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On the coefficients of entire harmonic functions

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Abstract: *A function which is harmonic in a neighbourhood of the origin in* R^3 *has there an expansion in spherical harmonics. For entire harmonic function* H*, measures of growth such as* (p,q)*-order,* (p,q)*-type, lower* (p,q)*-order and lower* (p,q)*-type are obtained in terms of the coefficients of spherical harmonic expansions. Alternative characterizations for* (p,q)*-order,* (p,q)*-type, lower* (p,q)*-order and lower* (p,q)*-type are also obtained in terms of the ratios of these successive coefficients. Some results obtained earlier by Srivastava become the particular cases of the results obtained in this paper.*

1 – Introduction

A function which is harmonic in a neighbourhood of origin in R^3 has there an expansion in spherical harmonics. Thus, if H is a function which is harmonic in a neighbourhood of origin in R^3 then H has the following expansion in spherical harmonic:

$$(1.1)\qquad H(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\{a_{nm}^{(1)}\cos m\varphi+a_{nm}^{(2)}\sin n\varphi\}r^n P_n^m(\operatorname{Cos}\theta)\,.$$

The series converges absolutely and uniformly on compact sets of the largest open ball centered at the origin which omits singularities of H [3].

Key Words and Phrases: *Entire harmonic function* – (p,q)*-order* – (p,q)*-type.*
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Here $x = r\cos\theta$, $y = r\sin\theta\cos\phi$ and $z = r\sin\theta\sin\phi$. $P_n^m(x)$ are associated Legendre functions of the first kind, n^{th} degree and order m.

For H entire, we define

$$M(r) \equiv M(r, H) = \max_{\theta,\phi} H(r, \theta, \phi)\,. \tag{1.2}$$

2 – Definitions and notations

Following the usual definitions of (p,q)-order, lower (p,q)-order, (p, q)-type and lower (p, q)-type of an entire function of a complex variable z, we define:

Definition 1. The (p, q)-order $\rho(H)$ of H is defined as

$$\rho(H) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r, H)}{\log^{[q]} r} \tag{2.1}$$

where p and q are integers such that $p \geq q \geq 1$, $\log^{[0]} x = x$, $\log^{[m]} x = \log(\log^{[m-1]} x)$ for $0 < \log^{[m-1]} x < \infty$ and $m = 1, 2, 3, \ldots$ Moreover $0 \leq \rho(H) \leq \infty$ if $p > q$ and $1 \leq \rho(H) \leq \infty$ if $p = q$.

Definition 2. An entire harmonic function H of index-pair (p, q) is said to be of lower (p, q)-order $\lambda(H)$ if

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(r, H)}{\log^{[q]} r} = \lambda(H)\,. \tag{2.2}$$

Definition 3. An entire harmonic function H having (p, q)-order $\rho(H)(b < \rho(H) < \infty)$ is said to be of (p, q)-type $T(H)$ and lower (p, q)-type $t(H)$ if

$$\lim_{r\to\infty} {}^{\sup}_{\inf} \frac{\log^{[p-1]} M(r, H)}{(\log^{[q-1]} r)^{\rho(H)}} = {}^{T(H)}_{t(H)} \tag{2.3}$$

where $b = 1$ if $p = q$ and $b = 0$ if $p > q$.

The following notations are frequently used in the sequel:

Notation 1.

$$P(\alpha)=\begin{cases}\alpha & \text{if } p>q\\ 1+\alpha & \text{if } p=q=2\\ 1 & \text{if } 3\le p=q<\infty\\ \infty & \text{if } p=q=\infty\end{cases}\tag{2.4}$$

where $0\le\alpha\le\infty$.

Notation 2.

$$M(F)=\begin{cases}(\rho(F)-1)^{(\rho(F)-1)}/\rho(F)^{\rho(F)} & \text{if } (p,q)=(2,2)\\ 1/(e\rho(F)) & \text{if } (p,q)=(2,1)\\ 1 & \text{if } p\ge 3\,.\end{cases}\tag{2.5}$$

Notation 3.

$$X=\begin{cases}1/\rho & \text{if } p=2\\ 1 & \text{if } p\ge 3\,.\end{cases}\tag{2.6}$$

Notation 4.

$$A=\begin{cases}1 & \text{if } (p,q)=(2,2)\\ 0 & \text{if } (p,q)\ne(2,2)\,.\end{cases}\tag{2.7}$$

