Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 203-215

On the Dirichlet problem for degenerate monotone operators

N. D'AURIA – O. FIODO

Abstract: *We prove an existence theorem for the Dirichlet problem in divergence form for degenerate monotone operators* (*also multivalued*) *with growth coefficients in* BMO. *We consider these problems in an open cube of* $\mathbb{R}^n$, $n \geq 2$, *and in Sobolev spaces with exponent* $2 \leq p < +\infty$.

1 – Introduction

The aim of this paper is to study the Dirichlet problem on $W_0^{1,p}(\Omega)$ for non linear monotone operators of the form

$$\mathcal{A}u = -\operatorname{div} a(x, \nabla u) \tag{1.1}$$

where Ω is an open cube of $\mathbb{R}^n$ and $2 \leq p < +\infty$.

We assume that the (possibly multivalued) map $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ which occurs in (1.1) is measurable on $\Omega \times \mathbb{R}^n$, maximal monotone on $\mathbb{R}^n$ for almost every $x \in \Omega$ and satisfies the following conditions:

i) $|\eta|^q \leq m_1(x) + K(x)(\eta, \xi) \qquad q = p(p-1)^{-1}$

Key Words and Phrases: *Multivalued function – Dirichlet problem – Monotone operators – Div-curl vector fields.*
A.M.S. Classification: 35J70 – 47H05

ii) $|\xi|^p \leq m_2(x) + K(x)(\eta, \xi)$
iii) $0 \in a(x, 0)$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and $\eta \in a(x, \xi)$.

The non negative functions $m_1(x)$ and $m_2(x)$ are integrable on Ω and $K = K(x) \geq 1$ is in BMO(Ω) with

$$\|K\|_{\mathrm{BMO}(\Omega)} \leq c\gamma^{-1} \tag{1.2}$$

where $c = c(n)$ is the Coifman-Rochberg constant (see [8]) and $\gamma = \gamma(n)$ sufficiently large. The class of all these maps will be denoted by $M_{K,\Omega}(\mathbb{R}^n)$.

The main examples have the form

$$a(x, \xi) = \partial_\xi \psi(x, \xi)$$

where ∂_ξ denotes the subdifferential with respect to ξ and $\psi : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is measurable in (x, ξ), convex in ξ and satisfies the inequality

$$|\xi|^p + |\eta|^q \leq (K(x) + K^{-1}(x))(\psi(x, \xi) + \psi^*(x, \eta))$$

where $\psi^*(x, \eta)$ is the Young conjugate of $\psi(x, \cdot)$ (see [15]).

In this case the operator (1.1) is the subdifferential of the functional

$$\Psi(u) = \int_\Omega \psi(x, \nabla u) dx \,.$$

The main result of this paper is the following

THEOREM A. *Under the above assumptions there exists a constant $\overline{\gamma} = \overline{\gamma}(n) > 0$ such that, if $\gamma > \overline{\gamma}$, for every function f, for which $Kf \in (L^q(\Omega))^n$, the problem*

$$\begin{cases} -\operatorname{div} f \in -\operatorname{div} a(x, \nabla u) \ \text{ on } \Omega \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{1.3}$$

admits a solution, or equivalently, there exists a function $u \in W_0^{1,p}(\Omega)$ and a vector- valued function $g(x) \in a(x, \nabla u(x))$ for a. e. $x \in \Omega$ and

$$\operatorname{div}(g - f) = 0 \,. \tag{1.4}$$

Our result is strictly related with the theory of Quasiharmonic fields introduced in [11]. In fact there are two vector fields associated with a solution of problem (1.3): the first one $E = \nabla u$ is curl free, the second $B = g - f$ is divergence free.

Moreover inequalities i), ii) imply

$$|E|^p + |B|^q \leq c(K\langle B, E\rangle + \theta)$$

where $\theta = \theta(x) = m_1(x) + m_2(x) + |K(x)f(x)|^q$ and $c = c(q)$ is a constant.

