On the Dirichlet problem
for degenerate monotone operators

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Abstract: We prove an existence theorem for the Dirichlet problem in divergence form for degenerate monotone operators (also multivalued) with growth coefficients in $\text{BMO}$. We consider these problems in an open cube of $\mathbb{R}^n$, $n \geq 2$, and in Sobolev spaces with exponent $2 \leq p < +\infty$.

1 – Introduction

The aim of this paper is to study the Dirichlet problem on $W^{1,p}_0(\Omega)$ for non linear monotone operators of the form

\begin{equation}
Au = - \text{div } a(x, \nabla u)
\end{equation}

where $\Omega$ is an open cube of $\mathbb{R}^n$ and $2 \leq p < +\infty$.

We assume that the (possibly multivalued) map $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ which occurs in (1.1) is measurable on $\Omega \times \mathbb{R}^n$, maximal monotone on $\mathbb{R}^n$ for almost every $x \in \Omega$ and satisfies the following conditions:

1) $|\eta|^q \leq m_1(x) + K(x)(\eta, \xi)$ \hspace{1cm} $q = p(p - 1)^{-1}$


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ii) $|\xi|^p \leq m_2(x) + K(x)(\eta, \xi)$

iii) $0 \in a(x, 0)$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and $\eta \in a(x, \xi)$.

The non negative functions $m_1(x)$ and $m_2(x)$ are integrable on $\Omega$ and $K = K(x) \geq 1$ is in $\text{BMO}(\Omega)$ with

\[(1.2) \quad \|K\|_{\text{BMO}(\Omega)} \leq c\gamma^{-1}\]

where $c = c(n)$ is the Coifman-Rochberg constant (see [8]) and $\gamma = \gamma(n)$ sufficiently large. The class of all these maps will be denoted by $M_{K,\Omega}(\mathbb{R}^n)$.

The main examples have the form

$$a(x, \xi) = \partial_\xi \psi(x, \xi)$$

where $\partial_\xi$ denotes the subdifferential with respect to $\xi$ and $\psi : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is measurable in $(x, \xi)$, convex in $\xi$ and satisfies the inequality

$$|\xi|^p + |\eta|^q \leq (K(x) + K^{-1}(x))(\psi(x, \xi) + \psi^*(x, \eta))$$

where $\psi^*(x, \eta)$ is the Young conjugate of $\psi(x, \cdot)$ (see [15]).

In this case the operator (1.1) is the subdifferential of the functional

$$\Psi(u) = \int_\Omega \psi(x, \nabla u) \, dx.$$ 

The main result of this paper is the following

**Theorem A.** Under the above assumptions there exists a constant $\overline{\gamma} = \overline{\gamma}(n) > 0$ such that, if $\gamma > \overline{\gamma}$, for every function $f$, for which $Kf \in (L^q(\Omega))^n$, the problem

\[(1.3) \quad \begin{cases} 
-\text{div} \ f \in \text{div} \ a(x, \nabla u) \text{ on } \Omega \\
u \in W^{1,p}_0(\Omega)
\end{cases}\]

admits a solution, or equivalently, there exists a function $u \in W^{1,p}_0(\Omega)$ and a vector-valued function $g(x) \in a(x, \nabla u(x))$ for a.e. $x \in \Omega$ and

\[(1.4) \quad \text{div}(g - f) = 0.\]
Our result is strictly related with the theory of Quasiharmonic fields introduced in [11]. In fact there are two vector fields associated with a solution of problem (1.3): the first one \( E = \nabla u \) is curl free, the second \( B = g - f \) is divergence free.

Moreover inequalities \( i), ii \) imply

\[
|E|^p + |B|^q \leq c(K(B, E) + \theta)
\]

where \( \theta = \theta(x) = m_1(x) + m_2(x) + |K(x)f(x)|^q \) and \( c = c(q) \) is a constant.

As consequence of Theorem A we have an existence and uniqueness result for the degenerate p-laplacian.

Let us emphasize explicity that we are dealing here with genuine non isotropic degenerate operators.

The main tool in the proof of Theorem A is an a-priori estimate for a solution of problem (1.3) (see Section 3). Moreover, it is very useful an existence result for problem (1.3), proved in [6] when \( K(x) \) is bounded.

Many different situations have been studied in linear and nonlinear theory of elliptic PDE’s with BMO coefficients (see [3], [4], [11], [12]).

