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Time-stepping methods for Volterra-Fredholm integral equations

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Abstract: *The semidiscretization in space of Volterra-Fredholm integral equations (arising, for example, as mathematical models of the spreading of epidemics) leads to large systems of Volterra integral equations. Here, we study inexpensive time-stepping methods using certain DQ (= direct quadrature) methods which are employed in a way that exploits the local superconvergence properties of spatial collocation. The performance of these methods is then compared with that of time-stepping based on collocation methods.*

1 – Introduction

In this paper we investigate time-stepping methods for spatially semi-discretized versions of linear and nonlinear Volterra-Fredholm integral equations of the form

$$u(t,x) = f(t,x) + \int_0^t \int_\Omega G(t,s,x,\xi,u(s,\xi))d\xi\, ds\,, \tag{1.1}$$

$t \in I := [0,T],\ x \in \Omega$, where $\Omega \subset \mathbf{R}^d$ $(d = 1, 2)$ is open and bounded. In

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the linear case we write

$$G(t,s,x,\xi,u) = H(t,s,x,\xi)u\,. \tag{1.2}$$

Equations of type (1.1), (1.2) often arise from the mathematical modeling of the spreading, in space and time, of some contagious disease in a population living in a habitat Ω ([3], [13]), in the theory of nonlinear parabolic boundary value problems ([12]), and in many physical and biological models.

In the following we assume that the given real-valued functions $f = f(t,x)$ and $G = G(t,s,x,\xi,u)$ are at least continuous on $D = [0,T] \times \Omega$ and $S \times \mathbf{R}$ (where $S := \{(t,s,x,\xi) : 0 \le s \le t \le T; (x,\xi) \in \Omega \times \Omega\}$). Existence and uniqueness results for (1.1) and (1.2) may be found in [3], [13], [12].

The literature on numerical methods for solving equations (1.1) and (1.2) mainly consists of the papers [2], [9], [7], [8], [14]. The great cost for solving the double discretization has always been an important problem the authors had to face.

The main goal in numerical solving equations (1.1) and (1.2) is to suitably combine discretization in space and in time in order to generate high order approximations at a reasonable computational cost. In [8] for the linear case and in [2] in the general case the numerical solution of integral equations of Volterra-Fredholm type is carried out by continuous-time and discrete-time spline collocation methods with the aim to produce high-order methods. It was indicated already in [2, p. 999] that time-stepping based on high-order collocation techniques is often too expensive, and hence suitable linear multistep methods may be more feasible.

In this paper we investigate time-stepping methods for solving the (large) systems of second-kind Volterra integral equations (VIEs) resulting from the spatial semidiscretization of (1.1) by collocation. These time-stepping methods are based on Direct Quadrature (DQ) methods (see, e.g., [1]). Moreover, the integration is advanced along the spatial collocation lines in order to exploit the local superconvergence for certain choices of the collocation parameters.

In Section 2 we give an illustration of the system of nonlinear Volterra integral equations (VIEs) which arises in the semi-discretization in space

by collocation techniques as described in [2]. In Section 3 we describe the DQ methods used in this paper to solve these systems. Section 4 is devoted to the analysis of the discretization error in space and time and to the study of the overall order of convergence. Section 5 contains numerical results for two examples taken from the literature, and in Section 6 some concluding remarks and open problems are reported.

2 – Spatial semidiscretization

For ease of exposition we shall restrict our description to the one dimensional spatial domain $\Omega = [0, b] \subset \mathbf{R}$. The extension to the spatial domains $\Omega \subset \mathbf{R}^2$ does not present essential difficulties (except that for Ω with irregular geometry it may be more advantageous to employ spatial semidiscretization based on finite element Galerkin or on discontinuous Galerkin methods).

