Rendiconti di Matematica, Serie VII
Volume 25, Roma (2005), 195-221

Quadrature formulae of non-standard type

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Dedicated to the 70th anniversary of Professor Laura Gori

Abstract: *We discuss quadrature formulae of highest algebraic degree of precision for integration of functions of one or many variables which are based on non-standard data, i.e., in which the information used is different from the standard sampling of function values. Among the examples given in this survey is a quadrature formula for integration over the disk, based on linear integrals on n chords, which integrates exactly all bivariate algebraic polynomials of degree* $2n - 1$.

1 – Introduction

The *standard information* used in univariate approximation methods consists of function values at a finite number of points. Important algorithms in numerical analysis dealing with recovery of functions, integrals, solutions of differential equations, zeros of functions and other quantities are based on data of function values. Most of the classical formulae are of this type. Famous examples are the Lagrange interpolation formula and the Newton-Cotes quadrature rules. It seems that, in the univariate case, sampling of function values is the most natural way of collecting information about a function. Besides, in practical problems, the standard outcome of experimental procedures and measurements is a function evaluation. That is why, studying approximation problems in the

Key Words and Phrases: *Gaussian formula – Numerical integration – Multivariate functions.*
A.M.S. Classification: 65D30 – 41A55

univariate case one naturally starts with: given a set of function values

$$f(x_1), \ldots , f(x_n)$$

find a method for approximate calculation of the quantity $L[f]$ such that ...

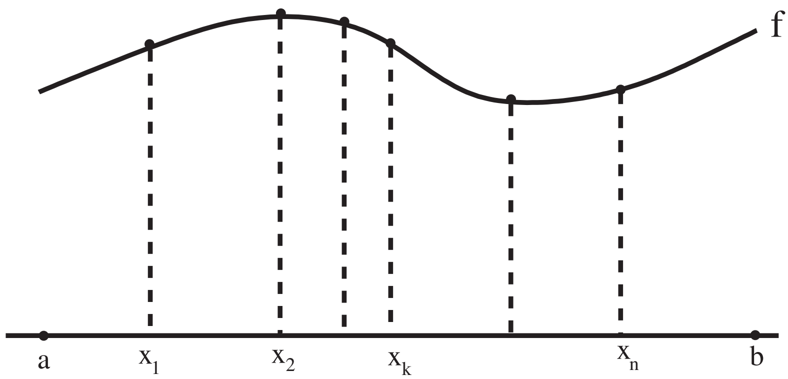

Fig. 1: Standard information

This natural statement of the problem often leads to a nice solution. A remarkable illustration is the Gauss formula. Namely the Gauss idea of constructing a formula which is the best one with respect to the class of algebraic polynomials, and its recent developments, is the topic of the present survey.

In 1814 Carl Friedrich Gauss [16] proved the following:

For every n there exists a unique quadrature formula of the form

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} A_k f(x_k)$$

which integrates exactly all algebraic polynomials of degree $\leq 2n-1$.

As is easily seen, $2n-1$ is the highest algebraic degree of precision that can be achieved by a formula using n values of the function. Gauss quadrature have been extended in various directions. The idea of constructing an approximation rule of a given type that has a highest degree of precision with respect to a certain class of functions became a dominating one in classical numerical analysis. The story was excellently presented in the extensive historical survey by Walter Gautschi [17].

The direct extension of Gauss formula to functions of many variables was a task of many generations of mathematicians. In order to formulate the corresponding multivariate problem let us recall first some definitions and notations.

The set of all real algebraic polynomials of *total degree* n of d variables will be denoted by $\pi_n(\mathrm{I\!R}^d)$, i.e.,

$$\pi_n(\mathrm{I\!R}^d) := \left\{ \sum_{|\mathbf{k}|=0}^{n} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \right\},$$

where

$$\mathbf{k} = (k_1, \ldots, k_d), \quad |\mathbf{k}| = k_1 + \ldots + k_d\,,$$
$$\mathbf{x} = (x_1, \ldots, x_d), \quad \mathbf{x}^{\mathbf{k}} = x_1^{k_1}, \ldots, x_d^{k_d}\,.$$

For example, the space of polynomials of total degree n in two variables x, y is

$$\pi_n(\mathrm{I\!R}^2) := \left\{ \sum_{0 \le i+j \le n} c_{ij} x^i y^j \;:\; c_{ij} \;\text{ are real} \right\}.$$

As usual, the *Algebraic Degree of Precision* (ADP) of a given quadrature rule Q is defined by

$$\mathrm{ADP}(Q) := \max \{ m \;:\; Q \text{ is exact for all } f \in \pi_m(\mathrm{I\!R}^d) \}\,.$$

The *multivariate Gaussian problem* could be stated in this way:

Given N, a domain Ω in $\mathrm{I\!R}^d$ and a weight function $\mu(\mathbf{x})$ on Ω, what is the highest ADP that can be achieved by a formula of the form

$$\int_\Omega \mu(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \approx \sum_{k=1}^{N} a_k f(\mathbf{x}_k)$$

with free coefficients and nodes? Characterize the optimal location of the nodes.

The following equivalent formulation is usually considered: given n, what is the minimal number $N^*(n)$ of nodes for which there exists a cubature formula Q based on n values of the integrated function such that ADP $(Q) = 2n - 1$?

Such a formula, if it exists, is called *minimal.*

Early works on the Gauss multivariate problem have been published by J. C. Maxwell [21] in 1877, who constructed a minimal formula for an integral over the square with 7 nodes and ADP $= 5$, and by Radon [35].

Bounds of $N^*(n)$ for special domains of integration have been obtained. For example, a general result due to Möller [25] and Mysovskih ([26], Theorem 9.2) for centrally symmetric domains and weight functions yields, in the case $d = 2$, the estimate

(1)
$$N^*(n) \geq \frac{1}{2}n(n+1) + \left[\frac{1}{2}n\right] .$$

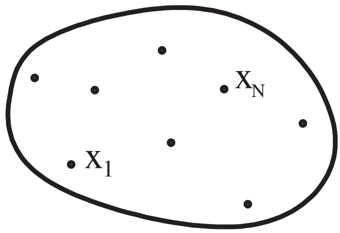

Fig. 2: Multivariate setting

Radon [34] used for the first time the orthogonal polynomials on the domain of integration to find the extremal nodes.

The relation between the nodes of the Gaussian cubature formulae and the common zeros of the polynomials that are orthogonal on Ω with respect to the weight have been described by A. H. Stroud and D. Secrest [38], and I. P. Mysovskih [26]. Further developments in this direction are summarized in the recently published book by Dunkl and Xu [15].

