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An efficient version of the direct method for planar grid generation

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ABSTRACT: *We consider the problem of the grid generation for a generic compact connected domain* Ω *of the two-dimensional real Euclidean space. This problem can be seen as a finite dimensional version of the extension of a given parameterization of the boundary of* Ω *to a parameterization of the whole domain* Ω*. We describe the direct method, where the grid generation problem is reformulated as an optimization problem, and two different modifications of this method. We present a large number of numerical results on standard test problems. From these results we can see that the new version has higher accuracy and lower computational cost than the usual version of the direct method for the grid generation problem.*

1 – Introduction

Grid generation is a basic ingredient for a large number of very important problems, such as for example the numerical solution of partial differential equation with finite difference methods or finite element methods (see [9], [15], [16], [20]). In the scientific literature many grid generation methods has been proposed (see [11], [12], [14], [17]); in particular KNUPP and STEINBERG in [11] introduced methods that use length, area and orthogonality continuous functionals, CASTILLO in [4] introduced methods

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that use length, area and orthogonality discrete functionals. This last approach is developed in [1], [2], [5], [6].

We begin fixing some notations. Let $\mathbf{N}$, $\mathbf{R}$ be the sets of natural and real numbers, respectively. Let $N \in \mathbf{N}$, we denote with $\mathbf{R}^N$ the N-dimensional real Euclidean space. Let $\underline{x} = (x_1,\ x_2, \dots,\ x_N)^t \in \mathbf{R}^N$ be a generic vector, where the superscript t denotes the transposition operation. Let $1 \leq p < \infty$, we denote with $\|\underline{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}$ the usual p-norm in $\mathbf{R}^N$. Let $A \subset \mathbf{R}^N$ be a subset of $\mathbf{R}^N$, we denote with ∂A the boundary of A.

Let $R = \{(x, y)\ :\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ be the unit square, $N,\ M \in \mathbf{N}$ and

$$I = \{(i, j),\ i = 0, 1, \dots, N,\ j = 0, 1, \dots, M\} \tag{1}$$

be a set of multi-indices for $(N + 1)(M + 1)$ different points of R, that is for each $(i, j) \in I$ we define $\underline{p}_{i,j}$ the point of R having coordinates $(\frac{i}{N}, \frac{j}{M})$. Moreover we define the following sets of multi-indices:

$$I_1 = \{(i, j),\ i = 0, 1, \dots, N - 1,\ j = 0, 1, \dots, M\}\,, \tag{2}$$

$$I_2 = \{(i, j),\ i = 0, 1, \dots, N,\ j = 0, 1, \dots, M - 1\}\,, \tag{3}$$

$$\partial I = \{(i, j) \in I\ :\ \underline{p}_{i,j} \in \partial R\}\,. \tag{4}$$

We note that points $\underline{p}_{i,j}$, $(i, j) \in I$ can be seen as the vertices of the $N \times M$ uniform rectangular grid $\mathbf{P}$ of R, where the edges of $\mathbf{P}$ are given by the horizontal segments joining $\underline{p}_{i,j}$ and $\underline{p}_{i+1,j}$, $(i, j) \in I_1$, and by the vertical segments joining $\underline{p}_{i,j}$ and $\underline{p}_{i,j+1}$, $(i, j) \in I_2$. The faces of $\mathbf{P}$ are the rectangles of the partition of R defined by the edges of $\mathbf{P}$, see Figure 1.

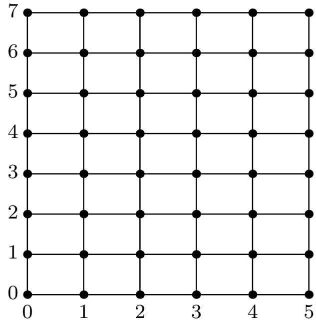


Fig. 1: An example of the N × M uniform rectangular grid $\mathbf{P}$ of R, where N = 5, M = 7.

The *grid generation problem* considered in this paper is roughly speaking a finite dimensional version of the following problem: find a bijective map $\underline{u}$ from R to a given domain Ω, such that the image of $\mathbf{P}$ through $\underline{u}$ is a grid $\mathbf{Q}$ of Ω having nice properties. More precisely, let $\Omega \subset \mathbf{R}^2$ be a two-dimensional compact and connected domain, we consider the following problem: from the knowledge of a parameterization

$$\underline{u}_\partial = (x_\partial, y_\partial)^t : \partial R \to \partial\Omega\,, \tag{5}$$

of the boundary $\partial\Omega$ of Ω, we want to compute a parameterization of Ω

$$\underline{u} = (x, y)^t : R \to \Omega\,, \tag{6}$$

such that

$$\underline{u}(\underline{p}) = \underline{u}_\partial(\underline{p}),\ \ \underline{p} \in \partial R\,. \tag{7}$$

Let $\mathbf{Q}$ be the grid of Ω obtained as image of $\mathbf{P}$ through the parameterization $\underline{u}$. In particular we have that vertices, edges and faces of $\mathbf{Q}$ can be defined as the images of vertices, edges and faces of $\mathbf{P}$. We note that there exists an infinite number of ways to solve the above mentioned problem, so that we can try to use these degrees of freedom to require suitable properties for the grid $\mathbf{Q}$. The quality of the grid $\mathbf{Q}$ can be defined according to some simple geometric properties of its edges and its faces. These properties are straightforward generalizations of the ones that hold for grid $\mathbf{P}$, that is:

- convexity of quadrilaterals,
- uniformity of the edges length,
- uniformity of the faces area,
- uniformity of the angles between edges having a common vertex.

In general the edges of $\mathbf{Q}$ are curvilinear segments, so that the faces of $\mathbf{Q}$ are not rectangles and they are neither polygons. We consider the approximation of $\mathbf{Q}$ obtained substituting these curvilinear edges with the line segments having the same ends. In the sequel we denote with $\mathbf{Q}$ this approximate grid, where we can easily express the length of edges of $\mathbf{Q}$, the area of faces of $\mathbf{Q}$ and the angle between edges of $\mathbf{Q}$ in terms of the coordinates of the vertices of $\mathbf{Q}$. Let $\underline{u}_{i,j} = \underline{u}(\underline{p}_{i,j}) = (x_{i,j}, y_{i,j})^t$,

$(i, j) \in I$ be the vertices of $\mathbf{Q}$. We denote with $\underline{x}$, $\underline{y} \in \mathbf{R}^{(N+1)(M+1)}$ the vectors having components $x_{i,j}$, $(i, j) \in I$, and $y_{i,j}$, $(i, j) \in I$, respectively. Let $F_L(\underline{x}, \underline{y})$, $F_A(\underline{x}, \underline{y})$, $F_O(\underline{x}, \underline{y})$, $\underline{x}, \underline{y} \in \mathbf{R}^{(N+1)(M+1)}$ be the total length of edges of $\mathbf{Q}$, the total area of faces of $\mathbf{Q}$, and the sum of the angles between edges of $\mathbf{Q}$, respectively.

