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On a class of differential equations with self-reference

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Abstract: *We give a result of local existence and uniqueness for equations of the type*

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u(x,t), t\right)$$

where k_1 and k_2 are given real numbers or real functions $k_i = k_i(x,t)$.

1 – Introduction

In previous notes we have started the study of a new type of differential equations that we have called self-referred and hereditary (see [4], [5]). The abstract scheme that we can think of in order to embed such equations can be stated as follows. Given two functionals $A : X \to R$ and $B : X \to R$, where X is a space of functions $u = u(x,t)$, we look for a solution of the equation:

$$(Au)(x,t) = u(Bu(x,t), t)$$

with $u(x,0)$ satisfying suitable conditions.

When Bu is an "hereditary" operator as for instance

$$Bu(x,t) = \int_0^t u(x,\tau)d\tau$$

and similar, the previous equation is said of hereditary and self-referred type. General equations with "hereditary" operator has been studied since the beginning of XX-th century (see [2], [3] and [8]).

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With the previous general scheme can be associated the class of equations that we will study in this note; in fact, the equations that we will consider are of the type:

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u(x,t), t\right), \tag{1.1}$$

where k_1, k_2 are given real numbers or also suitable real functions. For such equations, arguing as in [5] and [4], we can establish a result of local existence and uniqueness, under suitable conditions on k_1, k_2 and on the initial datum $u(x,0)$. We can also establish similar results for more general equations than the previous one, as for instance for equations of the type:

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + k_3 u(x,t), t\right), t\right), \tag{1.2}$$

The given general scheme is also related to recent papers as [1], [6] and [7].

2 – The local existence and uniqueness result

If we consider the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u(x,t), t\right) \tag{2.3}$$

we will assume, for the sake of simplicity, that $|k_1| = |k_2| = 1$. Let $\alpha, \beta \in L^\infty(\mathbb{R},\mathbb{R}) \bigcap Lip(\mathbb{R},\mathbb{R})$ be given functions, and let us denote by L_α, L_β their Lipchitz contants; we will also assume that $L_\alpha < \frac{1}{2}$. We define:

$$u_0(x,t) = \alpha(x) + t\beta(x)$$

and we consider

$$u_1(x,t) = \alpha(x) + t\beta(x) + k_1 \int_0^t \int_0^\tau u_0\left(k_2 u_0(x,s)\right) ds d\tau$$

and then, for all $n \in \mathbb{N}$:

$$\begin{aligned} u_{n+1}(x,t) = {} & \alpha(x) + t\beta(x) + \\ & + k_1 \int_0^t \int_0^\tau u_n\left(\frac{\partial^2}{\partial s^2}u_n(x,s) + k_2 u_n(x,s), s\right) ds d\tau. \end{aligned}$$

We want to prove that the sequence u_n is a Cauchy sequence and that the limit function u_∞ is a solution of equation (2.3). It is easy to prove that

$$|u_n(x,t)| \le (||\alpha||_\infty + ||\beta||_\infty)e^t \quad \forall x \in> \mathbb{R}, t \ge 0, n \in \mathbb{N}.$$

We notice that:

$$|u_1(x,t) - u_0(x,t)| \leq \left(\|\alpha\|_\infty \frac{t^2}{2!} + \|\beta\|_\infty \frac{t^3}{3!} \right) = A_1(t).$$

Moreover, for the second derivative the following estimate holds:

$$\left| \frac{\partial^2}{\partial t^2} u_1(x,t) - \underbrace{\frac{\partial^2}{\partial t^2} u_0(x,t)}_{=0} \right| \leq (\|\alpha\|_\infty + t\|\beta\|_\infty) = B_1(t)$$

and there holds:

$$A_1(t) = \int_0^t \int_0^\tau B_1(s) ds d\tau.$$

Arguing in the same way, for the second step of the sequence, there holds:

$$\begin{aligned}
|u_2(x,t) - u_1(x,t)| \leq & \int_0^t \int_0^\tau \Big| u_1 \left(\frac{\partial^2}{\partial s^2} u_1(x,s) + k_2 u_1(x,s), s \right) + \\
& - u_0 \left(\frac{\partial^2}{\partial s^2} u_1(x,s) + k_2 u_1(x,s) \right) + \\
& + u_0 \left(\frac{\partial^2}{\partial s^2} u_1(x,s) + k_2 u_1(x,s) \right) + \\
& - u_0 \left(\frac{\partial^2}{\partial s^2} u_0(x,s) + k_2 u_0(x,s) \right) \Big| ds d\tau \leq \\
& \leq \int_0^t \int_0^\tau \Big(A_1(s) + \\
& + L_0(s) \Big(\Big| \frac{\partial^2}{\partial s^2} u_1(x,s) - \frac{\partial^2}{\partial s^2} u_0(x,s) \Big| + \\
& + A_1(s) \Big) \Big) ds d\tau,
\end{aligned}$$

where we have defined $L_0(t) = L_\alpha + t L_\beta$. Then, by induction, if we define;

$$A_{n+1}(t) = \int_0^t \int_0^\tau \Big(A_n(s)(1 + L_{n-1}(s)) + L_{n-1}(s) B_n(s) \Big) ds, \tag{2.4}$$

it is now easy to notice that:

$$|u_{n+1}(x,t) - u_n(x,t)| \leq A_{n+1}(t).$$

In addition, if we define;

$$B_{n+1}(t) = A_n(t)(1 + L_{n-1}(t)) + L_{n-1}(t) B_n(t) \tag{2.5}$$

there holds:

$$A_{n+1}(t) = \int_0^t \int_0^\tau B_{n+1}(s) ds d\tau,$$

and we obtain that:

$$\left| \frac{\partial^2}{\partial t^2} u_{n+1}(x,t) - \frac{\partial^2}{\partial t^2} u_n(x,t) \right| \leq B_{n+1}(t).$$

We want now prove that the $(u_n)_n$ is an equi-Lipschitz sequence. Let us then consider;

$$\begin{aligned} |u_1(x,t) - u_1(y,t)| \leq (L_\alpha + tL_\beta)|x-y| + \\ + \Big| \int_0^t \int_0^\tau u_0 \left(\frac{\partial^2}{\partial s^2} u_0(x,s) + k_2 u_0(x,s) \right) + \\ - u_0 \left(\frac{\partial^2}{\partial s^2} u_0(y,s) + k_2 u_0(y,s) \right) \Big| \leq \\ \leq \left(L_0(t) + \int_0^t \int_0^\tau L_0(s)^2 ds d\tau \right) |x-y| \end{aligned}$$

Then, if we define, by recurrence:

$$\begin{cases} C_0(t) \equiv 0 \\ C_{n+1}(t) = L_n(t)(L_n(t) + C_n(t)), \quad \forall n \in \mathbb{N} \end{cases}$$

and

$$L_{n+1}(t) = L_0(t) + \int_0^t \int_0^\tau L_n(s)(L_n(s) + C_n(s)) ds d\tau,$$

we obtain that, for every $n \in \mathbb{N}$:

$$|u_{n+1}(x,t) - u_n(x,t)| \leq L_{n+1}(t)|x-y|;$$

Moreover:

$$\begin{aligned} \left| \frac{\partial^2}{\partial t^2} u_1(x,t) - \frac{\partial^2}{\partial t^2} u_1(y,t) \right| \leq \Big| u_0 \Big(\frac{\partial^2}{\partial t^2} u_0(x,t) + k_2 u_0(x,t) \Big) + \\ - u_0 \Big(\frac{\partial^2}{\partial t^2} u_0(y,t) + k_2 u_0(y,t) \Big) \Big|. \end{aligned}$$