2 – Preliminary results

Let \( X \) and \( Y \) be two sets. A multivalued function \( F \) from \( X \) to \( Y \) is a map that associates with any \( x \in X \) a subset \( Fx \) of \( Y \). The subsets \( Fx \) are called the images or values of \( F \).

The sets

\[
D(F) = \{ x \in X : Fx \neq \emptyset \} \quad G(F) = \{ [x, y] \in X \times Y : y \in Fx \}
\]

are called the domain and the graph of \( F \) respectively.

The inverse \( F^{-1} \) of \( F \) is the multivalued map from \( Y \) to \( X \) defined by \( x \in F^{-1}y \) if and only if \( y \in Fx \).

Let \( (X, T) \) be a measurable space and let \( F : X \to \mathbb{R}^n \) be a multivalued function. For every \( B \subseteq \mathbb{R}^n \) the inverse image of \( B \) under \( F \) is denoted by

\[
F^{-1}(B) = \{ x \in X : B \cap Fx \neq \emptyset \}
\]

Let \( \mathcal{B}(\mathbb{R}^n) \) be the \( \sigma \)-field of all Borel subsets of \( \mathbb{R}^n \), we say that a multivalued function \( F : X \to \mathbb{R}^n \) is measurable (with respect to \( T \) and \( \mathcal{B}(\mathbb{R}^n) \)) if \( F^{-1}(C) \in T \) for every closed set \( C \subseteq \mathbb{R}^n \).
In the following we assume that $G(F) \in \mathcal{T} \times \mathcal{B}(\mathbb{R}^n)$ because we define a complete $\sigma$-finite measure on $\mathcal{T}$.

It is useful to recall the following theorem:

**Projection Theorem.** Let $(X, \mathcal{T}, \mu)$ be a measurable space, where $\mu$ is a complete $\sigma$-finite measure defined on $\mathcal{T}$. If $G$ belongs to $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$, then the projection $\text{pr}_x G$ belongs to $\mathcal{T}$.

For the proof of this theorem and properties of measurable multivalued functions we remain to [5] (Theorem III.23).

Let $F : X \to Y$ be a multivalued function, a function $\sigma : X \to Y$ is a *selection* of $F$ if $\sigma(x) \in Fx$ for every $x$.

Let us give a general theorem for the existence of measurable selection of a multivalued function due to Aumann and von Neumann (see [5] Theorem III.22).

**Theorem 2.1.** Let $(X, \mathcal{T})$ be a measurable space and let $F$ be a multivalued function from $X$ to $\mathbb{R}^n$ with non-empty values. If the graph $G(F)$ belongs to $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$ and exists a complete $\sigma$-finite measure defined on $\mathcal{T}$, then $F$ has a measurable selection.

Now, let $X$ be a real Banach space and let $X^*$ its dual space. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between $X^*$ and $X$. A set $M \subseteq X \times X^*$ is monotone if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \forall [x_1, y_1], [x_2, y_2] \in M.$$

A monotone set $M$ is *maximal monotone* if it is not a proper subset of a monotone set in $X \times X^*$ i.e. $[x, y] \in X \times X^*$ such that

$$\langle y - \eta, x - \xi \rangle \geq 0 \quad \forall [\xi, \eta] \in M$$

then $[x, y] \in M$.

A multivalued operator $F : X \to X^*$ is maximal monotone if its graph is maximal monotone.

We recall some properties of maximal monotone mappings that are very useful in the following (see [9], [2]). If $F : X \to X^*$ is a maximal monotone mapping, then

**Theorem 2.2.** For every $\lambda > 0$, $\lambda F$ is maximal monotone mapping.
Theorem 2.3. For any $x \in D(F)$ the image $Fx$ is closed convex subset of $X^*$. 

Theorem 2.4. If $X$ is a reflexive Banach space, then $F$ is surjective if and only if $F^{-1}$ is locally bounded on $X^*$. 

Theorem 2.5. If $0 \in \text{Int } D(F)$, $F$ is quasi-bounded. 

Theorem 2.6. If $F_1$ and $F_2$ are two maximal monotone mappings with $0 \in D(F_1) \cap D(F_2)$ and $F_1$ is quasi-bounded, we have $F_1 + F_2$ maximal monotone. 

Theorem 2.7. If $\varphi : X \rightarrow \mathbb{R}$ is a proper convex lower semi-continuous function, then the multivalued operator (subdifferential of $\varphi$) $\partial \varphi : X \rightarrow X^*$ is a maximal monotone mapping. 

