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A perturbative method for direct scattering problems

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ABSTRACT: *We present a numerical method to compute the solution of direct scattering problems, that is boundary-value problems for the Helmholtz equation in unbounded domains of the three dimensional real Euclidean space. Such problems arise, for example, from wave equation problems when the solution is assumed to be time-harmonic. We consider the T-matrix method for the solution of the direct scattering problems, which is a very classical numerical method for such a kind of problems. This method is based on the explicit construction of an operator T mapping the data of the problem to the solution of the problem. We propose a perturbative approach for the numerical approximation of the operator T. Finally we report the results of our numerical experience on a large number of test problems using the numerical method proposed here. This numerical experience shows very interesting results and it justifies further theoretical investigations.*

1 – Introduction

Let us begin with some basic definitions. Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of natural numbers, real numbers and complex numbers, respectively. Let $n \in \mathbb{N}$, we denote with $\mathbb{R}^n$, $\mathbb{C}^n$ the n-dimensional real Euclidean space and the n-dimensional complex Euclidean space, respectively. We denote with $(\cdot,\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, with $\|\cdot\|$ the corresponding Euclidean norm. Let $S^n = \{\underline{x} \in \mathbb{R}^{n+1} : \|\underline{x}\| = 1\}$. Let $\imath$ be the imaginary unit. Let $z \in \mathbb{C}$, we denote with $|z|$ the modulus of z and with $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ the real and imaginary part of z respectively.

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Let $D \subset \mathbb{R}^3$ be a bounded simply connected open set with boundary ∂D and let $\overline{D}$ be its closure. We suppose that $\underline{0} \in D$. From the physical point of view we consider D as the position of an obstacle, or equivalently a scatterer, for the acoustic waves propagating in $\mathbb{R}^3 \setminus D$, in particular we suppose this scatterer contained in a homogeneous isotropic medium filling $\mathbb{R}^3 \setminus D$. Moreover for such a medium we suppose a constant pressure field P. Let $U^i(\underline{x}, t)$, $\underline{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$ be an incident acoustic wave, where $\underline{x} \in \mathbb{R}^3$ denotes the space variables and $t \in \mathbb{R}$ denotes the time variable. Let $U^s(\underline{x}, t)$, $\underline{x} \in \mathbb{R}^3 \setminus D$, $t \in \mathbb{R}$ be the scattered acoustic wave generated by the interaction of U^i and the obstacle D. These waves can be considered as perturbations for the pressure field P; when such perturbations are small compared to P we have that U^i and U^s solve the wave equation, see [1], page 243 for details.

We suppose that U^i and U^s are time-harmonic, that is:

$$U^i(\underline{x}, t) = u^i(\underline{x}) e^{i\omega t}, \ \underline{x} \in \mathbb{R}^3, \ t \in \mathbb{R}, \tag{1}$$

$$U^s(\underline{x}, t) = u^s(\underline{x}) e^{i\omega t}, \ \underline{x} \in \mathbb{R}^3 \setminus D, \ t \in \mathbb{R}, \tag{2}$$

where u^i, u^s are suitable functions of the space variables and $\omega > 0$ is the time-frequency.

From the wave equation for U^i, U^s and from formulas (1), (2) we obtain the Helmholtz equation for u^i and u^s, that is

$$\Delta u^i(\underline{x}) + k^2 u^i(\underline{x}) = 0, \ \underline{x} \in \mathbb{R}^3, \tag{3}$$

$$\Delta u^s(\underline{x}) + k^2 u^s(\underline{x}) = 0, \ \underline{x} \in \mathbb{R}^3 \setminus \overline{D}, \tag{4}$$

where Δ is the Laplace operator with respect to the $\underline{x}$ variables, $k = \frac{\omega}{c} > 0$ is the wave number and $c > 0$ is the wave propagation velocity. We assume that D is an impenetrable acoustically soft obstacle, so that u^s satisfies the following boundary conditions:

$$u^s(\underline{x}) = -u^i(\underline{x}), \ \underline{x} \in \partial D, \tag{5}$$

see [2] page 67 for details. We note that impenetrable acoustically hard obstacles satisfy Neumann boundary condition, and obstacles having more complicated acoustic behaviour satisfy a boundary condition that can be given in terms of an acoustic surface impendance. Moreover we assume that the scattered acoustic wave u^s has the asymptotic behaviour of an outgoing spherical wave, so that u^s satisfies the Sommerfeld radiation condition, that is

$$\frac{\partial u^s}{\partial \hat{\underline{x}}}(\underline{x}) - \imath k u^s(\underline{x}) = o\left(\frac{1}{\|\underline{x}\|}\right), \ \|\underline{x}\| \to \infty, \tag{6}$$

where $\hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|} \in S^2$, $\|\underline{x}\| \neq 0$ and $o(\cdot)$ is the Landau symbol, see [3] page 189 for a more detailed discussion on the radiation condition.

Boundary-value problem (4)-(6) is uniquely solvable provided u^i in (5) is a continuous function and D is a class C^2 domain with connected complement, see [2] page 83, [4] pages 13 and 262 for details. Let us consider the following problem: from the knowledge of D, k and u^i compute the solution u^s of problem (4)-(6).

We consider the numerical approximation of such a problem. Many different methods for the solution of problem (4)-(6), or similar scattering problems, have been proposed in the scientific literature, see for example [5], [6], [7], for finite difference approaches or [8], [9], [10], [11] for finite element approaches. We note that these general purpose methods cannot be applied directly to problem (4)-(6) being this problem defined on an unbounded domain. A quite common technique to avoid this difficulty is to consider this problem in a domain $\mathcal{D} \setminus D$, where $\mathcal{D} \subset \mathbb{R}^3$ is a bounded open domain containing $\overline{D}$, and to substitute the Sommerfeld radiation condition with an auxiliary condition on the artificial boundary $\partial\mathcal{D}$. This condition is usually called transparent boundary condition or absorbing boundary condition, see [12] and the references therein. However some specialized numerical methods allow to deal with the unbounded domain of problem (4)-(6), see for example [2], [13], [14], [15], [16], [17], [18], [19], [20], [21] for integral equation approaches and [22], [23], [24], [25], [26], [27] for T-matrix approaches.

