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On the stability of the first eigenvalue of $A_p u + \lambda\, g(x) \mid u \mid^{p-2} u = 0$ with varying p

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Abstract: *The stability with respect to p of the nonlinear eigenvalue problem*

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[\left(\sum_{m,k=1}^{N} a_{m,k}(x)\frac{\partial u}{\partial x_m}\frac{\partial u}{\partial x_k}\right)^{\frac{p-2}{2}} a_{i,j}(x)\frac{\partial u}{\partial x_j}\right] + \lambda g(x) \mid u \mid^{p-2} u = 0,$$

is studied.

1 – Introduction and notations

In this paper we study the continuity (stability) of the eigenvalue problem

$$\text{(1.1)} \qquad \begin{cases} -A_p u = \lambda g(x) \mid u \mid^{p-2} u & \text{in } \Omega \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

with respect to p which varies continuously in $(1, \infty)$. Here Ω is a bounded domain in $\mathbb{R}^N$ and $g \in L^\infty_{\text{loc}}(\Omega) \cap L^r(\Omega)$ is an indefinite weight function. The exponent $r = r(N, p)$ satisfies the following conditions

$$\text{(1.2)} \qquad \begin{cases} r \geq \frac{Np}{p-1} & \text{when } 1 < p \leq N \\ r = 1 & \text{when } p > N, \end{cases}$$

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and g can change its sign in Ω, we assume only that $\Omega^+ = \{x \in \Omega, g(x) > 0\}$ has positive measure. The so-called A_p-Laplacian operator is defined by

$$A_p u = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left[\left(\sum_{m,k=1}^{N} a_{m,k}(x) \frac{\partial u}{\partial x_m} \frac{\partial u}{\partial x_k} \right)^{\frac{p-2}{2}} a_{i,j}(x) \frac{\partial u}{\partial x_j} \right].$$

Where $A = (a_{i,j})_{i,j}$ is a matrix satisfying the conditions

$$(1.3) \quad \begin{cases} \text{(i)} \quad a_{i,j} \equiv a_{j,i} \in L^\infty(\Omega) \cap \mathcal{C}^1(\Omega) \\ \text{(ii)} \quad |\, \xi \,|_a^2 \equiv \displaystyle\sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq |\, \xi \,|^2 \quad \text{when } x \in \Omega \text{ for all } \xi \in \mathbb{R}^N. \end{cases}$$

We will use the norm

$$\| v \|_{1,p} = \| \, |\nabla v|_a \|_p = \left(\int_\Omega |\nabla v|_a^p \; dx \right)^{\frac{1}{p}}.$$

We also define an inner product

$$\langle \xi, \zeta \rangle_a \equiv \sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \zeta_j.$$

The A_p-Laplacian operator defined above was studied by Yu. G. Reshetnyak [13] and J. Mossino [11] and used in [8]. Many elliptic operators are particular cases of the A_p-Laplacian operator. For example, the p-Laplacian

$$\Delta_p u = \operatorname{div} (|\nabla u|^{p-2} \nabla u)$$

and the linear operator

$$A_2 u = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial u}{\partial x_i} \right).$$

These operators,with $p \neq 2$, are used for non-Newtonian fluids (dilatant fluids have $p > 2$, pseudo-plastics have $1 < p < 2$), and appear in some reaction-diffusion problems as well as in nonlinear elasticity, and in glaciology ($p = \frac{3}{4}$).

Under various conditions the simplicity of the first eigenvalue for the above case Δ_p were obtained by various authors. When $g \equiv 1$ the first eigenvalue for the A_p-Laplacian is simple as in the case of the ordinary p-Laplacian, see [3, 12, 14] for more general g. These results were extended to our problem in [15].

Recently, for $g \equiv 1$ and without any assumptions of regularity on the domain Ω, the simplicity of the first eigenvalue was proved in [9] for the p-Laplacian Δ_p. Its stability (continuity) with respect to p was studied in [10]. In some other cases, it was studied in [6].

The principal eigenvalue $\lambda_p(g)$ of the A_p-Laplacian with indefinite weight g is here defined as the least positive real number $\lambda > 0$ for which the problem (1.1) has a nontrivial solution.

We now describe some main results of this paper. We study the convergence of the first eigenfunctions in connection with the inequalities

$$\lim_{s \to p_-} \lambda_s(g) \le \lambda_p(g) = \lim_{s \to p_+} \lambda_s(g),$$

proved in Theorem 3.2 and Corollary 3.1. In other words we explore the behavior of the principal eigenfunction $u_s \in W_0^{1,s}(\Omega)$ (required to be positive and $\int_\Omega g(x)|u_s|^s\, dx = 1$) to the equation

$$A_s u_s + \lambda_s(g)|u_s|^{s-2} u_s = 0,$$

as s varies continuously in $(1, \infty)$. This is why we are interested in the stability to the right.

In very irregular domains with $p \le N$, the situation $\lim_{s \to p_-} \lambda_s(g) < \lambda_p(g)$ is possible. An example is given by [10] in the case $A_p = \Delta_p$ and $g \equiv 1$. This situation is as a consequence of a strange convergence phenomenon: The principal eigenfunctions u_s, $s < p$, converge to a positive solution of the first equation (1.1).

