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A survey of pseudo Runge-Kutta methods

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Abstract: *This survey collects the theoretical results in the area of pseudo Runge-Kutta methods (PRK) for ordinary differential equations and it is a vehicle for a current bibliography from 1966 to 2002.*
PRK methods require fewer functional evaluations than Runge-Kutta methods of the same order. Byrne and Lambert (1966-1967) was the first who considered PRK methods in significative forms. Afterwards Costabile (1968-1975) introduced PRK methods of II and III species as an alternative to the first ones. The latter methods are also autostarting and reduce the cost by 50 percent compared with the similar Runge-Kutta ones. Nakashima (1982-1999) improved PRK methods of II species and introduced the implicit methods. Jackiewicz, Tracogna, Bartoszewki, Zennaro, Wanner, Hairer (1991-2000) introduced the modern theory of order, also with variable step-size and embedded and continuous formulas. Finally Bartoszewki and Jackiewicz (2000) introduced a PRK code for nonstiff differential systems. PRK methods for special second order differential equations are also studied.

1 – Introduction

For the numerical solution of the initial value problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \tag{1}$$

where $y : [x_0, b] \longrightarrow \mathbb{R}^M$, $f : [x_0, b] \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$, $y_0 \in \mathbb{R}^M$ and f satisfies all hypotesis for existence and unicity of solution, the explicit

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Runge-Kutta methods (ERKm in what follows) are well-known:

$$(2)\qquad \begin{cases} y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_{i,n} & n = 0, 1, \dots N-1 \\ y(x_0) = y_0 \end{cases}$$

where

$$\begin{aligned} & h_n = x_{n+1} - x_n = \sigma_n h \qquad \sigma_n < +\infty \qquad n = 0, 1, \dots N-1 \\ & y_n \approx y(x_n) \\ (3)\qquad & k_{i,n} = f\Big(x_n + c_i h_n, y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_{j,n}\Big) \quad i = 1, \dots, s \\ & c_1 = 0 \quad \sum_{j=1}^{0} \equiv 0\,. \end{aligned}$$

With the papers of Butcher ([6], [7]) it became customary to simbolize the method (2), called s-stage ERKm, by Table 1.

Table 1: s-stage ERKm

0			
c_2	a_{21}		
$\vdots$	$\vdots$	$\ddots$	
c_s	a_{s1}	$\cdots$	$a_{s,s-1}$
	b_1	$\cdots$	b_s

It is known that if $p(s)$ denotes the highest order that can be attained by an s-stage method, we have

$$(4)\qquad p(s) = s \quad \text{for } s = 1, 2, 3, 4$$

and, as Butcher proved ([8], [10]),

$$(5)\qquad \begin{aligned} & p(5) = 4, \qquad p(6) = 5, \quad p(7) = 6, \quad p(8) = 6, \\ & p(9) = 7, \qquad p(10) = 7, \quad p(11) = 8 \\ & p(s) \le s - 2, \quad 12 \le s\,. \end{aligned}$$

As we can see, the s-stage ERKm (2) requires almost s functional evaluations for step. We shall look for other ERK type methods which have the same order of (2), but which requires less functional evaluations than (2).

Such methods have been discussed in [61] and in many other papers. For example, Byrne and Lambert ([13]) have defined two-step Runge-Kutta methods and after them several authors have studied similar methods, the so called Pseudo Runge-Kutta methods (PRKm in what follows).

Now, stimulated also by commemoration of Wolf Gross ([36]), who introduced me in this topics, I am proposing a survey of the mean results on PRKm.

2 – Pseudo Runge-Kutta methods of I species

Byrne and Lambert ([13]) were the first people who considered two-step Runge-Kutta methods:

$$(6) \qquad \begin{cases} y_{n+1} = y_n + h\Big(\sum_{i=0}^{s} \alpha_i k_{i,n} + \sum_{i=0}^{s} \beta_i k_{i,n-1}\Big) \quad n = 1, 2, \dots N-1 \\ y(x_0) = y_0 \end{cases}$$

where

$$k_{i,n} = f\Big(x_n + c_i h, y_n + h\sum_{j=0}^{i-1} b_{ij} k_{j,n}\Big) \quad i = 0, \dots, s$$

$$c_0 = 0, \quad \sum_{j=0}^{-1} \equiv 0, \quad c_i = \sum_{j=0}^{i-1} b_{ij}, \quad i = 1, \dots N-1,$$

$$x_{n+1} - x_n = h = \frac{b - x_0}{N}\,.$$

In [13] there is the family of formulas of order three and four respectively for $s = 1$ and $s = 2$.