We study the T-matrix method which is a very classical method for the solution of scattering problems. This method consists in the construction of an operator $T = T(D, k)$, depending only on D and k, such that:

$$u^s = T\, u^i \tag{7}$$

for every continuous function $u^i : \partial D \to \mathbb{C}$. Usually functions u^i and u^s are expanded with respect to particular bases of functions defined in terms of the spherical harmonics, so that the operator T looks like a matrix with an infinite number of rows and an infinite number of columns. In practical situations we consider only a finite number of entries of T, whose computation foresees the solution of several linear systems where the entries of the coefficient matrix are obtained by the evaluation of several surface integrals on ∂D. We denote with $Q = Q(D, k)$ the matrix coefficient of this linear system. Usually Q is a dense matrix and, depending on D and k, it can be quite ill-conditioned, so that the solution of the corresponding linear system can produce a large error in the final solution. Moreover having in mind an efficient implementation of this method via parallel computations the step of the solution of such a linear system is an unpleasant step since it lowers considerably the parallel efficiency of the whole method.

To avoid a linear system solution in the T-matrix method we propose a perturbative method for the computation of the operator T, where the perturbation is made with respect to the boundary ∂D of the obstacle D. As base point of this perturbation is considered the boundary ∂B of a generic obstacle B; in such a case for the construction of the operator T we have to solve several linear systems where the matrix coefficient is $Q(B, k)$. So that also in the perturbative method we really have to solve some linear systems, but now the matrix coefficient $Q(B, k)$ can be chosen in terms of B. We note that when the base point B is chosen as an axial-symmetric obstacle the matrix $Q(B, k)$, arising in the construction of the operator T, has a particular block-structure; when B is chosen as a sphere the matrix $Q(B, k)$ is a diagonal matrix, so that the solution of the corresponding linear system can be performed accurately, quickly and efficiently in a sequential computation as well as in a parallel computation. However in general we can compute, for example, the LU factorization of the matrix $Q(B, k)$ and we can use this factorization everytime the boundary ∂B of B is used as base point in the perturbative procedure.

Finally we report some of the results of our numerical experience obtained using the numerical method proposed here. We consider a large number of test problems, where we take into account axial-symmetric and non-axial-symmetric obstacles, convex and non-convex obstacles. In the numerical results convergence and stabilization features of the perturbative method proposed are outlined. This numerical experience shown very interesting results, so that we deserve further theoretical investigations to this introductory study.

The paper is organized as follows. In Section 2 we provide a brief introduction to the T-matrix method and we give some useful formulas for the development of the method proposed here. In Section 3 we present the perturbative method. In Section 4 we report some results of our numerical experience using the method presented in the previous section. In Section 5 we give some conclusions and the possible developments of the work.

2 – The T-matrix method

The construction of the operator T is usually given in terms of suitable bases of functions for the expansion of u^i, i.e. the datum of problem (4)-(6), and u^s, i.e. the unknown solution of problem (4)-(6). We denote with:

$$(8)\quad Y^{\sigma}_{l,m}(\underline{\hat{x}}(\theta,\phi)) = \gamma_{l,m}\begin{cases} P^m_l(\cos\theta)\cos(m\phi), & \sigma=0,\ l=0,1,\dots,\ m=0,1,\dots,l, \\ P^m_l(\cos\theta)\sin(m\phi), & \sigma=1,\ l=1,2,\dots,\ m=1,2,\dots,l, \end{cases}$$

the spherical harmonics, where $\underline{\hat{x}}(\theta,\phi) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)^t \in S^2$, $\theta \in [0,\pi]$, $\phi \in [0, 2\pi)$, and for $l = 0, 1, \dots,\ m = 0, 1, \dots, l$, P^m_l are the Legendre functions of order m and degree l and $\gamma_{l,m}$ are normalization coefficients, that is we have:

$$(9)\qquad \int_{S^2} (Y^{\sigma}_{l,m}(\underline{\hat{x}}))^2 ds(\underline{\hat{x}}) = 1,$$

where ds is the surface measure on S^2, see [28] page 331 for details. In the sequel we denote with ν the multi-index (σ, l, m) and we denote with I the set of all possible values of ν given by formula (8), i.e. $I = \{\nu = (\sigma, l, m)\colon \sigma = 0, 1, l = \sigma, \sigma+1, \ldots, m = \sigma, \sigma+1, \ldots, l\}$. We note that the spherical harmonics verify an orthogonality property, that can be seen as a generalization of property (9), that is we have:

$$\int_{S^2} Y^{\sigma}_{l,m}(\underline{\hat{x}}) Y^{\sigma'}_{l',m'}(\underline{\hat{x}}) ds(\underline{\hat{x}}) = \delta_{\sigma,\sigma'}\delta_{l,l'}\delta_{m,m'}, \quad \nu,\ \nu' = (\sigma', l', m') \in I, \tag{10}$$

where δ denotes the kronecker delta.

In the construction of the operator T, introduced in (7), we use two bases of functions $\{\psi_\nu,\ \nu \in I\}$, $\{\mathrm{Re}\,\psi_\nu,\ \nu \in I\}$, which are defined as follows:

$$\psi_\nu(k\underline{x}) = h^{(1)}_l(k\,\|\underline{x}\|) Y^{\sigma}_{l,m}(\underline{\hat{x}}),\ \underline{x} \in \mathbb{R}^3 \setminus \{\underline{0}\},\ \nu \in I, \tag{11}$$

$$\mathrm{Re}\,\psi_\nu(k\underline{x}) = j_l(k\,\|\underline{x}\|) Y^{\sigma}_{l,m}(\underline{\hat{x}}),\ \underline{x} \in \mathbb{R}^3,\ \nu \in I, \tag{12}$$

where j_l denotes the spherical Bessel function of order l, $h^{(1)}_l$ denotes the spherical Hankel function of first kind and order l, see [28] page 435 for details. We note that for each $\nu \in I$ the complex-valued function ψ_ν is singular at the origin of the coordinate system, while the real-valued function $\mathrm{Re}\,\psi_\nu$ is regular at the origin of the coordinate system. Moreover from the separation of the Helmholtz operator in spherical coordinates it is easy to see that, for each $\nu \in I$, function $\mathrm{Re}\,\psi_\nu$ satisfies the Helmholtz equation in $\mathbb{R}^3$, function ψ_ν satisfies equation (4), being $\underline{0} \in D$, and it satisfies equation (6), for a detailed discussion see [29] page 1462.