The limit function is in the Sobolev space $W^{1,p}(\Omega)$ and in every $W_0^{1,p-\epsilon}(\Omega)$, $\epsilon > 0$ small enough, but is not in the required $W_0^{1,p}(\Omega)$. If Ω satisfies the segment property then it follows from Theorem 2.1, that

$$W_0^{1,q}(\Omega) \cap W^{1,p}(\Omega) = W_0^{1,p}(\Omega), 1 < q < p.$$

In this case we show in Corollary 3.2 and Corollary 3.3 our main results related to the stability.

In Theorem 3.6 we show that the eigenfunctions and their gradients converge locally uniformly to a positive solution of the first equation problem (1.1), by the $\mathcal{C}^{1,\alpha}_{\mathrm{loc}}$-regularity, see [4], and the L^∞-estimate established in the Appendix.

The paper is organized as follows: In Section 2, we establish some definitions and basic properties. In Section 3, we first give some general stability results with respect to p for the first positive eigenvalue of problem (1.1) and we restrict ourselves to bounded domain Ω having the segment property. This class of domains is fairly large. Then we prove the global stability using some results established in Section 2 and in Appendix. The segment property is needed here to guarantee the right boundary values of the limit function.

2 – Preliminary results

In defining the eigenvalues of the A_p-Laplacian operator with weight (in a given bounded domain $\Omega \subset \mathbb{R}^N$), we shall interpret Equation (1.1) in the weak sense.

Definition 2.1. We say that $\lambda \in \mathbb{R}$ is an eigenvalue, if there exists a function $u \in W_0^{1,p}(\Omega), u \not\equiv 0$, such that

$$\int_\Omega |\nabla u|_a^{p-2} \langle \nabla u, \nabla\varphi\rangle_a \, dx = \lambda \int_\Omega g(x) \, | \, u \, |^{p-2} \, u\varphi \, dx, \tag{2.1}$$

whenever $\varphi \in W_0^{1,p}(\Omega)$. The function u is called an eigenfunction.

2.1 – Basic properties

Under our conditions on $a_{i,j}$ and g, it is well-known that the problem (1.1) possesses at least a sequence of positive eigenvalues $\lambda_n, \lambda_n \nearrow^{+\infty}$, as $n \to +\infty$. These can obtained by the Ljusternick-Schnirelmann theory minimizing the energy functional,

$$\Phi(u) = \left(\frac{1}{p}|||\nabla u|_a||_p^p\right)^2 - \frac{1}{p}\int_\Omega g(x)|u|^p \, dx,$$

on $W_0^{1,p}(\Omega)$. See [2], see also [8] or [15].

Let now $\lambda_p(g)$ denote the first positive eigenvalue of (1.1). We recall that $\lambda_p(g)$ can be variationally characterized as

$$\begin{aligned}\lambda_p(g) &= \min\left\{\int_\Omega |\nabla u|_a^p \; dx; \; u \in W_0^{1,p}(\Omega), \int_\Omega g(x) \, | \, u \, |_a^p \; dx = 1\right\} = \\ &= \min\left\{\frac{\int_\Omega |\nabla u|_a^p \; dx}{\int_\Omega g(x) \, | \, u \, |^p \; dx}; \; u \in W_0^{1,p}(\Omega), \int_\Omega g(x) \, | \, u \, |^p \; dx > 0\right\}.\end{aligned} \tag{2.2}$$

Throughout this paper, the first eigenfunctions are those corresponding to $\lambda_p(g)$. The principal eigenfunction, denoted u_p, is the first eigenfunction normalized by $\int_\Omega g(x) \, | \, u_p \, |^p \; dx = 1$, and required to be positive. Hence

$$\lambda_p(g) \; = \int_\Omega |\nabla u_p|_a^p \; dx.$$

We end this paragraph by recalling some fundamental properties, found in [8], [15], which valid under our assumptions.

1) The first eigenfunctions are essentially unique in any bounded domain, i.e., they are merely constant multiples of each other.
2) The principal eigenfunction has no zeros in the domain the first eigenfunctions are only those not changing sign.
3) The solutions of problem (1.1) are known to be of class $C_{\mathrm{loc}}^{1,\alpha}(\Omega)$ for some $\alpha > 0$ depending on p and N, see [4].

2.2 – The segment property

We begin with defining a sharp class of domains for which the boundary is sufficiently regular to guarantee that

$$W^{1,p}(\Omega) \cap_{q<p} W_0^{1,q}(\Omega) = W_0^{1,p}(\Omega).$$

Definition 2.2. An open subset Ω of $\mathbb{R}^N$ is said to have the segment property if, given any $x \in \partial\Omega$, there exist an open set G_x in $\mathbb{R}^N$ with $x \in G_x$ and y_x of $\mathbb{R}^N \setminus \{0\}$ such that, if $z \in \overline{\Omega} \cap G_x$ and $t \in]0,1[$, then $z + ty_x \in \Omega$.

This property allows us by a translation to push the support of a function u in Ω. The following result is essential here.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ having the segment property. If $u \in W^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ for some $q \in]1,p[$, then $u \in W_0^{1,p}(\Omega)$.*

Proof. The following technique is inspired by [1, Theorem 3.18]. The function

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

is in $W^{1,p}(\mathbb{R}^N)$. Indeed, we have $u \in W_0^{1,q}(\Omega)$, and so $\tilde{u} \in W^{1,q}(\mathbb{R}^N)$; moreover $\nabla\tilde{u} = \widetilde{\nabla u}$ weakly and a.e. on $\mathbb{R}^N$. On the other hand, $\tilde{u} \in L^p(\mathbb{R}^N)$ and $\widetilde{\nabla u} \in (L^p(\mathbb{R}^N))^N$, because $u \in W^{1,p}(\Omega)$. Finally, we conclude that $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$). Let $K = \text{supp } u =: \overline{\{x \in \Omega, u(x) \neq 0\}}^{\mathbb{R}^N}$, (closure in $\mathbb{R}^N$). Thus K is compact and $K \subset \overline{\Omega}$.