The case of (6) for $s = 3$ has been studied by W. G. Gruttke ([37]), obtaining a family of formulas of order five. It is clear that the method (6) requires for $s = 1, 2, 3$ one functional evaluation fewer than (2) for the same order, but there is a stage of previous step.

It is possible to reduce ulteriorly the number of functional evaluations for step by previous steps; infact in [19] there is a family of formulas of

the fourth order with only two functional evaluations for step, but with stages of the two previous steps. It is:

$$(7)\qquad \begin{cases} y_{n+1} = y_n + h\Big(\sum_{i=0}^{1}\alpha_i k_{i,n} + \sum_{i=0}^{1}\beta_i k_{i,n-1} + \sum_{i=0}^{1}\gamma_i k_{i,n-2}\Big) \\ y(x_0) = y_0 \qquad n = 2, 3, \dots N-1 \end{cases}$$

where coefficients and stages have the same meaning as in (6).

We note that methods (6) and (7) have the step-size constant and are not autostarting: it is necessary to know y_1 for (6) and y_1, y_2 for (7) to start the calculation.

The parameters α_i, β_i, c_i, b_{ij} in (6) and (7) have been determined in [13], [37], [19], by using expansion in Taylor series about the point (x_n, y_n), and the differential operator

$$(8)\qquad D^n\Psi = \sum_{k=0}^{n}\binom{n}{k} f^{n-k}\frac{\partial^n\Psi}{\partial x^k \partial y^{n-k}}\,.$$

The stability intervals of these methods have been determined in [49], [21] and they are comparable to ERKm.

However, computational experiments show that the local accuracy is frequently inferior to that of the ERKm (2).

3 – Pseudo Runge-Kutta methods of II species

To improve the local accuracy of methods (6) and (7) COSTABILE in [20] proposed the following pseudo Runge-Kutta methods called of second species:

$$(9)\qquad \begin{cases} y_{n+1} = y_n + h\sum_{i=0}^{s}\alpha_i k_{i,n} + \beta k_{0,n-1} \\ y(x_0) = y_0 \qquad n = 1, 2, \dots N-1 \end{cases}$$

where

$$k_{i,n} = f\Big(x_n + c_i h, y_n + h\Big(\sum_{j=0}^{i-1} b_{ij}k_{j,n} + bk_{0,n-1}\Big)\Big) \quad i = 0, \dots, s$$

$$c_0 = 0, \quad \sum_{j=0}^{-1} \equiv 0, \quad c_i = \sum_{j=0}^{i-1} b_{ij} + b, \quad h = \frac{b - x_0}{N}\,.$$

In [20] there are formulas of order three and four respectively for $s = 1, 2$; whereas in ([74]) there are formulas of five order for $s = 3$.

Method (9) has the same cost as (6) and therefore less than (2) for the same order.

Using the classic hypothesis

$$|f(x,y)| < M, \quad \Big|\frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}\Big| \leq \frac{L^{i+j}}{M^{j-1}} \quad 0 < L, M < +\infty \tag{10}$$

for truncation local error we have the bounds

$$|R_n(h)| \leq T(c_i) h^{r+1} L^r M \quad r = 3, 4, 5 \tag{11}$$

and then the values of parameters c_i have been determined which minimizes $T(c_i)$. The relative formulas have been called optimal in [23]. There are also formulas which for $f(x,y) \equiv f(x)$ give the quadrature formulas of Radau and Lobatto.

The computational experiments in [20], [23] show that methods (9) are more accurate than methods (6) and (7).

4 – Pseudo Runge-Kutta methods autostarting or of III species

Methods (9) and (6) are not autostarting: it is necessary to know $y_1 \approx y(x_0 + h)$. Costabile in [22] and [24] eliminated this defect. In particular in [24] there is the method

$$\begin{cases} y_{n+1} = y_n + \sum_{i=0}^{s} \alpha_i k_{i,n} & n = 0, 1, \ldots N-1 \\ y(x_0) = y_0 \end{cases} \tag{12}$$

where

$$k_{i,n} = hf\Big(x_n + c_i h, y_n + \sum_{j=0}^{i-1} b_{ij} k_{j,n} + \sum_{j=0}^{s} \overline{b}_{ij} k_{j,n-1}\Big)$$

$$n = 0, \ldots, N-1$$

$$c_0 \neq 0, \quad c_i = \sum_{j=0}^{i-1} b_{ij} + \sum_{j=0}^{p} \overline{b}_{ij}, \quad i = 0, \ldots, s, \quad \sum_{j=0}^{-1} b_{0j} = b_{00}\,;$$

$k_{i,-1}$ are initial conditions to be determined for particular formulae.