The *direct method* formulates the grid generation problem as an optimization problem where the independent variables are $\underline{x}, \underline{y} \in \mathbf{R}^{(N+1)(M+1)}$, and the objective function is given by a suitable linear combination of the functions F_L, F_A, F_O. We note that this is an elegant and easy formulation of the grid generation problem. However the main drawback of this method is that the quality of the computed grid $\mathbf{Q}$ usually depends very much on the value of the coefficients used, in the objective function, for the linear combination of F_L, F_A and F_O. Note that the right value of such coefficients usually must be chosen according to the particular shape of Ω and the number N, M of subdivisions considered. On the contrary when these coefficients are not right we obtain low quality grids $\mathbf{Q}$ or even strange structures $\mathbf{Q}$ that are not really grids of Ω, that is grids having vertices outside Ω or folded grids.

The authors (see [8]) have proposed a new optimization problem for the formulation of the grid generation problem. This optimization problem differs from the usual formulation for the following two reasons: the use of non-Euclidean norms in the objective functions and the use of different variables. In particular the use of p-norm, with $p \geq 2$, is an attempt to impose a strong requirement for the uniformity properties of the grid $\mathbf{Q}$, so that we expect an improvement of the quality of the computed grid as consequence of such a modification. Moreover the new variables used in the proposed formulation can be seen as suitable differences of vertices coordinates of $\mathbf{Q}$. These variables give some simplifications in the expression of F_L, F_A and F_O, so that we expect an improvement in the computational cost of the method as consequence of such a modification.

We present some numerical results obtained for various planar regions. These planar regions are proposed in a classical set of test problems, which is called *Rogue's Gallery of Grids*, see [11]. From these numerical results we can see that the new formulation improves the usual formulation of the direct method for the grid generation problem; in particular this improvement is relative to the computational cost of the method and to the grid quality.

In Section 2 we describe the direct method for the grid generation problem and the corresponding generalizations. In Section 3 we present some numerical results obtained with the usual formulation and the new formulation of the direct method for the grid generation problem. In Section 4 we conclude giving the possible future developments of this paper.

2 – The Direct Method For The Grid Generation Problem

We describe the various addenda of the objective function in the direct method for the grid generation problem. We begin by defining new variables that are useful to express these addenda. Let

$$\xi_{1,i,j} = x_{i+1,j} - x_{i,j},\ \eta_{1,i,j} = y_{i+1,j} - y_{i,j} \quad (i,j) \in I_1\,, \tag{8}$$

$$\xi_{2,i,j} = x_{i,j+1} - x_{i,j},\ \eta_{2,i,j} = y_{i,j+1} - y_{i,j} \quad (i,j) \in I_2\,. \tag{9}$$

Let $\underline{\xi}_1$, $\underline{\eta}_1 \in \mathbf{R}^{N(M+1)}$, $\underline{\xi}_2$, $\underline{\eta}_2 \in \mathbf{R}^{(N+1)M}$, be the vectors having components $\xi_{\nu,i,j}$, $(i,j) \in I_\nu$, and $\eta_{\nu,i,j}$, $(i,j) \in I_\nu$, $\nu = 1,2$, respectively.

We note that $\xi_{\nu,i,j}$, $(i,j) \in I_\nu$, $\nu = 1,2$ are the discrete derivatives of the discrete function $x_{i,j}$, $(i,j) \in I$, and $\eta_{\nu,i,j}$, $(i,j) \in I_\nu$, $\nu = 1,2$ are the discrete derivatives of the discrete function $y_{i,j}$, $(i,j) \in I$. The discrete vector field $(\xi_{1,i,j}, \xi_{2,i,j})^t$, $(i,j) \in I_1 \cap I_2$ can be seen as the discrete gradient of $x_{i,j}$, $(i,j) \in I$, and the discrete vector field $(\eta_{1,i,j}, \eta_{2,i,j})^t$, $(i,j) \in I_1 \cap I_2$ can be seen as the discrete gradient of $y_{i,j}$, $(i,j) \in I$. As a consequence they must satisfy the irrotational property:

$$\xi_{2,i+1,j} - \xi_{2,i,j} = \xi_{1,i,j+1} - \xi_{1,i,j},\ (i,j) \in I_1 \cap I_2\,, \tag{10}$$

$$\eta_{2,i+1,j} - \eta_{2,i,j} = \eta_{1,i,j+1} - \eta_{1,i,j},\ (i,j) \in I_1 \cap I_2\,. \tag{11}$$

On the other hand if $\xi_{\nu,i,j}$ and $\eta_{\nu,i,j}$, $(i,j) \in I_\nu$, $\nu = 1,2$ are discrete functions satisfying (10), (11), then they are the discrete derivatives of some functions $x_{i,j}$ and $y_{i,j}$, $(i,j) \in I$. These functions are univocally determined by their value on a point of I and by the corresponding discrete gradient. Moreover, as in the classical theory of first order differential forms, we can explicitly compute the discrete function. Let us suppose, for example, to know $x_{0,0}$, and $\underline{\xi}_\nu$, $\nu = 1, 2$, we have:

$$x_{i,j} = x_{0,0} + \sum_{s=0}^{i-1} \xi_{1,s,0} + \sum_{t=0}^{j-1} \xi_{2,i,t}, \ (i,j) \in I\,, \tag{12}$$

where the first sum is equal to zero when $i = 0$ and the second sum is equal to zero when $j = 0$. An analogous formula is valid for $y_{i,j}$, $(i, j) \in I$.

The direct method formulates the grid generation problem as an optimization problem. In particular we describe each addendum of the objective function of this optimization problem, that is the addendum corresponding to the length of edges of $\mathbf{Q}$, the addendum corresponding to the area of faces of $\mathbf{Q}$ and the addendum corresponding to the angle between the edges of $\mathbf{Q}$. Each addendum is given in terms of variables $x_{i,j}$, $y_{i,j}$, $(i, j) \in I$ and in terms of $\xi_{\nu,i,j}$, $\eta_{\nu,i,j}$, $(i, j) \in I_\nu$, $\nu = 1, 2$.

Let $p_L, p_A, p_O \in \mathbf{N}$, with $p_L, p_A, p_O \geq 2$, be the parameters that define the norms used.