Now we introduce a class of multivalued monotone operators on Sobolev spaces of the type $-\text{div}(a(x, \nabla u))$. Let $2 \leq p < +\infty$, we denote by $q$ the dual exponent of $p$, $p^{-1} + q^{-1} = 1$. We fix an open cube $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), two non negative functions $m_1, m_2 \in L^1(\Omega)$ and a function $K(x), K'(x) \geq 1$ a.e. in $\Omega$. We denote by $\mathcal{L}(\Omega)$ the $\sigma$-field of all Lebesgue measurable subsets of $\Omega$; if $E \in \mathcal{L}(\Omega)$, $|E|$ denotes the measure of $E$. 

The Euclidean norm and the scalar product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $(\cdot, \cdot)$ respectively. 

Definition 2.8. $M_{K, \Omega}(\mathbb{R}^n)$ is the class of all multivalued functions $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with closed values, measurable with respect to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$ such that are maximal monotone with respect to $\xi \in \mathbb{R}^n$ and satisfying 

\begin{itemize}
  \item[i)] $|\eta|^q \leq m_1(x) + K(x)(\eta, \xi)$ 
  \item[ii)] $|\xi|^p \leq m_2(x) + K(x)(\eta, \xi)$ 
  \item[iii)] $0 \in a(x, 0)$ for a.e. $x \in \Omega$. 
\end{itemize}
**Remark 2.9.** For 2.3 the set \( a(x, \xi) \) is closed and convex for a.e. \( x \in \Omega \) and \( \forall \xi \in \mathbb{R}^n \). For Theorem 2.1 the graph of \( a(x, \xi) \) belongs to \( \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n) \). By \( i \) we have that there is a non negative function \( m_3(x) \in L^q(\Omega) \) such that

\[
|\eta| \leq m_3(x) + K^{p-1}(x)|\xi|^{p-1} \quad \text{for a.e. } x \in \Omega.
\]

By (2.1) for a.e. \( x \in \Omega \) the maximal monotone operator \( a(x, \xi) \) is locally bounded, hence by 2.4 \( a^{-1}(x, \cdot) \) is surjective. This implies \( a(x, \xi) \neq \emptyset \) for a.e. \( x \in \Omega \) and \( \forall \xi \in \mathbb{R}^n \).

Let \( K(x) \in L^\infty(\Omega) \) and \( a \in M_{K, \Omega}(\mathbb{R}^n) \), we consider the Dirichlet boundary value problem

\[
\begin{aligned}
&-\operatorname{div} f \in -\operatorname{div} a(x, \nabla u) \quad \text{on } \Omega, \ f \in (L^q(\Omega))^n \\
u &\in W^{1,p}_0(\Omega). \quad p \geq 2
\end{aligned}
\]

We say that \( u \in W^{1,p}_0(\Omega) \) is a solution of problem (2.2) (as well as (1.3)) if

\[
\exists g \in (L^q(\Omega))^n \text{ such that } g(x) \in a(x, \nabla u(x)) \text{ and } \int_\Omega g(x) \nabla \varphi(x) dx = \int_\Omega f(x) \nabla \varphi(x) dx \quad \forall \varphi \in C_0^\infty(\Omega)
\]

By using the same arguments of [6] it is easy to verify that

**Theorem 2.10.** If \( K(x) \in L^\infty(\Omega) \) and \( a \in M_{K, \Omega}(\mathbb{R}^n) \) the Dirichlet problem (2.2) has a solution.

**3 – A priori estimates for the solution of problem (1.3)**

If \( \Omega \) is an open cube of \( \mathbb{R}^n \), by \( H^1(\Omega) \) and \( \text{BMO}(\Omega) \) we denote the Hardy space and the space of all functions of bounded mean oscillation respectively, we have see [7], [10], [13], [14], [16], [17]):

**Proposition 3.1.** Let \( E \in (L^p(\Omega))^n \) and \( B \in (L^q(\Omega))^n \) be two vector fields such that \( \operatorname{div} B = 0 \) and \( \operatorname{curl} E = 0 \), where \( p \) and \( q \) are
conjugate. Then their scalar product \((E, B)\) belongs to \(H^1(\Omega)\) and there is a constant \(c = c(n)\) such that

\[
\|(E, B)\|_{H^1(\Omega)} \leq c \|E\|_{L^p(\Omega)} \|B\|_{L^q(\Omega)}.
\]