As consequence of Theorem A we have an existence and uniqueness result for the degenerate p-laplacian.

Let us emphasize explicetly that we are dealing here with genuine non isotropic degenerate operators.

The main tool in the proof of Theorem A is an a-priori estimate for a solution of problem (1.3) (see Section 3). Moreover, it is very useful an existence result for problem (1.3), proved in [6] when $K(x)$ is bounded.

Many different situations have been studied in linear and nonlinear theory of elliptic PDE's with BMO coefficients (see [3], [4], [11], [12]).

2 – Preliminary results

Let X and Y be two sets. A multivalued function F from X to Y is a map that associates with any $x \in X$ a subset Fx of Y. The subsets Fx are called the *images* or *values* of F.

The sets

$$D(F) = \{x \in X : Fx \neq \varnothing\} \qquad G(F) = \{[x, y] \in X \times Y : y \in Fx\}$$

are called the *domain* and the *graph* of F respectively.

The *inverse* F^{-1} of F is the multivalued map from Y to X defined by $x \in F^{-1}y$ if and only if $y \in Fx$.

Let $(X, \mathcal{T})$ be a measurable space and let $F : X \to \mathbb{R}^n$ be a multivalued function. For every $B \subseteq \mathbb{R}^n$ the inverse image of B under F is denoted by

$$F^{-1}(B) = \{x \in X : B \cap Fx \neq \varnothing\}$$

Let $\mathcal{B}(\mathbb{R}^n)$ be the σ-field of all Borel subsets of $\mathbb{R}^n$, we say that a multivalued function $F : X \to \mathbb{R}^n$ is *measurable* (with respect to $\mathcal{T}$ and $\mathcal{B}(\mathbb{R}^n)$) if $F^{-1}(C) \in \mathcal{T}$ for every closed set $C \subset \mathbb{R}^n$.

In the following we assume that $G(F) \in \mathcal{T} \times \mathcal{B}(\mathbb{R}^n)$ because we define a complete σ-finite measure on $\mathcal{T}$.

It is usefull to recall the following theorem:

Projection Theorem. Let $(X, \mathcal{T}, \mu)$ be a measurable space, where μ is a complete σ-finite measure defined on $\mathcal{T}$. If G belongs to $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$, then the projection $\mathrm{pr}_x G$ belongs to $\mathcal{T}$.

For the proof of this theorem and properties of measurable multivalued functions we remaind to [5] (Theorem III.23).

Let $F : X \to Y$ be a multivalued function, a function $\sigma : X \to Y$ is a *selection* of F if $\sigma(x) \in Fx$ for every x.

Let us give a general theorem for the existence of measurable selection of a multivalued function due to Aumann and von Neumann (see [5] Theorem III.22).

Theorem 2.1. *Let $(X, \mathcal{T})$ be a measurable space and let F be a multivalued function from X to $\mathbb{R}^n$ with non-empty values. If the graph $G(F)$ belongs to $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n)$ and exists a complete σ-finite measure defined on $\mathcal{T}$, then F has a measurable selection.*

Now, let X be a real Banach space and let X^* its dual space. By $\langle , \rangle$ we denote the duality pairing between X^* and X. A set $M \subseteq X \times X^*$ is *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0 \qquad \forall [x_1, y_1], [x_2, y_2] \in M \,.$$

A monotone set M is *maximal monotone* if it is not a proper subset of a monotone set in $X \times X^*$ i.e. $[x, y] \in X \times X^*$ such that

$$\langle y - \eta, x - \xi \rangle \geqslant 0 \qquad \forall [\xi, \eta] \in M$$

then $[x, y] \in M$.

A multivalued operator $F : X \to X^*$ is *maximal monotone* if its graph is maximal monotone.