If $K \subset \Omega$, let j_ϵ be defined as in Section 2.17 of [1], thus the convolution $j_\epsilon * u \in C_0^\infty(\Omega)$, provided $0 < \epsilon < \text{dist}(K, \partial\Omega)$, and $j_\epsilon * u \to u$ in $W^{1,p}(\Omega)$, as $\epsilon \to 0^+$. This shows that $u \in W_0^{1,p}(\Omega)$. We shall therefore suppose that $K \cap \partial\Omega \neq \emptyset$. From Definition 2.2, to each $x \in \partial\Omega$, there corresponds a neighborhood G_x and a vector $y_x \in \mathbb{R}^N \setminus \{0\}$. Put $F = K \cap (\overline{\Omega} \setminus \bigcup_{x\in\partial\Omega} G_x)$; then F is compact and $F \subset \Omega$. Thus there is an open set G_0 such that $F \subset G_0 \subset \Omega$, with $\overline{G}_0 \subset \Omega$. On the other hand, $K \cap \partial\Omega$ is compact in $\mathbb{R}^N$ and covered by the open sets G_x,$x \in \partial\Omega$. Therefore $K \cap \partial\Omega$ may be covered by finitely may of the G_x, say $G_1, G_2, ..., G_k$, and also the sets $G_0, G_1, ..., G_k$ form an open covering of K. By a similar argument as that in the proof of Theorem 3.18. of [1, p.55], we can construct open sets $G'_0, G'_1, ..., G'_k$ which form an open covering of K with $\overline{G'_j} \subset G_j$ for each j. Now, let $\Theta = \{\theta_j, 0 \leq j \leq k\}$ be a partition of unity subordinate to covering $\{G'_j, 0 \leq j \leq k\}$ and put $u_j = \theta_j u, \forall j = 0, ..., k$. We have $u = \sum_{j=0}^N u_j$ and supp $u_j \subset G'_j$, for each $j = 0, ..., k$. Therefore, it suffices to show that each $u_j \in W_0^{1,p}(\Omega \cap G_j)$. Since $\overline{G'_0} \subset \Omega$, our discussion of the case $K \subset \Omega$ above shows that $u_0 \in W_0^{1,p}(\Omega)$. For $j \geq 1$, we have $u_j \in W^{1,p}(\Omega \cap G_j)$

and $\tilde{u}_j \in W^{1,p}(\mathbb{R}^N)$. Put $K_j = \operatorname{supp} u_j$ and let $u_{j,t} = \tilde{u}_j(x - ty_j)$, with $0 < t < \min\{1, |\, y_j \,|^{-1} \operatorname{dist}(G'_j, G^c_j)\}$, y_j denoting the element associated with G_j as in Definition 2.2. Thus we have

$$\operatorname{supp} u_{j,t} \subset \Omega \cap G_j, \tag{2.3}$$

for each t satisfying $0 < t < \min\{1, |\, y_j \,|^{-1} \operatorname{dist}(G'_j, G^c_j)\}$. Indeed, we have

$$\operatorname{supp} u_{j,t} \;=\; K_j + ty_j \;\subset G_j \cap \overline{\Omega} + ty_j \;\subset \Omega$$

by the segment property. On the other hand, let $x \in \operatorname{supp} u_{j,t}$. Then

$$\operatorname{dist}(x, G'_j) \;\leq \operatorname{dist}(x, x - ty_j) + \operatorname{dist}(x - ty_j, K_j) + \operatorname{dist}(K_j, G'_j) = \operatorname{dist}(x, x - ty_j).$$

We obtain

$$\operatorname{dist}(x, G'_j) \;\leq \operatorname{dist}(x, x - ty_j) = |\, ty_j \,| \,.$$

Therefore, $\operatorname{dist}(x, G'_j) < \operatorname{dist}(G'_j, G^c_j)$ by the choice of t. Hence $x \in G_j$. This completes the proof of (2.3). We also have $u_{j,t} \in W^{1,p}(\mathbb{R}^N)$, because $\widetilde{u_j} \in W^{1,p}(\mathbb{R}^N)$; especially, we have $u_{j,t} \in W^{1,p}(\Omega \cap G_j)$ and from (2.3), we deduce that $u_{j,t} \in W^{1,p}_0(\Omega \cap G_j)$, for $t > 0$ sufficiently small. Translation is continuous in $L^p(\Omega \cap G_j)$ so $u_{j,t} \to u_j$ in $L^p(\Omega \cap G_j)$ and $\nabla u_{j,t} \to \nabla u_j$ in $(L^p(\Omega \cap G_j))^N$, as $t \to 0^+$ (note that $\nabla \widetilde{u_j} = \widetilde{\nabla u_j}$). Hence $u_{j,t} \to u_j$ in $W^{1,p}(\Omega \cap G_j)$. This together with the fact that $u_{j,t} \in W^{1,p}_0(\Omega \cap G_j)$, for $t > 0$ small enough, ends the proof. $\square$

Remark 2.1. A bounded domain $\Omega \subset \mathbb{R}^N$ has the segment property if, and only if, it is in the class $\mathcal{C}$, cf. [5]. This means that locally the boundary has the continuous equation $x_N = f(x_1, x_2, ..., x_{N-1})$, after a notation of the coordinate axis.