Supposing that the functions u^i and u^s have the following expansion:

$$u^i(\underline{x}) = \sum_{\nu \in I} a_\nu \,\mathrm{Re}\,\psi_\nu(k\underline{x}),\ \underline{x} \in \mathbb{R}^3, \tag{13}$$

$$u^s(\underline{x}) = \sum_{\nu \in I} f_\nu \psi_\nu(k\underline{x}),\ \underline{x} \in \mathbb{R}^3 \setminus \overline{D}, \tag{14}$$

we obtain that the operator $T = T_{\nu;\nu'}(D,k)$, ν, $\nu' \in I$, depending on the obstacle D and the wave number k, can be rewritten in a more practical way than formula (7), that is

$$f_\nu = \sum_{\nu' \in I} T_{\nu;\nu'}(D,k) a_{\nu'},\ \nu \in I. \tag{15}$$

We briefly recall the formulas useful for the computation of operator T. Let us define the following operator:

$$Q_{\nu;\nu'}(D,k) = -\frac{\imath}{2}\delta_{\sigma,\sigma'}\delta_{l,l'}\delta_{m,m'} + \\ + \frac{k}{2}\int_{\partial D}\frac{\partial}{\partial\underline{\hat{n}}(\underline{x})}(\psi_\nu(k\underline{x})\operatorname{Re}\psi_{\nu'}(k\underline{x}))d\sigma(\underline{x}),\ \nu,\nu'\in I, \tag{16}$$

where $\underline{\hat{n}}(\underline{x})$ denotes the unit outward normal to ∂D at the point $\underline{x}\in\partial D$ and $d\sigma$ denotes the surface measure on ∂D. Let $\operatorname{Re} Q_{\nu;\nu'}(D,k) = \operatorname{Re}(Q_{\nu;\nu'}(D,k))$, ν, $\nu'\in I$. The operator T is defined as the solution of the following equation:

$$\sum_{\nu'\in I} T_{\nu;\nu'}(D,k)\, Q_{\nu';\nu''}(D,k) = -\operatorname{Re} Q_{\nu;\nu''}(D,k),\ \nu,\nu''\in I. \tag{17}$$

Formula (16) and equation (17) are the results of simple but quite involved mathematical manipulations, which are mainly based on a representation formula for the solutions of the Helmholtz equation and on an expansion formula, with respect to the bases $\{\psi_\nu,\, \nu\in I\}$, $\{\operatorname{Re}\psi_\nu,\, \nu\in I\}$, of the free space Green's function of the Helmholtz operator with the Sommerfeld radiation condition at infinity, see [22] for a complete derivation of these formulas.

We note that in practical situations we consider only a finite number of elements for the operators Q and T previously defined. Given $L_{\max}\in\mathbb{N}$ we define the following finite set of multi-indices $I_{L_{\max}} = \{\nu = (\sigma,\, l,\, m)\colon\ \sigma = 0,1,$ $l = \sigma,\sigma+1,\ldots,L_{\max},\ m = \sigma,\sigma+1,\ldots,l\}$ and in (16), (17) we consider $I_{L_{\max}}$ in place of I. So that, in particular, from (17) we have:

$$\sum_{\nu'\in I_{L_{\max}}} T_{\nu;\nu'}(D,k)\, Q_{\nu';\nu''}(D,k) = -\operatorname{Re} Q_{\nu;\nu''}(D,k),\ \nu,\nu''\in I_{L_{\max}}. \tag{18}$$

We abuse the notations Q and T for the matrices obtained from the corresponding operators. We note that the rows of matrix T can be computed as solutions of the linear system (18), where we have multiple right-hand sides, that is each row of matrix T corresponds to a different row of matrix $\operatorname{Re} Q$ through linear system (18).

We note that the T-matrix method is an interesting technique to solve problem (4)-(6), in fact matrix T depends only on D and k. Thus once matrix T is computed the solution of problem (4)-(6) can be easily obtained from formulas (14), (15) for every different incident acoustic wave u^i using the same matrix T.

Usually in problem (4)-(6) is considered an acoustic plane wave as the incident acoustic wave u^i, that is

$$u^i(\underline{x}) = e^{\imath k(\underline{x},\underline{\alpha})},\ \underline{x}\in\partial D, \tag{19}$$

where $\underline{\alpha} \in S^2$ is the wave propagation direction. We note that function u^i given in (19) is a solution of equation (3) for every $\underline{\alpha} \in S^2$. When the choice (19) is made the expansion (13) can be given explicitely, that is we have:

$$e^{\imath k(\underline{x},\underline{\alpha})} = 4\pi \sum_{\nu \in I} \imath^l Y^{\sigma}_{l,m}(\underline{\alpha}) \operatorname{Re}\psi_\nu(k\underline{x}), \ \underline{x} \in \mathbb{R}^3, \ \underline{\alpha} \in S^2, \ k > 0, \tag{20}$$

see [29] page 1466 for details.

3 – The perturbative method

For the computation of matrix T we must perform two different steps: (i) computation of the entries of matrix Q using formula (16), (ii) solution of the linear system (18). Step (i) can be performed accurately and efficiently using parallel computations, in fact it consists in the approximation of several integrals that are independent one from the other. On the contrary step (ii) must be performed with special care since the ill-conditioning of the matrix Q can make the computation of matrix T not well accurate. We note that the condition number of Q depends on D, k and the value chosen for the truncation parameter $L_{\max}$. Moreover we note that step (ii) is not well suited for parallel computations, being the solution of a linear system with, in general, a dense matrix coefficient.

We propose a perturbative method to avoid the solution of the linear system (18). We note that similar perturbative techniques have been already used for the solution of Fredholm integral equations of the first kind that formulate problem (4)-(6), or similar problems. In such cases it has been noted that perturbative techniques take care of the ill-posedness of the corresponding problem, solving the difficulty of the problem at the various perturbative orders, see for example [13], [14], [15], [16], [17], [21].

