On the coefficients of entire harmonic functions

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Abstract: A function which is harmonic in a neighbourhood of the origin in $\mathbb{R}^3$ has there an expansion in spherical harmonics. For entire harmonic function $H$, measures of growth such as $(p, q)$-order, $(p, q)$-type, lower $(p, q)$-order and lower $(p, q)$-type are obtained in terms of the coefficients of spherical harmonic expansions. Alternative characterizations for $(p, q)$-order, $(p, q)$-type, lower $(p, q)$-order and lower $(p, q)$-type are also obtained in terms of the ratios of these successive coefficients. Some results obtained earlier by Srivastava become the particular cases of the results obtained in this paper.

1 – Introduction

A function which is harmonic in a neighbourhood of origin in $\mathbb{R}^3$ has there an expansion in spherical harmonics. Thus, if $H$ is a function which is harmonic in a neighbourhood of origin in $\mathbb{R}^3$ then $H$ has the following expansion in spherical harmonic:

\[
H(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left\{ a_{nm}^{(1)} \cos m\varphi + a_{nm}^{(2)} \sin n\varphi \right\} r^n P_m^m(\cos \theta).
\]

The series converges absolutely and uniformly on compact sets of the largest open ball centered at the origin which omits singularities of $H$ [3].

Key Words and Phrases: Entire harmonic function – $(p, q)$-order – $(p, q)$-type.

A.M.S. Classification: 30D20
Here \( x = r \cos \theta \), \( y = r \sin \theta \cos \phi \) and \( z = r \sin \theta \sin \phi \). \( P^m_n(x) \) are associated Legendre functions of the first kind, \( n^{th} \) degree and order \( m \).

For \( H \) entire, we define

\[
M(r) \equiv M(r, H) = \max_{\theta, \phi} H(r, \theta, \phi).
\]

2 – Definitions and notations

Following the usual definitions of \((p,q)\)-order, lower \((p,q)\)-order, \((p, q)\)-type and lower \((p, q)\)-type of an entire function of a complex variable \( z \), we define:

**Definition 1.** The \((p, q)\)-order \( \rho(H) \) of \( H \) is defined as

\[
\rho(H) = \limsup_{r \to \infty} \frac{\log[p] M(r, H)}{\log[q] r}
\]

where \( p \) and \( q \) are integers such that \( p \geq q \geq 1 \), \( \log^{[0]} x = x \), \( \log^{[m]} x = \log(\log^{[m-1]} x) \) for \( 0 < \log^{[m-1]} x < \infty \) and \( m = 1, 2, 3, \ldots \). Moreover \( 0 \leq \rho(H) \leq \infty \) if \( p > q \) and \( 1 \leq \rho(H) \leq \infty \) if \( p = q \).

**Definition 2.** An entire harmonic function \( H \) of index-pair \((p, q)\) is said to be of lower \((p, q)\)-order \( \lambda(H) \) if

\[
\liminf_{r \to \infty} \frac{\log[p] M(r, H)}{\log[q] r} = \lambda(H).
\]

**Definition 3.** An entire harmonic function \( H \) having \((p, q)\)-order \( \rho(H)(b < \rho(H) < \infty) \) is said to be of \((p, q)\)-type \( T(H) \) and lower \((p, q)\)-type \( t(H) \) if

\[
\limsup_{r \to \infty} \frac{\log^{[p-1]} M(r, H)}{(\log^{[q-1]} r)^{\rho(H)}} = \frac{T(H)}{t(H)}
\]

where \( b = 1 \) if \( p = q \) and \( b = 0 \) if \( p > q \).

The following notations are frequently used in the sequel:
Notation 1.

\[ P(\alpha) = \begin{cases} 
\alpha & \text{if } p > q \\
1+\alpha & \text{if } p = q = 2 \\
1 & \text{if } 3 \leq p = q < \infty \\
\infty & \text{if } p = q = \infty 
\end{cases} \]

where \( 0 \leq \alpha \leq \infty \).

