xi \geq g_j(\xi) \cdot \xi \geq m_j.$$

Multiplying this inequality by $|x|$, we arrive at (55).

Now using (55) we may write

$$J_j^+ \leq m_j^{-s} \int_{\Sigma_j^+} |(I_U u)_j(x_j)|^{2-s}((I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j))^s \, d\mu_j(x_j).$$

By Hölder's inequality we get

$$J_j^+ \leq m_j^{-s} \left(\int_{\Sigma_j^+} |(I_U u)_j(x_j)|^{\frac{2-s}{1-s}} \, d\mu_j(x_j) \right)^{1-s} \times$$
$$\times \left(\int_{\Sigma_j^+} (I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j)) \, d\mu_j(x_j) \right)^s .$$

By (11) and the assumption (52) (since $\alpha = \frac{2-s}{1-s}$) we conclude that

$$J_j^+ \leq c_8(\mathcal{E}(0) - \mathcal{E}(T))^s, \tag{56}$$

where $c_8 > 0$ depends on T, α and $\|u_0\|_{D(\mathcal{A})}$ (since Komura-Kato's theorem (see for instance Proposition IV.3.1 of [42] and Lemma 2.1 guarantee that $\|u(t)\|_{D(\mathcal{A})} \leq \|u_0\|_{D(\mathcal{A})}$).

As before the estimates (50), (54) and (56) into the estimate (47) and the monotonicity of G give

$$\mathcal{E}(T) \leq c_9 \left\{ \mathcal{E}(0) - \mathcal{E}(T) + G\left(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T\mu}\right) + (\mathcal{E}(0) - \mathcal{E}(T))^s \right\},$$

for some positive constant c_9 depending on T, $\mu_j(X_j)$, $j = 1, \cdots, J$, α and $\|u_0\|_{D(\mathcal{A})}$. The concavity of G and of the mapping $x \to x^s$ yields

$$\mathcal{E}(0) \leq c_3 G_1 \left(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T\mu}\right).$$

The conclusion follows as previously. ∎

REMARK 5.5. In (42) (resp. (43)) the proviso $|x| \geq 1$ (resp. $|x| \leq 1$) may be replaced by $|x| \geq \eta$ (resp. $|x| \leq \eta$), for some $\eta > 0$ without changing the conclusion of Theorem 5.3 or Theorem 5.4. ∎

Examples of functions g_j leading to an explicit decay rate (44) or (53) are given in [7]. Let us give the following illustrations.

EXAMPLE 5.6. Suppose that g_j satisfies (35) to (37) and (42) as well as

$$x \cdot g_j(x) \geq c_0 |x|^{p+1},\ |g_j(x)| \leq C_0 |x|^\alpha, \forall |x| \leq 1, \tag{57}$$

for some positive constants c_0, C_0, $\alpha \in (0,1]$ and $p \geq \alpha$. Then g_j satisfies (43) with $G(x) = x^{\frac{2}{q+1}}$ and $q = \frac{p+1}{\alpha} - 1$ (which is ≥ 1). If $p = \alpha = 1$ (then $q = 1$) and under the other assumptions of Theorem 5.3 we get an exponential decay (since $\psi^{-1}(t) = e^{-t}$). On the contrary if $p + 1 > 2\alpha$ then we get the decay $t^{-\frac{2\alpha}{p+1-2\alpha}}$ (since $\psi^{-1}(t) = t^{\frac{2}{1-q}}$). A function g satisfying all these assumptions is given by

$$g(x) = \begin{cases} |x|^{\alpha-1}x & \text{if } |x| \leq 1, \\ x & \text{if } |x| \geq 1, \end{cases}$$

for some $\alpha \in (0,1]$. In that case (57) holds for $p = \alpha$.

In the setting of Theorem 5.4 it suffices to take g_j satisfying (35) to (37) and (57) to get the decay rate $t^{-\frac{2}{q'-1}}$ with $q' = \min\{q, \frac{2}{s} - 1\}$. Such a g is given by

$$g(x) = \begin{cases} |x|^{\alpha-1}x & \text{if } |x| \le 1, \\ \dfrac{x}{|x|} & \text{if } |x| \ge 1, \end{cases}$$

for some $\alpha \in (0,1]$, which satisfies (57) for $p = \alpha$.