We limit our discussion to star-like obstacles, that is we suppose there exists a function $r : S^2 \to \mathbb{R}$, such that:

$$\partial D = \{\underline{x} \in \mathbb{R}^3 : \underline{x} = r(\hat{\underline{x}})\hat{\underline{x}}, \ \hat{\underline{x}} \in S^2\}, \tag{21}$$

so that from (16) we have that the entries of matrix Q can be rewritten as follows:

$$\begin{aligned} Q_{\nu;\nu'}(D,k) = &-\frac{\imath}{2}\delta_{\sigma,\sigma'}\delta_{l,l'}\delta_{m,m'} + \\ &+\frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \Bigg(\rho^2 \frac{d(j_{l'}(\rho)h^{(1)}_l(\rho))}{d\rho} Y^{\sigma}_{l,m}(\hat{\underline{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\hat{\underline{x}}(\theta,\phi)) + \\ &- \frac{\partial\rho}{\partial\theta} j_{l'}(\rho) h^{(1)}_l(\rho) \frac{\partial}{\partial\theta}\Big(Y^{\sigma}_{l,m}(\hat{\underline{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\hat{\underline{x}}(\theta,\phi))\Big) + \\ &- \frac{1}{\sin^2\theta}\frac{\partial\rho}{\partial\phi} j_{l'}(\rho) h^{(1)}_l(\rho) \frac{\partial}{\partial\phi}\Big(Y^{\sigma}_{l,m}(\hat{\underline{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\hat{\underline{x}}(\theta,\phi))\Big)\Bigg), \quad \nu,\nu' \in I_{L_{\max}}, \end{aligned} \tag{22}$$

where $\rho(\hat{\underline{x}}) = kr(\hat{\underline{x}})$, $\hat{\underline{x}} \in S^2$.

In the perturbative approach we consider the obstacle D as a perturbation of a given obstacle B, where we suppose that there exists a function $r_B : S^2 \to \mathbb{R}$, such that:

$$\partial B = \{\underline{x} \in \mathbb{R}^3 : \underline{x} = r_B(\hat{\underline{x}})\hat{\underline{x}},\ \hat{\underline{x}} \in S^2\}. \tag{23}$$

Let $\epsilon \in \mathbb{R}$ with $0 \leq \epsilon \leq 1$, let

$$R(\hat{\underline{x}}, \epsilon) = r_B(\hat{\underline{x}}) + \epsilon H(\hat{\underline{x}}),\ \hat{\underline{x}} \in S^2, \tag{24}$$

where $H(\hat{\underline{x}}) = r(\hat{\underline{x}}) - r_B(\hat{\underline{x}})$, $\hat{\underline{x}} \in S^2$. Let D_ϵ be the star-like obstacle having boundary ∂D_ϵ parametrized by the function $R(\cdot, \epsilon)$. We note that $r_B(\hat{\underline{x}}) = R(\hat{\underline{x}}, 0)$, $r(\hat{\underline{x}}) = R(\hat{\underline{x}}, 1)$, $\hat{\underline{x}} \in S^2$, so that we have $B = D_0$ and $D = D_1$ and a similar relation for the matrices Q defined in (22), that is $Q(B, k) = Q(D_0, k)$, $Q(D, k) = Q(D_1, k)$.

Now, given $N \in \mathbb{N}$, we consider the approximation of $Q(D_\epsilon, k)$ given by a series in powers of ϵ truncated to the order N-th, that is:

$$Q(D_\epsilon, k) \approx Q^{(0)} + Q^{(1)}\epsilon + \cdots + \frac{1}{N!}Q^{(N)}\epsilon^N,\ 0 \leq \epsilon \leq 1, \tag{25}$$

where, for $n = 0, 1, \ldots, N$, $Q^{(n)}$ denotes the formal derivative of order n-th of $Q(D_\epsilon, k)$ with respect to ϵ and evaluated at $\epsilon = 0$. Moreover for the matrix $T(D_\epsilon, k)$ we suppose a similar approximation, that is

$$T(D_\epsilon, k) \approx T^{(0)} + T^{(1)}\epsilon + \cdots + \frac{1}{N!}T^{(N)}\epsilon^N,\ 0 \leq \epsilon \leq 1, \tag{26}$$

where $T^{(n)}$, $n = 0, 1, \ldots, N$ are suitable square matrices having the same order as of matrix T. So that substituting the approximations (25), (26) in equation (18) we obtain:

$$\begin{aligned}&\left(T^{(0)} + T^{(1)}\epsilon + \cdots + \frac{1}{N!}T^{(N)}\epsilon^N\right)\left(Q^{(0)} + Q^{(1)}\epsilon + \cdots + \frac{1}{N!}Q^{(N)}\epsilon^N\right) = \\ &\qquad = -\operatorname{Re}\left(Q^{(0)} + Q^{(1)}\epsilon + \cdots + \frac{1}{N!}Q^{(N)}\epsilon^N\right),\end{aligned} \tag{27}$$

which is an equation for matrices $T^{(n)}$, $n = 0, 1, \ldots, N$. Solving this equation order by order with respect to the powers of ϵ, for matrices $T^{(n)}$, $n = 0, 1, \ldots, N$ we obtain the following expression:

$$\begin{aligned}&T^{(0)} = -\operatorname{Re}(Q^{(0)})(Q^{(0)})^{-1}\\ &T^{(n)} = -\left(\operatorname{Re}(Q^{(n)}) + \sum_{l=1}^{n}\binom{n}{l}T^{(n-l)}Q^{(l)}\right)(Q^{(0)})^{-1},\ n = 1, 2, \ldots, N,\end{aligned} \tag{28}$$

where $\binom{n}{l} = \frac{n!}{(n-l)!l!}$, $n, l \in \mathbb{N}$, $l \leq n$, is the binomial coefficient. Formula (28) gives an explicit expression for $T^{(n)}$, $n = 0, 1, \ldots, N$. More precisely, from the knowledge of $Q^{(0)}$ we can compute matrix $T^{(0)}$, then from the knowledge of $Q^{(0)}$, $Q^{(1)}$ and $T^{(0)}$ we can compute matrix $T^{(1)}$; we can compute the generic matrix $T^{(n)}$ from the knowledge of $Q^{(0)}, Q^{(1)}, \ldots, Q^{(n)}$ and $T^{(0)}, T^{(1)}, \ldots, T^{(n-1)}$ computed previously. The approximation of matrix $T(D, k)$ is obtained evaluating in $\epsilon = 1$ the truncated power series given in formula (26), where the matrices $T^{(n)}$, $n = 0, 1, \ldots, N$ are computed by formula (28) as explained.