We recall some properties of maximal monotone mappings that are very usefull in the following (see [9], [2]). If $F : X \to X^*$ is a maximal monotone mapping, then

Theorem 2.2. *For every $\lambda > 0$, λF is maximal monotone mapping.*

Theorem 2.3. *For any $x \in D(F)$ the image Fx is closed convex subset of X^*.*

Theorem 2.4. *If X is a reflexive Banach space, then F is surjective if and only if F^{-1} is locally bounded on X^*.*

Theorem 2.5. *If $0 \in \operatorname{Int} D(F)$, F is quasi-bounded.*

Theorem 2.6. *If F_1 and F_2 are two maximal monotone mappings with $0 \in D(F_1) \cap D(F_2)$ and F_1 is quasi-bounded, we have $F_1 + F_2$ maximal monotone.*

Theorem 2.7. *If $\varphi : X \to \overline{R}$ is a proper convex lower semi-continuous function, then the multivalued operator (subdifferential of φ) $\partial\varphi : X \to X^*$ is a maximal monotone mapping.*

Now we introduce a class of multivalued monotone operators on Sobolev spaces of the type $-\operatorname{div}(a(x, \nabla u))$. Let $2 \leq p < +\infty$, we denote by q the dual exponent of p, $p^{-1} + q^{-1} = 1$. We fix an open cube $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, two non negative functions $m_1, m_2 \in L^1(\Omega)$ and a function $K(x)$, $K(x) \geq 1$ a.e. in Ω. We denote by $\mathcal{L}(\Omega)$ the σ-field of all Lebesgue measurable subsets of Ω; if $E \in \mathcal{L}(\Omega)$, $|E|$ denotes the measure of E.

The Euclidean norm and the scalar product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $(\cdot, \cdot)$ respectively.

Definition 2.8. $M_{K,\Omega}(\mathbb{R}^n)$ is the class of all multivalued functions $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ with closed values, measurable with respect to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$ such that are maximal monotone with respect to $\xi \in \mathbb{R}^n$ and satisfying

i) $|\eta|^q \leq m_1(x) + K(x)(\eta, \xi)$
ii) $|\xi|^p \leq m_2(x) + K(x)(\eta, \xi)$
for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$, $\eta \in a(x, \xi)$,
iii) $0 \in a(x, 0)$ for a.e. x $\in \Omega$.

REMARK 2.9. For 2.3 the set $a(x,\xi)$ is closed and convex for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^n$. For Theorem 2.1 the graph of $a(x,\xi)$ belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$. By *i)* we have that there is a non negative function $m_3(x) \in L^q(\Omega)$ such that

$$|\eta| \le m_3(x) + K^{p-1}(x)|\xi|^{p-1} \qquad \text{for a.e. } x \in \Omega\,. \tag{2.1}$$

By (2.1) for a.e. $x \in \Omega$ the maximal monotone operator $a(x,\xi)$ is locally bounded, hence by 2.4 $a^{-1}(x,\cdot)$ is suriective. This implies $a(x,\xi) \neq \varnothing$ for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^n$.

Let $K(x) \in L^\infty(\Omega)$ and $a \in M_{K,\Omega}(\mathbb{R}^n)$, we consider the Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div} f \in -\operatorname{div} a(x,\nabla u) & \text{on } \Omega, f \in (L^q(\Omega))^n \\ u \in W_0^{1,p}(\Omega). & p \ge 2 \end{cases} \tag{2.2}$$

We say that $u \in W_0^{1,p}(\Omega)$ is a solution of problem (2.2) (as well as (1.3)) if

$$\begin{gathered} \exists\, g \in (L^q(\Omega))^n \text{ such that } g(x) \in a(x,\nabla u(x)) \text{ and} \\ \int_\Omega g(x)\nabla\varphi(x)dx = \int_\Omega f(x)\nabla\varphi(x)dx \qquad \forall\,\varphi \in C_0^\infty(\Omega) \end{gathered} \tag{2.3}$$