Tables of nodes and coefficients of particular cubature formulae with a small number of nodes for the disk, the simplex, and the square are published in books and web pages. But still there is no general multivariate result, even for the simplest domains, like the disk, which can be considered as an analog of the famous quadrature of Gauss. More precisely, there is no result which says that for a certain domain Ω and every n there exists a cubature formula, based on n values of the integrated function, having highest degree of precision and the location of the nodes of this cubature is described completely. Only recently a formula for integration on the square was derived that could be interpreted as an analog of the Gauss formula. It concerns integration over the unit square with the Tchebycheff weight

$$w_T(x, y) := \{(1 - x^2)(1 - y^2)\}^{-1/2} .$$

The theorem below describes a set of Gaussian nodes for every even number of nodes n. It was proved by Morrow and Patterson [23], and later, in a different way, also by Yuan Xu [40], and in [10].

Let

$$\eta_j := \cos j\pi/n, \quad j = 0, \ldots, n\,,$$

be the extremal points of the Tchebycheff polynomial of the first kind

$$T_n(x) := \cos n \arccos x, \quad x \in [-1,1]\,.$$

Let us set $R := [-1,1] \times [-1,1]$.

Theorem of Morrow and Patterson. *For each even n the cubature formula*

$$\int_R w_T(x,y) f(x,y)\,dxdy \approx \frac{\pi}{n} \sum_{\text{odd}\,k}{}' \sum_{\text{even}\,j}{}' f(\eta_k, \eta_j) \tag{2}$$

is exact for all polynomials $f \in \pi_{2n-1}(\mathbb{R}^2)$. Moreover, (2) *is minimal.*

For example, if we take $n = 4$ then, by (1), $N^*(4) \geq 12$, and thus the cubature based on the 12 nodes, marked by small squares on Fig. 3, is minimal.

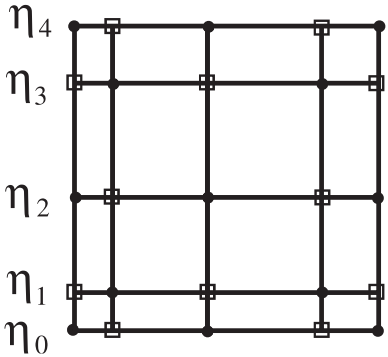

Fig. 3: Gaussian nodes for the square

The following more general result which yields the above theorem as a corollary was proved in [10].

Consider the univariate quadrature formula

$$\int_{-1}^{1} \mu(t) f(t)\,dt \approx \sum_{j=1}^{s} a_j f(t_j) =: S[f]\,, \tag{3}$$

with a certain positive integer s, and set

$$O[f] := \sum_o := \text{ the odd part of } S[f]\,,$$
$$E[f] := \sum_e := \text{ the even part of } S[f]\,.$$

Let $\{P_k\}$ be the sequence of orthogonal polynomials, associated with the weight $\mu(t)$ on $[-1,1]$.

THEOREM 1.1. *Assume that the quadrature formula* (3) *is exact for all polynomials of degree* $2n-1$ *and satisfies the conditions*

$$E[P_k] = O[P_k], \quad k = 0, \ldots, n-1\,.$$

Then the cubature formulae

$$\int_R \mu(x)\mu(y) f(x,y)\,dx\,dy \approx 2(O_x O_y + E_x E_y)[f]\,,$$

$$\int_R \mu(x)\mu(y) f(x,y)\,dx\,dy \approx 2(O_x E_y + E_x O_y)[f]\,,$$

are exact for every bivariate polynomial f *of total degree less than or equal to* $2n-1$.

In the above $E_x O_y$ means that first we apply the operator $O[\cdot]$ to $f(x,y)$ as a function of y and then we apply $E[\cdot]$ to the expression obtained as a function of x.

Choosing now $\mu(t) = (1-t^2)^{-1/2}$ and the Lobatto quadrature formula in the role of (3), we obtain the result of Morrow and Patterson.

Note that in the last example about integration on the square the optimal nodes are described for even n but the uniqueness of the extremal configuration is not yet proved. The optimal nodes for odd n are still not known.

The difficulties in extending the Gauss formula to the multivariate case in its classic form, keeping as basic information the sampling of function values, should have given a hint to the researchers that one might have a look at other, eventually more natural setting of the question in the space of multivariate polynomials. Hakopian's interpolation formula is an encouraging example in this direction. As is well known, the Lagrange interpolation problem for multivariate polynomials is not always regular. The regularity depends on the location of the nodes. Hakopian considered in [18] interpolation of data consisting of integrals over intersections of hyperplanes and a given convex body in $\mathbb{R}^d$. In particular, for interpolation on the plane, the result reduces to the following.

For any given $n+2$ distinct points $M_0, \dots, M_{n+1}$ on the boundary Γ of a convex body D (say, a disk), the set of integrals of f over all the linear segments $[M_i, M_j]$, determines uniquely every polynomial f of degree n. Note that, in the bivariate case, Hakopian's interpolation coincides with another multivariate interpolation, considered by Cavaretta, Micchelli and Sharma [13].

In this survey we present some recent results which are analogs of the Gauss formula and which are based on evaluation of averaged values of the integrated function over linear segments or circles.

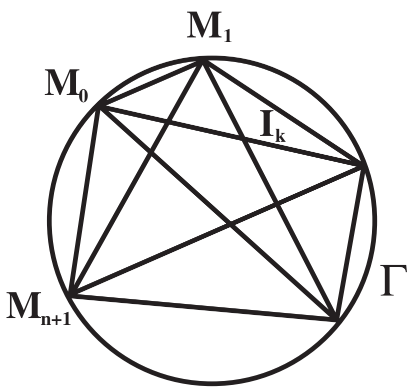

Fig. 4: Hakopian's interpolation

2 – A cubature formula for integration over the disk

In this section we give an example of a cubature formula on the unit ball which has a highest algebraic degree of precision among all formulae of a prescribed type; it is unique, and the optimal location of the "nodes" is completely characterized. Thus, it supplies the only known natural extension of the classical Gauss formula to the multivariate case. Instead of point evaluations $\{f(\mathbf{x}_k)\}$ we shall use averaged values of f over intersections of the ball with n hyperplanes.

For the sake of simplicity, we shall present the result first for integration on the disk

$$D := \{(x,y) : x^2 + y^2 \le 1\}$$

in the plane. In this important case, besides its theoretical value, the formula is of practical interest since the information used, integrals $\{\int_{I_k} f\}$ over linear segments, is a natural outcome in tomography and electronic microscopy.

The parameters

$$(\theta_k, t_k), \quad \theta_k \in [0,\pi),\ 0 \le t_k \le 1, \quad k = 1, \dots, n\,,$$

define the chords I_k by

$$I_k := \{(x, y) : x\cos\theta_k + y\sin\theta_k = t_k\} \cap D .$$

The line integrals

$$\int_{I_k} f := \int_{-\sqrt{1-t_k^2}}^{\sqrt{1-t_k^2}} f(t_k\cos\theta_k - s\sin\theta_k, t_k\sin\theta_k + s\cos\theta_k)\, ds ,$$

for $k = 1, \ldots, n$, constitute the information set to be used for approximate evaluation of the integral over the disk D. In other words, the goal is to construct a cubature formula of the form

$$\int_D f(x,y)\, dxdy \approx \sum_{k=1}^{n} C_k \int_{I_k} f \tag{4}$$

of highest ADP with respect to the space of bivariate polynomials. The coefficients $\{C_k\}$, the angles $\{\theta_k\}$, and the numbers $\{t_k\}$, (which describe the location of the chords I_k, are free parameters.