Since $0 < L_\alpha < 1/2$, there follows that there exists $M > 0$ such that:

$$L_\alpha < M \leq \frac{1}{2}.$$

Let us fix $T_0 > 0$ in such a way that:

$$0 \le t \le T_0 \Rightarrow \begin{cases} L_\alpha + tL_\beta + M\frac{t^2}{2} \le M \\ M + (1+M)\frac{t^2}{2} \le h < 1 \end{cases}$$

We observe what follows:

$$L_1(t) = L_\alpha + tL_\beta + L_\alpha^2 \frac{t^2}{2} \le L_\alpha + tL_\beta + M\frac{t^2}{2} \le M.$$

Since

$$C_1(t) = L_0^2(t) = (L_\alpha + tL_\beta)^2 \le L_\alpha + tL_\beta \le M,$$

there follows

$$C_2(t) = L_1^2(t) + L_1(t)C_1(t) \le M^2 + M \cdot M = 2M^2 \le M.$$

Moreover

$$\begin{aligned} L_2(t) = L_0(t) + \int_0^t \int_0^\tau (L_1^2(s) + L_1(s)C_1(s)) ds d\tau = \\ = L_\alpha + tL_\beta + \int_0^t \int_0^\tau (M^2 + M^2) ds d\tau = \\ \le L_\alpha + tL_\beta + M\frac{t^2}{2} \le M; \end{aligned}$$

and then

$$C_3(t) = L_2^2(t) + L_2(t)C_2(t) \le M^2 + M^2 \le M.$$

In conclusion, it is possible to prove, by induction:

$$0 \le L_n(t) \le M, \quad \forall n \in \mathbb{N}, \forall t \in [0, T_0].$$

Using formulae (2.4) and (2.5), we obtain, for all $t \in [0, T_0]$ and $n \in \mathbb{N}$:

$$\begin{aligned} 0 \le B_{n+1}(t) \le A_n(t)(1+M) + MB_n(t) \le \\ \le \|B_n\|_\infty \left(\frac{t^2}{2}(1+M) + M \right). \end{aligned}$$

Hence, there follows that for all $n \in \mathbb{N}$:

$$\frac{\|B_{n+1}\|_\infty}{\|B_n\|_\infty} \le h < 1.$$

Then the series

$$\sum B_{n+1}(t)$$

is totally convergent and then there exists v_∞ such that:

$$\frac{\partial^2}{\partial t^2} u_n(x,t) \rightrightarrows v_\infty(x,t), \quad x \in \mathbb{R}, t \in [0, T_0],$$

and, since

$$\|A_{n+1}\|_\infty \le \|B_{n+1}\|_\infty \frac{T_0^2}{2}$$

we can deduce that

$$\sum A_{n+1}(t)$$

is totally convergent. This implies that there exists u_∞ such that:

$$u_n(x,t) \rightrightarrows u_\infty(x,t), \quad x \in \mathbb{R}, t \in [0, T_0].$$

Since

$$\left| u_n \left(\frac{\partial^2}{\partial t^2} u_n(x,t) + k_2 u_n(x,t), t \right) - u_\infty(v_\infty(x,t) + k_2 u_\infty(x,t), t) \right|$$

$$\le \|u_n - u_\infty\|_\infty + L_\infty(t) \left(\|u_n - u_\infty\|_\infty + \left\| \frac{\partial^2}{\partial t^2} u_n - v_\infty \right\|_\infty \right) \to 0,$$

we can deduce that:

$$u_\infty(x,t) = \alpha(x) + t\beta(x) + \int_0^t k_1 \int_0^\tau u_\infty(v_\infty(x,s) + k_2 u_\infty(x,s), s) ds.$$