By using the same arguments of [6] it is easy to verify that

THEOREM 2.10. *If $K(x) \in L^\infty(\Omega)$ and $a \in M_{K,\Omega}(\mathbb{R}^n)$ the Dirichlet problem* (2.2) *has a solution.*

3 – A priori estimates for the solution of problem (1.3)

If Ω is an open cube of $\mathbb{R}^n$, by $H^1(\Omega)$ and $\mathrm{BMO}(\Omega)$ we denote the Hardy space and the space of all functions of bounded mean oscillation respectively, we have see [7], [10], [13], [14], [16], [17]):

PROPOSITION 3.1. *Let $E \in (L^p(\Omega))^n$ and $B \in (L^q(\Omega))^n$ be two vector fields such that* $\operatorname{div} B = 0$ *and* $\operatorname{curl} E = O$, *where p and q are* *conjugate. Then their scalar product* (E,B) *belongs to* $H^1(\Omega)$ *and there is a constant* $c = c(n)$ *such that*

$$\|(E,B)\|_{H^1(\Omega)} \le c\|E\|_{L^p(\Omega)}\|B\|_{L^q(\Omega)}\,.$$

Lemma 3.2. (Div-Curl) *Suppose the vector fields* E_k, B_k *respectively in* $(L^p(\Omega))^n$, $(L^q(\Omega))^n (1 < p < \infty,\ n \ge 2)$ *verify* $\operatorname{div} B_k = 0 \operatorname{curl} E_k = O$ *and* E_k, B_k *converge weakly respectively in* $(L^p(\Omega))^n$, $(L^q(\Omega))^n$ *to some* E, B *then*

$$(E_k, B_k) \rightharpoonup (E,B) \quad \textit{in } D'(\Omega)\,.$$

Theorem 3.3. *For every* $K \in \mathrm{BMO}(\Omega)$ *there exists a unique* $T \in (H^1(\Omega))^*$ *such that*

$$T(h) \;= \int_\Omega h(x)K(x)dx \qquad \forall\, h \in H^1(\Omega)\,. \tag{3.1}$$

Conversely, for every $T \in (H^1(\Omega))^*$ *there exists a unique* $g \in \mathrm{BMO}(\Omega)$ *which satisfies* (3.1)*. The corrispondence* $T \longrightarrow g$ *determined by* (3.1) *is a Banach space isomorphism between* $(H^1(\Omega))^*$ *and* $\mathrm{BMO}(\Omega)$.

Now let $K \in \mathrm{BMO}(\Omega)$, $K(x) \ge 1$ a.e. in Ω, we prove the following

Lemma 3.4. *Let* $u \in W^{1,p}_o(\Omega)$ *a solution of the problem* (1.3) *with* $Kf \in (L^q(\Omega))^n$*. There exists* γ_o *such that whenever* K *satisfies* (1.2) *with* $\gamma > \gamma_o$*, we have*

$$\int_\Omega |\nabla u|^p + |g|^q dx \le c(\|m_1\|_{L^1} + \|m_2\|_{L^1} + \|Kf\|^q_{L^q}) \tag{3.2}$$

where $c = c(n) > 0$

Proof. From $i)$ and $ii)$

$$\int_\Omega |\nabla u(x)|^p + |g(x)|^q dx \le \|m_1\|_{L^1} + \|m_2\|_{L^1} + $$
$$+ 2\Big|\int_\Omega K(x)(g(x)-f(x),\nabla u(x))\ dx\Big| + 2\Big|\int_\Omega K(x)(f(x),\nabla u(x))dx\Big|$$

so, by Theorem 3.3 there exists a positive constant c_1 independent of $\|K\|_{\mathrm{BMO}}$ such that

$$\int_\Omega |\nabla u|^p + |g|^q\ dx \le \|m_1\|_{L^1} + \|m_2\|_{L^1} + 2c_1\|K\|_{\mathrm{BMO}}\|(g-f,\nabla u)\|_{H^1(\Omega)} +$$
$$+ 2\Big|\int_\Omega K(x)(f(x),\nabla u(x))dx\Big|\,.$$