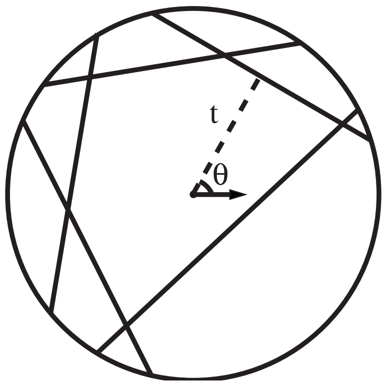

Fig. 5: A general configuration of chords

There is a great variety of such formulae. The extremal one, that one of highest ADP will be refereed to as a formula of *Gaussian type* or *Gaussian cubature*.

Let us make some simple observations concerning the Gaussian cubature.

(i) Clearly, ADP $((4)) \le 2n-1$.

PROOF. Let $l_k(x,y)$ be the linear polynomials associated with the chords I_k, that is,

$$l_k(x,y) := x\cos\theta_k + y\sin\theta_k - t_k\,.$$

Take the polynomial

$$\omega^2(x,y) := l_1^2(x,y)\dots l_n^2(x,y)\,,$$

which is of degree $2n$. Then, evidently, formula (4) is not exact for $\omega^2(x,y)$ since the integral is strictly positive while the approximating sum vanishes.

(ii) If ADP ((4)) $= 2n-1$, then $\omega(x,y)$ is orthogonal on the disk D to every bivariate polynomial of total degree less than or equal to n (i.e., $\omega(x,y) \perp \pi_{n-1}(\mathbb{R}^2)$).

PROOF. Let ADP ((4)) $= 2n-1$. If $P \in \pi_{n-1}(\mathbb{R}^2)$, then $\omega P \in \pi_{2n-1}(\mathbb{R}^2)$ and thus, by (4),

$$\int_D P\omega\,dx\,dy = \sum_{k=1}^{n} C_k \int_{I_k} P\omega = 0 \quad \text{since} \quad P\omega = 0 \quad \text{on} \quad I_k\,.$$

Therefore, $\omega(x,y) \perp P$.

According to (i), the highest ADP one may hope to attain by a formula of type (4) is $2n-1$. On the one hand, it seems unbelievable to get such a precision on the basis of only n evaluations since the space of bivariate polynomials of degree $2n-1$ is of dimension $n(2n+1)$, i.e., a number much bigger than the number of free parameters in the formula (4), which is $3n$. On the other hand, the symmetry of the region of integration, the disk, would be very helpful in solving the problem. So, the chance of constructing a formula (4) of $ADP = 2n-1$ does not seem bad. According to (ii), in order to construct such a formula, one should study the orthogonal polynomials on the disk. In difference to the univariate case, where the orthogonal polynomial of a given degree n is determined uniquely by its leading coefficient, there are many bivariate polynomials of total degree n which are orthogonal on the disk D to every polynomial of degree $n-1$. In view of (ii) we have to study those that can be factorized in linear factors. Namely, these linear factors would eventually determine the chords of the optimal cubature formula. Hence, like in the univariate case, the Gaussian problem in this new setting is again related to the theory of orthogonal polynomials. In our study with PETROVA [11], [12] of the Gaussian formula (4) we use the ridge orthogonal polynomials, defined by the *Tchebycheff polynomials* of the second kind

$$U_m(t) := \frac{1}{\sqrt{\pi}}\frac{\sin(m+1)\theta}{\sin\theta}, \quad t = \cos\theta\,.$$

It is well known (see, for example, [36]) that $U_m(t)$ is orthogonal to all polynomials of degree less than m on $[-1,1]$ with respect to the weight function $\sqrt{1-t^2}$.

Every univariate function $\rho(t)$ generates a *ridge function* $\rho(\theta;x,y)$ with a profile ρ and direction θ in the following way

$$\rho(\theta;x,y) := \rho(x\cos\theta + y\sin\theta)\,.$$

In other words, the surface of the ridge function $\rho(\theta;x,y)$ is obtained from the graph of the univariate function $\rho(t)$, first rotating it in angle θ, and then translating it in constant direction $\theta+\pi/2$.

The ridge polynomials

$$U_m(\theta;\mathbf{x}) := U_m(x\cos\theta + y\sin\theta)$$

play a fundamental role in the class of bivariate polynomials. They are orthogonal on the unit disk D to every polynomial from $\pi_{m-1}(\mathbb{R}^2)$, i.e., for each θ we have

$$\int_D U_m(\theta;\mathbf{x})P(\mathbf{x})\,d\mathbf{x} = 0 \qquad \text{for every } P\in\pi_{m-1}(\mathbb{R}^2)\,.$$

In particular, the ridge polynomials $U_m(\theta;t)$ with equally spaced angles θ form an orthogonal basis on D in the space of bivariate polynomials, as can be seen from the following proposition from [11], Lemma 6.

Theorem 2.1. *Let*

$$\theta_{mj} := \frac{j\pi}{m+1}, \qquad m\in\mathbb{N},\ j=0,\dots,m\,,$$

and

$$U_{mj}(\mathbf{x}) := U_m(\theta_{mj};\mathbf{x})\,.$$

The ridge polynomials $\{U_{mj}\}$ form an orthonormal basis in $\pi_n(\mathbb{R}^d)$ on D.

Using essentially the properties of $\{U_{mj}(\mathbf{x})\}$ we constructed in [11] a cubature formula of form (4) with ADP $=2n-1$. The extremal formula is described in the next theorem.

Let us denote by $\eta_1,\dots,\eta_n$ the zeros of $U_n(x)$. More precisely,

$$\eta_k := \cos\frac{k\pi}{n+1}, \qquad k=1,\dots,n\,.$$

Theorem 2.2. *The quadrature formula*

$$(*) \qquad \int_D f(x,y)\,dx\,dy \approx \sum_{k=1}^n A_k \int_{-\sqrt{1-\eta_k^2}}^{\sqrt{1-\eta_k^2}} f(\eta_k,y)\,dy\,,$$

with

$$A_k = \frac{\pi}{n+1} \sin \frac{k\pi}{n+1}, \quad k = 1, \ldots, n,$$

is exact for each polynomial $f \in \pi_{2n-1}(\mathbb{R}^2)$.

Making use of the high precision of the formula, one can show in the usual way that the coefficients A_k are positive.