Let us consider the p_L-norm for the length of the edges of $\mathbf{Q}$. Thus the length of the edge of $\mathbf{Q}$ joining the vertices $\underline{u}_{i,j}$ and $\underline{u}_{i+1,j}$ is given by:

$$\begin{aligned} &L^{p_L}_{1,i,j}(x_{i,j}, x_{i+1,j}, y_{i,j}, y_{i+1,j}) = \\ &= |x_{i+1,j} - x_{i,j}|^{p_L} + |y_{i+1,j} - y_{i,j}|^{p_L}, (i,j) \in I_1\,, \end{aligned} \tag{13}$$

when we consider variables $\underline{x}$, $\underline{y}$, and is given by:

$$\mathcal{L}^{p_L}_{1,i,j}(\xi_{1,i,j}, \eta_{1,i,j}) = |\xi_{1,i,j}|^{p_L} + |\eta_{1,i,j}|^{p_L}, \ (i,j) \in I_1\,, \tag{14}$$

when we consider variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu = 1, 2$. The length of the edge joining the vertices $\underline{u}_{i,j}$ and $\underline{u}_{i,j+1}$ is given by:

$$\begin{aligned} &L^{p_L}_{2,i,j}(x_{i,j}, x_{i,j+1}, y_{i,j}, y_{i,j+1}) = \\ &= |x_{i,j+1} - x_{i,j}|^{p_L} + |y_{i,j+1} - y_{i,j}|^{p_L}, \ (i,j) \in I_2\,, \end{aligned} \tag{15}$$

when we consider variables $\underline{x}$, $\underline{y}$, and is given by:

$$\mathcal{L}^{p_L}_{2,i,j}(\xi_{2,i,j}, \eta_{2,i,j}) = |\xi_{2,i,j}|^{p_L} + |\eta_{2,i,j}|^{p_L}, \ (i,j) \in I_2\,, \tag{16}$$

when we consider variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu = 1, 2$. We note that (13)-(16) really give the power p_L-th of the length of the corresponding edge. So that the sum of these quantities gives an estimation of the total length of the edges of $\mathbf{Q}$; this estimation can be seen as a function of $\underline{x}$ and $\underline{y}$:

$$
\begin{aligned}
F_L(\underline{x},\underline{y}) = &\sum_{(i,j)\in I_1} L_{1,i,j}^{p_L}(x_{i,j},x_{i+1,j},y_{i,j},y_{i+1,j})+ \\
&+\sum_{(i,j)\in I_2} L_{2,i,j}^{p_L}(x_{i,j},x_{i,j+1},y_{i,j},y_{i,j+1}),
\end{aligned}
\tag{17}
$$

or equivalently as a function of $\underline{\xi}_1$, $\underline{\xi}_2$, $\underline{\eta}_1$ and $\underline{\eta}_2$:

$$
\Phi_L(\underline{\xi}_1,\underline{\xi}_2,\underline{\eta}_1,\underline{\eta}_2)=\sum_{(i,j)\in I_1}\mathcal{L}_{1,i,j}^{p_L}(\xi_{1,i,j},\eta_{1,i,j})+\sum_{(i,j)\in I_2}\mathcal{L}_{2,i,j}^{p_L}(\xi_{2,i,j},\eta_{2,i,j}). \tag{18}
$$

Let us consider the p_A-norm for the area of the faces of $\mathbf{Q}$. The area of the quadrilateral of $\mathbf{Q}$ having vertices $\underline{u}_{i,j}$, $\underline{u}_{i,j+1}$, $\underline{u}_{i+1,j+1}$, $\underline{u}_{i+1,j}$, is given by:

$$
\begin{aligned}
&A_{i,j}(x_{i,j},x_{i,j+1},x_{i+1,j+1},x_{i+1,j},y_{i,j},y_{i,j+1},y_{i+1,j+1},y_{i+1,j}) = \\
&=\frac{1}{2}|(x_{i,j}-x_{i+1,j+1})(y_{i+1,j}-y_{i,j+1})+ \\
&\quad -(y_{i,j}-y_{i+1,j+1})(x_{i+1,j}-x_{i,j+1})|,\ (i,j)\in I_1\cap I_2,
\end{aligned}
\tag{19}
$$

when we consider variables $\underline{x}$, $\underline{y}$, and is given by:

$$
\begin{aligned}
&\mathcal{A}_{i,j}(\xi_{1,i,j},\xi_{1,i,j+1},\xi_{2,i,j},\eta_{1,i,j},\eta_{1,i,j+1},\eta_{2,i,j}) = \\
&=\frac{1}{2}|(\xi_{2,i,j}+\xi_{1,i,j+1})(\eta_{2,i,j}-\eta_{1,i,j})+ \\
&\quad -(\eta_{2,i,j}+\eta_{1,i,j+1})(\xi_{2,i,j}-\xi_{1,i,j})|,\ (i,j)\in I_1\cap I_2,
\end{aligned}
\tag{20}
$$

when we consider variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu=1,2$; we note that (19) follows from simple geometric arguments and (20) is obtained from simple manipulations before substituting (8), (9) in (19). So that the power p_A-th of the p_A-norm for the area of the quadrilaterals of $\mathbf{Q}$ is given by

$$
\begin{aligned}
&F_A(\underline{x},\underline{y}) = \\
&=\sum_{(i,j)\in I_1\cap I_2} A_{i,j}^{p_A}(x_{i,j},x_{i,j+1},x_{i+1,j+1},x_{i+1,j},y_{i,j},y_{i,j+1},y_{i+1,j+1},y_{i+1,j}),
\end{aligned}
\tag{21}
$$

when we consider variables $\underline{x}$, $\underline{y}$ and it is given by:

$$\Phi_A(\underline{\xi}_1,\underline{\xi}_2,\underline{\eta}_1,\underline{\eta}_2) = = \sum_{(i,j)\in I_1\cap I_2} \mathcal{A}^{p_A}_{i,j}(\xi_{1,i,j},\xi_{1,i,j+1},\xi_{2,i,j},\eta_{1,i,j},\eta_{1,i,j+1},\eta_{2,i,j}), \tag{22}$$

when we consider variables $\underline{\xi}_1,\underline{\xi}_2,\underline{\eta}_1,\underline{\eta}_2$.

Let us consider the p_O-norm to measure the angles between edges of $\mathbf{Q}$. For $(i,j)\in I_1\cap I_2$, in the quadrilateral having vertices $\underline{u}_{i,j}$, $\underline{u}_{i,j+1}$, $\underline{u}_{i+1,j+1}$, $\underline{u}_{i+1,j}$ we can give a measure of each interior angle. This measure is represented by the absolute value of the cosine of the angle multiplied by the lengths of the adjacent edges. In particular for the interior angle corresponding to the vertex $\underline{u}_{i,j}$ we consider:

$$O_{1,i,j}(x_{i,j},x_{i,j+1},x_{i+1,j},y_{i,j},y_{i,j+1},y_{i+1,j}) = = |(x_{i+1,j}-x_{i,j})(x_{i,j+1}-x_{i,j}) + (y_{i+1,j}-y_{i,j})(y_{i,j+1}-y_{i,j})|, \tag{23}$$

$$\mathcal{O}_{1,i,j}(\xi_{1,i,j},\xi_{2,i,j},\eta_{1,i,j},\eta_{2,i,j}) = |\xi_{1,i,j}\xi_{2,i,j}+\eta_{1,i,j}\eta_{2,i,j}|, \tag{24}$$

for the interior angle corresponding to the vertex $\underline{u}_{i,j+1}$ we consider:

$$O_{2,i,j}(x_{i,j},x_{i,j+1},x_{i+1,j+1},y_{i,j},y_{i,j+1},y_{i+1,j+1}) = = |(x_{i,j}-x_{i,j+1})(x_{i+1,j+1}-x_{i,j+1}) + +(y_{i,j}-y_{i,j+1})(y_{i+1,j+1}-y_{i,j+1})|, \tag{25}$$