By Proposition 3.1, applying Hölder and Young inequalities we have, for every $\varepsilon > 0$,

$$\begin{aligned}&\|\nabla u\|^p_{L^p} + \|g\|^q_{L^q} \le \|m_1\|_{L^1} + \|m_2\|_{L^1} + \\ &+ 4c_1c(n)\|K\|_{\mathrm{BMO}}(\|\nabla u\|^p_{L^p} + \|g\|^q_{L^q}) + \\ &+ \varepsilon(\|\nabla u\|^p_{L^p} + \|g\|^q_{L^q}) + 2\ c_1c(n)\|K\|_{\mathrm{BMO}}\|f\|^q_{L^q} + c_\varepsilon\|Kf\|^q_{L^q}\end{aligned} \tag{3.3}$$

where c_ε is independent of $\|K\|_{\mathrm{BMO}}$.

Now, for all $K \in \mathrm{BMO}(\Omega)$ such that $4c_1c(n)\|K\|_{\mathrm{BMO}} \le \rho$ with $0 < \rho < 1$, we have that, for every fixed $0 < \varepsilon < 1-\rho$, there is a costant c independent of $\|K\|_{\mathrm{BMO}}$ such that

$$\|\nabla u\|^p_{L^p} + \|g\|^q_{L^q} \le c(\|m_1\|_{L^1} + \|m_2\|_{L^1} + \|Kf\|^q_{L^q})\,.$$

Applying condition (1.2), the Lemma is proved provided $\gamma_0 > 4c_1c\,c(n)$. □

4 – Proof of Theorem A

In order to prove Theorem A, for every $\varepsilon \in]0,1]$ we consider the multivalued function $a_\varepsilon(x,\xi) : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$a_\varepsilon(x,\xi) = \frac{a(x,\xi) + \varepsilon K^{p-1}(x)|\xi|^{p-2}\xi}{1+\varepsilon K^{p-1}(x)} \tag{4.1}$$

where $a(x,\xi) \in M_{K,\Omega}(\mathbb{R}^n)$ and $K(x) \in \mathrm{BMO}(\Omega)$, $K(x) \ge 1$.

The multivalued function $a_\varepsilon(x,\xi)$ with closed values is measurable with respect to $\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$. Moreover, by properties 2.2, 2.5-2.7, it is maximal monotone with respect to ξ a.e. $x\in\Omega$. From (4.1), using (2.1) we have

$$|\eta_\varepsilon| \le m_3(x) + \frac{K^{p-1}(x)(1+\varepsilon)}{1+\varepsilon K^{p-1}(x)}|\xi|^{p-1} \qquad \text{a.e. } x\in\Omega$$

while using $ii)$

$$|\xi|^p \le m_2(x) + \frac{K^p(x)(1+\varepsilon)}{1+\varepsilon K^p(x)}(\eta_\varepsilon,\xi) \qquad \text{a.e. } x\in\Omega$$

for every $\varepsilon\in]0,1]$, $\forall\xi\in\mathbb{R}^n$ and $\eta_\varepsilon\in a_\varepsilon(x,\xi)$.

We can easily verify that the following lemmas yeld

Lemma 4.1. *Let be $p\ge 2$. For every $\varepsilon\in]0,1]$ the map $a_\varepsilon(x,\xi)\in M_{(\frac{2}{\varepsilon})q+3,\Omega}(\mathbb{R}^n)$.*

Lemma 4.2. *Let be $p\ge 2$. For every $\varepsilon\in]0,1]$ the map $a_\varepsilon(x,\xi)\in M_{4K,\Omega}(\mathbb{R}^n)$.*

Now we are in position to prove the main result

Proof of Theorem A. For every $\varepsilon\in]0,1]$, let us consider the problem

$$\text{(4.2)} \qquad \begin{cases} -\operatorname{div} f \in -\operatorname{div} a_\varepsilon(x,\nabla u) & \text{on } \Omega \\ u\in W_o^{1,p}(\Omega) & (p\ge 2) \end{cases}$$

with data f such that $Kf\in(L^q(\Omega))^n$. From Lemma 4.1 and Theorem 2.10 it follows that problem (4.2) admits at least a solution $u_\varepsilon\in W_o^{1,p}(\Omega)$.