Let us consider the computation of matrices $Q^{(n)}$, $n = 0, 1, \ldots, N$. The entries of these matrices are given by the derivatives of order n with respect to ϵ of the corresponding entries of matrix $Q(D_\epsilon, k)$ and these derivatives are evaluated at $\epsilon = 0$, that is

$$Q^{(0)}_{\nu;\nu'} = Q_{\nu;\nu'}(D_\epsilon, k)\Big|_{\epsilon=0}, \quad \nu, \nu' \in I_{L_{\max}}, \tag{29}$$

$$Q^{(n)}_{\nu;\nu'} = \frac{d^n}{d\epsilon^n} Q_{\nu;\nu'}(D_\epsilon, k)\Big|_{\epsilon=0}, \quad \nu, \nu' \in I_{L_{\max}}, \ n \geq 1. \tag{30}$$

When the differentiation operator with respect to ϵ can be exchanged with the integral operators appearing in the expression of $Q(D_\epsilon, k)$ and when also the limit as $\epsilon \to 0$ can be exchanged with these integral operators we obtain a more practical expression for the entries of matrices $Q^{(n)}$, $n = 0, 1, \ldots, N$, in fact we have:

$$Q^{(0)} = Q(B, k), \tag{31}$$

$$\begin{aligned} Q^{(n)}_{\nu;\nu'} = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{d^n}{d\epsilon^n}\Bigg(& \eta^2 \frac{d(j_{l'}(\eta) h^{(1)}_l(\eta))}{d\eta} Y^{\sigma}_{l,m}(\hat{\underline{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\hat{\underline{x}}(\theta,\phi)) + \\ & - \frac{\partial\eta}{\partial\theta} j_{l'}(\eta) h^{(1)}_l(\eta) \frac{\partial}{\partial\theta}\left(Y^{\sigma}_{l,m}(\hat{\underline{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\hat{\underline{x}}(\theta,\phi))\right) + \\ & - \frac{1}{\sin^2\theta} \frac{\partial\eta}{\partial\phi} j_{l'}(\eta) h^{(1)}_l(\eta) \frac{\partial}{\partial\phi}\left(Y^{\sigma}_{l,m}(\hat{\underline{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\hat{\underline{x}}(\theta,\phi))\right)\Bigg)\Bigg|_{\epsilon=0}, \\ & \nu, \nu' \in I_{L_{\max}}, \ n \geq 1, \end{aligned} \tag{32}$$

where $\eta = kR(\cdot, \epsilon)$, $0 \leq \epsilon \leq 1$, is the unique function in (32) that depends on ϵ. From formulas (31), (32) we can easily seen that the computation of the entries of matrices $Q^{(n)}$, $n = 0, 1, \ldots, N$ can be performed accurately and efficiently by a parallel computation being these entries defined as integrals of functions that are independent one from the other. But now also the computation of matrices $T^{(n)}$, $n = 0, 1, \ldots, N$ can be performed accurately, in fact from formula (28) we can easily see that it consists in sums and products of matrices. However formula (28) foresees also the computation of $(Q(B,k))^{-1}$, but this matrix does not depend on the particular obstacle D. We note that the computation of $(Q(B,k))^{-1}$ can be quite easier than the computation of $(Q(D,k))^{-1}$; for example, when B is chosen as an axial-symmetric obstacle the matrix $Q(B,k)$ is a 2×2 block-diagonal matrix, so that $(Q(B,k))^{-1}$ can be given in terms of the inverses of its two diagonal blocks, see [22] for details. However the computation of $(Q(B,k))^{-1}$ can be performed only one time since $(Q(B,k))^{-1}$ can be stored and it can be used back for all the obstacles D that we decide to express in terms of B in the perturbative procedure. Moreover formula (28) is well suited for parallel computations, in fact for $n = 0, 1, \dots, N$ the computation of $T^{(n)}$ consists in $n+1$ matrix-matrix multiplications, where n of these multiplications are independent one from the other. Finally we note that the choice of B cannot be completely independent from D, in fact we expect that fast and accurate approximations of $T(D,k)$ can be obtained from formula (26) when B is close to D in a suitable normed space. This normed space, essential for an eventual investigation of the convergence properties of the approximation (26), is useless for the purpose of the present paper thus its definition is omitted.

We conclude describing the computational consequences of a particular choice for B, that seems quite interesting. In fact when B is chosen as a sphere of radius $r_S > 0$, that is $r_B(\underline{\hat{x}}) = r_S$, $\underline{\hat{x}} \in S^2$, the matrix $Q(B,k)$ becomes a diagonal matrix. More precisely, we have:

$$Q^{(0)}_{\nu;\nu'} = \left(-\frac{\imath}{2} + \frac{\rho_S^2}{2} \frac{d(j_l(\rho) h_l^{(1)}(\rho))}{d\rho} \bigg|_{\rho=\rho_S} \right) \delta_{\sigma,\sigma'} \delta_{l,l'} \delta_{m,m'}, \nu, \nu' \in I_{L_{\max}}, \tag{33}$$

$$\begin{aligned} Q^{(n)}_{\nu;\nu'} = {} & \frac{1}{2} \frac{d^n}{d\rho^n} \left(\rho^2 \frac{d(j_{l'}(\rho) h_l^{(1)}(\rho))}{d\rho} \right) \bigg|_{\rho=\rho_S} \cdot \\ & \cdot \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta H^n(\underline{\hat{x}}(\theta,\phi)) Y^{\sigma}_{l,m}(\underline{\hat{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\underline{\hat{x}}(\theta,\phi)) + \\ & - \frac{n}{2} \frac{d^{n-1}}{d\rho^{n-1}} \left(j_{l'}(\rho) h_l^{(1)}(\rho) \right) \bigg|_{\rho=\rho_S} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \cdot H^{n-1}(\underline{\hat{x}}(\theta,\phi)) \cdot \\ & \cdot \left(\frac{\partial H(\underline{\hat{x}}(\theta,\phi))}{\partial\theta} \frac{\partial}{\partial\theta} \left(Y^{\sigma}_{l,m}(\underline{\hat{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\underline{\hat{x}}(\theta,\phi)) \right) + \right. \\ & \left. + \frac{1}{\sin^2\theta} \frac{\partial H(\underline{\hat{x}}(\theta,\phi))}{\partial\phi} \frac{\partial}{\partial\phi} \left(Y^{\sigma}_{l,m}(\underline{\hat{x}}(\theta,\phi)) Y^{\sigma'}_{l',m'}(\underline{\hat{x}}(\theta,\phi)) \right) \right), \\ & \nu, \nu' \in I_{L_{\max}}, \; n \geq 1, \end{aligned} \tag{34}$$

where $\rho_S = kr_S$. We note that (33) follows from straightforward calculations using formulas (10), (22), (31) and formula (34) follows from formula (32). This is an interesting case since the fact that $Q^{(0)}$ is a diagonal matrix can be used effectively in formula (28) for the computation of $(Q^{(0)})^{-1}$, and besides the evident gain in the accuracy and in the computational cost of $(Q^{(0)})^{-1}$ we have that it improves the parallel efficiency of formula (28).