In [24] the cases $s = 0, s = 1$ have been studied and formulae of order respectively two and four have been derived.

Hence, these methods, compared to ERKm of the same order, reduce the number of stages by 50 percent.

EXAMPLE. For $s = 0$ in (12) we have the simplex formula ([24]):

$$\begin{cases} y_{n+1} = y_n + k_{0,n} & n = 0, 1, \dots N-1 \\ y(x_0) = y_0 \end{cases} \tag{13}$$

$$k_{0,n} = hf\Big(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_{0,n-1}\Big)$$
$$k_{0,-1} = hf(x_0, y_0)\,.$$

This method has the same cost and the same interval of stability of Euler's classic method, but (13) has order two.

The general theory of order method (12) has been showed in [31], infact it is observed that (12) belongs to the class of "general linear methods" and the order and convergence theory of SKELL ([64]) is easily applicable.

For the order $p\,(s)$ we have the following bounds:

$$s < p(s) \leq 2s$$

and $p = 2s$ for $s = 1, 2, 3$; the last case has been examined in ([27]).

We can observe that for $b_{ij} = 0$, $i = 0, s$, $j = 0, \dots, i-1$ (12) it is a parallel algorithm.

5 – Generalization of Pseudo Runge Kutta methods of II species

NAKASHIMA in [53] generalized the PRKmII of Costabile and obtained a further reduction of the cost. He considered the method

$$y_{n+1} = b_{-2}y_{n-1} + b_{-1}y_n + \sum_{i=0}^{s} \alpha_i k_{i,n} + \beta k_{0,n-1} \tag{14}$$

$n = 1, \dots N-1$, where

$$k_{i,n} = hf\Big(x_n + c_i h, y_n + b_i(y_n - y_{n-1}) + \sum_{j=0}^{i-1} b_{ij} k_{j,n} + b k_{0,n-1}\Big)$$

$$c_0 = 0, \quad c_i = b_i + b + \sum_{j=0}^{i-1} b_{ij} \quad i = 0, \dots, s\,,$$

and obtained formulas of order 4, 5, 6.

The idea of Nakashima was the complete utilization of previous point y_{n-1}, and not only the previous stage $k_{0,n-1}$.

Comparing the Nakashima method and the previous PRKm for the same order, these methods require less functional evaluations than the other methods. The derivation of the formulae is classic.

The methods (14), like (6), (9), (12), are fix-step and therefore convenient procedures are required to estimate the local truncation error, which is important from the point of view of step-size control policy.

Nakashima in [58] proposed a new algorithm similar to (14) with change of steplength during the calculation:

$$y_{n+\sigma} = b_{-2} y_{n-1} + b_{-1} y_n + \sum_{i=0}^{s} \alpha_i k_{i,n} + \beta k_{0,n-1} \tag{15}$$

where $k_{i,n}$ has the same form as in (14).

The proposed method (15) requires that the constants b_{-2}, b_{-1}, b_i, b, b_{ij}, c_i are chosen so that the expansion of the right-hand side of (15) is equivalent to Taylor expansion of $y(x_n + \sigma h)$ up to $p-$th powers of h, and the coefficients b_i, b, c_i, are independent of the factor σ. Thus, the new algorithm is designed to compute the value $y_{n+\sigma}$ at the desired point $x_{n+\sigma} = x_n + \sigma h$ without additional function evaluations. In [58] formulas for $s = 1, 2, 3$ are studied, obtaining four order for $s = 1, 2$ and five order for $s = 3$. Now, the automatic step size control is possible.

A useful idea is the control of estimation local truncation error by embedded methods. Nakashima in [59] uses a $(4-5)$ order formulas; the stability is also considered in [59].