From Lemmas 4.2 and 3.4, if K verifies (1.2) with $\gamma>4\gamma_0$, the pair $[\nabla u_\varepsilon, g_\varepsilon]$ satisfy the estimate (3.2) uniformely with respect to $\varepsilon\in]0,1]$. So we can construct two subsequences again denoted by $\{u_\varepsilon\}$ and $\{g_\varepsilon\}$ such that as $\varepsilon\to 0^+$

$$\text{(4.3)} \qquad u_\varepsilon \to u \text{ weakly in } W_o^{1,p}(\Omega),,$$

$$\text{(4.4)} \qquad g_\varepsilon \to g \text{ weakly in } (L^q(\Omega))^n\,,$$

and

$$\int_\Omega g_\varepsilon \nabla\varphi dx = \int_\Omega f \nabla\varphi dx \qquad \forall\,\varphi \in C_o^\infty(\Omega)\,. \tag{4.5}$$

The equality in (2.3) is obtained by passing to limit as $\varepsilon \to 0^+$ in (4.5).

To conclude the proof we show that $g(x) \in a(x, \nabla u(x))$ for a.e. in Ω. Put

$$M = \{x \in \Omega : \exists \xi \in \mathbb{R}^n\, \exists\, \eta \in a(x,\xi) : (g(x) - \eta, \nabla u(x) - \xi) < 0\}\,,$$

according to the maximal monotonicity of the map a it is enough to show that $|M| = 0$. To prove that let us write $M = \{x \in \Omega : Gx \neq \emptyset\}$ where

$$Gx = \{[\xi,\eta] \in \mathbb{R}^n \times \mathbb{R}^n : \eta \in a(x,\xi) \text{ and } (g(x) - \eta, \nabla u(x) - \xi) < 0\}\,.$$

By Remark 2.9 the graph of G belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$ thus $M \in \mathcal{L}(\Omega)$ by the projection Theorem.

The Theorem 2.1 assures that there is a measurable selection $[\xi(x), \eta(x)]$ of G defined on M, then $\eta(x) \in a(x, \xi(x))$ and

$$(g(x) - \eta(x), \nabla u(x) - \xi(x)) < 0 \qquad \forall x \in M\,.$$

If $|M| > 0$, there are two measurable subset M', M'' of M, $M'' \subset \overline{M''} \subset M'$, with $0 < |M''| < |M'| < +\infty$ such that $[\xi(x), \eta(x)]$ is bounded on M'. Now, for every $\varepsilon \in]0,1]$ and $x \in M'$ we consider

$$\eta_\varepsilon(x) = \frac{\eta(x) + \varepsilon K^{p-1}(x)|\xi(x)|^{p-2}\xi(x)}{1 + \varepsilon K^{p-1}(x)}\,.$$

The sequence $\{\eta_\varepsilon(x)\}$ is bounded on M' and converges to $\eta(x)$ a.e. in M'. Since $|M'| < +\infty$, the sequence $\{\eta_\varepsilon(x)\}$ converges in measure to $\eta(x)$, hence

$$\eta_\varepsilon(x) \longrightarrow \eta(x) \qquad \text{in} \qquad (L^q(M'))^n\,. \tag{4.6}$$