Example 5.6 (Logarithmic decay). Take $g_j(\xi) = \exp(-\frac{1}{|\xi|^{2p_j}})\frac{\xi}{|\xi|^2}$ for $|\xi|$ small enough and for $p_j > 0$. Then by Example 2.4 of [7] (43) holds with

$$G(x) = \frac{C}{|\log x|^{\frac{1}{p}}}$$

and $p = \max_j p_j$ and some constant $C > 0$. In the setting of Theorem 5.3 or Theorem 5.4 we will get the decay

$$\mathcal{E}(t) \le \frac{C}{|\log t|^{\frac{1}{p}}},$$

since ψ^{-1} is bounded from below.

Example 5.8 (Log-Log decay). Take $g_j(\xi) = \exp(-\exp(1/|\xi|^{2p}))\frac{\xi}{|\xi|^2}$ for $|\xi|$ small enough and for $p > 0$. Then by Example 2.5 of [7] (43) holds with

$$G(x) = \frac{C}{|\log|\log x||^{\frac{1}{p}}}$$

and some constant $C > 0$. In the setting of Theorem 5.3 or Theorem 5.4 we will get the decay

$$\mathcal{E}(t) \le \frac{C}{|\log|\log t||^{\frac{1}{p}}}.$$

Note that combinations of the above examples give rise to the worse decay rate.

6 – Examples

6.1 – Second order evolution equations

Some examples given below enter in the following framework: Let H and V be two real separable Hilbert spaces such that V is densely and continuously embedded into H. Define the linear operator A_2 from V into V' by

$$\langle A_2u, v\rangle_{V'-V} = (u, v)_V, \forall u, v \in V, \tag{58}$$

and suppose given a (nonlinear) mapping B_2 from V into V'.

Consider now the second order evolution equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + A_2u + B_2\dfrac{\partial u}{\partial t} = 0 \text{ in } V', t \geq 0, \\ u(0) = u_0, \dfrac{\partial u}{\partial t}(0) = u_1. \end{cases}$$

This system is reduced to the first order system (9) using the standard argument of reduction of order: setting $\mathcal{H} = V \times H$, $\mathcal{V} = V \times V$ with natural inner products,

$$x = (u, z),$$

with $z = \frac{\partial u}{\partial t}$ (from now on we use the letter x for generic elements of $\mathcal{H}$ since the letter u is already used in (59) as usual) and introducing the operators

$$A_1x = (-z, A_2u), Bx = (0, B_2z).$$

Under appropriate assumptions on B_2, we can prove the

Theorem 6.1. *If B_2 is monotone, hemicontinuous, bounded and satisfies $B_20 = 0$, then the assumptions* (3) *to* (8) *hold for the pair* (A_1, B).

Proof. In the above setting we see that

$$D(\mathcal{A}^{\pm}) = \{x = (u, z) \in \mathcal{V} | \pm A_2u + B_2z \in H\}.$$

To check the assumptions (3) and (4), from the definitions of A_1, A_2 and the inner product in $\mathcal{H}$ we easily verify that

$$(\mathcal{A}^{\pm}(u, z) - \mathcal{A}^{\pm}(u', z'), (u, z) - (u', z'))_{\mathcal{H}} = \langle B_2z - B_2z', z - z'\rangle_{V'-V}.$$

The monotonicity of $\mathcal{A}^{\pm}$ then follows from the same property on B_2.

Let us pass to the maximality of $\mathcal{A}^{\pm}$: for all $(f, g) \in \mathcal{H}$ we are looking for $(u, z) \in D(\mathcal{A}^{\pm})$ such that

$$u \mp z = f \text{ in } V,$$
$$z \pm A_2 u + B_2 z = g \text{ in } H.$$

The first identity is equivalent to

$$u = \pm z + f \text{ in } V,$$

and eliminating u in the second identity we obtain

$$z + A_2 z + B_2 z = g \mp f \text{ in } V'.$$

The solvability of this problem is equivalent to the surjectivity of the operator

$$A : V \to V' : z \to z + A_2 z + B_2 z.$$

For that purpose we make use of Corollary 2.2 of [42] which proves that A is surjective if A is monotone, hemicontinuous, bounded and coercive. The first three properties easily follows from the same property of B_2. The coercivity also easily follows from the fact that

$$\langle Az, z\rangle_{V'-V} = \|z\|_H^2 + \|z\|_V^2 + \langle B_2 z, z\rangle_{V'-V} \geq \|z\|_V^2,$$

this last inequality following from the property $\langle B_2 z, z\rangle_{V'-V} \geq 0$ consequence of the monotonicity of B_2 and the property $B_2 0 = 0$.