Notation 2.

\[ M(F) = \begin{cases} 
(\rho(F) - 1)(\rho(F)^{-1})/\rho(F)^{\rho(F)} & \text{if } (p, q) = (2, 2) \\
1/(e\rho(F)) & \text{if } (p, q) = (2, 1) \\
1 & \text{if } p \geq 3 .
\end{cases} \]

Notation 3.

\[ X = \begin{cases} 
1/\rho & \text{if } p = 2 \\
1 & \text{if } p \geq 3 .
\end{cases} \]

Notation 4.

\[ A = \begin{cases} 
1 & \text{if } (p, q) = (2, 2) \\
0 & \text{if } (p, q) \neq (2, 2) .
\end{cases} \]

To avoid unnecessary repetition, we shall denote throughout the paper:

\[ \lim_{r \to \infty} \sup_{n} \inf \frac{\log^{p-1} n}{\log^{q-1}(n^{-1} \log |a_n|^{-1})} = L(F) \psi(F) , \]

\[ \lim_{r \to \infty} \sup_{n} \inf \frac{\log^{p-2} n}{\log^{q-1}(a_n/a_{n+1})^{\rho(F)-A}} = Q(F) R(F) , \]

\[ \lim_{r \to \infty} \sup_{n} \inf \frac{\log^{p-2} n}{(\log^{q-1}(a_n|^{-1/n})^{\rho(F)-A}} = V(F) v(F) , \]

and

\[ \lim_{r \to \infty} \sup_{n} \inf \frac{\log^{p-1} n}{\log^{q}(|a_n/a_{n+1}|)} = L^*(F) \psi^*(F) , \]

for the entire function \( F(z) = \sum_{n=0}^{\infty} a_n z^n \).
In this paper we define \((p, q)\)-order, \((p, q)\)-type, lower \((p, q)\)-order and lower \((p, q)\)-type of an entire harmonic function \(H\) and obtain various characterizations in terms of \(\{\alpha_n\}\) defined in equation (3.2). We also obtain alternative characterizations for \((p, q)\)-order, \((p, q)\)-type, lower \((p, q)\)-order and lower \((p, q)\)-type in terms of the ratio \((\alpha_n/\alpha_{n+1})\). Some results obtained earlier by Srivastava become the particular cases of the results obtained in this paper.

3 – Known results

We have the following lemmas:

**Lemma 1** (Fryant, [2]). The harmonic function \(H(r, \theta, \varphi)\) having expansion (1.1) is entire if and only if

\[
\lim_{n \to \infty} (\alpha_n)^{1/n} = 0
\]

where \(\alpha_n\) is given by

\[
\alpha_n = \max_{m,i} \left\{ \frac{(n+m)!}{(n-m)!} \right\}^{1/2} |a_{nm}^{(i)}|.
\]

**Lemma 2** (Fryant, [2]). If \(H\) is entire harmonic function then for all \(r > 0\) and for all \(n\),

\[
(2(1 + 2n)^{1/2})^{-1} \alpha_n r^n \leq M(r, H) \leq 2 \sum_{n=0}^{\infty} (1 + n^{-1/2})^n \alpha_n r^n.
\]

Now, we define

\[
f(z) = \sum_{n=0}^{\infty} (1 + n^{-1/2})^n \alpha_n z^n = \sum_{n=0}^{\infty} \beta_n z^n
\]

and

\[
g(z) = \sum_{n=0}^{\infty} (1 + 2n)^{-1/2} \alpha_n z^n = \sum_{n=0}^{\infty} \gamma_n z^n
\]

where \(\alpha_n\) is given by (3.2).
On using (3.1), we obtain
\[ \lim_{n \to \infty} \left\{ (1 + n^{-1/2})^n \alpha_n \right\}^{1/n} = \lim_{n \to \infty} \left\{ (1 + 2n)^{-1/2} \alpha_n \right\}^{1/n} = \lim_{n \to \infty} \alpha_n^{1/n} = 0. \]

Hence we get the following:

**Lemma 3.** If \( \有待研究 \) is entire harmonic function then \( f \) and \( g \) are also entire functions of complex variable \( z \). Further, if \( m(r, g) = \max_n \{ (1 + 2n)^{-1/2} \alpha_n r^n \} \) and \( M(r, f) = \max_{|z| \leq r} |f(z)| \) then from (3.3), we get

\[ (3.6) \quad \frac{m(r, g)}{2} \leq M(r, H) \leq 2M(r, f). \]

**Lemma 4.** If \( \alpha_n/\alpha_{n+1} \) forms a non-decreasing function of \( n \) for \( n > n_0 \) then \( \beta_n/\beta_{n+1} \) and \( \gamma_n/\gamma_{n+1} \) also form a non-decreasing function of \( n \) for \( n > n_0 \), where \( \beta_n = (1 + n^{-1/2})^n \alpha_n \) and \( \gamma_n = (1 + 2n)^{-1/2} \alpha_n \).

**Proof.** We have
\[ \frac{\beta_n}{\beta_{n+1}} = \frac{(1 + n^{-1/2})^n \alpha_n}{\{1 + (n + 1)^{-1/2}\}^{n+1} \alpha_{n+1}}. \]

Let
\[ \eta(x) = \frac{(1 + x^{-1/2})^x}{\{1 + (x + 1)^{-1/2}\}^{x+1}} = \frac{G(x)}{G(x + 1)} \]
then we can easily prove that
\[ \frac{d}{dx} \left[ \log G(x) \right] > \frac{d}{dx} \left[ \log G(x + 1) \right] \]
so, \( \frac{d\eta(x)}{dx} > 0 \) for \( x > 0 \). Thus \( \beta_n/\beta_{n+1} \) is a non-decreasing function of \( n \) for \( n > n_0 \) if \( \alpha_n/\alpha_{n+1} \) is a non-decreasing function of \( n \) for \( n > n_0 \). Similarly, we can prove for \( \gamma_n/\gamma_{n+1} \).

**Lemma 5.** Let \( f(z) = \sum_{n=0}^\infty (1 + n^{-1/2})^n \alpha_n z^n \) and \( g(z) = \sum_{n=0}^\infty (1 + 2n)^{-1/2} \alpha_n z^n \) be two entire functions. Then for \( p \geq 2 \), \( q \geq 1 \), \( p \geq q \), \( (p,q) \)-orders and \( (p,q) \)-types of \( f(z) \) and \( g(z) \) are equal.
Proof. Let \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of \((p, q)\)-order \( \rho(F) \) and \((p, q)\)-type \( T(F) \). Then for \( p \geq 2, q \geq 1, p \geq q \), we have [4]

\[
\rho(F) = P(L(F))
\]

and

\[
T(F) = M(F)V(F).
\]

Now, for the function \( f(z) \), we have

\[
L(f) = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{n^{-1} \log \{ (1 + n^{-1/2}) n^\alpha \}^{-1} \}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{n^{-1} \log \alpha_n^{-1} - \log(1 + n^{-1/2}) \}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{ (n^{-1} \log \alpha_n^{-1}) \left(1 - \frac{\log(1 + n^{-1/2})}{\log(\alpha_n^{-1})} \right) \}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{n^{-1} \log \alpha_n^{-1} (1 - o(1)) \}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{ (n^{-1} \log \alpha_n^{-1}) \}}.
\]

Similarly, for the function \( g(z) \), we have

\[
L(g) = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{n^{-1} \log \{ (1 + 2n)^{-1/2} \alpha_n \}^{-1} \}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{ (n^{-1} \log \alpha_n^{-1}) \left(1 - \frac{\log(1 + 2n)^{-1/2}}{\log(\alpha_n^{-1})} \right) \}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{ (n^{-1} \log \alpha_n^{-1}) \}}.
\]
(3.7), (3.9) and (3.10) give \( \rho(f) = \rho(g) \). Since \( f \) and \( g \) have same order, using (2.10) and (3.8) we can easily prove that

\[
(3.11) \quad T(f) = T(g)
\]

This completes the proof of Lemma 5.