$$\mathcal{O}_{2,i,j}(\xi_{1,i,j+1},\xi_{2,i,j},\eta_{1,i,j+1},\eta_{2,i,j}) = |\xi_{2,i,j}\xi_{1,i,j+1}+\ \eta_{2,i,j}\eta_{1,i,j+1}|, \tag{26}$$

for the interior angle corresponding to the vertex $\underline{u}_{i+1,j+1}$ we consider:

$$O_{3,i,j}(x_{i,j+1},x_{i+1,j+1},x_{i+1,j},y_{i,j+1},y_{i+1,j+1},y_{i+1,j}) = = |(x_{i,j+1}-x_{i+1,j+1})(x_{i+1,j}-x_{i+1,j+1}) + +(y_{i,j+1}-y_{i+1,j+1})(y_{i+1,j}-y_{i+1,j+1})|, \tag{27}$$

$$\mathcal{O}_{3,i,j}(\xi_{1,i,j+1},\xi_{2,i+1,j},\eta_{1,i,j+1},\eta_{2,i+1,j}) = = |\xi_{1,i,j+1}\xi_{2,i+1,j}+\eta_{1,i,j+1}\eta_{2,i+1,j}|, \tag{28}$$

and for the interior angle corresponding to the vertex $\underline{u}_{i+1,j}$ we consider:

$$O_{4,i,j}(x_{i,j},x_{i+1,j+1},x_{i+1,j},y_{i,j},y_{i+1,j+1},y_{i+1,j}) = =|(x_{i+1,j+1}-x_{i+1,j})(x_{i,j}-x_{i+1,j})+(y_{i+1,j+1}-y_{i+1,j})(y_{i,j}-y_{i+1,j})|, \tag{29}$$

$$\mathcal{O}_{4,i,j}(\xi_{1,i,j},\xi_{2,i+1,j},\eta_{1,i,j},\eta_{2,i+1,j}) = |\xi_{2,i+1,j}\xi_{1,i,j}+\eta_{2,i+1,j}\eta_{1,i,j}|, \tag{30}$$

we note that (23), (25), (27), (29) follow from simple geometric arguments and formulas (24), (26), (28), (30) are obtained substituting (8), (9) in (23), (25), (27), (29), respectively. Thus taking the power p_O-th of the p_O-norm of the quantities (23), (25), (27), (29) we define the following orthogonality function:

$$
\begin{aligned}
F_O(\underline{x},\underline{y}) = \sum_{(i,j)\in I_1\cap I_2} &\big(O^{p_O}_{1,i,j}(x_{i,j},x_{i,j+1},x_{i+1,j},y_{i,j},y_{i,j+1},y_{i+1,j})+\\
&+O^{p_O}_{2,i,j}(x_{i,j},x_{i+1,j+1},x_{i+1,j},y_{i,j},y_{i+1,j+1},y_{i+1,j})+\\
&+O^{p_O}_{3,i,j}(x_{i,j+1},x_{i+1,j+1},x_{i+1,j},y_{i,j+1},y_{i+1,j+1},y_{i+1,j})+\\
&+O^{p_O}_{4,i,j}(x_{i,j},x_{i,j+1},x_{i+1,j+1},y_{i,j},y_{i,j+1},y_{i+1,j+1})\big)\,,
\end{aligned} \tag{31}
$$

when we consider variables $\underline{x}$, $\underline{y}$ or:

$$
\begin{aligned}
\Phi_O(\underline{\xi}_1,\underline{\xi}_2,\underline{\eta}_1,\underline{\eta}_2) = \sum_{(i,j)\in I_1\cap I_2} &\big(\mathcal{O}^{p_O}_{1,i,j}(\xi_{1,i,j},\xi_{2,i,j},\eta_{1,i,j},\eta_{2,i,j})+\\
&+\mathcal{O}^{p_O}_{2,i,j}(\xi_{1,i,j},\xi_{2,i+1,j},\eta_{1,i,j},\eta_{2,i+1,j})+\\
&+\mathcal{O}^{p_O}_{3,i,j}(\xi_{1,i,j+1},\xi_{2,i+1,j},\eta_{1,i,j+1},\eta_{2,i+1,j})+\\
&+\mathcal{O}^{p_O}_{4,i,j}(\xi_{1,i,j+1},\xi_{2,i,j},\eta_{1,i,j+1},\eta_{2,i,j})\big)\,,
\end{aligned} \tag{32}
$$

when we consider variables $\underline{\xi}_1,\underline{\xi}_2,\underline{\eta}_1,\underline{\eta}_2$.

Functions F_L, F_A and F_O are usually called functional of length, functional of area and functional of orthogonality, respectively. We use the same name for the corresponding function of variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu=1,2$, i.e. Φ_L, Φ_A and Φ_O.

We consider the following problem:

$$
\min_{\underline{x},\underline{y}\in\mathbf{R}^{(N+1)(M+1)}} \{w_L F_L(\underline{x},\underline{y})+w_A F_A(\underline{x},\underline{y})+w_O F_O(\underline{x},\underline{y})\}\,, \tag{33}
$$

$$
\begin{aligned}
(x_{i,0},y_{i,0})^t &= \Big(x_\partial\Big(\frac{i}{N},0\Big),y_\partial\Big(\frac{i}{N},0\Big)\Big)^t, && i=0,1,\ldots,N\,,\\
(x_{i,M},y_{i,M})^t &= \Big(x_\partial\Big(\frac{i}{N},1\Big),y_\partial\Big(\frac{i}{N},1\Big)\Big)^t, && i=0,1,\ldots,N\,,\\
(x_{N,j},y_{N,j})^t &= \Big(x_\partial\Big(1,\frac{j}{M}\Big),y_\partial\Big(1,\frac{j}{M}\Big)\Big)^t, && j=1,2,\ldots,M-1\,,\\
(x_{0,j},y_{0,j})^t &= \Big(x_\partial\Big(0,\frac{j}{M}\Big),y_\partial\Big(0,\frac{j}{M}\Big)\Big)^t, && j=1,2,\ldots,M-1\,,
\end{aligned} \tag{34}
$$

where w_L, w_A and w_O are weights for the functionals F_L, F_A and F_O, respectively. The minimum $\underline{x}^*, \underline{y}^* \in \mathbf{R}^{(N+1)(M+1)}$ of problem (33)-(34) defines the vertices of the computed grids.

We note that the constraints (34) require that condition (7) holds. Problem (33)-(34) is a generalization of the direct method for the grid generation problem. This generalization reduces to the usual formulation when $p_L = p_A = p_O = 2$ is chosen. In the next section we see that this generalization improves the quality of grids generated by the method. We expect such an improvement since the norms defined by parameters p_L, p_A, p_O, with $p_L, p_A, p_O \geq 2$, can be seen as approximations of the uniformity norm. So that, according to the discussion given in the previous section, we expect that the resulting grid has higher quality than the one obtained using $p_L = p_A = p_O = 2$.