The assumptions (5) and (6) are reduced to the density of $D(\mathcal{A})$ since we easily check that $(u, z) \in D(\mathcal{A})$ if and only if $(-u, z) \in D(\mathcal{A}^-)$. Let us now fix (u, z) in $\mathcal{H}$, then let $\tilde{u} \in V$ be the unique solution of

$$A_2 \tilde{u} = -B_2 z,$$

whose existence follows from Lax-Milgram's lemma. Applying Theorem III.2.B of [41] there exists a sequence of $u_n \in D(\mathcal{A}_2)$ such that

$$u_n \to u - \tilde{u} \text{ in } V, \text{ as } n \to \infty,$$

where $\mathcal{A}_2$ is the Friedrichs extension of A_2. We conclude by remarking that $(\tilde{u}+u_n, z)$ belongs to $D(\mathcal{A})$ and tends to (u, z) in $\mathcal{H}$.

The assumption (7) follows from the identity

$$\langle A_1 x, x\rangle = -(z,u)_V + \langle A_2 u, z\rangle_{V'-V},$$

and the definition of A_2. Finally the assumption (8) follows from the identity

$$\langle Bx, x\rangle = \langle B_2 z, z\rangle_{V'-V},$$

and the positiveness of B_2. □

In view of this theorem the assumptions (3) to (8) are reduced to the verification of the above properties of B_2 that we now check for different systems.

In the rest of the section Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with a Lipschitz boundary Γ. Some restrictions will be specified later on when they will be necessary. We further denote by ν the unit outward normal vector along Γ.

6.2 – Nonlinear stabilization of the wave equation

Consider the wave equation

$$\text{(60)}\qquad \begin{cases} \partial_t^2 u - \Delta u + f(\partial_t u) = 0 \text{ in } Q := \Omega\times]0,+\infty[,\\ u = 0 \text{ on } \Sigma_0 := \Gamma_0\times]0,+\infty[,\\ \partial_\nu u + au + g(\partial_t u) = 0 \text{ on } \Sigma_1 := \Gamma_1\times]0,+\infty[,\\ u(0) = u_0, \partial_t u(0) = u_1 \text{ in } \Omega, \end{cases}$$

where Γ_0 is a open subset of Γ and $\Gamma_1 = \Gamma \setminus \bar{\Gamma}_0$ is the remainder. The functions f and g are two nondecreasing continuous functions from $\mathbb{R}$ into itself such that $f(0) = g(0) = 0$ and finally a is a nonnegative real number. For the sake of simplicity we suppose that

$$\text{(61)}\qquad \text{either } \Gamma_0 \text{ is not empty or } a > 0,$$

and that

$$\text{(62)}\qquad \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset.$$

The stability of this problem was extensively studied in the litterature, let us cite the papers [18], [19], [20], [23], [22], [43], [26], [10] and the references cited there. Both papers are restricted to some particular choices of Γ_0, a, f and g leading to some exponential or polynomial decay rates of the energy of the solution of (60). In [25], [29], [31], [32], [33], [34], some arbitrary decay rates are obtained for different f and g (even with degenerate or local dissipations). Using the results of the previous sections, we also obtain arbitrary decay rates for a large class of f and g.