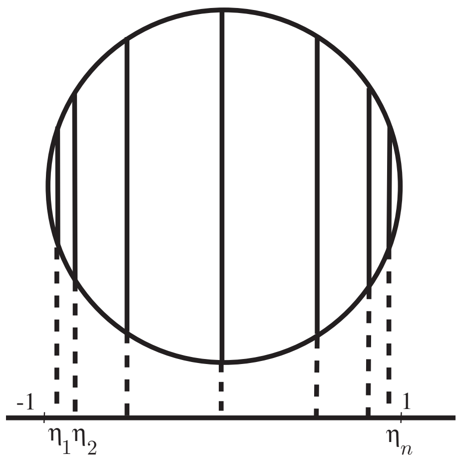

Fig. 6: Optimal location of the chords

Note that the classical univariate Gauss quadrature formula is based on n values of the function and it integrates exactly $2n$ basic polynomials, namely:

$$1, x, x^2, \ldots, x^{2n-1}.$$

The cubature we presented above is based also on n evaluations (n linear integrals) and it integrates exactly $n(2n+1) = 2n^2 + n$ basis functions:

$$1, x, y, x^2, xy, y^2, \ldots, x^{2n-1}, x^{2n-2}y, \ldots, y^{2n-1},$$

that is, much more than the number of free parameters one has at disposal in a cubature formula of type (4). Of course, this is due to the symmetry of the region of integration, but it also shows the power of the integration rule given in Theorem 2.2.

As we have mentioned already, there are practical problems, like in tomography, where the output is a collection of linear integrals. Thus, formula $(*)$

would be useful there. But even if we have to approximate an integral over the disk on the basis of function values, we could still use $(*)$ in the following way. First we evaluate the linear integrals over the chords using the corresponding Gauss formula with n nodes and then use $(*)$ to compute the integral over the disk. In this way we would need the value of the function at n^2 points from the disk. This is approximately half the number $n(2n+1)$ of values needed to calculate the integral by an interpolatory type cubature formula with the same ADP equal to $2n-1$. Therefore, even in such a situation formula $(*)$ excels the standard interpolatory type formulae.

It was quite a challenging problem to show that the constructed formula is unique up to rotation. It was done in [12]. Moreover, the result was extended to the multivariate case in the following form.

For every $\mathbf{x} = (x_1, \dots, x_d)$ from $\mathbb{R}^d$ we set

$$\|\mathbf{x}\| := (x_1^2 + \ldots + x_d^2)^{1/2},$$

$$\mathbf{B}^d(r) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \le r\}.$$

In particular, $\mathbf{B}^d := \mathbf{B}^d(1)$. We consider quadrature formulae on $\mathbf{B}^d$ in the case of a ridge weight function μ. Recall that a function G on $\mathbb{R}^d$ is called a *ridge function* if $G(\mathbf{x}) = g(\xi \cdot \mathbf{x})$ with a certain univariate function g and some ξ on the unit sphere

$$S^{d-1} := \partial \mathbf{B}^d = \{\mathbf{x} \in \mathbb{R}^d \,:\, \|\mathbf{x}\| = 1\}.$$

With every vector $\xi \in S^{d-1}$, $\xi = (\xi_1, \dots, \xi_d)$, $\xi_d \ge 0$, and a number t we associate the hyperplane $\beta(\xi, t)$ that is perpendicular to ξ and passes through the point $t\xi$.

We consider a class of cubature formulae for the ball $\mathbf{B}^d$ based on integrals

$$\mathcal{R}(f; \xi_k, t_k) := \int_{\beta(\xi_k, t_k) \cap \mathbf{B}^d} f(\mathbf{y})\, d\mathbf{y}, \quad -1 \le t_k \le 1, \quad k = 1, \dots, n,$$

of f over intersections of $\mathbf{B}^d$ with n hyperplanes $\{\beta(\xi_k, t_k)\}$.

For a given weight $\mu(\mathbf{x})$ on $\mathbf{B}^d$ we look for a cubature formula of the form

$$\int_{\mathbf{B}^d} \mu(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \approx \sum_{k=1}^{n} A_k \mathcal{R}(f; \xi_k, t_k) \tag{5}$$

of highest ADP. We allow some of the t_k's to be equal to 1. Then we interpret $\mathcal{R}(f; \xi_k, t_k)$ as $f(\xi_k)$. Such an interpretation is justified by the continuity argument

$$\lim_{t_k \to 1} \frac{1}{\mathrm{Vol}_{d-1}\{\beta(\xi_k, t_k) \cap \mathbf{B}^d\}} \int_{\beta(\xi_k, t_k) \cap \mathbf{B}^d} f(\mathbf{y})\, d\mathbf{y} = f(\xi_k).$$

It can be easily derived that

$$\mathrm{ADP}\,((5)) < n + n_0$$

where n_0 is the number of t_k's for which $|t_k| < 1$.

The construction of a formula of ADP equal to $n + n_0 - 1$ was given in [12]. The extremal cubature is based on intersections with n parallel hyperplanes $\{B(\tau_k)\}$, where $B(t)$ is the intersection of $\mathbf{B}^d$ with the hyperplane in $\mathbb{R}^d$ which is perpendicular to the Ox_1 axis and passes through the point $(t, 0, \dots, 0)$. The optimal hyperplanes are defined by the zeros $\{\tau_k\}$ of the Gegenbauer orthogonal polynomial $C_n^{(d)}$ of degree n. Recall that

$$\int_{-1}^{1} C_n^{(\lambda)}(t) C_m^{(\lambda)}(t)(1-t^2)^{\lambda-1/2}\,dt = 0, \quad n \neq m\,.$$

The construction of the cubature relies on the following theorem which reveals a one-to-one correspondence between the univariate quadratures on $[-1, 1]$ and a class of cubature formulae on $\mathbf{B}^d$.

THEOREM 2.3. *Assume that μ is an arbitrary weight function on $[-1, 1]$ and d is a natural number. The formula*

$$\int_{\mathbf{B}^d} \mu(x_1) f(\mathbf{x})\,d\mathbf{x} \approx \sum_{k=1}^{n} \frac{a_k}{(1-\zeta_k^2)^{(d-1)/2}} \int_{B(\zeta_k)} f(\mathbf{x})\,d\mathbf{x}$$

is exact for all elements in $\pi_N(\mathbb{R}^d)$ if and only if the quadrature formula

$$\int_{-1}^{1} \mu(t)(1-t^2)^{(d-1)/2} p(t)\,dt \approx \sum_{k=1}^{n} a_k p(\zeta_k), \quad -1 \leq \zeta_1 < \ldots < \zeta_n \leq 1$$

integrates exactly all polynomials from $\pi_N(\mathbb{R})$.

If $\zeta_1 = -1$ (or $\zeta_n = 1$), the corresponding term in the formula has to be interpreted as

$$a_1 \operatorname{Vol} \mathbf{B}^{d-1} f(-1, 0, \dots, 0) \quad (\text{or } a_n \operatorname{Vol} \mathbf{B}^{d-1} f(1, 0, \dots, 0))\,.$$

In order to prove the uniqueness we first show that every cubature with highest ADP must use parallel intersections. This is the most difficult part of the proof. The main ingredient is a recurrence relation between the set of orthogonal polynomials on the unit ball in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$. Then as a corollary of the above theorem and the uniqueness of the univariate orthogonal polynomials we derive the uniqueness of the Gaussian cubature.