3 – Stability of $s \longrightarrow \lambda_s(g)$

The first positive eigenvalue $\lambda_s(g)$ of the A_s-Laplacian with weight $g \in L^\infty_{\text{loc}}(\Omega) \cap L^r(\Omega)$, where $r = r(N, p)$ satisfies (1.2), exists for each $s \in (1, \infty)$ which is near enough to p. Indeed, observe that (1.2) yields the following conditions: $r > \frac{N}{p}$ if $1 < p < N, r > N$ if $p = N$ and $r = 1$ if $p > N$, which imply the existence of $\lambda_s(g)$, (cf. [15]).

We will assume throughout this section that our conditions on g and $a_{i,j}$ are satisfied.

3.1 – Some inequalities

Theorem 3.1. *The eigenvalues $\lambda_s(g)$ and λ_s satisfy*

$$p\lambda_p^{\frac{1}{p}}(g) \leq s\lambda_s^{\frac{1}{s}}\left(\frac{\lambda_s(g)}{\lambda_s}\right)^{\frac{1}{p}}, \tag{3.1}$$

when $1 < p < s$ and p, s are close enough.

Proof. Let $\varphi = u_s^{\frac{s}{p}}$. Then $\varphi \in W_0^{1,p}(\Omega)$, because $s > p$. Moreover $\int_\Omega g|\varphi|^p dx = \int_\Omega g u_s^s dx = 1$, and $\nabla\varphi = \frac{s}{p} \mid u_s \mid^{\frac{s}{p}-1} \nabla u_s$. Observe that φ is admissible to compute $\lambda_s(g)$ in (2.2). Hence

$$\lambda_p^{\frac{1}{p}}(g) \leq \left(\int_\Omega \mid \nabla\varphi \mid_a^p dx\right)^{\frac{1}{p}} = \frac{s}{p}\left(\int_\Omega u_s^{s-p} \mid \nabla u_s \mid_a^p dx\right)^{\frac{1}{p}}.$$

From Hölder's inequality, we obtain the estimate

$$\lambda_p^{\frac{1}{p}}(g) \leq \frac{s}{p}\left(\int_\Omega u_s^s dx\right)^{\frac{1}{p}-\frac{1}{s}} \lambda_s^{\frac{1}{s}}(g).$$

On the other hand, we have

$$\lambda_s \int_\Omega u_s^s dx \leq \int_\Omega \mid \nabla u_s \mid_a^s dx = \lambda_s(g)$$

by (1.3 ii) and the minimizing property of λ_s. Hence

$$\lambda_p^{\frac{1}{p}}(g) \leq \frac{s}{p}\left(\frac{\lambda_s(g)}{\lambda_s}\right)^{\frac{1}{p}-\frac{1}{s}} \lambda_s^{\frac{1}{s}}(g) = \frac{s}{p}\lambda_s^{\frac{1}{s}}\left(\frac{\lambda_s(g)}{\lambda_s}\right)^{\frac{1}{p}}. \qquad \square$$

Remark 3.1. If $g \in L^\infty(\Omega)$, the inequality (3.1) holds for each $1 < p < s < \infty$. Notice that the one-sided limits

$$\lim_{s\to p_-} \lambda_s(g) \text{ and } \lim_{s\to p_+} \lambda_s(g)$$

exist.

Corollary 3.1. *We have*

$$\limsup_{s\to p_-} \lambda_s(g) \leq \lambda_p(g) \leq \liminf_{s\to p_+} \lambda_s(g).$$

Proof. • When $s \to p_+$, we have $p < s < p+1$. Thus

$$s\lambda_s^{\frac{1}{s}} \le (p+1)\lambda_{p+1}^{\frac{1}{p+1}}.$$

Hence the set $\{\lambda_s \mid p < s < p+1\}$ is bounded. Thus $\lambda_s^{\frac{p}{s}-1} \to 1$, as $s \to p_+$. Finally, from the inequality (3.1), we deduce that

$$\lambda_p(g) \le \liminf_{s \to p_+} \lambda_s(g).$$

• For $s \to p_-$, with $1 < s < p$, we have from (3.1), the following inequalities

$$\left[\left(\frac{s}{p}\right)^{p-s} \lambda_s^{\frac{p}{s}-1}\right] \lambda_s(g) \le \lambda_s(g)\lambda_s^{1-\frac{s}{p}} \le \left(\frac{s}{p}\right)^s \lambda_p(g).$$

The first inequality is (3.1) for $g \equiv 1$. Hence

$$\left(\frac{s}{p}\right)^p \lambda_s^{\frac{p}{s}-1}\lambda_s(g) \le \lambda_p(g).$$

Therefore

$$\limsup_{s \to p_-} \left[\left(\frac{s}{p}\right)^p \lambda_s^{\frac{p}{s}-1}\lambda_s(g)\right] \le \lambda_p(g).$$

On the other hand, since $\lambda_s^{\frac{p}{s}-1} \to 1$, as $s \to p_-$, we obtain that

$$\limsup_{s \to p_-} \lambda_s(g) \le \lambda_p(g).$$

□

Remark 3.2. Observe that if $\lim_{s \to p} \lambda_s(g)$ exists, then this limit is necessarily equal to $\lambda_p(g)$. Therefore we will study the different cases $s \to p_+$ and $s \to p_-$.