Furthermore in problem (33)-(34) we can consider variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu = 1, 2$ in place of variables $\underline{x}$, $\underline{y}$. Performing this change of variables we obtain:

$$\begin{aligned}\min_{\substack{\underline{\xi}_1,\underline{\eta}_1 \in \mathbf{R}^{N(M+1)} \\ \underline{\xi}_2,\underline{\eta}_2 \in \mathbf{R}^{(N+1)M}}} \Big\{ & w_L \Phi_L(\underline{\xi}_1, \underline{\xi}_2, \underline{\eta}_1, \underline{\eta}_2) + \\ & + w_A \Phi_A(\underline{\xi}_1, \underline{\xi}_2, \underline{\eta}_1, \underline{\eta}_2) + w_O \Phi_O(\underline{\xi}_1, \underline{\xi}_2, \underline{\eta}_1, \underline{\eta}_2) \Big\},\end{aligned} \tag{35}$$

$$\begin{aligned}
(\xi_{1,i,0}, \eta_{1,i,0})^t &= \Big(x_\partial\Big(\frac{i+1}{N}, 0\Big) - x_\partial\Big(\frac{i}{N}, 0\Big), y_\partial\Big(\frac{i+1}{N}, 0\Big) - y_\partial\Big(\frac{i}{N}, 0\Big)\Big)^t, \\
&\qquad i = 0, 1, \dots, N-1, \\
(\xi_{1,i,M}, \eta_{1,i,M})^t &= \Big(x_\partial\Big(\frac{i+1}{N}, 1\Big) - x_\partial\Big(\frac{i}{N}, 1\Big), y_\partial\Big(\frac{i+1}{N}, 1\Big) - y_\partial\Big(\frac{i}{N}, 1\Big)\Big)^t, \\
&\qquad i = 0, 1, \dots, N-1, \\
(\xi_{2,N,j}, \eta_{2,N,j})^t &= \Big(x_\partial\Big(1, \frac{j+1}{M}\Big) - x_\partial\Big(1, \frac{j}{M}\Big), y_\partial\Big(1, \frac{j+1}{M}\Big) - y_\partial\Big(1, \frac{j}{M}\Big)\Big)^t, \\
&\qquad j = 0, 1, \dots, M-1, \\
(\xi_{2,0,j}, \eta_{2,0,j})^t &= \Big(x_\partial\Big(0, \frac{j+1}{M}\Big) - x_\partial\Big(0, \frac{j}{M}\Big), y_\partial\Big(0, \frac{j+1}{M}\Big) - y_\partial\Big(0, \frac{j}{M}\Big)\Big)^t, \\
&\qquad j = 0, 1, \dots, M-1,
\end{aligned} \tag{36}$$

$$\begin{aligned}
\xi_{2,i+1,j} - \xi_{2,i,j} - \xi_{1,i,j+1} + \xi_{1,i,j} = 0, \ (i,j) \in I_1 \cap I_2, \\
\eta_{2,i+1,j} - \eta_{2,i,j} - \eta_{1,i,j+1} + \eta_{1,i,j} = 0, \ (i,j) \in I_1 \cap I_2,
\end{aligned} \tag{37}$$

where the objective function follows from the equivalence of formulas (17), (21), (31) and formulas (18), (22), (32), respectively. The constraints (36) require that condition (7) holds. Moreover constraints (37) are consequences of the irrotational conditions (10), (11), they state that variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu = 1, 2$ are discrete gradients of unknown discrete functions. Let $\underline{\xi}_\nu^*$, $\underline{\eta}_\nu^*$, $\nu = 1, 2$ be the minimizer of problem (35)-(37), the vertices coordinates $\underline{x}^*$, $\underline{y}^*$ of the computed grid are obtained from formula (12).

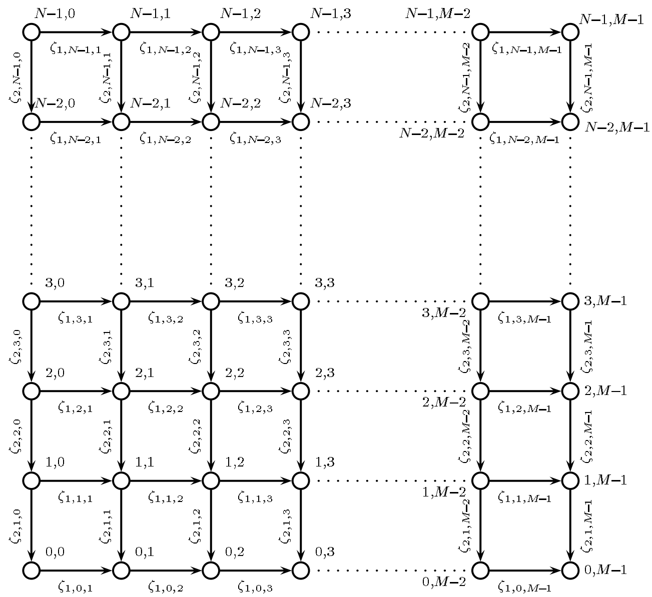


Fig. 2: Graph related to the minimum cost flow problem (35)-(37), where variables $\zeta_{\nu,i,j}$, $(i, j) \in I_\nu$, $\nu = 1, 2$ are generic variables to represent either variables $\xi_{\nu,i,j}$, $(i, j) \in I_\nu$, $\nu = 1$, 2 or variables $\eta_{\nu,i,j}$, $(i, j) \in I_\nu$, $\nu = 1, 2$.

We note that the constraints (34) in problem (33)-(34) and the constraints (36) in problem (35)-(37) are easy to treat, in fact these constraints fix to a given value some variables of the corresponding problem. At a first glance it seems that problem (35)-(37) is more complicated than problem (33)-(34), as consequence of the presence of constraints (37), nevertheless these constraints are the flow conservation conditions on the nodes of a particular graph, see Figure 2. Problem (35)-(37) is usually called minimum cost flow problem; we note that there exist very efficient methods for the solution of such networks problems, see [3], [7] for a detailed discussion. Note that the objective functions of problem (33)-(34) and of problem (35)-(37) are given by the sum of several addenda. These addenda are functions of a limited number of variables, however from formulas (13)-(16), (19), (20), (23)-(30) it can be easily seen that the use of variables $\underline{\xi}_\nu$, $\underline{\eta}_\nu$, $\nu = 1, 2$ reduces the number of the variables in each addendum. This yields a remarkable gain in the computational cost of the method used to solve the optimization problem. In particular we expect that this benefit is predominant with respect to the disadvantages of the introduction of these new variables, i.e. flow conservation conditions (37). This expectation is confirmed by the numerical results presented in the next section.

3 – The Numerical Experience

In the numerical experience we consider the planar regions proposed in the *Rogue's Gallery of Grids*, see [11] for details. Note that this is a classical set of test problems.

In particular, we use the following examples:

Ω_1 - *Annulus*;
Ω_2 - *Airfoil*;
Ω_3 - *Backstep*;
Ω_4 - *Swan*;
Ω_5 - *Valley*;
Ω_6 - *C*;
Ω_7 - *Chevron*;
Ω_8 - *Horseshoe*;
Ω_9 - *Dome*;
Ω_{10} - *S*.

The boundary of such regions are reported in Figure 3, for a detailed description of the equation of $\underline{u}_\partial$ see [11]. We note that only the domain Ω_{10} is not present in the *Rogue's Gallery of Grids*, but it is proposed in [6]. We have considered also this region since it seems to be a difficult region to grid.