Finally, we note that the actual validation of the perturbative method proposed in this paper needs a rigorous convergence analysis of series

$$\sum_{n=0}^{\infty} \frac{T^{(n)}}{n!}\epsilon^n, \; 0 \le \epsilon \le 1, \tag{35}$$

generated by formulas (28)-(30). This theoretical analysis deserves to be considered with further investigations, so, at present we provide only some convincing numerical results for the experimental validation of the proposed method.

4 – Numerical results

We present some results extracted from our numerical experience using the perturbative method proposed in the previous section. The numerical results are relative to ten different obstacles and they show mainly the convergence and the stabilization features of the perturbative method. In particular, we consider star-like obstacles whose boundary is parametrized by the following functions:

$$\text{Oblate Ellipsoid}: \; r_1(\underline{\hat{x}}(\theta,\phi)) = \frac{1}{\sqrt{(\frac{2}{3}\sin\theta)^2 + \cos^2\theta}}, \tag{36}$$

$$\text{Prolate Ellipsoid}: \; r_2(\underline{\hat{x}}(\theta,\phi)) = \frac{1}{\sqrt{\sin^2\theta + (\frac{2}{3}\cos\theta)^2}}, \tag{37}$$

$$\text{Pseudo Apollo}: \; r_3(\underline{\hat{x}}(\theta,\phi)) = \frac{3}{5}\sqrt{\frac{17}{4} + 2\cos(3\theta)}\,, \tag{38}$$

$$\text{Reverse Platelet}: \; r_4(\underline{\hat{x}}(\theta,\phi)) = 1 + \frac{1}{2}\sin^2\theta, \tag{39}$$

$$\text{Short Cylinder}: \; r_5(\underline{\hat{x}}(\theta,\phi)) = \frac{1}{\sqrt[10]{(\frac{2}{3}\sin\theta)^{10} + \cos^{10}\theta}}, \tag{40}$$

$$\text{Long Cylinder}: \; r_6(\underline{\hat{x}}(\theta,\phi)) = \frac{1}{\sqrt[10]{\sin^{10}\theta + (\frac{2}{3}\cos\theta)^{10}}}, \tag{41}$$

$$\text{Vogel's Nut}: \; r_7(\underline{\hat{x}}(\theta,\phi)) = \frac{3}{2}\sqrt{1 - \frac{3}{4}\sin^2\theta}, \tag{42}$$

$$(43) \quad \text{Generic Ellipsoid :} \quad r_8(\underline{\hat{x}}(\theta,\phi)) = \frac{1}{\sqrt{\left(\left(\frac{3}{2}\sin\phi\right)^2 + \cos^2\phi\right)\sin^2\theta + \left(\frac{2}{3}\cos\theta\right)^2}},$$

$$(44) \quad \text{Corrugated Sphere :} \ r_9(\underline{\hat{x}}(\theta,\phi)) = \left(1 + \frac{1}{20}\cos(4\theta) + \frac{1}{40}\cos(8\theta)\right) \cdot$$

$$\cdot \left(1 + \frac{1}{20}\cos(4\phi) + \frac{1}{40}\cos(8\phi)\right),$$

$$(45) \quad \text{Cuboid :} \quad r_{10}(\underline{\hat{x}}(\theta,\phi)) = \frac{1}{\sqrt[10]{(\sin^{10}\phi + \cos^{10}\phi)\sin^{10}\theta + \cos^{10}\theta}},$$

where $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$. We note that obstacles (36)-(42) are axial-symmetric obstacles, that is the corresponding parametrization of the boundary is a function independent from variable ϕ, obstacles (43)-(45) have not particular symmetry properties; all the obstacles are convex excepting (38), (42), (44) that are non-convex obstacles. In Figure 1 are shown the ten obstacles defined in (36)-(45). Finally, in problem (4)-(6) we always consider $k = 1$, and in equation (5) we choose function (19) with $\underline{\alpha} = \underline{\hat{x}}(\frac{\pi}{3}, \frac{\pi}{6})$.

The numerical results corresponding to obstacles (36)-(45) are reported in Table 1. For the computation of these results we have performed the sum in formula (26) using the arithmetic mean methods for the summation of divergent series. The simpler arithmetic mean method is the usual Cesàro means. This method can be generalized in several different ways obtaining, for example, the method of Hölder, the method of Cesàro, the method of Riesz; all these methods depend on a parameter usually called order of the method and they reduce to the usual Cesàro means when the order is equal to one, see [30] page 94 for a more detailed discussion. In particular for the results reported in Table 1 we have considered the Riesz method, that is given $\tau \in \mathbb{N}$ we define $\Sigma^{(N,\tau)}$ to be the sum of the matrices $T^{(n)}$, $n = 0, 1, \ldots, N$ according to the Riesz method of order τ, that is

$$(46) \qquad \Sigma^{(N,\tau)} = \sum_{n=0}^{N} \left(1 - \frac{n}{N+1}\right)^{\tau} \frac{T^{(n)}}{n!}.$$

The Riesz method is regular, that is, it does not modify the sum of convergent series. Thus, supposing that (35) is a convergent series we can compute $T(D,k)$ using either series (35) or $\Sigma^{(N,\tau)}$, as $N \to \infty$. In practice methods for the summation of divergent series are usually used for transforming slowly convergent into rapidly convergent series. From numerical results not reported in this paper series (35) seems to be convergent for all the considered obstacles, but the rate of convergence is quite dependent on the difficulty of the particular obstacle taken