Let $\Pi_M : 0 = x_0 < x_1 < \ldots < x_M = b$ be a partition of Ω with diameter $h := x_{m+1} - x_m,\ m = 0, \ldots, M-1$. The semidiscretization in space, as described in [2], consists in approximating the solution $u(t, x)$ of (1.1) by a function $U_h(t, x) \in [0, T] \times S_{p-1}^{-1}(\Pi_M)$, where $S_{p-1}^{-1}(\Pi_M) := \{\nu : \nu_{|(x_m, x_{m+1}]} \in \pi_{p-1}, 0 \le m \le M-1\}$ is the real polynomial spline space of degree $p-1$ and continuity class -1. $U_h(t, x)$ is determined by solving the Volterra-Fredholm integral equation (1.1) on the set of Mp collocation points $X(M) := \{x_{mi} := x_m + c_i h,\ 1 \le i \le p,\ 0 \le m \le M-1\}$, where $0 \le c_1 < c_2 < \ldots < c_p \le 1$ are the collocation parameters. Hence $U_h(t, x)$ is such that

$$U_h(t, x_{mi}) = f(t, x_{mi}) + \int_0^t \int_0^b G(t, s, x_{mi}, \xi, U_h(s, \xi)) d\xi\, ds\,, \tag{2.1}$$

$m = 0, \ldots, M-1,\ i = 1, \ldots, p$. A suitable change of variables in the spatial integral leads to

$$\begin{aligned} U_{h,mi}(t) &= f_{h,mi}(t) + \\ &\quad + h \int_0^t \sum_{l=0}^{M-1} \int_0^1 G(t, s, x_{mi}, x_l + \xi h, U_h(s, x_l + \xi h)) d\xi\, ds, \end{aligned} \tag{2.2}$$

where $m = 0, \ldots, M-1,\ i = 1, \ldots, p$. We set

$$U_{h,mi}(t) := U_h(t, x_{mi}) \quad \text{and} \quad f_{h,mi}(t) := f(t, x_{mi})$$

for $m = 0, \dots, M-1,\ i = 1 \dots p$. Equation (2.2) represents a system of Mp nonlinear VIEs of the second kind.

It is possible to show ([2]) that the convergence behavior on $[0,T] \times \Omega$ of the collocation solution $U_h(t,x)$ is described by

$$\sup\{|e(t,x)| : (t,x) \in [0,T] \times \Omega\} = O(h^p)\,,$$

where $e(t,x) = u(t,x) - U_h(t,x)$ represent the collocation error. However, if $\{c_i,\ i = 1, \dots, p\}$ are the p Gauss(-Legendre) points for $(0,1)$, the order of convergence along the collocation lines becomes higher, more precisely

$$\max\{|e(t,x_{mi})| : t \in [0,T],\ m = 0, \dots, M-1,\ i = 1, \dots, p\} = O(h^{2p})\,.$$

REMARK. In general, we need quadrature formulas to approximate

$$\int_0^1 G(t,s,x_{mi}, x_l + \xi h, U_{h,l}(s, x_l + \xi h))d\xi;\ l = 0, \dots, M-1\,.$$

In practice, interpolatory quadrature formulas based on the collocation points $\{x_l + c_k h,\ k = 1, \dots, p\}$ are used. This choice produces fully discretized approximation methods that do not lower the optimal orders of global convergence (and local superconvergence) of the "exact" schemes described above.

By replacing each of the above integrals by the corresponding $p-$point quadrature formula based on the abscissas $\{x_{lk}\}$ and on the weights γ_k we are led to

$$\hat{U}_{h,mi}(t) = f_{h,mi}(t) + h \int_0^t \sum_{l=0}^{M-1} \sum_{k=1}^{p} \gamma_k G(t,s,x_{mi}, x_{lk}, \hat{U}_{h,lk}(s))ds\,, \tag{2.3}$$

where $m = 0, \dots, M-1,\ i = 1, \dots, p$. Equivalently, (2.3) can be written in the compact form

$$\hat{U}_h(t) = f_h(t) + h \int_0^t \mathcal{G}(t,s,\hat{U}_{h,lk}(s)) \cdot \underline{e} ds\,, \tag{2.4}$$

where $\underline{e} := (1, \dots, 1)^T$,

$$\hat{U}_h(t) = (\hat{U}_{h,01}(t), \dots, \hat{U}_{h,M-1p}(t))^T$$

and $\mathcal{G}(t,s,\hat{U}_h(s))$ is the following $M\cdot p\times M\cdot p$ matrix

$$\begin{pmatrix} \gamma_1 G(t,s,x_{01},x_{01},\hat{U}_{h,01}(s)) & \dots & \gamma_p G(t,s,x_{01},x_{M-1p},\hat{U}_{h,M-1p}(s)) \\ \vdots & \ddots & \vdots \\ \gamma_1 G(t,s,x_{M-1p},x_{01},\hat{U}_{h,01}(s)) & \dots & \gamma_p G(t,s,x_{M-1p},x_{M-1p},\hat{U}_{h,M-1p}(s)) \end{pmatrix}$$