6 – Implicit Pseudo Runge-Kutta methods

BUTCHER ([7]) was the first who proposed implicit Runge-Kutta processes:

$$\begin{cases} y_{n+1} = y_n + \sum_{i=1}^{s} \alpha_i k_i \qquad n = 0, 1, \dots N-1 \\ y(x_0) = y_0 \end{cases} \tag{16}$$

where

$$k_i = hf\Big(x_n + c_i h, y_n + \sum_{j=1}^{s} \lambda_{ij} k_j\Big) \quad i = 0, \dots, N-1$$

$$c_i = \sum_{j=1}^{s} \lambda_{ij} \, .$$

Their advantage over ERKm is the highest order which can be attained by s-stage methods and the property of A-stability ([39]).

In [7] it is proved that, if $p(s)$ is the highest order attained with s stages, then

$$p(s) = 2s \qquad s = 1, 2, \dots$$

In order to apply IRKm, it is necessary, at each step, to solve non linear equations in some way.

Nakashima was the first who considered implicit Pseudo Runge-Kutta methods ([54]). Their advantage over IRKm lies in the fact that they are less expensive in terms of functional evaluations for a given order. In [54] there are the following s-stage IPRKm:

$$\begin{cases} y_{n+1} = y_{n-1} + v(y_{n-1} - y_n) + \sum_{i=0}^{s} \alpha_i k_{i,n} + \beta k_{0,n-1} \\ y(x_0) = y_0 \end{cases} \tag{17}$$

where

$$k_{i,n} = hf\Big(x_n + c_i h, (1 + b_i) y_n - b_i y_{n-1} + h \sum_{j=0}^{s} b_{ij} k_{j,n} + \overline{b}_i k_{0,n-1}\Big)$$

$$i = 0, \dots, p$$

$$c_0 = 0, \quad c_i = b_i + \sum_{j=0}^{s} b_{ij} + \overline{b}_i \, .$$

He examined the case $p = 1$ and $p = 2$, obtaining the family of formulas of order four and six. There is also the analysis of stability and formulas with very large interval of stability, but not A-stable.

Nakashima in [55] considered a different class of IPRKm: he replaced x_{n-1}, x_n, y_{n-1}, y_n on the right-hand side of (17) with x_n, x_{n+1}, y_n, y_{n+1} respectively and obtained the following IPRKm:

$$y_{n+1} = y_n + v(y_n - y_{n+1}) + \sum_{i=0}^{s} \alpha_i k_{i,n} + \beta k_{0,n-1} \tag{18}$$

$$k_{i,n} = hf\Big(x_{n+1} + c_i h, y_{n+1} + b_i(y_{n+1} - y_n) + \\ + \sum_{j=0}^{i-1} b_{ij} k_{j,n} + \overline{b}_i k_{0,n-1}\Big)$$

$$c_0 = 0, \quad c_i = b_i + \overline{b}_i + \sum_{j=1}^{i-1} b_{ij}\,.$$

We note that (18) is implicit in y_{n+1} but not in $k_{i,n}$. The previous implicit methods (16) and (17), however, suffer a serious practical disadvantage: if an s-stage method is applied to an m-dimensional system of ordinary equations, then a system of ms non linear simultaneous algebraic equations will have to be solved exactly for the functions k_i, $i = 1, \ldots, r$ at each step by some iterative processes. But the algorithm (18) reduces the effort to solve non linear equations, since there is only one m-dimensional system of equations.

Moreover, the fully IRKm requires a suitable starting approximation of k_i, $i = 1, \ldots, r$ to converge, especially if the derivative $f(x, y)$ varies rapidly at $x = x_n$ and $y = y_n$; for algorithm (18) is easy to obtain a suitable initial approximation y_{n+1}.

Before Nakashima, Cash in [14] and Cash-Moore in [15] have also proposed some methods which are closely related to (18) and are A-stable ([14]) and L-stable ([15]).

In [55] Nakashima examined the cases $s = 2, 3$ of (18) and determined a family of formulas of order 4 and 5; the derivation of calculation is classic. There is also the analysis of stability and convergence of iterative processes; particular formulas are A-stable.

7 – Generalization of Pseudo Runge Kutta methods of I species

Jackiewicz-Renaut-Feldstein in [42] proposed a generalization of PRKm of first species. Their formulation is the following:

$$y_{n+1} = (1-\theta)y_n + \theta y_{n-1} + \sum_{i=0}^{s}(\alpha_i k_{i,n} + \beta_i k_{i,n-1}) \tag{19}$$

$$k_{i,n} = h\Big(x_n + c_i h, y_n + h\sum_{j=1}^{s}\alpha_{ij}k_{j,n}\Big)\,.$$

These methods are implicit and belong to the class of general linear methods ([39]).