To avoid unnecessary repetition, we shall denote throughout the paper:

$$\lim_{r\to\infty}{}^{\sup}_{\inf}\frac{\log^{[p-1]}n}{\log^{[q-1]}(n^{-1}\log|a_n|^{-1})}={}^{L(F)}_{\psi(F)}\,,\tag{2.8}$$

$$\lim_{r\to\infty}{}^{\sup}_{\inf}\frac{\log^{[p-2]}n}{\log^{[q-1]}|a_n/a_{n+1}|^{\rho(F)-A}}={}^{Q(F)}_{R(F)}\,,\tag{2.9}$$

$$\lim_{r\to\infty}{}^{\sup}_{\inf}\frac{\log^{[p-2]}n}{(\log^{[q-1]}|a_n|^{-1/n})^{\rho(F)-A}}={}^{V(F)}_{v(F)}\,,\tag{2.10}$$

and

$$\lim_{r\to\infty}{}^{\sup}_{\inf}\frac{\log^{[p-1]}n}{\log^{[q]}(|a_n/a_{n+1}|)}={}^{L^*(F)}_{\psi^*(F)}\,,\tag{2.11}$$

for the entire function $F(z)=\sum_{n=0}^{\infty}a_nz^n$.

In this paper we define (p, q)-order, (p, q)-type, lower (p, q)-order and lower (p, q)-type of an entire harmonic function H and obtain various characterizations in terms of $\{\alpha_n\}$ defined in equation (3.2). We also obtain alternative characterizations for (p, q)-order, (p, q)-type, lower (p, q)-order and lower (p, q)-type in terms of the ratio (α_n/α_{n+1}). Some results obtained earlier by Srivastava become the particular cases of the results obtained in this paper.

3 – Known results

We have the following lemmas:

Lemma 1 (Fryant, [2]). *The harmonic function $H(r, \theta, \varphi)$ having expansion* (1.1) *is entire if and only if*

$$\lim_{n\to\infty} (\alpha_n)^{1/n} = 0 \tag{3.1}$$

where α_n is given by

$$\alpha_n = \max_{m,i} \left\{ \frac{(n+m)!}{(n-m)!} \right\}^{1/2} |a_{nm}^{(i)}| \, . \tag{3.2}$$

Lemma 2 (Fryant, [2]). *If H is entire harmonic function then for all $r > 0$ and for all n,*

$$(2(1+2n)^{1/2})^{-1} \alpha_n r^n \le M(r, H) \le 2 \sum_{n=0}^{\infty} (1 + n^{-1/2})^n \alpha_n r^n \, . \tag{3.3}$$

Now, we define

$$f(z) = \sum_{n=0}^{\infty} (1 + n^{-1/2})^n \alpha_n z^n = \sum_{n=0}^{\infty} \beta_n z^n \tag{3.4}$$

and

$$g(z) = \sum_{n=0}^{\infty} (1 + 2n)^{-1/2} \alpha_n z^n = \sum_{n=0}^{\infty} \gamma_n z^n \tag{3.5}$$

where α_n is given by (3.2).

On using (3.1), we obtain

$$\lim_{n\to\infty}\{(1+n^{-1/2})^n\alpha_n\}^{1/n}=\lim_{n\to\infty}\{(1+2n)^{-1/2}\alpha_n\}^{1/n}=\lim_{n\to\infty}\alpha_n^{1/n}=0\,.$$

Hence we get the following:

Lemma 3. *If H is entire harmonic function then f and g are also entire functions of complex variable z. Further, if $m(r,g)=\max_n\{(1+2n)^{-1/2}\alpha_n r^n\}$ and $M(r,f)=\max_{|z|\le r}|f(z)|$ then from (3.3), we get*

$$\frac{m(r,g)}{2}\le M(r,H)\le 2M(r,f)\,. \tag{3.6}$$

Lemma 4. *If α_n/α_{n+1} forms a non-decreasing function of n for $n>n_0$ then β_n/β_{n+1} and γ_n/γ_{n+1} also form a non-decreasing function of n for $n>n_0$, where $\beta_n=(1+n^{-1/2})^n\alpha_n$ and $\gamma_n=(1+2n)^{-1/2}\alpha_n$.*

Proof. We have

$$\frac{\beta_n}{\beta_{n+1}}=\frac{(1+n^{-1/2})^n}{\{1+(n+1)^{-1/2}\}^{n+1}}\frac{\alpha_n}{\alpha_{n+1}}\,.$$

Let

$$\eta(x)=\frac{(1+x^{-1/2})^x}{\{1+(x+1)^{-1/2}\}^{x+1}}=\frac{G(x)}{G(x+1)}$$

then we can easily prove that

$$\frac{d}{dx}[\log G(x)]>\frac{d}{dx}[\log G(x+1)]$$

so, $\frac{d\eta(x)}{dx}>0$ for $x>0$. Thus β_n/β_{n+1} is a non-decreasing function of n for $n>n_0$ if α_n/α_{n+1} is a non-decreasing function of n for $n>n_0$. Similarly, we can prove for γ_n/γ_{n+1}.