3 – Time-stepping

The collocation equation (2.3) represents a system of Mp nonlinear Volterra integral equations for the unknown continuous functions

$$\hat{U}_{h,0},\dots,\hat{U}_{h,(M-1)\cdot p}\,,$$

where, for ease of notation we identify $\hat{U}_{h,mp+i}$ with $\hat{U}_{h,mi}$. Let

$$\Pi_N := \{t_n:\ 0=t_0<t_1<\ldots<t_N=b\}$$

be a partition of the time interval $[0,T]$ with stepsize $\tau=t_{n+1}-t_n,\ n=0,\dots,N-1$. The discretization in time of the semidiscrete problem (2.6) can in principle be realized by any numerical method for approximating the solution of a system of VIEs.

In [2], [8], the system (2.3) is solved by discrete time collocation in $S^{-1}_{q-1}(\Pi_N)$ at the points $t_n+d_i\tau,\quad i=1,\dots,q$. A drawback of this approach is the high cost for archieving high-order of convergence (on a given time interval $[t_n,t_{n+1}]$, we have to solve a system of qMp nonlinear algebraic equations as opposed to a system of Mp nonlinear Volterra integral equations in the case (2.3) of continuous time collocation).

Here we use VLM (Volterra Linear Multistep) methods for the time integration process. In order to exploit the local superconvergence properties in the Gauss(-Legendre) spatial collocation points and to obtain high-order approximations for $t=t_n$, we advance the solution along these spatial lines. Numerical methods based on this approach will require considerably less computational effort. In this section one of the most commonly used VLM methods is first described in the scalar case and then applied to the system (2.3).

3.1 – Direct Quadrature methods

Consider first the general scalar nonlinear second kind Volterra integral equation

$$y(t) = g(t) + \int_0^t K(t, s, y(s))ds, \quad t \in I\,. \tag{3.1}$$

A simple way of solving (3.1) is to write down the equation for a set of equidistant points $t_n = n\tau$, $n = 1, \dots, N$, and to approximate the integral terms by appropriate quadrature formulas. This procedure yields the general DQ (Direct Quadrature) method

$$y_n = g(t_n) + \tau \sum_{j=0}^{n} w_{n,j} K(t_n, t_j, y_j), \quad n \geq k\,, \tag{3.2}$$

where $y_0 = g(0)$, $y_1, \dots y_{k-1}$ are given starting values and $\{w_{n,j},\ n = 0, \dots, N;\ j = 0, \dots, N\}$, are the weights of the DQ method.

DQ methods which are constructed by means of linear multistep (LM) methods for ordinary differential equations are known as $(\rho, \sigma)-$reducible DQ methods (see [15] for a detailed description and analysis) and are very attractive because of their easy and inexpensive implementation.

The most commonly used DQ methods are based either on Adams Moulton (DQ(AM)) or on Backward Differentiation (DQ(BD)) formulae. In general, DQ(BD) methods show better stability properties than DQ(AM) methods but, with the exception of the two step case, their weigths are not known in an explicit form and the numerical derivation gives rise to rather complicated implementation. For DQ(AM) methods however, the weights can be given explicitly and the methods, under the same order conditions, turn out to be more accurate because of their smaller error constants.

Application to the system (2.4) yields

$$\hat{U}_h^n = f_h(t_n) + h\tau \sum_{j=0}^{n} w_{n,j} \mathcal{G}(t_n, t_j, \hat{U}_h^j)e, \quad n \geq 2\,, \tag{3.3}$$

where $\hat{U}_h^n = (\hat{U}_{h,0}^n, \hat{U}_{h,2}^n, \dots, \hat{U}_{h,(M-1)\cdot p}^n)^T \approx \hat{U}_h(t_n)$, or equivalently

$$\hat{U}_{h,mi}^n = f_{h,mi}(t_n) + h\tau \sum_{j=0}^{n} w_{n,j} \sum_{l=0}^{M-1} \sum_{k=0}^{p} \gamma_k G(t_n, t_j, x_{mi}, x_{lk}, \hat{U}_{h,lk}^j)$$
$$m = 0, \dots, M-1;\ i = 1, \dots, p \quad (n \geq 2)\,.$$

The convergence properties of DQ methods are summarized in the following theorem ([15].)