**Lemma 3.2.** (Div-Curl) Suppose the vector fields \(E_k, B_k\) respectively in \((L^p(\Omega))^n, (L^q(\Omega))^n\) \((1 < p < \infty, n \geq 2)\) verify \(\text{div} \ B_k = 0\) \(\text{curl} \ E_k = 0\) and \(E_k, B_k\) converge weakly respectively in \((L^p(\Omega))^n, (L^q(\Omega))^n\) to some \(E, B\) then

\[
(E_k, B_k) \rightharpoonup (E, B) \quad \text{in} \ D'(\Omega).
\]

**Theorem 3.3.** For every \(K \in \text{BMO}(\Omega)\) there exists a unique \(T \in (H^1(\Omega))^*\) such that

\[
(3.1) \quad T(h) = \int_{\Omega} h(x)K(x)dx \quad \forall \ h \in H^1(\Omega).
\]

Conversely, for every \(T \in (H^1(\Omega))^*\) there exists a unique \(g \in \text{BMO}(\Omega)\) which satisfies (3.1). The correspondence \(T \longrightarrow g\) determined by (3.1) is a Banach space isomorphism between \((H^1(\Omega))^*\) and \(\text{BMO}(\Omega)\).

Now let \(K \in \text{BMO}(\Omega), K(x) \geq 1\) a.e. in \(\Omega\), we prove the following

**Lemma 3.4.** Let \(u \in W^{1,p}_o(\Omega)\) a solution of the problem (1.3) with \(Kf \in (L^q(\Omega))^n\). There exists \(\gamma_o\) such that whenever \(K\) satisfies (1.2) with \(\gamma > \gamma_o\), we have

\[
(3.2) \quad \int_{\Omega} |\nabla u|^p + |g|^q dx \leq c(\|m_1\|_{L^1} + \|m_2\|_{L^1} + \|Kf\|_{L^q})
\]

where \(c = c(n) > 0\)
Proof. From $i$) and $ii$)

$$
\begin{align*}
\int_\Omega |\nabla u(x)|^p + |g(x)|^q dx & \leq \|m_1\|_{L^1} + \|m_2\|_{L^1} + \\
+ 2 \int_\Omega K(x)(g(x) - f(x), \nabla u(x)) dx & + 2 \int_\Omega K(x)(f(x), \nabla u(x)) dx
\end{align*}
$$

so, by Theorem 3.3 there exists a positive constant $c_1$ independent of $\|K\|_{\text{BMO}}$ such that

$$
\int_\Omega |\nabla u|^p + |g|^q dx \leq \|m_1\|_{L^1} + \|m_2\|_{L^1} + 2c_1\|K\|_{\text{BMO}}\|g - f, \nabla u\|_{H^1(\Omega)} + \\
+ 2 \int_\Omega K(x)(f(x), \nabla u(x)) dx.
$$

By Proposition 3.1, applying Hölder and Young inequalities we have, for every $\varepsilon > 0$,

$$
\begin{align*}
\|\nabla u\|_{L^p}^p + \|g\|_{L^q}^q & \leq \|m_1\|_{L^1} + \|m_2\|_{L^1} + \\
+ 4c_1c(n)\|K|_{\text{BMO}}(\|\nabla u\|_{L^p}^p + \|g\|_{L^q}^q) + \\
+ \varepsilon(\|\nabla u\|_{L^p}^p + \|g\|_{L^q}^q) + 2c_1c(n)\|K\|_{\text{BMO}}\|f\|_{L^q}^q + c_\varepsilon\|Kf\|_{L^q}^q
\end{align*}
$$

(3.3)

where $c_\varepsilon$ is independent of $\|K\|_{\text{BMO}}$.

Now, for all $K \in \text{BMO}(\Omega)$ such that $4c_1c(n)\|K\|_{\text{BMO}} \leq \rho$ with $0 < \rho < 1$, we have that, for every fixed $0 < \varepsilon < 1 - \rho$, there is a constant $c$ independent of $\|K\|_{\text{BMO}}$ such that

$$
\|\nabla u\|_{L^p}^p + \|g\|_{L^q}^q \leq c(\|m_1\|_{L^1} + \|m_2\|_{L^1} + \|Kf\|_{L^q}^q).
$$

Applying condition (1.2), the Lemma is proved provided $\gamma_0 > 4c_1c(n)$. \hfill \Box

4 – Proof of Theorem A

In order to prove Theorem A, for every $\varepsilon \in [0,1]$ we consider the multivalued function $a_\varepsilon(x, \xi) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$
a_\varepsilon(x, \xi) = \frac{a(x, \xi) + \varepsilon K^{p-1}(x)|\xi|^{p-2}\xi}{1 + \varepsilon K^{p-1}(x)}
$$

(4.1)

where $a(x, \xi) \in M_{K,\Omega}(\mathbb{R}^n)$ and $K(x) \in \text{BMO}(\Omega)$, $K(x) \geq 1$. 