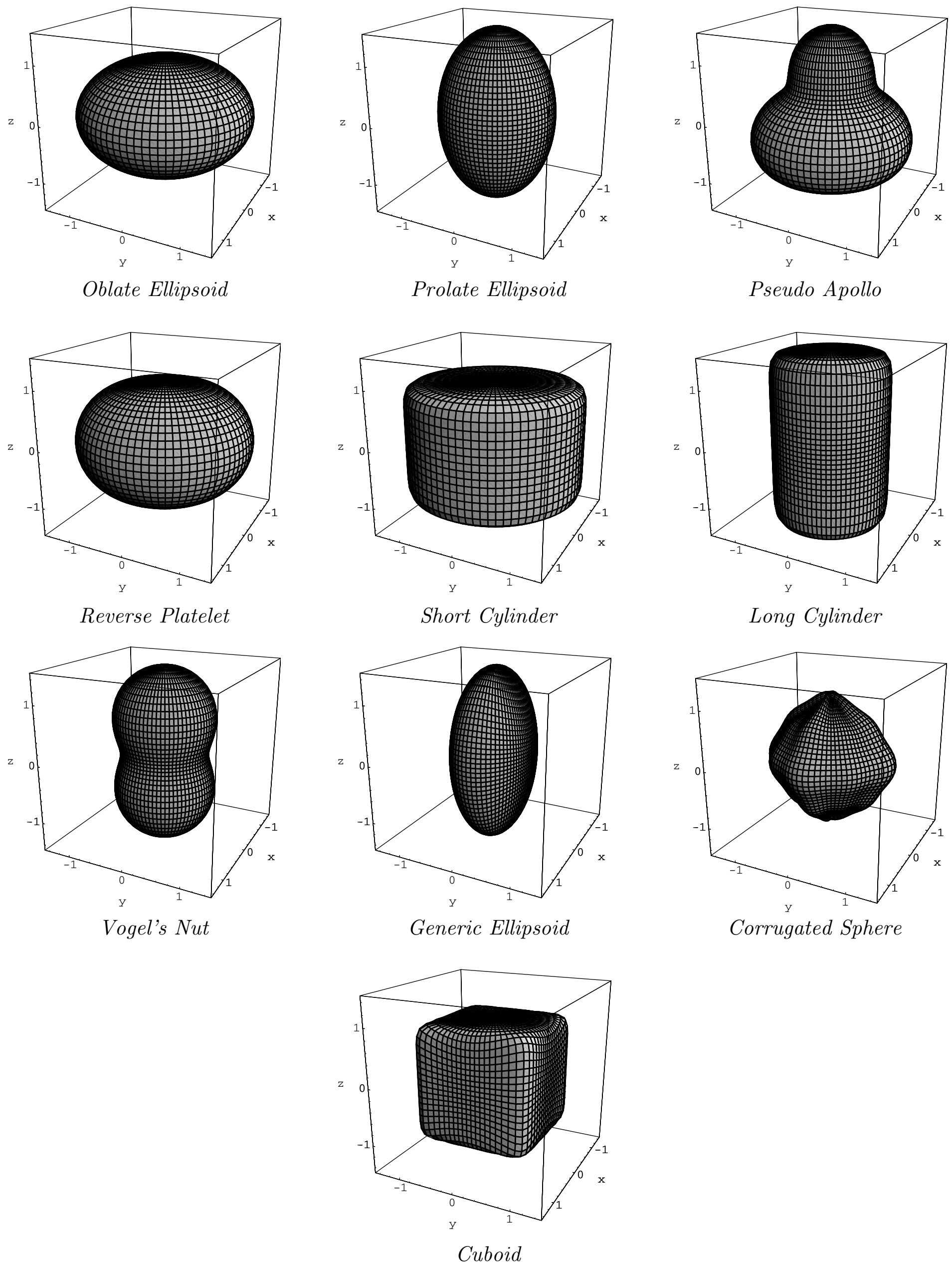

Fig. 1: The obstacles defined in (36)-(45).

into account. This unpleasant property of approximation (26) is attenuated by using the above mentioned methods for the summation of divergent series; in particular, we have that the method of Cesàro gives results similar to the ones obtained with the method of Riesz. The method of Hölder gives usually better results with respect to the method of Riesz when we consider hard obstacles, such as for example *Reverse Platelet*, but it gives much worse results when we consider easy obstacles, such as for example ellipsoids. So that given $N, \tau \in \mathbb{N}$ matrix $\Sigma^{(N,\tau)}$ is the computed approximation of the matrix $T(D,k)$; Table 1 shows the convergence properties of the sum $\Sigma^{(N,\tau)}$ to the matrix $T(D,k)$. We define the following performance index:

$$E_T^{(N,\tau)} = \frac{\left\|\Sigma^{(N,\tau)} - T(D,k)\right\|_\infty}{\|T(D,k)\|_\infty}. \tag{47}$$

where $\|\cdot\|_\infty$ denotes the operator matrix norm associated with the vector maximum norm. Moreover the approximation $\tilde{u}^{s,(N,\tau)}$ of the solution u^s of problem (4)-(6) is computed from formulas (14), (15) substituting $T(D,k)$ with $\Sigma^{(N,\tau)}$. Table 1 also shows a comparison between the approximation $\tilde{u}^s$ of the solution u^s of problem (4)-(6) obtained using the usual T-matrix method and the approximation $\tilde{u}^{s,(N,\tau)}$ obtained using the perturbative method. As a consequence of the discussion following formula (12) this comparison takes into account only the error in the approximation of condition (5), so that we consider the following two performance indices:

$$\begin{aligned} E_u = \frac{1}{92}\Bigg(&\left|\tilde{u}^s(\underline{\xi}_{0,0}) + u^i(\underline{\xi}_{0,0})\right| + \left|\tilde{u}^s(\underline{\xi}_{10,0}) + u^i(\underline{\xi}_{10,0})\right| + \\ &+ \sum_{i=1}^{9}\sum_{j=0}^{9}\left|\tilde{u}^s(\underline{\xi}_{i,j}) + u^i(\underline{\xi}_{i,j})\right| \Bigg), \end{aligned} \tag{48}$$

$$\begin{aligned} E_u^{(N,\tau)} = \frac{1}{92}\Bigg(&\left|\tilde{u}^{s,(N,\tau)}(\underline{\xi}_{0,0}) + u^i(\underline{\xi}_{0,0})\right| + \left|\tilde{u}^{s,(N,\tau)}(\underline{\xi}_{10,0}) + u^i(\underline{\xi}_{10,0})\right| + \\ &+ \sum_{i=1}^{9}\sum_{j=0}^{9}\left|\tilde{u}^{s,(N,\tau)}(\underline{\xi}_{i,j}) + u^i(\underline{\xi}_{i,j})\right| \Bigg), \end{aligned} \tag{49}$$

where $\underline{\xi}_{i,j} = r(\hat{\underline{x}}(\frac{\pi}{10}i, \frac{\pi}{5}j))\hat{\underline{x}}(\frac{\pi}{10}i, \frac{\pi}{5}j)$, $j, i = 0, 1, \ldots, 10$, and r is the parametrization of the boundary ∂D of the obstacle D under consideration. The indices E_u, $E_u^{(N,\tau)}$ can be seen as relative errors computed on a regular grid of ∂D; note that number 92, appearing in formulas (48), (49), represents the sum of the absolute values of u^i, defined in (19), on such a grid. We note that the results shown in Table 1 are relative to the following choice of the parameters previously described: $L_{\max} = 6$, $\tau = 5, 10$, $N = 5, 10$. For a generic obstacle D the choice of the base point B in the perturbative procedure is given by the sphere having the nearest boundary ∂B to ∂D in the least-squares sense. Moreover the integrals appearing in formulas (22), (34) are approximated by a composite Gauss-Legendre formula and the solution of equation (18) is computed by the LU factorization of matrix Q with partial pivoting.