3.2 – Stability to the right

Theorem 3.2. *For an arbitrary bounded domain we have*

$$\lim_{s \to p_+} \lambda_s(g) = \lambda_p(g).$$

Proof. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be such that

$$\int_\Omega g \mid \varphi \mid^p \, dx \, > \, 0, \tag{3.2}$$

and let $\epsilon > 0$ (small). Applying the Dominated Convergence Theorem, we find

$$\lim_{\epsilon \to 0_+} \int_\Omega g \mid \varphi \mid^{p+\epsilon} \, dx = \int_\Omega g \mid \varphi \mid^p \, dx \, > \, 0.$$

Hence, there is $\epsilon_0 > 0$ sufficiently small such that

$$\int_\Omega g \mid \varphi \mid^{p+\epsilon} \, dx \, > 0, \text{ when } \; 0 < \epsilon < \epsilon_0.$$

On the other hand, we have

$$\lambda_{p+\epsilon}(g) \, \leq \frac{\int_\Omega \mid \nabla\varphi \mid_a^{p+\epsilon} \, dx}{\int_\Omega g \mid \varphi \mid^{p+\epsilon} \, dx}.$$

It follows from the Dominated Convergence Theorem that

$$\limsup_{\epsilon \to 0_+} \lambda_{p+\epsilon}(g) \, \leq \frac{\int_\Omega \mid \nabla\varphi \mid_a^p \, dx}{\int_\Omega g \mid \varphi \mid^p \, dx}. \tag{3.3}$$

This, and the fact that φ is an arbitrary function satisfying (3.2), yield

$$\limsup_{\epsilon \to 0_+} \lambda_{p+\epsilon}(g) \leq \lambda_p(g).$$

Now the result follows from Corollary 3.1. ∎

Theorem 3.3. *The principal eigenfunctions u_s associated with $\lambda_s(g)$ satisfy*

$$\lim_{s \to p_+} \int_\Omega \mid \nabla u_s - \nabla u_p \mid_a^p \, dx = 0. \tag{3.4}$$

PROOF. For $1 < p < s$ with s near p. Hölder's inequality implies that

$$\int_\Omega |\nabla u_s|_a^p \, dx \leq |\Omega|^{1-\frac{p}{s}} (\lambda_s(g))^{\frac{p}{s}}. \tag{3.5}$$

This shows that $\{u_s,\, s > p\}$ is a bounded set in $W_0^{1,p}(\Omega)$. Hence there is a sequence $s_1, s_2, ...,$ converging to p_+ and there is a function $u \in W_0^{1,p}(\Omega)$ such that $u_{s_j} \rightharpoonup u$ (weakly) in $W_0^{1,p}(\Omega)$, as $j \to +\infty$. Using the Rellich-Kondrachov Compactness Theorem, (cf.[1, p.144]), we obtain that $u_{s_j} \to u$ in $L^{p+\frac{1}{N}}(\Omega)$, as $j \to +\infty$; in particular, $u_{s_j} \to u$ in $L^p(\Omega)$, as $j \to +\infty$. Passing to a subsequence if necessary, we can assume that $u_{s_j} \to u$ a.e. in Ω. We will prove that $u \equiv u_p$. The weak lower semicontinuity of the norm and (3.5) yield

$$\int_\Omega |\nabla u|_a^p \, dx \leq \lambda_p(g). \tag{3.6}$$

It suffices to have

$$\int_\Omega g u^p \, dx = 1.$$

Indeed, if we set $M_s = \max_\Omega u_s$, then from Lemma 4.1., we have $\max_{s \in [a,b]} M_s < M < \infty$. Here M is a constant not depending on s, and $[a, b]$ is any small interval containing p. Thus $0 < u_{s_j} \leq M$, and $0 \leq u \leq M$ a.e. on Ω. Hence

$$|g||u_{s_j}^{s_j} - u^p| \leq |g| u_{s_j}^{s_j} + |g| u^p \leq |g| M^{s_j} + |g| u^p$$

a.e. on Ω. On the other hand, $M^{s_j} \leq M^{p+1} + 1$. Thus a.e. on Ω, we have

$$|g||u_{s_j}^{s_j} - u^p| \leq |g| (M^{p+1} + 1 + M^p) \in L^1(\Omega).$$

The Dominated Convergence Theorem yields

$$\left| \int_\Omega g(u_{s_j}^{s_j} - u^p) dx \right| \leq \int_\Omega |g||u_{s_j}^{s_j} - u^p| \, dx \to 0,$$

as $j \to +\infty$, since $g(u_{s_j}^{s_j} - u^p) \to 0$, a.e. in Ω, as $j \to +\infty$. From this it follows easily that $\int_\Omega g |u|^p \, dx = 1$. Finally, (3.6) and the variational characterization of $\lambda_p(g)$ yield

$$\int_\Omega |\nabla u|_a^p \, dx = \lambda_p(g).$$

By the uniqueness of the principal eigenfunction we have $u = u_p$. Thus the limit function u does not depend on the particular (sub)sequence $s_1, s_2, ...$ Therefore $u_s \to u_p$ at least in $L^p(\Omega)$, as $s \to p_+$.