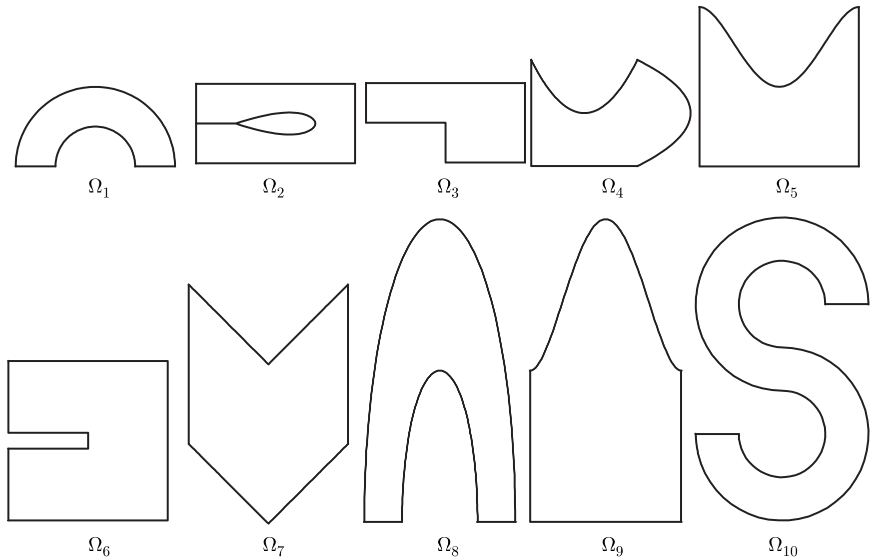


Fig. 3: The boundary $\partial\Omega$ of regions used for the test.

We propose the results obtained using two different implementations of the direct method for the grid generation problem:

$\mathcal{I}_1$- Language code: FORTRAN90; solution of the nonlinear minimization problem (35)-(37): the software package `LSNNO` (see [18], [19]),

$\mathcal{I}_2$- Language code: FORTRAN90; solution of the nonlinear minimum cost flow problem (33)-(34): routine E04HEF of the `NAG` software library (see [13]).

The numerical experience is given by two different parts. The former has the purpose to show the improvement from the point of view of the computational cost given by formulation (35)-(37); as explained in the previous section we expect such an improvement as consequence of the introduction of variables (8), (9). The second part of the numerical experience has the purpose to show the improvement in the quality of the computed grid given by formulation (35)-(37), or equivalently formulation (33)-(34), with respect to the classical formulation of the direct method; we expect such an improvement as consequence of the introduction of the non-Euclidean norms defined by p_L, p_A, p_O.

3.1 – The computational cost

We consider the computational cost of the direct method. In particular, we want to emphasize the improvement due to the introduction of variables $\underline{\xi}_{\nu}$, $\underline{\eta}_{\nu}$, $\nu = 1, 2$ in the corresponding optimization problem; so that we compare the computation cost of the method given by formulation (33)-(34) and the computation cost of the method given by formulation (35)-(37). This comparison is made on the ground of explicit computations with implementation $\mathcal{I}_1$ and $\mathcal{I}_2$.

In Table 1, we report the elapsed times $t_{\mathcal{I}_1}$, $t_{\mathcal{I}_2}$ for grid computation on the planar regions from Ω_1 to Ω_{10} using the implementations $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Moreover we denote $E_F = |F^* - \Phi^*|$, where F^* and Φ^* denote the computed optimal value of the objective function of problem (33)-(34) and problem (35)-(37), respectively. We denote with E_u the 2-norm of the difference between the coordinates of the vertices computed by the two implementations $\mathcal{I}_1$ and $\mathcal{I}_2$. In these experiments we have fixed the other previously defined parameters as follows: $p_L = 2$, $p_A = 2$, $p_O = 2$, $w_L = 1$, $w_A = 1$, $w_O = 1$.

Table 1 shows very convincing results about the improvements due to the introduction of variables $\underline{\xi}_{\nu}$, $\underline{\eta}_{\nu}$, $\nu = 1, 2$; in fact we note that from the values of E_F and E_u we can state that the obtained results are approximately the same, but we usually have $t_{\mathcal{I}_1}$ smaller than $t_{\mathcal{I}_2}$ and the difference between such values increases as N and M increase. We note that planar region S requires M quite larger than N; however the choice $N = M$ generates unstability in problem (33)-(34) as well as in problem (35)-(37) having two set of edges with very different lengths; the quite large values for E_F and E_u in Table 1 are consequence of this unstability.

3.2 – The quality of the computed grid

We consider the quality of the grids obtained by the direct method. In particular, we want to emphasize the improvements due to the introduction, in problems (33)-(34) or equivalently in problems (35)-(37), of the non-Euclidean norms defined by parameters p_L, p_A, p_O. Thus we compare the quality of the grids obtained using $p_L = p_A = p_O = 2$ and using a different choice for p_L, p_A, p_O, that is $p_L = 8$, $p_A = p_O = 2$. This

Table 1: *The comparison of the efficiency of method* (33)-(34) *and method* (35)-(37)*; for each planar region and for each choice of parameters N, M it is reported: the elapsed times $t_{\mathcal{I}_1}$ and $t_{\mathcal{I}_2}$ for the various grid computations with implementation $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively; E_F, i.e. the absolute value of the difference between optimal values of problem* (33)-(34) *and of problem* (35)-(37)*; E_u the 2-norm of the difference between the coordinates of vertices of the two grids computed by $\mathcal{I}_1$ and by $\mathcal{I}_2$.*

Example	$M = N$	E_F	E_u	$t_{\mathcal{I}_1}$	$t_{\mathcal{I}_2}$
Annulus	5	0.8988E-12	0.9550E-07	2.132	0.1820
	10	0.3201E-11	0.8669E-07	10.56	11.53
	15	0.1300E-11	0.3780E-07	63.82	189.9
Airfoil	5	0.1531E-10	0.4844E-06	0.1190	0.1890
	10	0.6540E-10	0.5039E-06	9.458	14.06
	15	0.1010E-10	0.2306E-06	48.58	165.0
Backstep	5	0.2380E-10	0.4744E-06	0.1000	0.1390
	10	0.9699E-11	0.1482E-06	2.4820	8.300
	15	0.4015E-12	0.2168E-07	2.3190	133.0
Swan	5	0.4820E-11	0.2221E-06	0.0670	0.1270
	10	0.8602E-12	0.4715E-07	0.2620	7.694
	15	0.6297E-12	0.2685E-07	1.121	122.4
Valley	5	0.3170E-11	0.1922E-06	0.0840	0.1060
	10	0.7100E-11	0.1373E-06	0.1970	7.687
	15	0.2195E-10	0.1602E-06	0.6010	121.9
C	5	0.2100E-11	0.1470E-06	0.1440	0.1300
	10	0.1243E-10	0.1753E-06	2.1120	7.676
	15	0.5110E-11	0.7479E-07	1.543	122.3
Chevron	5	0.1998E-13	0.1093E-07	0.0920	0.1310
	10	0.2529E-10	0.2557E-06	0.2650	7.430
	15	0.1399E-12	0.1273E-07	1.028	122.2
Horseshoe	5	0.7760E-09	0.2079E-05	2.071	0.1850
	10	0.1228E-08	0.1487E-05	1.587	20.49
	15	0.6112E-08	0.2531E-05	16.74	389.9
Dome	5	0.5540E-11	0.1713E-06	0.1320	0.1050
	10	0.3569E-11	0.6730E-07	0.6640	7.934
	15	0.5979E-11	0.5801E-07	1.014	122.1
S	5	0.9424E-07	0.1050E-04	3.457	0.6090
	10	0.1028E-07	0.4988E-05	20.67	11.69
	15	0.4010E-01	0.7537E-02	35.69	467.2