4 – Theorems

In this section we shall prove the following theorems for \( p \geq 2 \), \( q \geq 1 \) and \( p \geq q \):

**Theorem 1.** Let \( H \) be an entire harmonic function of \((p, q)\)-order \( \rho(H) \), lower \((p, q)\)-order \( \lambda(H) \), \((p, q)\)-type \( T(H) \) and lower \((p, q)\)-type \( t(H) \). If \( f \) and \( g \) are two entire functions defined by (3.4) and (3.5) respectively, then

\[
(4.1) \quad \rho(f) = \rho(g) = \rho(H) \\
(4.2) \quad T(f) = T(g) = T(H) \\
(4.3) \quad \lambda(g) \leq \lambda(H) \leq \lambda(f) \\
(4.4) \quad t(g) \leq t(H) \leq t(f).
\]

**Proof.** From (3.6), we have

\[
\limsup_{r \to \infty} (\inf) \frac{\log^{[p]} m(r, g)}{\log^{[q]} r} \leq \limsup_{r \to \infty} (\inf) \frac{\log^{[p]} M(r, H)}{\log^{[q]} r} \leq \limsup_{r \to \infty} (\inf) \frac{\log^{[p]} M(r, f)}{\log^{[q]} r}.
\]

Also for an entire function \( \xi \), we have \([1]\)

\[
\log M(r, \xi) \approx \log m(r, \xi) \quad \text{as} \quad r \to \infty.
\]

Hence,

\[
\rho(g) \leq \rho(H) \leq \rho(f)
\]
and
\[ \lambda(g) \leq \lambda(H) \leq \lambda(f). \]

Since \( \rho(g) = \rho(f) \), we thus obtain (4.1) and (4.3).

Let
\[ \rho(g) = \rho(H) = \rho(f) = \rho \]

then from (3.6), we have
\[ (4.5) \]
\[ \limsup_{r \to \infty} \frac{\log^{[p-1]} m(r, g)}{(\log^{[q-1]} r)^\rho} \leq \limsup_{r \to \infty} \frac{\log^{[p-1]} M(r, H)}{(\log^{[q-1]} r)^\rho} \leq \limsup_{r \to \infty} \frac{\log^{[p-1]} M(r, f)}{(\log^{[q-1]} r)^\rho}. \]

(3.11) and (4.5) give (4.2). (4.4) follows in a similar manner.

**Theorem 2.** Let \( H \) be an entire harmonic function of \((p, q)\)-order \( \rho(H)(b < \rho(H) < \infty) \), \((p, q)\)-type \( T(H) \) and lower \((p, q)\)-type \( t(H) \), then
\[ (4.6) \]
\[ XU \leq t(H) \leq T(H) \leq XW \]

where
\[ U = \liminf_{n \to \infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]} (\alpha_n/\alpha_{n+1})\}^{\rho(H)-A}}, \]
\[ W = \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{\{\log^{[q-1]} (\alpha_n/\alpha_{n+1})\}^{\rho(H)-A}} \]

and \( b = 1 \) if \( p = q \), \( b = 0 \) if \( p > q \). \( X \) and \( A \) are given by (2.6) and (2.7), respectively.

**Proof.** If \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) is entire function of \((p, q)\)-order \( \rho(F)(b < \rho(F) < \infty) \), \((p, q)\)-type \( T(F) \) and lower \((p, q)\)-type \( t(F) \), then, we have [5]
\[ (4.7) \]
\[ XR(F) \leq t(F) \leq T(F) \leq XQ(F). \]