The first point is that problem (60) enters in the framework of problem (59) from Subsection 6.1 once we take:

$$\begin{aligned} H &= L^2(\Omega), \\ V &= \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\}, \\ (u, v)_V &= \int_\Omega \nabla u \cdot \nabla v \, dx + a \int_{\Gamma_1} u \cdot v \, d\sigma, \\ \langle B_2 u, v \rangle_{V'-V} &= \int_\Omega f(u) v \, dx + \int_{\Gamma_1} g(u) v \, d\sigma, \forall u, v \in V. \end{aligned}$$

Let us remark that the assumption (61) implies that the inner product $(\cdot, \cdot)_V$ induces a norm on V equivalent to the usual one. In order to give a meaning to B_2 we simply require

$$|f(x)| \leq C(1 + |x|^\alpha), \forall x \in \mathbb{R}, \tag{63}$$

$$|g(x)| \leq C(1 + |x|^\beta), \forall x \in \mathbb{R}, \tag{64}$$

for some positive constant C, where $\alpha = \frac{n+2}{n-2}$ and $\beta = \frac{n}{n-2}$ if $n \geq 3$ and $\alpha, \beta \geq 1$ if $n = 2$.

Now we readily check that these assumptions guarantee that B_2 fulfils all the assumptions of Theorem 6.1. Consequently the corresponding pair (A_1, B) satisfies the assumptions (3) to (8). In order to deduce stability results for our system (60) we need to check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate (note that we just check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the assumptions (3) to (8)), where the control space U is clearly defined by

$$U = L^2(\Omega) \times L^2(\Gamma_1).$$

This stability estimate was proved in Theorem 1.2 of [10] under the assumption that there exists $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu > 0 \text{ on } \Gamma_1, m \cdot \nu \leq 0 \text{ on } \Gamma_0, \tag{65}$$

$$\frac{1}{R^2}\max\{n-2, n/3\} \leq a(m \cdot \nu) < \frac{n}{R^2} \text{ on } \Gamma_1, \tag{66}$$

where as usual m is the standard multiplier defined by

$$m(x) = x - x_0, \forall x \in \mathbb{R}^n,$$

and $R = \max_{x\in\Omega} |m(x)|$. Under these assumptions, appropriated conditions on f and g lead to exponential, polynomial, logarithmic or other decays. Note that bounded feedbacks are allowed since $D(\mathcal{A}) \hookrightarrow H^1(\Omega) \times H^1(\Omega) \hookrightarrow L^\alpha(\Omega) \times L^\alpha(\Gamma_1)$, for some $\alpha > 2$ consequently Theorem 5.4 may be applied.

For $f = 0$ or $g = 0$ similar results hold (changing the control space U) with less restrictions on Γ_0 and Γ_1, using the exponential decay with linear feedbacks established in [18], [19], [20], [23], [22], [43], [26].

6.3 – Nonlinear stabilization of the elastodynamic system

With the notation of the above subsection, we consider the following elastodynamic system:

$$\begin{cases} \partial_t^2 u - \nabla\sigma(u) + F(\partial_t u) = 0 \text{ in } Q, \\ u = 0 \text{ on } \Sigma_0, \\ \sigma(u)\cdot\nu + au + G(\partial_t u) = 0 \text{ on } \Sigma_1, \\ u(0) = u_0, \partial_t u(0) = u_1 \text{ in } \Omega. \end{cases} \tag{67}$$

As usual $u(x,t)$ is the displacement field at the point $x \in \Omega$ at time t and $\sigma(u) = (\sigma_{ij}(u))_{i,j=1}^3$ is the stress tensor given by (here and in the sequel we shall use the summation convention for repeated indices)

$$\sigma_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u),$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))_{i,j=1}^3$ is the strain tensor given by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and the tensor $(a_{ijkl})_{i,j,k,l=1,2,3}$ is made of $W^{1,\infty}(\Omega)$ entries such that

$$a_{ijkl} = a_{jikl} = a_{klij},$$

and satisfying the ellipticity condition

$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \alpha\varepsilon_{ij}\varepsilon_{ij},$$

for every symmetric tensor (ε_{ij}) and some $\alpha > 0$. Hereabove and below $\nabla\sigma(u)$ is the vector field defined by

$$\nabla\sigma(u) = (\partial_j\sigma_{ij}(u))_{i=1}^{3}.$$

The mappings F and G from $\mathbb{R}^n$ into itself satisfy the assumptions (35) to (37). Finally a is a nonnegative real number.

As before we suppose that (61) and (62) hold, but here we further assume that

$$F = 0 \text{ or } G = 0. \tag{68}$$

This last assumption means that we stabilizate our system either by boundary feedback or by internal feedback.

The stability of the system (67) was considered in [11], [13], [15], [1], [4] under some particular hypotheses on Γ_0, Γ_1, a, F and G leading to exponential or polynomial decay of the energy of the solution of (67).