The rest of the proof, i.e., the strong convergence (3.4) can be obtained from Clarkson's inequalities, (cf.[1]); but with the $|| \, | \, |_a ||_p$-norm in $W_0^{1,p}(\Omega)$. ∎

3.3 – Stability to the left

This case is more difficult, because if $u \in W_0^{1,p-\epsilon}(\Omega)$ then it is possible that $u \notin W_0^{1,p}(\Omega)$.

THEOREM 3.4. *Let Ω be an arbitrary bounded domain. If we suppose that*

$$\lim_{s \to p_-} \int_\Omega |\nabla u_s - \nabla u_p|_a^s \, dx = 0, \tag{3.7}$$

then we have

$$\lim_{s \to p_-} \lambda_s(g) = \lambda_p(g).$$

PROOF. (3.7) and the Hölder inequality imply

$$\lim_{s \to p_-} \int_\Omega |\nabla u_s - \nabla u_p|_a^{p-\epsilon} \, dx = 0,$$

for any $\epsilon > 0$ sufficiently small so that $0 < p - s < \epsilon < p - 1$. Therefore $\nabla u_s \to \nabla u_p$ in $(L^{p-\epsilon}(\Omega))^N$, as $s \to p_-$. For $\epsilon > 0$ small enough, the Hölder's inequality implies that

$$|| \, |\nabla u_s|_a ||_{p-\epsilon} \leq |\Omega|^{\frac{s+\epsilon-p}{s(p-\epsilon)}} || \, |\nabla u_s|_a ||_s \, .$$

Hence

$$|| \, |\nabla u_p|_a ||_{p-\epsilon} \leq |\Omega|^{\frac{\epsilon}{p(p-\epsilon)}} \liminf_{s \to p_-} \lambda_s^{\frac{1}{s}}(g). \tag{3.8}$$

Letting $\epsilon \to 0^+$, the Fatou lemma yields

$$\lambda_p^{\frac{1}{p}}(g) = || \, |\nabla u_p|_a ||_p \leq \liminf_{s \to p_-} \lambda_s^{\frac{1}{s}}(g).$$

This completes the proof, in view of Corollary 3.1. ∎

REMARK 3.3. The converse of the theorem is an open question in the case $p \leq N$.

However, we have the following partial result for any bounded domain and every p in $(1, +\infty)$.

THEOREM 3.5. *Under the same assumptions, suppose that $\lim_{s \to p_-} \lambda_s(g) = \lambda_p(g)$. Then each sequence of real numbers tending to p from below contains a subsequence $s_1, s_2, \ldots$ such that*

$$\lim_{j \to +\infty} \int_\Omega |\nabla u_{s_j} - \nabla u|_a^{s_j} \, dx = 0, \tag{3.9}$$

for some function $u \in W^{1,p}(\Omega) \cap W_0^{1,p-\epsilon}(\Omega)$, whenever $\epsilon > 0$, $\int_\Omega g \, |u|^p \, dx = 1$, $u \geq 0$ a.e. on Ω and $\int_\Omega |\nabla u|_a^p \, dx \leq \lambda_p(g)$. The function u may be depend on the sequence, but it is a weak solution to the equation

$$A_p u + \lambda_p(g)|u|^{p-2} u = 0.$$

Proof. Let us fix $\epsilon_0 > 0$ small enough, so that $0 < p - \epsilon_0 < s < p$ and that $(p+\epsilon) < (p-\epsilon)^*$ for $0 < \epsilon < \epsilon_0$; where for $t \in (1, +\infty)$, $t^* = \frac{Nt}{N-t}$ if $1 < t < N$ and $t^* = +\infty$ if $t \geq N$. Using Hölder's inequality, we obtain

$$||\mid \nabla u_s \mid_a||_{p-\epsilon} \leq \mid \Omega \mid^{\frac{s+\epsilon-p}{s(p-\epsilon)}} \lambda_s^{\frac{1}{s}}(g),$$

when $0 < \epsilon < \epsilon_0$. From (3.1), we conclude that the norms $||\mid \nabla u_s \mid_a||_{p-\epsilon}, 0 < \epsilon < \epsilon_0$, are uniformly bounded, in view of the assumption $\lim\limits_{s \to p_-} \lambda_s(g) = \lambda_p(g)$. Thus we can find a function $u \in W_0^{1,p-\epsilon}(\Omega), 0 < \epsilon < \epsilon_0$; and find a sequence $s_1, s_2, ...$ converging to p_- such that $u_{s_j} \rightharpoonup u$ (weakly) in $W_0^{1,p-\epsilon}(\Omega)$, as $j \to +\infty$, for each $\epsilon \in (0, \epsilon_0)$ and hence $u_{s_j} \to u$ in $L^{p+\epsilon}(\Omega)$. Passing to a subsequence if necessary, we can assume that $u \geq 0$ a.e. on Ω. Clearly $u \in L^p(\Omega)$ and is independent of ϵ. On the other hand, the weak lower semicontinuity of the norm and the assumption $\lim\limits_{j \to +\infty} \lambda_{s_j}(g) = \lambda_p(g)$ imply that

$$||\mid \nabla u \mid_a||_{p-\epsilon} \leq \mid \Omega \mid^{\frac{\epsilon}{p(p-\epsilon)}} \lambda_p^{\frac{1}{p}}(g).$$