THEOREM 2.4. *For every natural n and dimension d there is a unique (up to rotation) cubature formula of the form* (5) *which integrates exactly all algebraic polynomials from $\pi_{2n-1}(\mathbb{R}^d)$.*

Detailed proof can be seen in [12].

3 – Integration of polyharmonic functions

One of the characteristic properties of the univariate algebraic polynomials of degree n is that they are annihilated by the operator of taking the $(n+1)$st derivative. This was used as a motivation to study approximation properties of other classes of functions defined as null spaces of important differential (and other) operators. In this way, many interesting classical results concerning algebraic polynomials have been extended to wider classes of generalized polynomials. This approach opens a vast and rich field of investigation in multivariate approximation theory. Here we shall concentrate on some results of numerical integration in the space of polyharmonic functions.

Recall that f is said to be a *polyharmonic function* of order m on the ball $\mathbf{B}^d(r)$ if

$$\Delta^m f = 0 \quad \text{on} \quad \text{on } \mathbf{B}^d(r)\,,$$

where, as usual, the iterates Δ^m of Laplace operator Δ in $\mathbb{R}^d$ are defined recursively by

$$\Delta f = \Delta^1 f := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}, \quad \Delta^m := \Delta\Delta^{m-1}\,.$$

Let us denote by $H^{(m)}(\mathbf{B}^d(r))$ the set of all functions u that are polyharmonic of order m in a domain containing $\mathbf{B}^d(r)$.

Note that the algebraic polynomials of degree $2m-1$ are polyharmonic functions of order m, i.e.,

$$\pi_{2m-1}(\mathbb{R}^d) \subset H^{(m)}(\mathbf{B}^d(r))\,.$$

Thus, any approximation rule that is exact for all polyharmonic functions of order m is exact also for the class of algebraic polynomials of degree $2m-1$. We shall use this to derive approximation formulae with high algebraic degree of precision constructing formulae that are good for polyharmonic functions. This approach is quite successful since the theory of polyharmonic functions is well developed and operates with powerful analytic tools.

We shall consider integration of polyharmonic functions on the ball $\mathbf{B}^d(r)$ in $\mathbb{R}^d$ of radius r and center at the origin.

Our goal is to construct cubature formulae for the integral

$$\int_{\mathbf{B}^d(r)} f(\mathbf{x})\, d\mathbf{x}$$

of highest possible *polyharmonic order of precision* (PHOP), that is, formulae that integrate exactly all polyharmonic functions of order as high as possible.

We shall follow the concept set up by the following famous examples treating harmonic functions (which are polyharmonic of order 1):

1. Gauss mean-value theorem for harmonic functions h:

$$\int_{\mathbf{B}^d(r)} h(\mathbf{x})\, d\mathbf{x} = \mathrm{Vol}\mathbf{B}^d(r)\, h(0)\,.$$

   Here
$$\mathrm{Vol}\,\mathbf{B}^d(r) = r^d \left[\pi^{d/2}/\Gamma(d/2+1)\right]\,.$$

2. A corollary from Green's formula

$$\int_{\mathbf{B}^d(r)} h(\mathbf{x})\, d\mathbf{x} = \frac{r}{d}\int_{S(r)} h(\xi)\, d\sigma(\xi)\,.$$

   Here $S(r)$ denotes the boundary of $\mathbf{B}^d(r)$, i.e.,

$$S(r) := \{\mathbf{x} \in \mathbb{R}^d \,:\, \|\mathbf{x}\| = r\}$$

   and $d\sigma$ is the $(d-1)$-dimensional surface measure on the sphere $S(r)$.
3. Note also the following immediate consequence of the above formulae

$$\int_{S(r)} h(\xi)\, d\sigma(\xi) = \gamma_d r^{d-1} h(0), \quad \gamma_d := d\pi^{d/2}/\Gamma(d/2+1)\,.$$

   The Gauss mean-value theorem has been extended by Pizzetti [33] to the following multiple node form.

Pizzetti formula. For every polyharmonic function u of degree m on the ball $\mathbf{B}^d(r)$ there holds the formula

$$\int_{\mathbf{B}^d(r)} u(\mathbf{x})\, d\mathbf{x} = \pi^{d/2}\, r^d \sum_{k=0}^{m-1} \frac{r^{2k}}{2^{2k}\Gamma(d/2+k+1)}\, \frac{\Delta^k u(0)}{k!}\,.$$

In [5] we considered the question of constructing a formula that uses integrals of the Laplacians over a sphere of any given radius ρ. In case $\rho = 0$ we get as a particular case the Pizzetti formula. Actually, we studied this problem in a more general class of functions which we call "harmonic" span of any given system $\bar{\varphi} := \{\varphi_0(t), \ldots, \varphi_{m-1}(t)\}$ of linearly independent integrable functions on $[0, r]$.

As usual,

$$\mathrm{span}\,\bar{\varphi} := \left\{ \sum_{k=0}^{m-1} c_k \varphi_k(t) \,:\, (c_0, \ldots, c_{m-1}) \in \mathbb{R}^m \right\}\,.$$

We denote by $\mathrm{Hspan}\,\bar{\varphi}$ the *harmonic span* of the functions $\{\varphi_0(t), \dots, \varphi_{m-1}(t)\}$, that is,

$$\mathrm{Hspan}\,\bar{\varphi} := \left\{ \sum_{k=0}^{m-1} h_k(\mathbf{x})\varphi_k(\|\mathbf{x}\|) \,:\, \{h_k\} \text{ are harmonic in } \mathbf{B}^d(r) \right\}.$$

The following fundamental result is due to Almansi [1] (the proof can be seen also in [27], [2]).

Almansi theorem. *If $u \in H^m(\mathbf{B}^d(r))$, then there exist unique functions $h_0(\mathbf{x}), h_1(\mathbf{x}), \dots, h_{m-1}(\mathbf{x})$, each harmonic in $\mathbf{B}^d(r)$, such that*

$$u(\mathbf{x}) = \sum_{j=0}^{m-1} \|\mathbf{x}\|^{2j} h_j(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \mathbf{B}^d(r).$$

It is known also that any function of the form above is polyharmonic of degree m. Thus, $H^m(\mathbf{B}^d(r))$ consists of all functions that admit such a representation with harmonic $\{h_j\}$. This fact is a basis for any study of polyharmonic functions.

Note that in case $\varphi_k(t) = t^{2k}$, $k = 0, \dots, m-1$, $\mathrm{Hspan}\,\bar{\varphi}$ coincides with the space of polyharmonic functions $H^m(\mathbf{B}(r))$. Thus, the results we are going to present below lead to theorems concerning polyharmonic functions for a particular choice of the basis functions.

The idea to consider the above extension of the polyharmonic functions comes from the next simple extension of the Gauss mean-value theorem (see Lemma 1 from [5]).