Using the modern theory of Hairer and Wanner ([39]) the authors derived order conditions.

They considered also the stability theory and gave particular A-stable formulas. It is observed in [60] that these formulas can be useful for the solution of systems of ODEs arising from the semidiscretization of partial differential equations of parabolic type.

Jackiewicz and Zennaro in [48] considered explicit methods of class (19) but with variable step-size:

$$y_{n+1} = (1-\xi\theta)y_n + \xi\theta y_{n-1} + \sum_{i=0}^{s}(\alpha_i k_{i,n} + \beta_i k_{i,n-1}) \tag{20}$$

where

$$k_{i,n} = h_n f\Big(x_n + c_i h, y_n + h_n\sum_{j=0}^{i-1}a_{ij}k_{j,n}\Big)$$

$$\xi = \frac{h_n}{h_{n-1}} \qquad x_{n+1} - x_n = h_n \quad n = 1,\dots,N-1\,.$$

The order theory is general and embedded pairs of continuous RK methods and PRK methods are determined.

Bellen, Jackiewicz and Zennaro in [5] showed that the local discretization error of s-stage singly implicit Runge-Kutta methods of order p can be estimated by embedding these methods into s-stage two points Runge-Kutta methods of class (19) of order $p+1$, where $p = s$ or

$p = s + 1$. These error estimates do not require any extra evaluation of right hand side of differential equations.

8 – Generalization of Pseudo Runge Kutta methods of III species

Jackiewicz and Tracogna in [46] give a generalization of PRKs of III species, introduced in [13]. Their formulation is

$$y_{n+1} = \theta y_{n-1} + (1-\theta) y_n + \sum_{i=0}^{s} (\alpha_i k_{i,n} + \beta_i k_{i,n-1}) \tag{21}$$

$$k_{i,n} = h f\Big(x_n + c_i h, v_i y_{n-1} + (1-v_i) y_n + \sum_{j=0}^{s} (a_{js} k_{j,n} + b_{js} k_{j,n-1})\Big) .$$

These methods reduce to (19) if $v_j = 0$ and $b_{j,s} = 0$, and reduce to (12) for $\theta = v_j = \beta_i = 0$, $a_{js} = 0$, $s \geq j + 1$.

The order theory is general and follows Albert's approach ([1], [2]). There are examples of methods up to order 5 too. These methods can be divided into four classes that are appropriate for the numerical solution of stiff or non stiff differential equations in sequential or parallel computing environments.

Bartoszewski and Jackiewicz in [3] consider also the class (21) and describe the construction of methods of order p and "stage order" $q = p$ with stability polynomial given in advance. This polynomial is chosen to have a large interval of absolute stability for explicit methods and to be A-stable and L-stable for implicit methods.

Bartoszewski and Jackiewicz in [4] propose a new code for non stiff equations based on methods of order five and stage order five. The numerical experiments presented show that the new code is competitive with the Matlab ode45 program for all tolerances.

9 – Economical Runge-Kutta methods

In [28] economical Runge-Kutta methods (EcRKm) are defined like ERKm with the first stage $k_1 = f(x_n, y_n)$ replaced by the last stage of previous step. Referring to 2, the class A_s^p is defined of ERKm of order p and s-stage with the following parameters.

TABLE 2:

$$\begin{array}{c|cccc} 0 & & & & \\ \vdots & a_{21} & & & \\ \vdots & \vdots & & & \\ 1 & a_{s1} & \cdots & a_{s,s-1} & \\ \hline & 0 & b_2 & \cdots & b_s \end{array}$$

Then it was proved that the class A_s^p for $p = s = 3, 4$ and $p = 5,\ s = 6$ is not empty. Associate to class A^p the authors defined the EcRKm such that

$$y_{n+1} = y_n + h_n \sum_{i=2}^{s} b_i k_{i,n} \tag{22}$$

$$k_{i,n} = f\Big(x_n + c_i h_n, y_n + h_n\Big(\sum_{j=2}^{i-1} a_{ij} k_{j,n} + a_{i1} k_{s,n-1}\Big)\Big)$$

$$k_{i,-1} = f(x_0, y_0)\,.$$

This method has all pecularities of ERKm but it saves one function call for the same order. Formulas of order 3, 4, 5 are been determined. In particular they are the formulae of Simpson, Radau and Lobatto. The practical local error estimation has been done by embedded pair formulas. The analysis of stability is also given and interval of stability are determined. Finally, numerical experiments show that the code for the non stiff equations is very comparable with classical Runge-Kutta Fleberg method.