Then letting $\epsilon \to 0^+$, we obtain with Fatou's lemma that $\nabla u \in (L^p(\Omega))^N$ and

$$||\mid \nabla u \mid_a||_p \leq \lambda_p^{\frac{1}{p}}(g). \tag{3.10}$$

The normalization: $\int_\Omega g \mid u \mid^p dx = 1$, is preserved and

$$\lim_{j \to +\infty} \int_\Omega g \left(\frac{u_{s_j} + u}{2} \right)^{s_j} dx = 1, \tag{3.11}$$

for a subsequence if necessary. On the other hand, we have

$$\lambda_{s_j}(g) \ \leq \ \frac{\int_\Omega \mid \frac{\nabla u_{s_j} - \nabla u}{2} \mid_a^{s_j} dx}{\int_\Omega g(\frac{u_{s_j}+u}{2})^{s_j} dx}, \tag{3.12}$$

for j sufficiently large, because by (3.11) there is an index j_0 so large that

$$\int_\Omega g \left(\frac{u_{s_j}+u}{2} \right)^{s_j} dx > 0,$$

when $j \geq j_0$. Clarkson's inequality yields

$$\int_\Omega \left| \frac{\nabla u_{s_j} - \nabla u}{2} \right|_a^{s_j} dx \leq \frac{1}{2} \lambda_{s_j}(g) + \frac{1}{2} \; ||\mid \nabla u \mid_a||_{s_j}^{s_j} - \lambda_{s_j}(g) \int_\Omega g \left(\frac{u_{s_j} + u}{2} \right)^{s_j} dx,$$

if $s_j \geq 2$. Now (3.11) and the assumption $\lim\limits_{s\to p_-} \lambda_s(g) = \lambda_p(g)$ imply

$$\limsup_{j\to+\infty} \int_\Omega \left|\frac{\nabla u_{s_j} - \nabla u}{2}\right|_a^{s_j} dx \leq \frac{1}{2} \,||\, |\nabla u\,|_a||_p^p - \frac{1}{2}\lambda_p(g).$$

From this and (3.10), it follows easily that

$$\lim_{j\to+\infty} \int_\Omega |\, \nabla u_{s_j} - \nabla u\,|_a^{s_j}\, dx \,=\, 0,$$

for the case $p > 2$.

For the case $1 \leq p \leq 2$, we argue as follows. There is $j_1 \in \mathbb{N}$ such that $1 \leq s_j \leq 2$, for each $j \geq j_1$. Let $j_2 = max(j_1, j_0)$. Then Clarckson's inequality associated with s_j and (3.12) yield

$$\left\{\int_\Omega |\frac{\nabla u_{s_j} - \nabla u}{2}|_a^{s_j} dx\right\}^{\frac{1}{s_j-1}} + \left\{\lambda_{s_j}(g)\int_\Omega g(\frac{u_{s_j}+u}{2})^{s_j} dx\right\} \leq$$
$$\leq \left\{\frac{1}{2}\lambda_{s_j}(g) + \frac{1}{2}\int_\Omega |\nabla u|_a^{s_j} dx\right\}^{\frac{1}{s_j-1}}.$$

On the other hand from Hölder's inequality and (3.10) we deduce that

$$\int_\Omega |\nabla u|_a^{s_j} dx \leq |\Omega|^{\frac{p-s_j}{p}} \lambda_p(g)^{\frac{s_j}{p}}.$$

Thus

$$\left\{\int_\Omega |\frac{\nabla u_{s_j} - \nabla u}{2}|_a^{s_j} dx\right\}^{\frac{1}{s_j-1}} \leq \left\{\frac{1}{2}\lambda_{s_j}(g) + \frac{1}{2}|\Omega|^{\frac{p-s_j}{p}} \lambda_p(g)^{\frac{s_j}{p}}\right\}^{\frac{1}{s_j-1}} +$$
$$-\left\{\lambda_{s_j}(g)\int_\Omega g\left(\frac{u_{s_j}+u}{2}\right)^{s_j} dx\right\}^{\frac{1}{s_j-1}}.$$

Now, (3.11) and the assumption $\lim\limits_{s\to p_-} = \lambda_p(g)$ imply that

$$\left\{\limsup_{j\to+\infty}\int_\Omega |\frac{\nabla u_{s_j} - \nabla u}{2}|_a^{s_j} dx\right\}^{\frac{1}{p-1}} \leq \left\{\frac{1}{2}\lambda_p(g) + \frac{1}{2}\lambda_p(g)\right\}^{\frac{1}{p-1}} - \lambda_p(g)^{\frac{1}{p-1}} = 0.$$

Hence

$$\lim_{j\to+\infty}\int_\Omega |\nabla u_{s_j} - \nabla u|_a^{s_j} dx.$$

Remark 3.4. (i) If the limit function $u \in W_0^{1,p}(\Omega)$, then $u \equiv u_p$ by the uniqueness of the principal eigenfunction and (3.10).