Since $\eta_\varepsilon(x) \in a_\varepsilon(x, \xi(x))$ we have

$$(g_\varepsilon(x) - \eta_\varepsilon(x), \nabla u_\varepsilon(x) - \xi(x)) \geq 0 \qquad \forall\, x \in M'\,.$$

Moreover, if $\phi \in C_o^\infty(M'), \phi(x) > 0 \ \forall x \in M''$ and $\phi(x) \geq 0 \ \forall x \in M' - M''$, we have

$$
\begin{aligned}
0 &\leq \int_{M''} (g_\varepsilon(x) - \eta_\varepsilon(x), \nabla u_\varepsilon(x) - \xi(x))\phi(x)dx \leq \\
&\leq \int_{M'} (g_\varepsilon(x) - f(x), \nabla u_\varepsilon(x))\phi(x)dx + \\
&\quad + \int_{M'} (g_\varepsilon(x) - f(x), -\xi(x))\phi(x)dx + \\
&\quad + \int_{M'} (f(x) - \eta_\varepsilon(x), \nabla u_\varepsilon(x) - \xi(x))\phi(x)dx = I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3 \,.
\end{aligned} \tag{4.7}
$$

By Lemma 3.2 , $I_\varepsilon^1 \to \int_{M'} (g(x) - f(x), \nabla u(x))\phi(x)\, dx$. Moreover by (4.4), (4.6) and (4.3) we get

$$
\begin{aligned}
I_\varepsilon^2 &\to \int_{M'} (g(x) - f(x), -\xi(x))\phi(x)dx \,, \\
I_\varepsilon^3 &\to \int_{M'} (f(x) - \eta(x), \nabla u(x) - \xi(x))\phi(x)dx \,.
\end{aligned}
$$

So, by passing to limit as $\varepsilon \longrightarrow 0^+$ in (4.7) we have

$$
\int_{M'} (g(x) - \eta(x), \nabla u(x) - \xi(x)) \ \phi(x) \ dx \geq 0 \,. \tag{4.8}
$$

Since $|M'| > 0$, (4.8) contradicts the definition of M. Therefore we conclude that $|M| = 0$. This completes the proof of Theorem A with $\overline{\gamma} = 4\gamma_0$. □

Now, we consider the class, S_K, of all single-valued maps $a \in M_{K,\Omega}(\mathbb{R}^n)$, such that

iv) $|a(x,\xi_1) - a(x,\xi_2)| \leq \ K^{p-1} \ (x) \ |\xi_1 - \xi_2|(|\xi_1|^{p-2} + |\xi_2|^{p-2})$,

v) $K(x)^{-1}|\xi_1 - \xi_2|^p \leq (a(x,\xi_1) - a(x,\xi_2), \ \xi_1 - \xi_2)$
for a.e. $x \in \Omega$, $\forall \xi_1, \xi_2 \in \mathbb{R}^n$.
By Theorem A and condictions *iv*), *v*) we have

Corollary 4.3. *Let $p \geqslant 2$. There is a positive constant $\overline{\gamma}$ such that if $a \in S_K$ and K satisfies* (1.2) *with $\gamma > \overline{\gamma}$, the problem*

$$\begin{cases} -\operatorname{div}\ f = -\operatorname{div} a(x, \nabla u) & \textit{on } \Omega \qquad Kf \in (L^q(\Omega))^n \\ u \in W_o^{1,p}(\Omega) \end{cases}$$

an unique solution.

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Lavoro pervenuto alla redazione il 26 febbraio 2002
ed accettato per la pubblicazione il 16 gennaio 2003.
Bozze licenziate il 29 gennaio 2004

Indirizzo degli Autori:

Nunzia D'auria – Ornella Fiodo – Dipartimento di Matematica e Applicazioni "R. Caccioppoli" – Università degli Studi di Napoli "Federico II" – Complesso Universitario Monte S. Angelo – Via Cintia – 80126 Napoli (Italia)
E-mail: dauria@matna2.dma.unina.it fiodo@matna2.dma.unina.it