As in the above subsection problem (67) may be expressed in the form (59) from Subsection 6.1 with the choices:

$$\begin{aligned}
H &= L^2(\Omega)^n,\\
V &= \{v \in H^1(\Omega)^n | v = 0 \text{ on } \Gamma_0\},\\
(u,v)_V &= \int_\Omega \nabla u \cdot \nabla v \, dx + a\int_{\Gamma_1} u \cdot v \, d\sigma,\\
\langle B_2 u, v\rangle_{V'-V} &= \int_\Omega F(u)\cdot v\, dx + \int_{\Gamma_1} G(u)\cdot v\, d\sigma, \forall u, v \in V.
\end{aligned}$$

The assumptions made on F and G imply that B_2 fulfils the assumptions of Theorem 6.1, consequently the corresponding pair (A_1, B) satisfies the assumptions (3) to (8). For the stability results we need to check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate, where the control space U is defined by

$$U = L^2(\Gamma_1)^n \text{ if } F = 0,$$
$$U = L^2(\Omega)^n \text{ if } G = 0.$$

In the first case the stability estimate was proved in [4] under the assumption (65) (a similar estimate was proved in [11], [1] under stronger assumptions on Γ_0 and Γ_1). If the tensor (a_{ijkl}) corresponds to the Lamé system, then the stability estimate was proved in Lemma 3.2 of [15] under the weaker assumption

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0.$$

In the second case (i.e. $G=0$), the stability estimate for the pair $(-A_1, \mathcal{I}_U)$ was proved in Lemma 3.6 of [13].

As in the previous subsection, these conditions (on Γ_0, Γ_1 and the coefficients (a_{ijkl})) and appropriated conditions on F and G lead to exponential, polynomial, logarithmic or other decays. Bounded feedbacks are also allowed due to the embedding $H^1(\Omega) \times H^1(\Omega) \hookrightarrow L^\alpha(\Omega) \times L^\alpha(\Gamma_1)$, for some $\alpha > 2$.

6.4 – Nonlinear stabilization of a coupled system

We consider the following coupled system in a bounded domain Ω with a C^4-boundary:

$$\left\{\begin{array}{l} \partial_t^2 u_1 + \Delta^2 u_1 + a u_2 + g_1(\partial_t u_1, \partial_t u_2) = 0 \text{ in } Q, \\ \partial_t^2 u_2 - \Delta u_2 + a u_1 + g_2(\partial_t u_1, \partial_t u_2) = 0 \text{ in } Q, \\ u_1 = \partial_\nu u_1 = u_2 = 0 \text{ on } \Sigma = \Gamma \times]0, \infty[, \\ u_i(0) = u_{0i}, \partial_t u_i(0) = u_{1i} \text{ in } \Omega, i = 1, 2. \end{array}\right. \tag{69}$$

Here g_i are mappings from $\mathbb{R}^2$ into $\mathbb{R}$ such that the mapping G from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$G(x, y) = (g_1(x, y), g_2(x, y)),$$

satisfies the assumptions (35) to (37). Finally a is a scalar function that we assume to be in $L^\infty(\Omega)$.

The above system was considered in [14] when g_1 (resp. g_2) only depends on $\partial_t u_1$ (resp. $\partial_t u_2$). In that case this author proves exponential or polynomial decay rates under appropriated conditions on a, g_1 and g_2. Let us notice that if $a = 0$ and if g_1 (resp. g_2) only depends on $\partial_t u_1$ (resp. $\partial_t u_2$), then the above system is splitted up into the wave equation considered in Subsection 6.2 and the standard Petrovsky system studied in [12]. Our subsequent analysis then covers the analysis of this last Petrovsky system.