Lemma 3.1. *Let $\varphi(t)$ be any integrable function on $[0, r]$. Then*

$$\int_{\mathbf{B}^d(r)} \varphi(\|\mathbf{x}\|)h(\mathbf{x})\,d\mathbf{x} = h(0) \int_{\mathbf{B}^d(r)} \varphi(\|\mathbf{x}\|)\,d\mathbf{x}$$

for every function h which is harmonic in $\mathbf{B}^d(r)$.

Using this lemma we proved in [5] the following.

Theorem 3.2. *Let $\bar{\varphi} := \{\varphi_0, \dots, \varphi_{m-1}\}$ be any system of functions from $C^{2m-2}[0, r]$ satisfying the condition*

$$V(\bar{\varphi};\rho) := \begin{bmatrix} \varphi_0(\rho^2) & \Delta\varphi_0(\rho^2) & \dots & \Delta^{m-1}\varphi_0(\rho^2) \\ \varphi_1(\rho^2) & \Delta\varphi_1(\rho^2) & \dots & \Delta^{m-1}\varphi_1(\rho^2) \\ \dots & \dots & \dots & \dots \\ \varphi_{m-1}(\rho^2) & \Delta\varphi_{m-1}(\rho^2) & \dots & \Delta^{m-1}\varphi_{m-1}(\rho^2) \end{bmatrix} \neq 0$$

for all $\rho \in [0, \sqrt{r}]$. *Then there exist coefficients* $\{a_k(\rho)\}_0^{m-1}$ *such that the cubature formula*

$$\int_{\mathbf{B}^d(r)} v(\mathbf{x})\, d\mathbf{x} = \sum_{k=0}^{m-1} a_k(\rho) \frac{1}{\gamma_d \rho^{d-1}} \int_{S(\rho)} \Delta^k v(\xi)\, d\sigma(\xi)$$

holds for every $v \in \operatorname{Hspan}\{\varphi_0(t^2), \ldots, \varphi_{m-1}(t^2)\}$. *The coefficients* $\{a_k(\rho)\}$ *are determined as solutions of a linear system.*

In the particular case when $\varphi_j(\tau) = \tau^j$, $j = 0, \ldots, m-1$, we have

$$\Delta^k \varphi_j(\rho^2) = 0 \quad \text{for} \quad k > j$$

and

$$\Delta^k[\varphi_k(\rho^2)] = c_{k0}\, k! = 2^{2k}\, k!\, \frac{\Gamma(d/2+k)}{\Gamma(d/2)} > 0\,.$$

This shows that the determinant $V[\bar{\varphi}; \rho]$ is a triangular one, with non-zero diagonal elements, and thus $V[\bar{\varphi}; \rho]$ is distinct from zero. Hence the important polyharmonic case is governed by Theorem 3.2. Let us formulate this corollary as a separate statement.

Extension of the Pizzetti formula. There exist coefficients $\{a_k(\rho)\}_0^{m-1}$ such that the cubature formula

$$\int_{\mathbf{B}^d(r)} u(\mathbf{x})\, d\mathbf{x} = \sum_{k=0}^{m-1} a_k(\rho) \frac{1}{\gamma_d \rho^{d-1}} \int_{S(\rho)} \Delta^k u(\xi)\, d\sigma(\xi)$$

holds for every polyharmonic function u of order m.

Moreover, in this case the coefficients $\{a_k(\rho)\}$ of the resulting cubature formula can be computed easily solving the triangular system

$$a_0(\rho)\rho^{2j} + a_1(\rho)\Delta\rho^{2j} + \ldots + a_j(\rho)\, \Delta^j \rho^{2j} = b_j, \quad j = 0, \ldots, m-1\,,$$

with

$$b_j := \gamma_d \int_0^r t^{2j+d-1}\, dt = \gamma_d\, \frac{r^{d+2j}}{d+2j}, \quad j = 0, \ldots, m-1\,.$$

For example, the first three equations of the system take the form

$$\begin{gathered} a_0(\rho) = b_0, \\ a_0(\rho)\rho^2 + a_1(\rho)2d = b_1 \\ a_0(\rho)\rho^4 + a_1(\rho)4(d+2)\rho^2 + a_2(\rho)8d(d+2) = b_2 \end{gathered}$$

and we obtain

$$a_0(\rho) = \gamma_d \frac{r^d}{d}$$

$$a_1(\rho) = \gamma_d \frac{r^d}{2d}\left(\frac{r^2}{d+2} - \frac{\rho^2}{d}\right)$$

$$a_2(\rho) = \gamma_d \frac{r^d}{8d(d+2)}\left[\frac{r^4}{d+4} - \frac{\rho^4}{d} - \frac{2(d+2)}{d}\left(\frac{r^2}{d+2} - \frac{\rho^2}{d}\right)\rho^2\right].$$

Having the coefficients we construct for $m = 2$ the formula

$$\int_{\mathbf{B}^d(r)} u(\mathbf{x})\,d\mathbf{x} = \frac{r^d}{d\,\rho^{d-1}}\left\{\int_{S(\rho)} u(\xi)\ d\sigma(\xi) + \frac{1}{2}\left(\frac{r^2}{d+2} - \frac{\rho^2}{d}\right)\int_{S(\rho)} \Delta u(\xi)\,ds\right\}$$

which holds for every biharmonic function $u(\mathbf{x})$ and every $\rho \in [0, r]$. In particular, choosing ρ such that $\frac{r^2}{d+2} - \frac{\rho^2}{d} = 0$, i.e.,

$$\rho = \rho_1 := \left(\frac{d}{d+2}\right)^{\frac{1}{2}} r\,,$$

we obtain the simpler cubature formula

$$\int_{\mathbf{B}^d(r)} u(\mathbf{x})\,d\mathbf{x} = \frac{r}{d}\left(\frac{d+2}{d}\right)^{\frac{d-1}{2}} \int_{S(\rho_1)} u(\xi)\,d\sigma(\xi)$$

which is exact for all biharmonic functions. This fact was established directly by Dimitrov in [14].

Similarly, for $m = 3$, we derive an extremal cubature formula for 3-harmonic functions, given by

$$\int_{\mathbf{B}^d(r)} u(\mathbf{x})\,d\mathbf{x} = \frac{r}{d}\left(\frac{d+4}{d}\right)^{\frac{d-1}{2}} \times \left\{\int_{S(\rho_2)} u(\xi)\,d\sigma(\xi) + \frac{r^2}{(d+2)(d+4)}\int_{S(\rho_2)} \Delta u(\xi)\,d\sigma(\xi)\right\}.$$

The Almansi representation of polyharmonic functions and the Gauss mean-value theorem allow us to reduce problems concerning integration over the ball to univariate problems. This relation was described in detail in the "lifting theorem" proved in [5]. Actually, it was exploited indirectly in an earlier paper [6] where a cubature formula of Gaussian type was constructed for the ball that is based on integrals over n hyperspheres.