For the entire function \( f(z) = \sum (1 + n^{-1/2})^n \alpha_n z^n \) we obtain
\[ Q(f) = \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{\left[\frac{\log^{[q-1]} \left\{\frac{(1 + n^{-1/2})^n \alpha_n}{(1 + (n + 1)^{-1/2})^n + 1 \alpha_{n+1}}\right\}}{\log^{[q-1]} (\alpha_n/\alpha_{n+1})}\right]^{\rho(F)-A}. \]
Now, \( \{ (1 + n^{-1/2})^n / (1 + (n+1)^{-1/2})^{n+1} \} \) → 1 as \( n \to \infty \) and \( \rho(f) = \rho(H) \) by (4.1), so
\[
Q(f) = \limsup_{n \to \infty} \frac{\log[p^{-2}]n}{\{\log[q^{-1}](\alpha_n/\alpha_{n+1})\}^{\rho(H)-A}} = W.
\]
Since \( T(f) = T(H) \), we get
\[
(4.8) \quad T(H) \leq XW.
\]
Similarly, for the entire function \( g(z) = \sum_{n=0}^{\infty} (1 + 2n)^{-1/2} \alpha_n z^n \) we find
\[
R(g) = \liminf_{n \to \infty} \frac{\log[p^{-2}]n}{\left[\log[q^{-1}] \left\{ (1 + 2n)^{-1/2} \frac{\alpha_n}{(1 + 2(n+1))^{-1/2} \alpha_{n+1}} \right\}^{\rho(g)-A} \right]}
\]
\[
= \liminf_{n \to \infty} \frac{\log[p^{-2}]n}{\{\log[q^{-1}](\alpha_n/\alpha_{n+1})\}^{\rho(g)-A}} = U.
\]
Using (4.4) and (4.7), we find
\[
XU \leq t(H).
\]
This completes the proof of Theorem 2.

**Theorem 3.** Let \( H \) be an entire harmonic function of \( (p, q) \)-order \( \rho(H)(b < \rho(H) < \infty) \) and \( (p, q) \)-type \( T(H) \). Further, if \( \alpha_n/\alpha_{n+1} \) forms a non-decreasing function of \( n \) for \( n > n_0 \), then
\[
(4.9) \quad T(H) \leq XW \leq XT(H)/M(H)
\]
and
\[
(4.10) \quad \rho(H) = P(E)
\]
where \( X \) and \( W \) are as in Theorem 2. \( P(E) \) and \( M(H) \) are given by (2.4) and (2.5) respectively and
\[
(4.11) \quad E = \limsup_{n \to \infty} \frac{\log[p^{-1}]n}{\log[q](\alpha_n/\alpha_{n+1})},
\]
Proof. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of $(p, q)$-order $\rho(F)(b < \rho(F) < \infty)$ and $(p, q)$-type $T(F)$. If $|a_n/a_{n+1}|$ forms a non-decreasing function of $n$ for $n > n_0$, then we know [5] that

\begin{equation}
T(F) \leq XQ(F) \leq XT(F)/M(F) \tag{4.12}
\end{equation}

and

\begin{equation}
\rho(F) = P(L^*(F)) \tag{4.13}
\end{equation}

Applying (4.12) and (4.13) to $f(z) = \sum_{n=0}^{\infty} \beta_n z^n$, we obtain

\[ T(f) \leq XQ(f) \leq XT(f)/M(f) \quad \text{and} \quad \rho(f) = P(L^*(f)) \]

but $\rho(f) = \rho(H)$, $T(f) = T(H)$ and $M(f) = M(H)$, so

\begin{equation}
T(H) \leq XQ(f) \leq XT(H)/M(H) \tag{4.14}
\end{equation}

and

\begin{equation}
\rho(H) = P(L^*(f)) \tag{4.15}
\end{equation}

Now, we have

\begin{equation}
Q(f) = W. \tag{4.16}
\end{equation}

Also,

\begin{equation}
L^*(f) = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} \left\{ \frac{(1 + n^{-1/2})^n \alpha_n}{(1 + (n + 1)^{-1/2})^{n+1} \alpha_{n+1}} \right\}} = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} \left( \frac{\alpha_n}{\alpha_{n+1}} \right)} = E. \tag{4.17}
\end{equation}

Using (4.16) in (4.14) and (4.17) in (4.15), we find the desired results.
We now state a theorem concerning lower \((p, q)\)-order and lower \((p, q)\)-type of an entire harmonic function \(H\):