First problem (69) is in the form (59) with the definitions (see [14]):

$$\begin{aligned}
H &= L^2(\Omega)^2,\\
V &= H_0^2(\Omega) \times H_0^1(\Omega),\\
((u_1, u_2), (v_1, v_2))_V &= \int_\Omega (\Delta u_1 \Delta u_2 + \nabla u_2 \cdot \nabla v_2)\, dx +\\
&\quad + \int_\Omega a\,(u_1 v_2 + u_2 v_1)\, d\sigma,\\
\langle B_2(u_1, u_2), (v_1, v_2)\rangle_{V'-V} &= \int_\Omega (g_1(u_1, u_2)v_1 + g_2(u_1, u_2)v_2)\, dx,\\
&\quad \forall (u_1, u_2), (v_1, v_2) \in V.
\end{aligned}$$

The assumptions made on g_1 and g_2 imply that B_2 fulfils the assumptions of Theorem 6.1, consequently the corresponding pair (A_1, B) satisfies the assumptions (3) to (8). For the stability results we need to check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate when the control space U is given by $U = L^2(\Omega)^2$. This stability estimate was proved in Lemma 3.1 of [14] under the assumption

$$\|a\|_{L^\infty(\Omega)} < \frac{1}{c'c"},$$

where $c', c" > 0$ are the constants appearing in the above Poincaré type inequalities:

$$\|u\|^2_{H^2(\Omega)} \le c' \int_\Omega (\Delta u)^2\, dx, \forall u \in H_0^2(\Omega),$$

$$\|u\|^2_{H^1(\Omega)} \le c" \int_\Omega |\nabla u|^2\, dx, \forall u \in H_0^1(\Omega).$$

This condition and appropriated conditions on g_1 and g_2 lead to exponential, polynomial, logarithmic or other decays. As before bounded feedbacks are also allowed.

6.5 – Nonlinear stabilization of Maxwell's equations

We consider Maxwell's equations in $\Omega \subset \mathbb{R}^3$ with a smooth boundary and a nonlinear internal feedback:

$$(70)\qquad \begin{cases} \varepsilon\dfrac{\partial E}{\partial t} - \mathbf{curl}\, H + g(E) = 0 \text{ in } Q := \Gamma\times]0,+\infty[,\\ \mu\dfrac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q,\\ \operatorname{div}(\mu H) = 0 \text{ in } Q,\\ E\times\nu = 0, H\cdot\nu = 0 \text{ on } \Sigma := \Gamma\times]0,+\infty[,\\ E(0) = E_0, H(0) = H_0 \text{ in } \Omega. \end{cases}$$

As usual ε and μ are real, positive functions of class $C^\infty(\bar\Omega)$. The function g from $\mathbb{R}^3$ into itself is assumed to satisfy the properties (35) to (37).

The stability of this system was studied in [39] with a linear feedback $g(E) = \sigma E$, with $\sigma \geq 0$. In particular the exponential decay was shown in that paper if $\sigma \geq \sigma_0 > 0$.

Contrary to the above examples this system is not a second order system but (compare with [7]) it enters in the setting of (9) once we set

$$\begin{aligned} \mathcal{H} &= L^2(\Omega)^3 \times \hat J(\Omega,\mu),\\ \hat J(\Omega,\mu) &= \{H \in L^2(\Omega)^3 : \operatorname{div}(\mu H) = 0 \text{ in } \Omega, H\cdot\nu = 0 \text{ on } \Gamma\},\\ ((E,H),(E',H'))_{\mathcal{H}} &= \int_\Omega (\epsilon E\cdot E' + \mu H\cdot H')\,dx,\\ \mathcal{V} &= V\times \hat J(\Omega,\mu),\\ V &= \{E\in L^2(\Omega)^3 : \mathbf{curl}\, E \in L^2(\Omega)^3, E\times\nu = 0 \text{ on } \Gamma\},\\ \langle A_1(E,H),(E',H')\rangle &= \int_\Omega (\mathbf{curl}\, E\cdot H' - H\cdot \mathbf{curl}\, E')\,dx,\\ \langle B(E,H),(E',H')\rangle &= \int_\Omega g(E\times\nu)\cdot(E'\times\nu)\,d\sigma. \end{aligned}$$

One readily checks (as in [7, Section 3]) that the assumptions (3) and (4) hold since the bilinear form

$$\int_\Omega (\mu^{-1}\mathbf{curl}\, E \cdot \mathbf{curl}\, E' + \epsilon E \cdot E')\, dx$$

is clearly coercive on V. Moreover Lemma 2.3 of [35] implies that (5) and (6) hold. Finally from the definition of A_1 (7) clearly holds, while from the definition of B and the properties (35) and (36) satified by g, (8) holds. As the results of Section 5 of [39] imply that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate when the control space U is given by $U = L^2(\Omega)^3$, we may conclude exponential, polynomial, logarithmic or other decays under appropriated conditions on g. Here bounded feedbacks are not allowed since V is not embedded into $L^\alpha(\Omega)^3$ for some $\alpha > 2$.