THEOREM 3.3. *Let $\mu(t)$ be any given weight function on $[0,1]$. There exists a unique sequence of distinct radii $0 < \rho_1 < \ldots < \rho_m \leq 1$ and real weights $A_k,\ k = 1, \ldots, m$, such that the extended cubature formula*

$$\int_{\mathbf{B^d}} u(\mathbf{x})\ \mu(\|\mathbf{x}\|)\,d\mathbf{x} \ \approx\ \sum_{k=1}^{m} A_k \int_{S(\rho_k)} u(\xi)\,d\sigma(\xi) \tag{6}$$

has polyharmonic order of precision $2m$. Moreover, the radii ρ_k coincide with the positive zeros of the polynomial $P_{2m}(t;\mu^*)$ of degree $2m$, which is orthogonal on $[-1,1]$ with respect to the weight function $\mu^*(t) = |t|^{d-1}\mu(|t|)$ to any polynomial of degree $2m-1$.

There is no extended cubature formula of the form (6) with $PHOP > 2m$.

The coefficients $\{A_k\}$ are explicitly determined as integrals of univariate polynomials and they are positive.

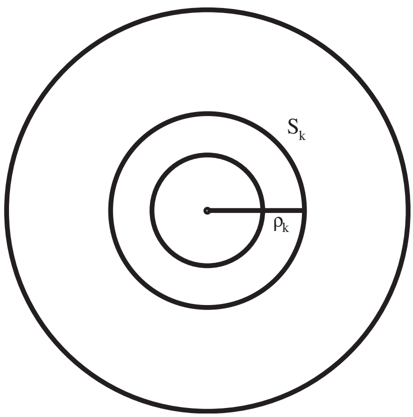

Fig. 7: A cubature based on circles

Note that the Gaussian extended cubature formula (6) approximates the integral of f over $\mathbf{B}^d$ in terms of m pieces of information about the integrand, namely, the “averaged” values of f over m hyperspheres. It integrates exactly all polyharmonic functions of order $2m$. Since the space H^{2m} contains the class $\pi_N(\mathbb{R}^d)$ of all algebraic polynomials of n variables of total degree $N := 4m-1$, the extended cubature (6) is precise for any polynomial in d variables of degree $4m-1$. And this is the highest algebraic degree of precision that can be achieved by a formula of this type. As mentioned in [26], p. 128, Kantarovich and Lusternik established the formula for $d = 2$. Mysovskih [26] extended the result to every dimension d and showed that the cubature (6) has algebraic degree of precision $4m-1$. In [6] the formula was derived in a different way, using the Almansi representation, and the fact was established that it is exact for the wider class than the polynomials of degree $4m-1$, namely for all polyharmonic functions of order $2m$. It follows also from the proof that there in no other cubature that uses m co-centered spheres which is of higher polyharmonic order of precision. The following question naturally arises: is formula (6) the best one even in the class of formulae that use integrals over scattered m spheres contained in the ball, not necessarily co-centered? In this new formulation one has at disposal additional m free parameters, the centers of the spheres. It seems that having the freedom to situate the centers all over in the ball one could get higher precision of the formula. This question was answered recently by Petrova [30]. She showed that the Gaussian cubature (6) is the best one even in this wider class of formulae with scattered spheres. Precisely she proved the following.

Theorem 3.4. *The cubature formula* (6) *is the only one of PHOP* $= 2m-1$ *among all cubature formulae of the form*

$$\int_{\mathbf{B}^d} u(\mathbf{x})\,d\mathbf{x} \approx \sum_{k=1}^{m} C_k \int_{S_k} u(\xi)\,d\sigma(\xi)$$

where

$$S_k := \{\mathbf{x} \in \mathbb{R}^d \,:\, \|\mathbf{x} - \mathbf{x}_k\| = r_k\}, \quad k = 1, \ldots, m\,,$$

are any m *spheres,* $S_k \subset \mathbf{B}^d$*, and* $d\sigma$ *denotes the* $(d-1)$*-dimensional surface measure on* S_k*.*

The proof is very elegant and relies on a new representation of polyharmonic functions which can be considered as an extension of the Almansi formula.

4 – Interval Gaussian quadrature formula

In the previous sections we gave examples of multivariate Gaussian cubature formulae which are based on averaged values of the integrand instead of the standard point evaluations. In addition to other reasons, this is a motivation to study the question of approximation also of univariate integrals on the basis of n averaged values. The problem can be formulated as follows.

Problem. Given $[a, b]$, the weight function $\mu(t)$ there, and the positive numbers $h_1, \ldots, h_n$ such that $h_1 + \ldots + h_n \le b-a$, does there exist a quadrature formula of the form

$$\int_a^b \mu(t) f(t)\, dt \approx \sum_{k=1}^{n} a_k \frac{1}{h_k} \int_{x_k}^{x_k+h_k} \mu(t) f(t)\, dt$$

which integrates exactly all polynomials of degree $2n-1$? If so, is the extremal formula unique?

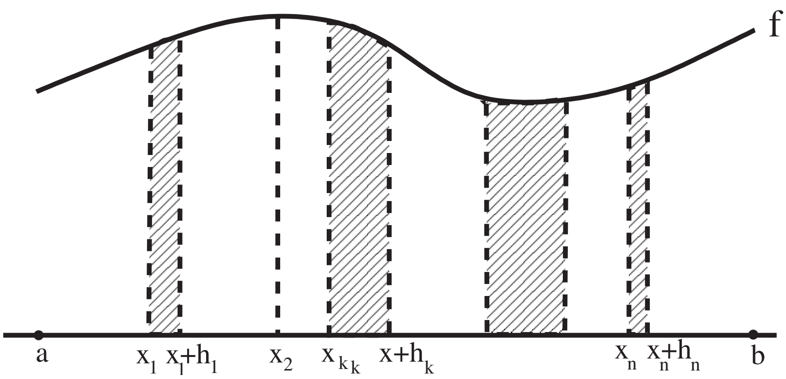

Fig. 8: Interval quadrature

It can be easily shown that $2n-1$ is the highest degree of precision that can be achieved by a formula of the above form. If it exists, such a formula would supply a nice extension of the classical Gauss formula since in case $h_k = 0$ ($k = 1, \ldots, n$) the scaled integral over the corresponding subinterval reduces to the value of f at x_k. The question of existence and construction of extremal interval quadrature formulae has been investigated by Omladic, Pahor and Suhadolc [28], Pitnauer and Reimer [31], [32], Sharipov [37], Kuz'mina [20], Babenko [3] and Motornyi [24]. Existence of Gaussian interval quadrature formulae was proved in [32] in the following sense: let $\tau_1, \ldots, \tau_n$ be the nodes of the Gauss formula (that is, the zeros of the orthogonal polynomial of degree n associated with the weight μ and $[a, b]$). Then for $c_k < \tau_k$, $k = 1, \ldots, n$, which are sufficiently close to $\tau_1, \ldots, \tau_n$, there exist points $d_k > \tau_k$, $k = 1, \ldots, n$, such that the interpolatory quadrature based on the integrals over $[c_k, d_k]$ is Gaussian. Note that the length of the subintervals $[c_k, d_k]$ is not specified. A similar result was obtained in [20] for a multiple-node case.