TABLE 1. The numerical results for the ten obstacles (36)-(45). For each obstacle the performance indices $E_T^{(N,\tau)}$, $E_u^{(N,\tau)}$, $N = 5, 10$, $\tau = 5, 10$ and E_u are reported.

		$E_T^{(N,\tau)}$		$E_u^{(N,\tau)}$		E_u
		$N = 5$	$N = 10$	$N = 5$	$N = 10$	
Oblate ellipsoid	$\tau = 5$	$2.29(-2)$	$3.94(-2)$	$7.24(-2)$	$9.34(-2)$	$9.26(-2)$
	$\tau = 10$	$2.29(-2)$	$3.94(-2)$	$7.41(-2)$	$9.38(-2)$	
Prolate ellipsoid	$\tau = 5$	$2.88(-2)$	$4.94(-2)$	$5.33(-2)$	$7.90(-2)$	$4.31(-2)$
	$\tau = 10$	$2.88(-2)$	$4.94(-2)$	$5.32(-2)$	$7.90(-2)$	
Pseudo apollo	$\tau = 5$	$4.75(-2)$	$7.36(-2)$	$1.31(-1)$	$1.03(-1)$	2.01(-1)
	$\tau = 10$	$4.65(-2)$	$7.36(-2)$	$1.54(-1)$	$1.04(-1)$	
Reverse platelet	$\tau = 5$	$3.45(-1)$	$3.69(-1)$	3.30	1.74	5.08
	$\tau = 10$	$3.33(-1)$	$3.67(-1)$	4.61	1.90	
Short cylinder	$\tau = 5$	$2.90(-2)$	$5.16(-2)$	$2.59(-1)$	$2.13(-1)$	$3.24(-1)$
	$\tau = 10$	$2.93(-2)$	$5.17(-2)$	$2.66(-1)$	$2.18(-1)$	
Long cylinder	$\tau = 5$	$4.69(-2)$	$8.63(-2)$	$2.36(-1)$	$1.91(-1)$	$3.03(-1)$
	$\tau = 10$	$4.78(-2)$	$8.64(-2)$	$2.33(-1)$	$1.90(-1)$	
Vogel's nut	$\tau = 5$	$1.34(-1)$	$1.67(-1)$	$1.81(-1)$	$1.43(-1)$	$1.89(-1)$
	$\tau = 10$	$1.27(-1)$	$1.66(-1)$	$3.46(-1)$	$1.58(-1)$	
Generic ellipsoid	$\tau = 5$	$4.77(-2)$	$8.11(-2)$	$2.77(-1)$	$2.02(-1)$	$4.69(-1)$
	$\tau = 10$	$4.81(-2)$	$8.13(-2)$	$2.52(-1)$	$1.97(-1)$	
Corrugated sphere	$\tau = 5$	$7.86(-3)$	$1.18(-2)$	$5.17(-2)$	$5.35(-2)$	$5.45(-2)$
	$\tau = 10$	$7.85(-3)$	$1.18(-2)$	$5.17(-2)$	$5.35(-2)$	
Cuboid	$\tau = 5$	$3.01(-2)$	$4.53(-2)$	$1.30(-1)$	$9.11(-2)$	$1.88(-1)$
	$\tau = 10$	$2.93(-2)$	$4.51(-2)$	$1.31(-1)$	$9.14(-2)$	

From Table 1 we can see very interesting results. In particular, we can note a quite rapid convergence, also due to the Riesz method, of the sum (46) to the matrix $T(D,k)$, in fact the indices $E_T^{(5,\tau)}$, $E_T^{(10,\tau)}$ are quite similar. Moreover, comparing $E_u^{(5,\tau)}$, $E_u^{(10,\tau)}$ we can see that high values of N need for obstacles having shape far from the sphere, such as for example *Long Cylinder*, *Vogel's Nut* and *Cuboid*, see Figure 1. We can also note that the use of a high value for parameter τ is usually useless and sometimes spoils the accuracy of the final approximation of $T(D,k)$. Moreover, comparing the indices E_u, $E_u^{(N,\tau)}$ reported in Table 1 it can be noted a quite general improvement of the solution obtained by the perturbative technique with respect to the one obtained by the usual T-matrix method. We can also note that the sensitivity of $E_u^{(N,\tau)}$ with respect to τ is larger than the one of $E_T^{(N,\tau)}$; furthermore it seems that the value of τ must be chosen according to the difficulty of the obstacle D, in fact for easy obstacles like ellipsoids we obtain better results for low values of τ, instead for the other obstacles we obtain better results for high values of τ.

5 – Conclusions

We consider the solution of direct scattering problems. These problems can be seen as boundary-value problems for the Helmholtz equation in unbounded domains. For the solution of these problems we study the so called T-matrix method, which is a very classical method for the solution of direct scattering problems. We propose a perturbative method based on the T-matrix method. From a large number of numerical experiments we have discussed the improvement in the accuracy of the T-matrix method due to the perturbative technique presented. In particular the numerical results shown in Section 4 are very interesting, so that we deserve further investigations of the method presented. The main question is, of course, the settlement of classes of obstacles for which the perturbative procedure proposed generates convergent approximations (see formula (26)) of the matrix $T(D,k)$. This investigation, unavoidable for a rigorous validation of the proposed method, can be integrated and completed with the study of the connection of formula (35) and the well known methods for the summation of divergent series. Another interesting question is also the development of versions of formulas (28), (31), (32), (33), (34) that are efficient for sequential computations and for parallel computations.

We conclude describing a possible very interesting application of the method proposed. The perturbative procedure presented here can deal in a natural way with the problem of scattering by random rough surface obstacles. This problem has been initially considered for the study of water waves on the ocean surface, but now it finds application in several different fields of engineering and natural sciences, such as for example detection of small defects in manufacturing processes or the study of the variations in height in natural ground surfaces, see [31], [32], [33] for a detailed discussion.

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