Lemma 5. *Let $f(z)=\sum_{n=0}^{\infty}(1+n^{-1/2})^n\alpha_n z^n$ and $g(z)=\sum_{n=0}^{\infty}(1+2n)^{-1/2}\alpha_n z^n$ be two entire functions. Then for $p\ge 2$, $q\ge 1$, $p\ge q$, (p,q)-orders and (p,q)-types of $f(z)$ and $g(z)$ are equal.*

PROOF. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(F)$ and (p,q)-type $T(F)$. Then for $p \geq 2$, $q \geq 1$, $p \geq q$, we have [4]

$$\rho(F) = P(L(F)) \tag{3.7}$$

and

$$T(F) = M(F)V(F)\,. \tag{3.8}$$

Now, for the function $f(z)$, we have

$$\begin{aligned} L(f) &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\{n^{-1}\log((1+n^{-1/2})^n\alpha_n)^{-1}\}} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\{n^{-1}\log\alpha_n^{-1} - \log(1+n^{-1/2})\}} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\left[(n^{-1}\log\alpha_n^{-1})\left\{1-\dfrac{\log(1+n^{-1/2})}{\log(\alpha_n^{-1/n})}\right\}\right]} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\{n^{-1}\log\alpha_n^{-1}(1-o(1))\}} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}(n^{-1}\log\alpha_n^{-1})}\,. \end{aligned} \tag{3.9}$$

Similarly, for the function $g(z)$, we have

$$\begin{aligned} L(g) &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}[n^{-1}\log\{(1+2n)^{-1/2}\alpha_n\}^{-1}]} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\left[(n^{-1}\log\alpha_n^{-1})\left\{1-\dfrac{\log(1+2n)^{-1/2}}{\log(\alpha_n)^{-1}}\right\}\right]} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}(n^{-1}\log\alpha_n^{-1})}\,. \end{aligned} \tag{3.10}$$

(3.7), (3.9) and (3.10) give $\rho(f) = \rho(g)$. Since f and g have same order, using (2.10) and (3.8) we can easily prove that

$$T(f) = T(g) \tag{3.11}$$

This completes the proof of Lemma 5.

4 – Theorems

In this section we shall prove the following theorems for $p \geq 2$, $q \geq 1$ and $p \geq q$:

Theorem 1. *Let H be an entire harmonic function of (p, q)-order $\rho(H)$, lower (p, q)-order $\lambda(H)$, (p, q)-type $T(H)$ and lower (p, q)-type $t(H)$. If f and g are two entire functions defined by* (3.4) *and* (3.5) *respectively, then*

$$\rho(f) = \rho(g) = \rho(H) \tag{4.1}$$

$$T(f) = T(g) = T(H) \tag{4.2}$$

$$\lambda(g) \leq \lambda(H) \leq \lambda(f) \tag{4.3}$$

$$t(g) \leq t(H) \leq t(f)\,. \tag{4.4}$$

Proof. From (3.6), we have

$$\limsup_{r\to\infty}\,(\inf)\frac{\log^{[p]} m(r,g)}{\log^{[q]} r} \leq \limsup_{r\to\infty}\,(\inf)\frac{\log^{[p]} M(r,H)}{\log^{[q]} r} \leq$$

$$\leq \limsup_{r\to\infty}\,(\inf)\frac{\log^{[p]} M(r,f)}{\log^{[q]} r}\,.$$

Also for an entire function ξ, we have [1]

$$\log M(r,\xi) \approx \log m(r,\xi) \quad \text{as} \quad r \to \infty\,.$$

Hence,

$$\rho(g) \leq \rho(H) \leq \rho(f)$$

and

$$\lambda(g) \leq \lambda(H) \leq \lambda(f)\,.$$

Since $\rho(g) = \rho(f)$, we thus obtain (4.1) and (4.3).