Let us finally notice that Maxwell's equations with a nonlinear boundary feedback

$$(71)\qquad \begin{cases} \varepsilon \dfrac{\partial E}{\partial t} - \mathbf{curl}\, H = 0 \text{ in } Q := \Gamma \times]0, +\infty[, \\ \mu \dfrac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q, \\ \operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q, \\ H \times \nu + g(E \times \nu) \times \nu = 0 \text{ on } \Sigma := \Gamma \times]0, +\infty[, \\ E(0) = E_0, H(0) = H_0 \text{ in } \Omega, \end{cases}$$

was studied in [3], [21], [39], [7], [36]. Different decay rates are avalaible under different conditions on ϵ, μ and Γ and appropriated assumptions on g. It was shown in [7] that (71) enters in the setting of (9), where the assumptions (3) and (5) are also checked under some conditions on Ω, ϵ and μ (similar arguments actually imply that (4) and (6) hold as well). The stability analysis following the point of view of our paper is given in [36]. We then refer to that paper for the details.

REFERENCES

[1] F. Alabau – V. Komornik: *Boundary observability, controllability and stabilization of linear elastodynamic systems*, SIAM J. Control Optim., **37** (1999), 521-542.

[2] C. Amrouche – C. Bernardi – M. Dauge – V. Girault: *Vector potentials in three-dimensional nonsmooth domains*, Math. Meth. Applied Sc., **21** (1998), 823-864.

[3] H. Barucq – B. Hanouzet: *Étude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller*, II, C. R. Acad. Sci. Paris, Série I, **316** (1993), 1019-1024.

[4] R. Bey – A. Heminna – J.-P. Lohéac: *A new approach for boundary stabilisation of the linear elastodynamic system*, Revista Matematica de la Universidad Complutense de Madrid, submitted.

[5] F. Conrad – B. Rao: *Decay of solutions of wave equations in a star-shaped domain with nonlinear feedback*, Asymptotic Analysis, **7** (1993), 159-177.

[6] G. Duvaut – J.-L. Lions: *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[7] M. Eller – J. E. Lagnese – S. Nicaise: *Decay rates for solutions of a Maxwell system with nonlinear boundary damping*, Comp. and Apppl. Math., **21** (2002), 135-165.

[8] V. Girault – P.-A. Raviart: *Finite element methods for Navier-Stokes equations*, Springer Series in Comp. Math., **5** Springer-Verlag, 1986.

[9] P. Grisvard: *Elliptic problems in nonsmooth Domains*, Monographs and Studies in Mathematics, **21** Pitman, Boston, 1985.

[10] A. Guesmia: *On the nonlinear stabilization of the wave equation*, Annales Polonici Mathematici, **68** (1998), 191-198.

[11] A. Guesmia: *Existence globale et stabilisation interne non linéaire d'un système d'élasticité*, Portugaliae Mathematica, **55** (1998), 333-347.

[12] A. Guesmia: *Existence globale et stabilisation interne non linéaire d'un système de Petrovsky*, Bull. Belg. Math. Soc. S. Stevin, **5** (1998), 583-594.

[13] A. Guesmia: *Existence globale et stabilisation frontière non linéaire d'un système d'élasticité*, Portugaliae Mathematica, **56** (1999), 361-379.

[14] A. Guesmia: *Energy decay for a damped nonlinear coupled system*, J. Math. Analysis Appl., **239** (1999), 38-48.

[15] M. A. Horn: *Implications of sharp trace regularity results on boundary stabilization of the system of linear elasticity*, J. Math. Analysis Appl., **223** (1998), 126-150.

[16] B. V. Kapitanov: *Stabilization and exact boundary controllability for Maxwell's equations*, SIAM J. Control Optim., **32** (1994), 408-420.

[17] B. V. Kapitanov – M. A. Raupp: *Exact boundary controllability in problems of transmission for the system of electromagneto-elasticity*, Math. Meth. Appl. Sci., **24** (2001), 193-207.