comparison is made on the ground of explicit computations using implementation on the previously defined planar regions. We describe briefly the performances indices used in Table 2. For the computed grids these indices give an information on the uniformity of edges length and on the angle between edges. In particular we consider the following performance indices:

$$r_h = \frac{\min_{(i,j)\in I_1}\{\sqrt{\xi_{1,i,j}^2 + \eta_{1,i,j}^2}\}}{\max_{(i,j)\in I_1}\{\sqrt{\xi_{1,i,j}^2 + \eta_{1,i,j}^2}\}}\,, \tag{38}$$

$$r_v = \frac{\min_{(i,j)\in I_2}\{\sqrt{\xi_{2,i,j}^2 + \eta_{2,i,j}^2}\}}{\max_{(i,j)\in I_2}\{\sqrt{\xi_{2,i,j}^2 + \eta_{2,i,j}^2}\}}\,. \tag{39}$$

We note that r_h and r_v give a measure of the uniformity of the edges length corresponding to horizontal edges of $\mathbf{P}$ and to vertical edges of $\mathbf{P}$, respectively. We can easily see that these indices are real numbers between 0 and 1; moreover for high quality grids we expect to have r_h and r_v close to 1 and for low quality grids we expect to have r_h and r_v close to 0.

Let us consider the set Θ given by all the quadrilaterals angles of the computed grid. We define the following performance index:

$$\theta_{\min} = \min \Theta\,, \tag{40}$$

that gives a measure of the uniformity of the angle between the edges of the computed grid. We note that $\theta_{\min}$ is always between $0°$ and $90°$, so that for high quality grids we expect to have $\theta_{\min}$ close to $90°$ and for low quality grids we expect to have $\theta_{\min}$ close to $0°$. In Table 2 we report the performance indices r_h, r_v, $\theta_{\min}$ obtained considering, for the planar regions from Ω_1 to Ω_{10}, the parameters $N \times M = 5 \times 10$, 5×15, 10×20, 10×30, and the parameters $p_L = p_A = p_O = 2$ and $p_L = 8$, $p_A = p_O = 2$.

Moreover, for the weights of problem (35)-(37), we have considered the simplest possible choice, that is $w_L = w_A = w_O = 1$.

TABLE 2: *The comparison of the grids obtained by the direct method; for each choice of the planar regions from* Ω_1 *to* Ω_{10}*, and for each choice of the parameters* $N \times M = 5 \times 10$, 5×15, 10×20, 10×30, *and* $p_L = p_A = p_O = 2$, $p_L = 8$, $p_A = p_O = 2$, *the performance indices* r_h, r_v, $\theta_{\min}$ *defined in* (38), (39), (40), *respectively, are reported. We note that the superscript of* $N \times M$ *shows the number of the corresponding figure.*

		$p_L = p_A = p_O = 2$			$p_L = 8, p_A = p_O = 2$		
Region	$N \times M$	r_h	r_v	$\theta_{\min}$	r_h	r_v	$\theta_{\min}$
Annulus	5×10	—	—	—	0.6599	0.5839	$70°30'$
	5×15	0.6339	0.2886	$68°30'$	0.6569	0.5867	$71°24'$
	10×20	—	—	—	0.5646	0.5408	$70°42'$
	10×30	0.5622	0.2285	$68°30'$	0.5620	0.5411	$70°48'$
Airfoil	5×10	—	—	—	0.3539	0.0962	$40°48'$
	5×15^4	—	—	—	0.3433	0.2212	$42°48'$
	10×20	—	—	—	0.2065	0.1598	$43°18'$
	10×30	—	—	—	0.1901	0.2004	$44°30'$
Backstep	5×10^5	—	—	—	—	—	—
	5×15	—	—	—	0.3418	0.1213	$4°12'$
	10×20	—	—	—	0.2606	0.0628	$16°12'$
	10×30	—	—	—	0.1968	0.0813	$17°42'$
Swan	5×10	0.3190	0.2254	$28°$	0.2492	0.1458	$29°6'$
	5×15	0.2393	0.3086	$28°12'$	0.2349	0.1515	$28°12'$
	10×20^6	0.2778	0.1539	$26°30'$	0.1888	0.0963	$27°48'$
	10×30	0.2046	0.2365	$26°30'$	0.1491	0.1068	$27°24'$
Valley	5×10	0.7348	0.1750	$35°36'$	0.3382	0.2728	$51°48'$
	5×15	0.6951	0.3153	$34°18'$	0.2656	0.2835	$51°30'$
	10×20^6	0.6470	0.0892	$33°48'$	0.2697	0.2301	$52°6'$
	10×30	0.6213	0.2665	$33°24'$	0.2230	0.2198	$49°6'$
C	5×10	—	—	—	0.3324	0.2651	$39°48'$
	5×15^7	0.3237	0.1446	$30°36'$	0.2941	0.2714	$34°54'$
	10×20	—	—	—	0.1969	0.2047	$24°$
	10×30	0.2740	0.0864	$7°24'$	0.1549	0.2095	$22°12'$
Chevron	5×10	0.7459	0.2211	$45°$	0.2563	0.0780	$45°$
	5×15	0.7079	0.5063	$45°$	0.1833	0.1379	$45°$
	10×20	0.7221	0.0789	$44°54'$	0.1741	0.0722	$42°6'$
	10×30	0.7132	0.2721	$45°$	0.1439	0.0479	$32°48'$
Horseshoe	5×10	—	—	—	0.1202	0.1029	$23°42'$
	5×15	0.1620	0.1322	$25°12'$	0.0869	0.1107	$24°42'$
	10×20	—	—	—	0.0580	0.0960	$22°48'$
	10×30	0.1452	0.1115	$26°24'$	0.0358	0.0843	$22°54'$

Table 2 (*continued*)

Region	$N \times M$	$p_L = p_A = p_O = 2$ r_h	r_v	$\theta_{\min}$	$p_L = 8, p_A = p_O = 2$ r_h	r_v	$\theta_{\min}$
S	5×10	—	—	—	0.5672	0.6698	$59^o12'$
	5×15	—	—	—	0.5365	0.6641	$19^o24'$
	10×20^4	—	—	—	0.1501	0.4967	$42^o12'$
	10×30	—	—	—	—	—	—
Dome	5×10	0.4781	0.3544	$19°16'$	0.1481	0.2574	$27°54'$
	5×15	0.4311	0.3817	$18°36'$	0.1140	0.2320	$28°12'$
	10×20	0.3905	0.2627	$18°18'$	0.1171	0.1383	$25°12'$
	10×30	0.3674	0.3392	$18°12'$	0.0872	0.1051	$21°48'$

Table 2 shows very interesting results, in fact we can note that the choice $p_L = p_A = p_O = 2$ sometimes produces folded grids, see Figure 4; on the contrary the choice $p_L = 8$, $p_A = p_O = 2$ produces only two folded grids: *Backstep* with $N \times M = 5 \times 10$ and S with $N \times M = 10 \times 30$.