THEOREM 3.1. *If the starting formulas and the (ρ, σ)-formulas are both based on reducible quadrature rules of order r, and if the errors in the starting values $y_0, \dots, y_{k-1}$ are of order s, then the method* (3.2) *is convergent of order r^*, where $r^* = \min\{r, s+1\}$.*

EXAMPLE 3.1. The reducible quadrature method DQ(BD2) generated by the 2−step backward differentiation formula and having and the trapezoidal rule as the starting formula is characterized by the following weights:

$$\begin{aligned} w_{n,j} &= \frac{3}{4}\Big(1 - \Big(\frac{1}{3}\Big)^n\Big), \quad n \geq 1,\ j = 0, 1 \\ w_{n,j} &= 1 - \Big(\frac{1}{3}\Big)^{n-j+1}, \quad n - j \geq 0,\ j \geq 2\,. \end{aligned} \tag{3.4}$$

EXAMPLE 3.2. The reducible quadrature method DQ(AM2) generated by the 2−step Adams Moulton and using and the trapezoidal rule as the starting formula is characterized by the weights

$$(w_{nj}) = \frac{1}{12}\begin{pmatrix} 5 & 14 & 5 & & & & & \\ 5 & 13 & 13 & 5 & & & & \\ 5 & 13 & 12 & 13 & 5 & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & & \\ 5 & 13 & 12 & \cdots & \cdots & 12 & 13 & 5 \end{pmatrix}$$

By Theorem 3.1, the order of convergence of the DQ(BD2) and DQ(AM2) methods is, respectively, 2 and 3.

3.2 – Solution of the nonlinear algebraic systems

It is clear from the previous section that time-stepping for solving (2.4) leads to a system of nonlinear algebraic equations for

$$\hat{U}_h^n = (\hat{U}_{h1}^n, \hat{U}_{h2}^n, \dots, \hat{U}_{h(M-1)\cdot p}^n)^T$$

of the form

$$\hat{U}_h^n = h\tau\beta\mathcal{G}(t_n, t_n, \hat{U}_h^n)e + \Psi_n\,, \tag{3.5}$$

where $\Psi_n = \Psi(\hat{U}_h^0, \dots, \hat{U}_h^{n-1})$ is a known vector of previously computed values. In the case of DQ methods

$$\Psi_n = f_h(t_n) + h\tau\sum_{j=0}^{n-1} w_{n,j}\mathcal{G}(t_n, t_j, U_h^j)e\,,$$

and $\beta = w_{n,n}$.

A conventional way of solving this nonlinear system is to apply fixed-point iteration,

$$\hat{U}_h^{n(j+1)} = h\tau\beta\mathcal{G}(t_n, t_n, \hat{U}_h^{n(j)})e + \Psi_n, \quad j = 0, 1, \dots \tag{3.6}$$

However, this *direct* iterative method will converge only if

$$h\tau < \frac{1}{|\beta|L}$$

holds, where L is the Lipschitz constant of $\mathcal{G}$ with respect to $\hat{U}$. Hence, when the Jacobian of $\mathcal{G}$ is "large", the restriction on $h\tau$ can be quite severe and impractical. In this case modified Newton iteration is to be preferred: its application to (3.5) yields

$$(I - h\tau\beta J)(\hat{U}_h^{n(j+1)} - \hat{U}_h^{n(j)}) = -R(\hat{U}_h^{n(j)}), \quad j = 0, 1, \dots\,, \tag{3.7}$$

where, for any $X \in \mathbf{R}^{Mp}$, $R(X) = X - h\tau\beta\mathcal{G}(t_n, X) - \widetilde{\Psi}_n$, J stands for an approximation to the Jacobian of $\mathcal{G}$ evaluated in $\hat{U}_h^{n(0)}$ and I is the $Mp \times Mp$ identity matrix.

In both of these the iteration processes $U_n^{(0)}$ is generated by an appropriate predictor formula (e.g. extrapolation or last point value), and the iteration is applied as many times as needed to obtain convergence of $U_h^{n(j)}$ to the solution of (3.5) within a prescribed tolerance.

4 – Analysis of discretization error

In this section we analize the global error of convergence $E(t,x)$ at the collocation points $\{x_{mi} := x_m + c_i h :\ m = 0, \dots, M-1;\ i = 1, \dots, p\}$.