The multivalued function \( a_\varepsilon(x, \xi) \) with closed values is measurable with respect to \( L(\Omega) \otimes B(\mathbb{R}^n) \) and \( B(\mathbb{R}^n) \). Moreover, by properties 2.2, 2.5-2.7, it is maximal monotone with respect to \( \xi \) a.e. \( x \in \Omega \). From (4.1), using (2.1) we have
\[
|\eta_\varepsilon| \leq m_3(x) + \frac{K^{p-1}(x)(1 + \varepsilon)}{1 + \varepsilon K^{p-1}(x)} |\xi|^{p-1} \quad \text{a.e. } x \in \Omega
\]
while using \( ii) \)
\[
|\xi|^p \leq m_2(x) + \frac{K^n(x)(1 + \varepsilon)}{1 + \varepsilon K^p(x)} (\eta_\varepsilon, \xi) \quad \text{a.e. } x \in \Omega
\]
for every \( \varepsilon \in ]0, 1[ \), \( \forall \xi \in \mathbb{R}^n \) and \( \eta_\varepsilon \in a_\varepsilon(x, \xi) \).

We can easily verify that the following lemmas yield

**Lemma 4.1.** Let be \( p \geq 2 \). For every \( \varepsilon \in ]0, 1[ \) the map \( a_\varepsilon(x, \xi) \in M_{\frac{2}{\varepsilon}q+3, \Omega}(\mathbb{R}^n) \).

**Lemma 4.2.** Let be \( p \geq 2 \). For every \( \varepsilon \in ]0, 1[ \) the map \( a_\varepsilon(x, \xi) \in M_{4K, \Omega}(\mathbb{R}^n) \).

Now we are in position to prove the main result

**Proof of Theorem A.** For every \( \varepsilon \in ]0, 1[ \), let us consider the problem
\[
(4.2) \begin{cases}
- \text{div } f \in - \text{div } a_\varepsilon(x, \nabla u) & \text{on } \Omega \\
u \in W^{1,p}_0(\Omega) & (p \geq 2)
\end{cases}
\]
with data \( f \) such that \( Kf \in (L^q(\Omega))^n \). From Lemma 4.1 and Theorem 2.10 it follows that problem (4.2) admits at least a solution \( u_\varepsilon \in W^{1,p}_0(\Omega) \).

From Lemmas 4.2 and 3.4, if \( K \) verifies (1.2) with \( \gamma > 4\gamma_0 \), the pair \( [\nabla u_\varepsilon, g_\varepsilon] \) satisfy the estimate (3.2) uniformly with respect to \( \varepsilon \in ]0, 1[ \).

So we can construct two subsequences again denoted by \( \{u_\varepsilon\} \) and \( \{g_\varepsilon\} \) such that as \( \varepsilon \to 0^+ \)
\[
(4.3) \quad u_\varepsilon \to u \text{ weakly in } W^{1,p}_0(\Omega),
(4.4) \quad g_\varepsilon \to g \text{ weakly in } (L^q(\Omega))^n,
\]
and

\[(4.5) \quad \int_{\Omega} g_{\varepsilon} \nabla \varphi dx = \int_{\Omega} f \nabla \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega).\]

The equality in (2.3) is obtained by passing to limit as \(\varepsilon \to 0^+\) in (4.5).

To conclude the proof we show that \(g(x) \in a(x, \nabla u(x))\) for a.e. in \(\Omega\). Put

\[M = \{x \in \Omega : \exists \xi \in \mathbb{R}^n \exists \eta \in a(x, \xi) : (g(x) - \eta, \nabla u(x) - \xi) < 0\},\]

according to the maximal monotonicity of the map \(a\) it is enough to show that \(|M| = 0\). To prove that let us write \(M = \{x \in \Omega : Gx \neq \emptyset\}\) where

\[Gx = \{[\xi, \eta] \in \mathbb{R}^n \times \mathbb{R}^n : \eta \in a(x, \xi) \text{ and } (g(x) - \eta, \nabla u(x) - \xi) < 0\}.\]

By Remark 2.9 the graph of \(G\) belongs to \(L(\Omega) \otimes B(\mathbb{R}^n) \otimes B(\mathbb{R}^n)\) thus \(M \in L(\Omega)\) by the projection Theorem.