Then the conclusion follows if we prove that $v_\infty = \partial^2 u / \partial t^2$; but, since

$$\begin{aligned}
\left| v_\infty(x,t) - \frac{\partial^2}{\partial t^2} u_\infty(x,t) \right| &\le \|v_\infty - \frac{\partial^2}{\partial t^2} u_n\|_\infty + \\
&+ \left| \frac{\partial^2}{\partial t^2} u_n(x,t) + \frac{\partial^2}{\partial t^2} u_\infty(x,t) \right| + \\
&\le \|v_\infty - \frac{\partial^2}{\partial t^2} u_n\|_\infty + \\
&+ \left| u_{n-1} \left(\frac{\partial^2}{\partial t^2} u_{n-1}(x,t) + k_2 u_{n-1}(x,t), t \right) + \right. \\
&\left. + u_\infty \left(v_\infty(x,t) + k_2 u_\infty(x,t), t \right) \right| \le \\
&\le \left\| v_\infty - \frac{\partial^2}{\partial t^2} u_{n-1} \right\|_\infty + \|u_{n-1} - u_\infty\|_\infty + \\
&+ L_\infty(t) \Big(\|u_{n-1} - u_\infty\|_\infty + \left\| \frac{\partial^2}{\partial t^2} u_{n-1} - v_\infty \right\|_\infty \Big) \\
&\longrightarrow 0,
\end{aligned}$$

we can deduce that:

$$u_\infty(x,t) = \alpha(x) + t\beta(x) + \\ + \int_0^t \int_0^\tau k_1 u_\infty \left(\frac{\partial^2}{\partial s^2} u_\infty(x,s) + k_2 u_\infty(x,s), s \right) ds d\tau,$$

with $x \in \mathbb{R}$, $t \in [0, T_0]$, and then u_∞ is a solution of (2.3). We prove now that (2.3) has a unique solution; let us then assume that there exists an other solution $v = v(x,t)$;

$$\begin{aligned} |v(x,t) - u_\infty(x,t)| \le \int_0^t \int_0^\tau \Big| & v\left(\frac{\partial^2}{\partial s^2} v(x,s) + k_2 v(x,s), s \right) + \\ & + u_\infty \left(\frac{\partial^2}{\partial s^2} v(x,s) + k_2 v(x,s), s \right) + \\ & + u_\infty \left(\frac{\partial^2}{\partial s^2} v(x,s) + k_2 v(x,s), s \right) + \\ & - u_\infty \left(\frac{\partial^2}{\partial s^2} u_\infty(x,s) + k_2 u_\infty(x,s), s \right) \Big| ds d\tau. \end{aligned}$$

Hence:

$$\begin{aligned} |v(x,t) - u_\infty(x,t)| \le \|v - u_\infty\|_\infty \int_0^t \int_0^\tau (1 + L_\infty(s)) ds d\tau + \\ + \int_0^t \int_0^\tau L_\infty(s) \left| \frac{\partial^2}{\partial s^2} u_\infty(x,s) - \frac{\partial^2}{\partial s^2} v(x,s) \right| ds d\tau. \end{aligned} \tag{2.6}$$

But:

$$\begin{aligned} \left| \frac{\partial^2}{\partial t^2} v(x,t) - \frac{\partial^2}{\partial t^2} u_\infty(x,t) \right| \le \Big| & v\left(\frac{\partial^2}{\partial t^2} v(x,t) + k_2 v(x,t), t \right) + \\ & - u_\infty \left(\frac{\partial^2}{\partial t^2} v(x,t) + k_2 v(x,t), t \right) + \\ & + u_\infty \left(\frac{\partial^2}{\partial t^2} v(x,t) + k_2 v(x,t), t \right) + \\ & - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x,t) + k_2 u_\infty(x,t), t \right) \Big| \le \\ & \le \|v - u_\infty\|_\infty + L_\infty \Big(\|v - u_\infty\|_\infty + \\ & + \left\| \frac{\partial^2}{\partial t^2} v - \frac{\partial^2}{\partial t^2} u_\infty \right\|_\infty \Big) \\ & \le (M+1) \|v - u_\infty\|_\infty + \\ & + M \left\| \frac{\partial^2}{\partial s^2} v - \frac{\partial^2}{\partial s^2} u_\infty \right\|_\infty . \end{aligned}$$