THEOREM 4.1. *Assume that $\hat{U}_h(t,x)$ is the continuous time collocation approximation in $S_{p-1}^{(-1)}(\Pi_M) \otimes C[0,T]$ and let $\hat{U}_{h,mi}^n$ be the discrete time solution to* (2.3) *in (t_n, x_{mi}) given by a DQ method of order r. Then, for $u \in C^{rp}([0,T] \times \Omega)$, the total error $E(t,x)$ satisfies*

$$|E(t,x)| = \mathcal{O}(h^p) + \mathcal{O}(\tau^r), \quad (t,x) \in \Pi_N \times X(M)\,. \tag{4.1}$$

The following corollary contains the motivation for our time-stepping approach.

COROLLARY 4.2. *Let the set $\{c_i :\ i = 1, \dots, p\}$ consist of the zeros of the shifted Legendre polynomial $P_p(2s-1)$ (Gauss points in $(0,1)$). Then the total error $E(t,x)$ satisfies*

$$|E(t,x)| = \mathcal{O}(h^{2p}) + \mathcal{O}(\tau^r), \quad (t,x) \in \Pi_N \times X(M)\,. \tag{4.2}$$

PROOF OF THEOREM 4.1. The total error at the points $(t_n, x_{mi}) \in \Pi_N \times X(M)$ is defined by

$$E(t_n, x_{mi}) = u(t_n, x_{mi}) - \hat{U}_{h,mi}^n$$

and can be written as

$$\begin{aligned} E(t_n, x_{mi}) &= u(t_n, x_{mi}) - \hat{U}_{h,mi}^n = \\ &= \hat{e}(t_n, x_{mi}) + \hat{U}_{h,mi}(t_n) - \hat{U}_{h,mi}^n\,, \end{aligned}$$

where $\hat{e}(t_n, x_{mi}) = u(t_n, x_{mi}) - \hat{U}_{h,mi}(t_n)$. From the remark in Section 2 it is clear that $\hat{e}(t, x)$ satisfies

$$|\hat{e}(t, x_{mi})| \leq Ch^p \,.$$

For the time-stepping part $\hat{U}_{h,mi}(t_n) - \hat{U}^n_{h,mi}$ of the total error, we have

$$\begin{aligned}\hat{U}_{h,mi}(t_n) - \hat{U}^n_{h,mi} = h \sum_{l=0}^{M-1} \sum_{k=1}^{p} \gamma_k \cdot \Big[\int_0^{t_n} G(t_n, s, x_{mi}, x_{lk}, \hat{U}_{h,lk}(s))ds + \\ - \tau \sum_{j=0}^{n} w_{nj} G(t_n, t_j, x_{mi}, x_{lk}, \hat{U}^j_{h,lk}) \Big] = \\ = h \sum_{l=0}^{M-1} \sum_{k=1}^{p} \gamma_k Q_{lk}\end{aligned}$$

where the terms Q_{lk} $(l = 0, \ldots, M-1;\ k = 1, \ldots, p)$ are the DQ errors. Here, we have used the fact that

$$\begin{aligned}\int_0^{t_n} G(t_n, s, x_{mi}, x_{lk}, \hat{U}_{h,lk}(s))ds = \\ = \tau \sum_{\nu=0}^{n-1} \int_0^1 G(t_n, t_\nu + s\tau, x_{mi}, x_{lk}, \hat{U}_{h,lk}(t_\nu + s\tau))ds \,.\end{aligned}$$

Thus,

$$|E(t_n, x_{mi})| \leq Ch^p + \max |Q_{lk}| h \sum_{l=0}^{M-1} \sum_{k=1}^{p} \gamma_k \tag{4.3}$$

with $\sum_{l=0}^{M-1} \sum_{k=1}^{p} \gamma_k = b < +\infty$.

In order to establish the result of Theorem 4.1, consider first (4.3). According to Theorem 3.1 the DQ errors Q_{lk} satisfy, for all sufficiently smooth integrands,

$$|Q_{lk}| \leq C_{lk} \tau^r$$

for some constant C_{lk}. Thus,

$$|E(t_n, x_{mi})| \leq Ch^p + C' \tau^r \,.$$

This yields the result stated in Theorem 4.1.