The Theorem 2.1 assures that there is a measurable selection \([\xi(x), \eta(x)]\) of \(G\) defined on \(M\), then \(\eta(x) \in a(x, \xi(x))\) and

\[(g(x) - \eta(x), \nabla u(x) - \xi(x)) < 0 \quad \forall x \in M.\]

If \(|M| > 0\), there are two measurable subset \(M'\), \(M''\) of \(M\), \(M'' \subset M'\), with \(0 < |M''| < |M'| < +\infty\) such that \([\xi(x), \eta(x)]\) is bounded on \(M'\). Now, for every \(\varepsilon \in ]0, 1]\) and \(x \in M'\) we consider

\[\eta_\varepsilon(x) = \frac{\eta(x) + \varepsilon K^{p-1}(x)|\xi(x)|^{p-2}\xi(x)}{1 + \varepsilon K^{p-1}(x)}.\]

The sequence \(\{\eta_\varepsilon(x)\}\) is bounded on \(M'\) and converges to \(\eta(x)\) a.e. in \(M'\). Since \(|M'| < +\infty\), the sequence \(\{\eta_\varepsilon(x)\}\) converges in measure to \(\eta(x)\), hence

\[(4.6) \quad \eta_\varepsilon(x) \longrightarrow \eta(x) \quad \text{in} \quad (L^q(M'))^n.\]

Since \(\eta_\varepsilon(x) \in a_\varepsilon(x, \xi(x))\) we have

\[(g_\varepsilon(x) - \eta_\varepsilon(x), \nabla u_\varepsilon(x) - \xi(x)) \geq 0 \quad \forall x \in M'.\]
Moreover, if $\phi \in C^\infty_c(M')$, $\phi(x) > 0 \ \forall x \in M''$ and $\phi(x) \geq 0 \ \forall x \in M' - M''$, we have

$$0 \leq \int_{M''} (g_\varepsilon(x) - \eta_\varepsilon(x), \nabla u_\varepsilon(x) - \xi(x))\phi(x)dx \leq$$

$$\leq \int_{M'} (g_\varepsilon(x) - f(x), \nabla u_\varepsilon(x))\phi(x)dx +$$

$$+ \int_{M'} (g_\varepsilon(x) - f(x), -\xi(x))\phi(x)dx +$$

$$+ \int_{M'} (f(x) - \eta_\varepsilon(x), \nabla u_\varepsilon(x) - \xi(x))\phi(x)dx = I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon. \tag{4.7}$$

By Lemma 3.2, $I^1_\varepsilon \rightarrow \int_{M'} (g(x) - f(x), \nabla u(x))\phi(x)dx$. Moreover by (4.4), (4.6) and (4.3) we get

$$I^2_\varepsilon \rightarrow \int_{M'} (g(x) - f(x), -\xi(x))\phi(x)dx,$$

$$I^3_\varepsilon \rightarrow \int_{M'} (f(x) - \eta(x), \nabla u(x) - \xi(x))\phi(x)dx.$$

So, by passing to limit as $\varepsilon \rightarrow 0^+$ in (4.7) we have

$$\int_{M'} (g(x) - \eta(x), \nabla u(x) - \xi(x))\phi(x)dx \geq 0. \tag{4.8}$$

Since $|M'| > 0$, (4.8) contradicts the definition of $M$. Therefore we conclude that $|M| = 0$. This completes the proof of Theorem A with $\gamma = 4\gamma_0$. \hfill \Box

Now, we consider the class, $S_K$, of all single-valued maps $a \in M_K, \Omega(\mathbb{R}^n)$, such that

iv) $|a(x, \xi_1) - a(x, \xi_2)| \leq K^{p-1} (x) \ |\xi_1 - \xi_2|(|\xi_1|^{p-2} + |\xi_2|^{p-2}),$

v) $K(x)^{-1}|\xi_1 - \xi_2|^p \leq (a(x, \xi_1) - a(x, \xi_2), \ \xi_1 - \xi_2)$

for a.e. $x \in \Omega, \ \forall \xi_1, \xi_2 \in \mathbb{R}^n$.

By Theorem A and conditions iv), v) we have
Corollary 4.3. Let $p \geq 2$. There is a positive constant $\gamma$ such that if $a \in S_K$ and $K$ satisfies (1.2) with $\gamma > \gamma$, the problem

$$\begin{cases}
- \text{div } f = - \text{div } a(x, \nabla u) & \text{on } \Omega \\
u \in W^{1,p}_0(\Omega)
\end{cases}$$

has a unique solution.

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