This implies that:

$$\left\|\frac{\partial^2}{\partial t^2}v-\frac{\partial^2}{\partial t^2}u_\infty\right\|_\infty(1-M)\leq(1+M)\|v-u_\infty\|_\infty.$$

Then

$$\left\|\frac{\partial^2}{\partial t^2}v-\frac{\partial^2}{\partial t^2}u_\infty\right\|_\infty\leq\frac{1+M}{1-M}\|v-u_\infty\|_\infty.$$

From (2.6) we obtain that, for every $t\in[0,T_0]$:

$$\begin{aligned}|v(x,t)-u_\infty(x,t)|&\leq\|v-u_\infty\|_\infty\left((1+M)\frac{t^2}{2}\right)+\frac{1+M}{1-M}\frac{t^2}{2}M\|v-u_\infty\|_\infty=\\&=\|v-u_\infty\|_\infty\left(\frac{t^2}{2}\Big(1+M+\frac{1+M}{1-M}M\Big)\right)=\\&=\|v-u_\infty\|_\infty\frac{t^2}{2}(1+M)\frac{1-M+M}{1-M}=\\&=\|v-u_\infty\|_\infty\frac{t^2}{2}\frac{1+M}{1-M}<\|v-u_\infty\|_\infty,\end{aligned}$$

whence the uniqueness.

In this way we have proved the following local existence and uniqueness Theorem.

THEOREM 1. *Let $\alpha,\beta\in L^\infty(\mathbb{R},\mathbb{R})\cap Lip(\mathbb{R},\mathbb{R})$ be given functions with Lipschitz constant L_α and L_β respectively and such that $L_\alpha<1/2$; then there exists $T_0>0$ and there exists a unique $u:\mathbb{R}\times[0,T_0]\to\mathbb{R}$ bounded, Lipschitz in the x variable, uniformly with respect to t(and Lipschitz in t uniformly with respect to x and continuous in both variables) with second derivative in t continuous and such that*

$$u(x,t)=\alpha(x)+t\beta(x)+\int_0^t\int_0^\tau k_1u\left(\frac{\partial^2}{\partial s^2}u(x,s)+k_2u(x,s),s\right)dsd\tau$$

for all $x\in\mathbb{R}$, $t\in[0,T_0]$ and then

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}u(x,t)=k_1u\left(\dfrac{\partial^2}{\partial t^2}u(x,t)+k_2u(x,t),t\right)\\u(x,0)=\alpha(x)\\\dfrac{\partial}{\partial t}u(x,0)=\beta(x).\end{cases}$$

Remark 2

1. From the previous proof it is clear that the condition $|k_1| = |k_2| = 1$ is not a restrictive hypothesis; in fact we can also assume that $k_i = k_i(x,t)$, beeing careful to insure that $|k_i(x,t)| \leq c \quad \forall x \in R, t \geq 0$ for some constant $c > 0$.
2. It is easy to notice that similar theorems of existence and uniqueness for small time can be established also for equations of the type:

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u(\frac{\partial^2}{\partial t^2}u(x,t) + k_3 u(x,t), t), t\right),$$

that we can denote by $E_{(k_1,k_2,k_3)}$. For these equations the existence time for the solution established using the proof of the Theorem is sensibly reduced.

In view of the previous remarks and from the boundedness of the approximating sequences used in the proof of the Theorem, we can state the following problems.

A) Find some condition assuring that the solution of $E_{(k_1,k_2,\ldots,k_i)}$ exists for all time.

B) An interesting problem is to give existence result for the equation when α and β are continuous but not Lipschitz functions.

C) Given a sequence (k_i) of real numbers with $|k_i| \leq 1 \quad \forall i$ and assuming that each equation $E_{(k_1,k_2,\ldots,k_n)}$ admits a unique solution $v_n = v_n(x,t)$, defined for every time and satisfying the initial condition $v_n(x,0) = \alpha(x)$, $\frac{\partial}{\partial t}v_n(x,0) = \beta(x)$, beeing α and β given, study the behaviour of the sequence (v_n).

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