**Theorem 4.** Let \(H\) be an entire harmonic function of \((p, q)\)-order \(\rho(H)(b < \rho(H) < \infty)\), lower \((p, q)\)-order \(\lambda(H)\) and lower \((p, q)\)-type \(t(H)\). If \(\alpha_n/\alpha_{n+1}\) forms a non-decreasing function of \(n\) for \(n > n_0\), then

\[
\lambda(H) = P(m) = P(m^*)
\]

and

\[
t(H) = M(H) \liminf_{n \to \infty} \frac{\log \frac{p-2}{n}}{\log \frac{\log \left[ \frac{q-1}{\log \alpha_n^{-1}} \right]}{\rho(H) - A}}
\]

where

\[m = \liminf_{n \to \infty} \frac{\log \frac{p-1}{n}}{\log \frac{\log \left[ \frac{q-1}{n^{-1} \log \alpha_n^{-1}} \right]}{\rho(H) - A}}\]

\[m^* = \liminf_{n \to \infty} \frac{\log \frac{p-1}{n}}{\log \frac{\log \left[ \frac{q-1}{\log \alpha_n/\alpha_{n+1}} \right]}{\rho(H) - A}}\]

\(P(m), M(H)\) and \(A\) are defined by (2.4), (2.5) and (2.7), respectively.

**Proof.** Let \(F(z) = \sum_{n=0}^{\infty} a_n z^n\) be an entire function of \((p, q)\)-order \(\rho(F)(b < \rho(F) < \infty)\), lower \((p, q)\)-order \(\lambda(F)\) and lower \((p, q)\)-type \(t(F)\) such that \(|a_n/a_{n+1}|\) forms as non-decreasing function of \(n\) for \(n > n_0\), then we have [4]

\[
\lambda(F) = P(\psi(F)) = P(\psi^*(F))
\]

and [5]

\[
t(F) = M(F) v(F).
\]

Applying (4.20) and Lemma 4 to the entire function \(f(z) = \sum_{n=0}^{\infty} (1+n^{-1/2})^n\), we find

\[
\lambda(f) = P(\psi(f)) = P(\psi^*(f)).
\]
Now,

\[
\psi(f) = \liminf_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{n^{-1} \log((1 + n^{-1/2})^n \alpha_n)^{-1}\}} = \]
\[
= \liminf_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} \{n^{-1} \log \alpha_n^{-1}(1 - o(1))\}} = \]
\[
= \liminf_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} (n^{-1} \log \alpha_n^{-1})} = m
\]

and

\[
\psi^*(f) = \liminf_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} \left\{\frac{(1 + n^{-1/2})^n \alpha_n}{(1 + (n + 1)^{-1/2})^{n+1} \alpha_{n+1}}\right\}} = \]
\[
= \liminf_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} (\alpha_n/\alpha_{n+1})} = \]
\[
= m^*.
\]

Similarly, from Lemma 4 and (4.20), for the entire function \( g(z) = \sum_{n=0}^{\infty} (1 + 2n)^{-1/2} \alpha_n z^n \), we have

\[
\lambda(g) = P(\psi(g)) = P(\psi^*(g)).
\]

Also, we can easily prove that

\[
\psi(g) = m \quad \text{and} \quad \psi^*(g) = m^*.
\]

The result (4.18) now follows in view of (4.3) and (4.22) to (4.26).

To prove (4.19) we apply Lemma 4 and (4.21) to the entire function \( f(z) = \sum_{n=0}^{\infty} \beta_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} \gamma_n z^n \) and we obtain

\[
t(f) = M(f)v(f) \quad \text{and} \quad t(g) = M(g)v(g).
\]

Now,

\[
M(g) = M(f) = M(H)
\]
because $\rho(g) = \rho(f) = \rho(H)$ and we can easily show that

\[(4.29) \quad v(g) = v(f) = \lim_{n \to \infty} \inf \frac{\log^{[p-2]} n}{\log^{[q-1]} \left(\frac{1}{n}\right) \rho(H) - A}.
\]

(4.4), (4.27), (4.28) and (4.29) together establish (4.19) and the proof of the theorem is complete.

**Remark.** For $p = 2$, $q = 1$, the results in Theorem 1, Theorem 2 and Theorem 4 are due to Srivastava [6].

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*Lavoro pervenuto alla redazione il 13 maggio 2003
ed accettato per la pubblicazione il 11 novembre 2003.
Bozze licenziate il 29 gennaio 2004*

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