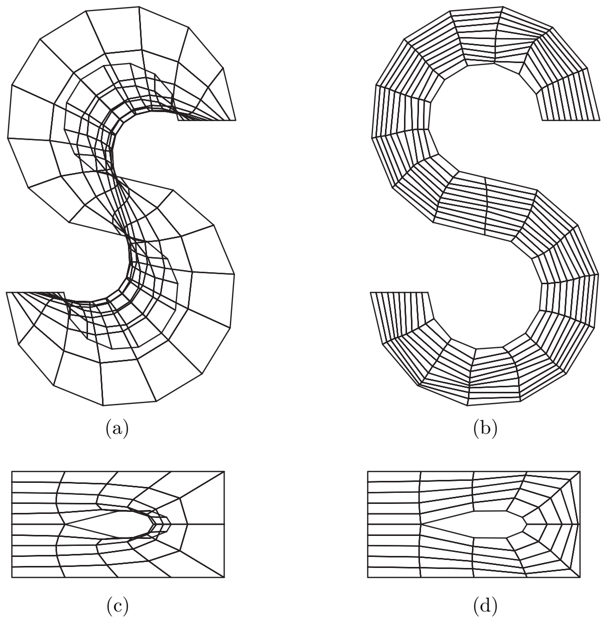


Fig. 4: The grids generated for region S with $\mathrm{N} \times \mathrm{M} = 10 \times 20$ and for region *Airfoil* with $\mathrm{N} \times \mathrm{M} = 5 \times 15$: folded grids (a), (c) are obtained using $\mathrm{p_L} = \mathrm{p_A} = \mathrm{p_O} = 2$, unfolded grids (b), (d) are obtained using $\mathrm{p_L} = 8$, $\mathrm{p_A} = \mathrm{p_O} = 2$.

Fig. 5: The grid generated for region *Backstep* with N = 5 and M = 10: (a) grid obtained with parameters $\mathrm{p_L} = \mathrm{p_A} = \mathrm{p_O} = 2$, (b) grid obtained with parameters $\mathrm{p_L} = 8$, $p_A = p_O = 2$. In both cases we have folded grids but in (a) the folding is more severe than in (b).

However when both choices of parameters p_L, p_A, p_O produce folded grids we can observe a more sever folding in the grid obtained with $p_L = p_A = p_O = 2$ with respect to the one obtained with $p_L = 8$, $p_A = p_O = 2$, see Figure 5. When these two choices of parameters p_L, p_A, p_O give unfolded grids we have that these grids are quite similar, see Figure 6. In particular we note that parameters $p_L = 8$, $p_A = p_O = 2$ give often index $\theta_{\min}$ better than the one given by parameter $p_L = p_A = p_O = 2$. We can also note that parameters $p_L = p_A = p_O = 2$ give sometimes indices r_h, r_v better than the ones given by parameters $p_L = 8$, $p_A = p_O = 2$, nevertheless in Figure 6 and in Figure 7, area of the faces in the grid computed using $p_L = 8$, $p_A = p_O = 2$ seems to be more uniform with respect to the one computed using $p_L = p_A = p_O = 2$.

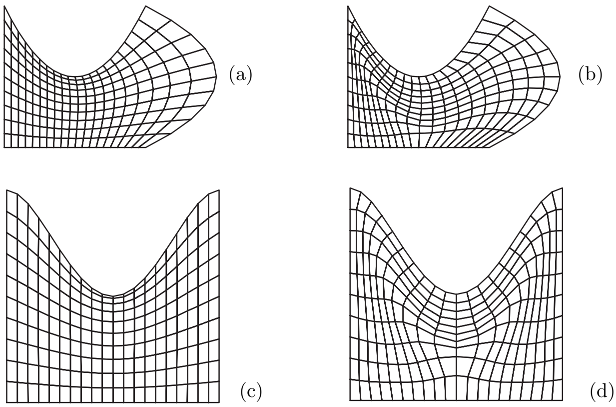
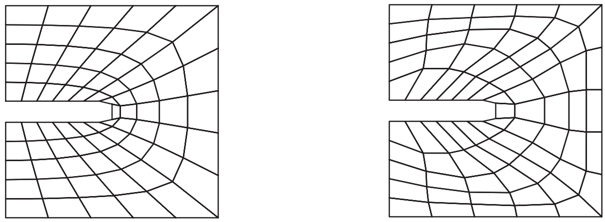


Fig. 6: The grids generated for region *Swan* with $\mathrm{N} \times \mathrm{M} = 10 \times 20$ and for region *Valley* with $\mathrm{N} \times \mathrm{M} = 10 \times 20$: (a), (c) are obtained using $\mathrm{p_L} = \mathrm{p_A} = \mathrm{p_O} = 2$, (b), (d) are obtained using $\mathrm{p_L} = 8$, $p_A = p_O = 2$. We note that edges length of grids (a), (c) is more uniform than the one of grids (b), (d), but these last grids have more uniform faces area with respect to (a), (c).



(a) (b)

Fig. 7: The grid generated for region C with $\mathrm{N} = 5$ and $\mathrm{M} = 15$: (a) the grid obtained with parameters $\mathrm{p_L} = \mathrm{p_A} = \mathrm{p_O} = 2$, (b) the grid obtained with parameters $\mathrm{p_L} = 8$, $\mathrm{p_A} = \mathrm{p_O} = 2$. In both cases we have unfolded grids but grid (b) is more uniform than grid (a).

4 – Conclusions

We consider the direct method for the grid generation problem and we describe some simple, but efficient modifications of this method. From various numerical experiments on a classical set of test problems we have shown the improvements due to such modifications in the quality of the computed grids as well as in the computational cost of the method.

The possible future developments of this work are various. For example, from the point of view of the computational cost of the method we have that formulation (35)-(37) seems to be an improvement with respect to formulation (33)-(34), but problem (35)-(37) is a minimum cost flow problem on a very particular graph, see Figure 2, so that we can take advantage on this fact to obtain further improvement of the computational cost of the method. This can be achieved specializing the usual algorithms for minimum cost flow problems to the particular graphs shown in Figure 2. Another important question is of course the quality of the computed grid, that is strictly related to the parameters chosen in the method. In particular the quality of the grids generated by the direct method is quite sensitive to the objective function of the corresponding optimization problem, and a very interesting question is to find an objective function that works well with a large number of domains. It seems that the use of non-Euclidean norms in the direct method goes in such a direction, so that a promising attempt is the use of the maximum norm in place of the usual Euclidean norm.

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