10 – Pseudo Runge-Kutta methods for second order differential equations

For discrete integration of initial value problem

$$\begin{cases} y''(x) = f(x, y(x)) \qquad x \in [-1, 1] \\ y(x_0) = y_0,\ y'(x_0) = y_0'\,. \end{cases} \tag{23}$$

PRKm were proposed too.

COSTABILE in [25] gives the following method

$$(24)\qquad \begin{cases} z_{n+1} = bz_{n-1} + ay_n + h^2[cf(x_n, y_n) + df(s_{n-1}, z_{n-1}) + \\ \quad + ef(x_{n-1}, y_{n-1})] \\ y_{n+1} = 2y_n - y_{n-1} + h^2[\alpha_0 f(x_n, y_n) + \alpha_1 f(s_n, z_n) + \\ \quad + \beta_0 f(x_{n-1}, y_{n-1}) + \beta_1 f(s_{n-1}, z_{n-1})] \end{cases}$$

where

$$y_n \approx y(x_n), \quad z_n \approx y(s_n), \quad s_n = x_n + \mu h, \quad x_n = x_0 + nh$$

$a, b, c, d, e, \alpha_0, \alpha_1, \beta_0, \beta_1$ are constants to be determined so that

$$(25)\qquad \begin{cases} y(s_n) - z_n = T(\mu)h^5 y^V(x_n) + O(h^6) \\ y(x_{n+1}) - y_{n+1} = T'(\mu)h^6 y^{VI}(x_n) + O(h^7)\,. \end{cases}$$

Method (24) in comparison with Nystrom methods of the same order reduces the cost by about 50% and the calculation of derivatives is not required.

In comparison with the well-known Numerov formula ([39]), (24) it is explicit and has a small coefficient of truncation error. In [34] there are the proofs of convergence and the analysis of stability for (24).

For a problem like (23) having periodic solutions, with a-priori unknown period, Dalquist in [35] and LAMBERT-WATSON in [51] defined P-stable methods. A two-step four order P-stable method for second order differential equations (23) is determined in [30]. This method is the following:

$$(26)\qquad \begin{cases} y_{n+1} - 2y_n + y_{n-1} = h^2\Big[\Big(\dfrac{1}{12} - \dfrac{1}{4}\beta_1\Big)(f_{n+1} + f_{n-1}) + \\ \quad + \Big(\dfrac{5}{6} - \dfrac{3}{2}\beta_1\Big)f_n + \beta_1(\tilde{f}_n + \tilde{f}_{n-1})\Big] \\ \tilde{y}_n - \dfrac{1}{2}(y_n + y_{n+1}) = -\dfrac{h^2}{16}(f_n + f_{n+1}) \end{cases}$$

where

$$f_n = f(x_n, y_n), \quad \tilde{f}_n = f(x_n + \mu h, \tilde{y}_n), \quad \tilde{y}_n \approx y(x_n + \mu h)\,.$$

For the local error we have

$$(27)\qquad \begin{cases} y(x_{n+1}) - y_{n+1} = \\ \quad = \dfrac{-4+15\beta_1}{960} h^6 y^{IV}(x_n) + \dfrac{5}{192}\beta_1 h^6 f_y y^{IV}(x_n) + O(h^7) \\ y(x_n+\mu h) - y_n = \dfrac{5}{384} h^4 y^{IV}(x_n) + O(h^5)\,. \end{cases}$$

This method is P-stable iff $\beta_1 \geq \frac{2}{3}$, and observing that the leading term in the local truncation error (27) is the smallest for $\beta_1 = \frac{2}{3}$, we have the final P-stable method:

$$\begin{cases} y_{n+1} - 2y_n + y_{n-1} = -\dfrac{1}{12} h^2 [f_{n+1} + f_{n-1} - 2f_n - 8(\tilde{f}_n + \tilde{f}_{n-1})] \\ y_{n+1} - 2\tilde{y}_n + y_{n-1} = \dfrac{1}{8} h^2 (f_n + f_{n-1})\,. \end{cases}$$

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