Let

$$\rho(g) = \rho(H) = \rho(f) = \rho$$

then from (3.6), we have

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[p-1]} m(r,g)}{(\log^{[q-1]} r)^{\rho}} &\leq \limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,H)}{(\log^{[q-1]} r)^{\rho}} \leq \\ &\leq \limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,f)}{(\log^{[q-1]} r)^{\rho}}\,.\end{aligned} \tag{4.5}$$

(3.11) and (4.5) give (4.2). (4.4) follows in a similar manner.

Theorem 2. *Let H be an entire harmonic function of (p,q)-order $\rho(H)(b < \rho(H) < \infty)$, (p,q)-type $T(H)$ and lower (p,q)-type $t(H)$, then*

$$XU \leq t(H) \leq T(H) \leq XW \tag{4.6}$$

where

$$U = \liminf_{n\to\infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]}(\alpha_n/\alpha_{n+1})\}^{\rho(H)-A}}\,,$$

$$W = \limsup_{n\to\infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]}(\alpha_n/\alpha_{n+1})\}^{\rho(H)-A}}$$

and $b = 1$ if $p = q$, $b = 0$ if $p > q$. X and A are given by (2.6) *and* (2.7), *respectively.*

Proof. If $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is entire function of (p,q)-order $\rho(F)(b < \rho(F) < \infty)$, (p,q)-type $T(F)$ and lower (p,q)-type $t(F)$, then, we have [5]

$$XR(F) \leq t(F) \leq T(F) \leq XQ(F)\,. \tag{4.7}$$

For the entire function $f(z) = \sum(1 + n^{-1/2})^n \alpha_n z^n$ we obtain

$$Q(f) = \limsup_{n\to\infty} \frac{\log^{[p-2]} n}{\left[\log^{[q-1]}\left\{\frac{(1+n^{-1/2})^n}{(1+(n+1)^{-1/2})^{n+1}} \frac{\alpha_n}{\alpha_{n+1}}\right\}\right]^{\rho(f)-A}}\,.$$

Now, $\{(1+n^{-1/2})^n/(1+(n+1)^{-1/2})^{n+1}\} \to 1$ as $n \to \infty$ and $\rho(f) = \rho(H)$ by (4.1), so

$$Q(f) = \limsup_{n\to\infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]}(\alpha_n/\alpha_{n+1})\}^{\rho(H)-A}} = W\,.$$

Since $T(f) = T(H)$, we get

$$T(H) \le XW\,. \tag{4.8}$$

Similarly, for the entire function $g(z) = \sum_{n=0}^{\infty}(1+2n)^{-1/2}\alpha_n z^n$ we find

$$\begin{aligned} R(g) &= \liminf_{n\to\infty} \frac{\log^{[p-2]} n}{\left[\log^{[q-1]}\left\{\dfrac{(1+2n)^{-1/2}}{(1+2(n+1))^{-1/2}}\dfrac{\alpha_n}{\alpha_{n+1}}\right\}\right]^{\rho(g)-A}} = \\ &= \liminf_{n\to\infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]}(\alpha_n/\alpha_{n+1})\}^{\rho(g)-A}} = \\ &= U\,. \end{aligned}$$

Using (4.4) and (4.7), we find

$$XU \le t(H)\,.$$

This completes the proof of Theorem 2.

THEOREM 3. *Let H be an entire harmonic function of (p,q)-order $\rho(H)(b < \rho(H) < \infty)$ and (p,q)-type $T(H)$. Further, if α_n/α_{n+1} forms a non-decreasing function of n for $n > n_0$, then*

$$T(H) \le XW \le XT(H)/M(H) \tag{4.9}$$

and

$$\rho(H) = P(E) \tag{4.10}$$

where X and W are as in Theorem 2. $P(E)$ and $M(H)$ are given by (2.4) and (2.5) respectively and

$$E = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}(\alpha_n/\alpha_{n+1})}\,. \tag{4.11}$$

PROOF. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p, q)-order $\rho(F)(b < \rho(F) < \infty)$ and (p, q)-type $T(F)$. If $|a_n/a_{n+1}|$ forms a non-decreasing function of n for $n > n_0$, then we know [5] that

$$T(F) \leq XQ(F) \leq XT(F)/M(F) \tag{4.12}$$

and

$$\rho(F) = P(L^*(F)) . \tag{4.13}$$

Applying (4.12) and (4.13) to $f(z) = \sum_{n=0}^{\infty} \beta_n z^n$, we obtain

$$T(f) \leq XQ(f) \leq XT(f)/M(f) \quad \text{and} \quad \rho(f) = P(L^*(f))$$

but $\rho(f) = \rho(H)$, $T(f) = T(H)$ and $M(f) = M(H)$, so

$$T(H) \leq XQ(f) \leq XT(H)/M(H) \tag{4.14}$$

and

$$\rho(H) = P(L^*(f)) . \tag{4.15}$$

Now, we have

$$Q(f) = W . \tag{4.16}$$

Also,

$$\begin{aligned} L^*(f) &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} \left\{ \dfrac{(1+n^{-1/2})^n}{(1+(n+1)^{-1/2})^{n+1}} \dfrac{\alpha_n}{\alpha_{n+1}} \right\}} = \\ &= \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}(\alpha_n/\alpha_{n+1})} = \\ &= E . \end{aligned} \tag{4.17}$$