(ii) When $p \leq N$, in a very irregular domain the defect $\lim_{s \to p_-} \lambda_s(g) < \lambda_p(g)$ is possible. See the counterpart in [10] for the case

$$\Delta_p u + \lambda |u|^{p-2} u = 0.$$

Corollary 3.2. *For any bounded domain Ω having the segment property, we have*

$$\lim_{s \to p_-} \lambda_s(g) = \lambda_p(g)$$

if and only if

$$\lim_{s \to p_-} \int_\Omega \mid \nabla u_s - \nabla u_p \mid_a^s dx \;=\; 0.$$

Proof. Suppose that $\lim_{s \to p_-} \lambda_s(g) = \lambda_p(g)$. From Theorem 3.5, the limit function u satisfies for $\epsilon > 0$ small enough $u \in W^{1,p}(\Omega) \cap W_0^{1,p-\epsilon}(\Omega)$ such that

$$u \geq 0 \;\text{ a.e. in }\; \Omega, \int_\Omega g \mid u \mid^p dx = 1 \;\text{ and }\; \int_\Omega \mid \nabla u \mid_a^p dx \leq \lambda_p(g).$$

Since Ω has the segment property, thus $u \in W_0^{1,p}(\Omega)$ by Theorem 2.1. Thus u is admissible in the definition of $\lambda_p(g)$. Consequently,

$$\lambda_p(g) = \int_\Omega \mid \nabla u \mid_a^p dx.$$

Hence $u \equiv u_p$ by the uniqueness of the principal eigenfunction. So by (3.9), we obtain

$$\lim_{s \to p_-} \int_\Omega \mid \nabla u_s - \nabla u_p \mid_a^s dx \;=\; 0,$$

since the limit function does not depend on the choice of the sequence. The converse is immediate, in view of Theorem 3.4. $\square$

Using the $C^{1,\alpha}_{\mathrm{loc}}$-regularity of the principal eigenfunctions u_s, s proved in [6] and the L^∞-estimate to be established in Lemma 4.1., we can state the following result generalizing [10].

Theorem 3.6. *Assume that the conditions on g and $a_{i,j}$ are satisfied. Then each sequence converging to p_-, contains a subsequence $s_1, s_2, \ldots$ such that $u_{s_j} \to u$ and $\nabla u_{s_j} \to \nabla u$ locally uniformly, where u is some function in $C^1(\Omega)$. Moreover, u is a weak solution of the equation*

$$(\mathcal{E}) \qquad A_p u + \lambda g(x) \mid u \mid^{p-2} u = 0 \quad ,$$

where $\lambda = \lim_{j \to +\infty} \lambda_{s_j}(g)$.

We know that only the first eigenfunctions are not changing signs. Thus if λ is an eigenvalue of $(\mathcal{E})$, then $\lambda = \lambda_p(g)$, and by normalization, we have $u \equiv u_p$. We have come to an important point: though the limit function u of $\{u_s\}$, as $s \to p_-$, is in $\in W^{1,p}(\Omega) \cap W_0^{1,p-\epsilon}(\Omega)$, for any $\epsilon > 0$ chosen sufficiently small, it is not always the right eigenfunction u_p, i.e., u is not necessary in $W_0^{1,p}(\Omega)$. Therefore u is not admissible in the definition of $\lambda_p(g)$. But, If Ω satisfies the segment property, then $u = u_p$, $\lambda = \lambda_p(g)$ and

$$\lim_{s \to p_-} \lambda_s(g) = \lambda_p(g).$$

So we have the following result.

Corollary 3.3. *For any bounded domain Ω having the segment property, we have*

$$\lim_{s \to p} \lambda_s(g) = \lambda_p(g).$$

4 – Appendix

The technique to uniformly bound u_p in an arbitrary domain is originally due to Ladyzhenskaya and Urlatseva, cf. [7].

Lemma 4.1. *Let the assumptions on g and $a_{i,j}$ be fulfilled. Then for any bounded domain Ω, $\max_\Omega u_p$ is bounded uniformly in p, (u_p denotes the normalized principal eigenfunction).*

Proof. If $p > N$, then from [5, Theorem 3.18., p.240] we have

$$\| u_p \|_\infty \leq C \mid \Omega \mid^{\frac{1}{N} - \frac{1}{p}} \| \nabla u_p \|_p \leq C \mid \Omega \mid^{\frac{1}{N} - \frac{1}{p}} \lambda_p^{\frac{1}{p}}(g),$$

where $C = \frac{1}{N}[\frac{1-\frac{1}{p}}{\frac{1}{N}-1}]^{\frac{1}{p'}} \omega_N^{\frac{1}{N}}$, and ω_N is the volume of the unit ball in $\mathbb{R}^N$.

For $1 < p \leq N$, we keep track of various "constants" in Proposition 2.16. of [15]; we obtain the lower bound

$$\|u_p\|_{\infty,\Omega} \leq b \|\nabla u_p\|_{1,\Omega} ;$$

where

$$b = (2^p \lambda_p(g) \|g\|_{r,\Omega})^{\frac{Nr}{pr-N}} (C^\theta + C^{\frac{\theta}{p}} \omega_N^{\frac{\theta(1-p)}{N}})^{1 - \frac{pr-N}{(p-1)Nr}},$$

$$\theta = \frac{Nr}{pr(1+N) - N(1+r)}$$

and $C = (\frac{(N-1)p}{2(N-p)\sqrt{N}})^p$ if $1 < p < N$, and $C = [\max\{N, \frac{r(N-1)}{r-1}\}]^{\frac{N}{r}}$ if $p = N$. This concludes the proof of the Lemma. □

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