PROOF OF COROLLARY 4.2. If $\{c_i : i = 1, \dots, p\}$ is the set of the p Gauss points in $(0, 1)$, then ([2])

$$|\hat{e}(t, x_{mi})| \leq Ch^{2p},$$

thus verifying (4.2).

5 – Numerical illustration

In this section we illustrate the performance of p-Gauss collocation/DQ methods (3.3). In our experiments we choose $r = 2p$ in order to match a convergence behavior like $\mathcal{O}(h^{2p}) + \mathcal{O}(\tau^{2p})$ and we discretize the integrals occurring in the collocation equation (2.2) by interpolatory quadrature formulas based on the p Gauss collocation points. The nonlinear algebraic systems resulting from the time-stepping are solved by modified Newton iteration and constant stepsizes $h = \tau$ are used.

We performed our experiments on two test problems taken from the literature: the linear Volterra-Fredholm integral equation given in [8],

$$u(t, x) = f(t, x) + \int_0^t \int_0^2 (-\cos(x - \xi)) \exp(-(t - s)) u(s, \xi) d\xi ds, \tag{5.1}$$

$t \in [0, 2]$, with $f(t, x)$ such that the exact solution of (5.1) is

$$u(t, x) = \cos(x) \exp(-t);$$

and the nonlinear Volterra-Fredholm integral equation given in [2],

$$u(t, x) = f(t, x) + \int_0^t \int_0^1 G(t, s, x, \xi)(1 - \exp(-u(s, \xi))) d\xi ds, \tag{5.2}$$

$t \in [0, 1]$. Here,

$$G(t, s, x, \xi) = \frac{x(1 - \xi^2)}{(1 + t)(1 + s^2)},$$

and $f(t, x)$ was chosen such that the exact solution of (5.1) is

$$u(t, x) = -\log\left(1 + \frac{xt}{1 + t^2}\right).$$

The tests were carried out for different values of p. The accuracy is defined by the number of correct digits cd at the end point (the maximal absolute end point error is written as 10^{-cd}). The values of cd^* represent the number of correct digits of the discrete-time collocation approximation in the space $S_q^{(-1)}(\Pi_N) \otimes S_p^{(-1)}(\Pi_M)$, at the Gauss points, followed by an iteration as developed in [8] and [2], with $p = q$ and $h = \tau$. This choice produces an $\mathcal{O}(h^{2p})$ convergence and requires the solution of a sequence of nonlinear systems of dimension Mp^2 plus a quadrature process. The method of this paper requires the solution of a sequence of nonlinear systems of dimension Mp. Hence, under the same order $2p$, a factor p is gained at each time-step point.

In Tables 1-4 the two methods are compared in terms of accuracy. The results listed in Tables 1-3 clearly show that our method produces slightly better accuracies with less computational effort (M systems of dimension Mp instead of M systems of dimension Mp^2). In Table 4 the order obtained by (3.3) is less than one for collocation, however the DQ methods are much less expensive to implement, especially when h is small and hence the number of spatial lines large.

TABLE 1: problem (5.1), $p = 1$.

$h = \tau = T/M$	cd	cd^*
0.4	2.8	2.6
0.2	3.4	3.2
0.1	4.0	3.8
0.05	4.6	4.4

TABLE 2: problem (5.1), $p = 3$.

$h = \tau = T/M$	cd	cd^*
0.2	9.8	9.2
0.1	11.7	10.9
0.05	13.5	

TABLE 3: problem (5.2), $p = 1$.

$h = \tau = T/M$	cd	cd^*
0.2	3.3	3.0
0.1	3.8	3.6
0.05	4.4	
0.025	5.0	

TABLE 4: problem (5.2), $p = 3$.

$h = \tau = T/M$	cd	cd^*
0.1	6.6	10.6
0.05	8.1	
0.025	9.8	

6 – Concluding remarks

In this paper direct quadrature (DQ) methods were used for the time-stepping in the large systems of second-kind Volterra integral equations resulting from the semidiscretization in space of linear and nonlinear Volterra-Fredholm integral equations by collocation. In order to exploit the resulting spatial superconvergence, we carried out the time-stepping along the spatial lines corresponding to the Gauss collocation points. This method was compared with the discrete time collocation approach proposed in [2] and [8]. The numerical experiments clearly indicate that time-stepping by collocation is very accurate, but rather expensive, while DQ time-stepping is much less expensive, but possibly not always quite as accurate, e.g. for $p = 3$.

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