Using (4.16) in (4.14) and (4.17) in (4.15), we find the desired results.

We now state a theorem concerning lower (p, q)-order and lower (p, q)-type of an entire harmonic function H:

Theorem 4. *Let H be an entire harmonic function of (p, q)-order $\rho(H)(b < \rho(H) < \infty)$, lower (p, q)-order $\lambda(H)$ and lower (p, q)-type $t(H)$. If α_n/α_{n+1} forms a non-decreasing function of n for $n > n_0$, then*

$$\lambda(H) = P(m) = P(m^*) \tag{4.18}$$

and

$$t(H) = M(H) \liminf_{n\to\infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]}(\alpha_n^{-1/n})\}^{\rho(H)-A}} \tag{4.19}$$

where

$$m = \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\{\log^{[q-1]}(n^{-1}\log \alpha_n^{-1})\}},$$

$$m^* = \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}(\alpha_n/\alpha_{n+1})},$$

$P(m)$, $M(H)$ and A are defined by (2.4), (2.5) *and* (2.7), *respectively.*

Proof. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p, q)-order $\rho(F)(b < \rho(F) < \infty)$, lower (p, q)-order $\lambda(F)$ and lower (p, q)-type $t(F)$ such that $|a_n/a_{n+1}|$ forms as non-decreasing function of n for $n > n_0$, then we have [4]

$$\lambda(F) = P(\psi(F)) = P(\psi^*(F)) \tag{4.20}$$

and [5]

$$t(F) = M(F)v(F)\,. \tag{4.21}$$

Applying (4.20) and Lemma 4 to the entire function $f(z)=\sum_{n=0}^{\infty}(1+n^{-1/2})^n$, we find

$$\lambda(f) = P(\psi(f)) = P(\psi^*(f))\,. \tag{4.22}$$

Now,

$$\psi(f) = \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\{n^{-1}\log((1+n^{-1/2})^n\alpha_n)^{-1}\}} =$$

(4.23)
$$= \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}\{n^{-1}\log \alpha_n^{-1}(1-o(1))\}} =$$

$$= \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}(n^{-1}\log \alpha_n^{-1})} = m$$

and

$$\psi^*(f) = \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}\left\{\dfrac{(1+n^{-1/2})^n}{(1+(n+1)^{-1/2})^{n+1}}\dfrac{\alpha_n}{\alpha_{n+1}}\right\}} =$$

(4.24)
$$= \liminf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]}(\alpha_n/\alpha_{n+1})} =$$

$$= m^* .$$

Similarly, from Lemma 4 and (4.20), for the entire function $g(z) = \sum_{n=0}^{\infty}(1+2n)^{-1/2}\alpha_n z^n$, we have

(4.25)
$$\lambda(g) = P(\psi(g)) = P(\psi^*(g)) .$$

Also, we can easily prove that

(4.26)
$$\psi(g) = m \;\text{ and }\; \psi^*(g) = m^* .$$

The result (4.18) now follows in view of (4.3) and (4.22) to (4.26).

To prove (4.19) we apply Lemma 4 and (4.21) to the entire function $f(z) = \sum_{n=0}^{\infty} \beta_n z^n$ and $g(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ and we obtain

(4.27)
$$t(f) = M(f)v(f) \;\text{ and }\; t(g) = M(g)v(g) .$$

Now,

(4.28)
$$M(g) = M(f) = M(H)$$

because $\rho(g) = \rho(f) = \rho(H)$ and we can easily show that

$$v(g) = v(f) = \liminf_{n\to\infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]}(\alpha_n^{-1/n})\}^{\rho(H)-A}} . \tag{4.29}$$

(4.4), (4.27), (4.28) and (4.29) together establish (4.19) and the proof of the theorem is complete.

Remark. For $p = 2$, $q = 1$, the results in Theorem 1, Theorem 2 and Theorem 4 are due to Srivastava [6].

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