[18] V. Komornik: *Rapid boundary stabilization of the wave equation*, SIAM J. Control Optim., **29** (1991), 197-208.

[19] V. Komornik: *Decay estimates for the wave equation with internal damping*, in: Proc. Conf. Control Theory, Vorau, 1993, Int. Ser. Numer. Anal., **118** (1994), 253-266.

[20] V. Komornik: *On the nonlinear boundary stabilization of the wave equation*, Chinese Ann. Math. Ser. B, **14** (1993), 153-164.

[21] V. Komornik: *Boundary stabilization, observation and control of Maxwell's equations*, PanAm. Math. J., **4** (1994), 47-61.

[22] V. Komornik: *Exact controllability and stabilization, the multiplier method*,**RMA 36**, Masson, Paris, 1994.

[23] V. Komornik – E. Zuazua: *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl., **69** (1990), 33-54.

[24] J. E. Lagnese: *Exact controllability of Maxwell's equations in a general region*, SIAM J. Control Optim., **27** (1989), 374-388.

[25] I. Lasiecka – D. Tataru: *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, J. Diff. Integr. Eq., **6** (1993), 507-533.

[26] I. Lasiecka – R. Triggiani: *Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim., **25** (1992), 189-224.

[27] J.-L. Lions: *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, tome 1, **RMA 8** Masson, Paris, 1988.

[28] W. J. Liu: *Partial exact controllability and exponential stability of higher dimensional linear thermoelasticity*, ESAIM: COCV, **3** (1998), 23-48.

[29] W. J. Liu – E. Zuazua: *Decay rates for dissipative wave equations*, Ric. Mat., **48** Suppl. (1999), 61-75.

[30] P. Martinez: *Uniform stabilization of the wave equation in almost star-shaped domains*, SIAM J. Control and Opt., **37** (1999), 673-694.

[31] P. Martinez: *A new method to obtain decay rate estimates for dissipative systems*, ESAIM: COCV, **4** (1999), 419-444.

[32] P. Martinez: *A new method to obtain decay rate estimates for dissipative systems with localized damping*, Rev. Mat. Complut., **12** (1999), 251-283.

[33] P. Martinez: *Decay of solutions of the wave equation with a local highly degenerate dissipation*, Asymptot. Anal., **19** (1999), 1-17.

[34] P. Martinez: *Precise decay rate estimates for time-dependent dissipative systems*, Israel J. Math., **119** (2000), 291-324.

[35] S. Nicaise: *Exact boundary controllability of Maxwell's equations in heteregeneous media and an application to an inverse source problem*, SIAM J. Control and Opt., **38** (2000), 1145-1170.

[36] S. Nicaise – M. Eller – J. E. Lagnese: *Stabilization of heteregeneous Maxwell's equations by nonlinear boundary feedbacks*, EJDE, **21** (2002), 1-26.

[37] S. Nicaise: *Stability and controllability of the electromagneto-elastic system*, Portugaliae Mathematica, **60** (2003), 37-70.

[38] C. Pignotti: *Observavility and controllability of Maxwell's equations*, Rendiconti di Mat., **19** (1999), 523-546.

[39] K. D. Phung: *Contrôle et stabilisation d'ondes électromagnétiques*, ESAIM: COCV, **5** (2000), 87-137.

[40] D. L. Russell: *The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region*, SIAM J. Control Optim., **24** (1986), 199-229.

[41] R. E. Showalter: *Hilbert space methods for partial differential equations*, Math. and Studies in Math., **1** Pitman, 1977.

[42] R. E. Showalter: *Monotone operators in Banach space and nonlinear partial differential equations*, Math. Surveys and Monographs, **49** AMS, 1997.

[43] E. Zuazua: *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM J. Control Optim., **28** (1990), 466-477.

Lavoro pervenuto alla redazione il 23 maggio 2002
ed accettato per la pubblicazione il 16 gennaio 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo dell'Autore:

S. Nicaise – Université de Valenciennes et du Hainaut Cambrésis – MACS – Institut des Sciences et Techniques de Valenciennes – F-59313 – Valenciennes Cedex 9 (France)
E-mail: snicaise@univ-valenciennes.fr