We first give an existence theorem for interval quadrature formulae of highest degree of precision in a more general setting, involving multiple nodes and considered with respect to an arbitrary Tchebycheff system. We need to recall some definitions.

Assume that $[a, b]$ is a fixed finite interval and $\mu(t)$ is a Lebesgue integrable non-negative function on $[a, b]$ such that $\int_\alpha^\beta \mu(t)\,dt > 0$ for each subinterval $[\alpha, \beta] \subset [a, b]$ of positive length. Let the set of continuous functions $U_N = \{u_0, u_1, \ldots, u_N\}$ constitute a Tchebycheff system on $[a, b]$, that is, any non-zero generalized polynomial $a_0 u_0(t) + \ldots + a_N u_N(t)$ has no more than N distinct zeros in $[a, b]$.

Assume further that $V_N = \{v_0, \ldots, v_N\}$ (with $v_0 \equiv 1$) is a given Markov system of continuous functions on $[a, b]$. Recall that V_N is said to be a *Markov system* if the functions $\{v_0, v_1, \ldots, v_k\}$ form a Tchebycheff system for each $k = 0, 1, \ldots, N$.

We shall use the notation H for the set of all admissible lengths, namely,

$$H := \left\{ \mathbf{h} := (h_1, h_2, \ldots, h_n) \,:\, h_1 \geq 0, \ldots, h_n \geq 0,\ \sum_{i=1}^{n} h_i \leq b - a \right\}.$$

For a given $\mathbf{h} \in H$, consider the class of multiple node interval quadrature formulae of the form

$$\int_a^b \mu(t) f(t) dt \approx \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} a_{k,\lambda} \mu_{k,\lambda}(f)\,, \tag{7}$$

where $\mu_{k,\lambda}$ are the moments of f with respect to V_N,

$$\mu_{k,\lambda}(f) := \frac{1}{d_k - c_k} \int_{c_k}^{d_k} \mu(t) f(t) v_\lambda(t) dt$$

and $[c_k, d_k]$, $k = 1, 2, \ldots, n$, are any non-overlapping subintervals such that

$$d_0 = a \leq c_1 < d_1 \leq \ldots \leq c_n < d_n \leq b = c_{n+1}\,,$$

and $h_k := d_k - c_k$.

The next existence theorem was proved in [7].

THEOREM 4.1. *Let $\nu_1, \ldots, \nu_n$ be arbitrary positive even integers and $N + 1 = \sum_{k=1}^{n} \nu_k$. Assume that the functions $u_0, \ldots, u_N$ are continuous and form a Tchebycheff system on $[a, b]$. For any weight function $\mu(t)$ on $[a, b]$ and any set of lengths $\mathbf{h} \in H$, there exists a generalized Gaussian interval quadrature formula of the form*

$$\int_a^b \mu(t) f(t) dt \approx \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 2} a_{k,\lambda} \mu_{k,\lambda}(f)\,,$$

whch is exact for every u_k, $k = 1, \dots, N$. Moreover, if $\sum_{k=1}^{n} h_k < b - a$, then its coefficients $a^*_{i,j}$ satisfy

$$a^*_{k,\nu_k-2}\begin{cases} > 0, & \nu_k = 2 \\ \neq 0, & \nu_k > 2 \end{cases}$$

for $k = 1, 2, \dots, n$.

The study of the uniqueness for any fixed lengths $\mathbf{h}$ from H turned out to be quite a difficult problem. The uniqueness was first prove for quadratures with equal subintervals (see [7]).

For convenience, choose $[a, b] = [0, 1]$.

Theorem 4.2. *Let $\mu(t) \equiv 1$ and $h = 1/m$, $m \geq n + 1$. The quadrature formula*

$$\int_0^1 f(x)dx \approx \sum_{k=1}^{n} a_k \int_{x_k}^{x_k+h} f(t)dt \,, \tag{14}$$

is exact for every $f \in \pi_{2n-1}$ if and only if the polynomial $\omega(x) := \prod_{k=1}^{n}(x - x_k)$ is orthogonal to any polynomial of degree $\leq n - 1$ with respect to the discrete inner product

$$(f, g) := \sum_{j=0}^{m-1} f(j/m)g(j/m) \,.$$

Next the uniqueness was proved in [8] for any finite interval and a constant weight $\mu(t) = 1$. The proof is based on a iterated application of a property of topological degree of non-linear mappings, described in the Homotopy Invariance theorem (see [29], p. 156). Later, Milovanović and Cvetković [22] modified the proof to cover integrals with Jacobi weights. Recently the uniqueness was shown in [9] for any weight function. Let us formulate this result in detail.

Actually, the theorem we are going to present below is an extension of the interval Gaussian formula along the lines of the Krein theorem about canonical representation of linear functionals in Tchebycheff spaces. Krein [19] proved in 1951 that if the functions $u_1, \dots, u_{2n}$ form a T-system on $[a, b]$, then there exists a unique quadrature formula of the form

$$\int_a^b u(x)\,dx \approx \sum_{k=1}^{n} c_k u(x_k)$$

which is exact for every $u \in \text{span}\,\{u_1, \dots, u_{2n}\}$. The theorem below is a further extension of this proposition concerning formulae based on averaged values.

With any given $\mathbf{h} \in H$ we associate the set

$$D = D(\mathbf{h}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, a < x_1 \le x_1 + h_1 < \ldots < x_n \le x_n + h_n < b\}$$

of admissible nodes. Let us set

$$m_k := \int_{x_k}^{x_k+h_k} \mu(t)\,dt, \quad k = 1, \ldots, n\,.$$

Recall that $u_1, \ldots, u_k$ is an ET system of order r on $[a, b]$ if any non-zero generalized polynomial $c_1u_1 + \ldots + c_ku_k$ has at most $k-1$ zeros in $[a, b]$ counting the multiplicities up to order r.

THEOREM 4.3. *Let $U_{2n} = \{u_1, \ldots, u_{2n}\}$ be any ET system of order* 2 *of continuously differentiable functions on $[a, b]$ and let μ be an integrable function on $[a, b]$ which is continuous and positive on (a, b). Then, for every given set of numbers $\mathbf{h} \in H$ there exists a unique set of nodes $\mathbf{x} \in D(\mathbf{h})$ such that*

$$\int_a^b \mu(t)f(t)\,dt = \sum_{k=1}^n a_k \frac{1}{m_k} \int_{x_k}^{x_k+h_k} \mu(t)f(t)\,dt$$

for every f from the space $\mathcal{U}_{2n} := \operatorname{span}\{u_1, \ldots, u_{2n}\}$.

Note that Theorem 4.3 holds also in the trivial case $h_1 + \ldots + h_n = b - a$ since the uniqueness of the best coefficients $a_k = 1$, $k = 1, \ldots, n$, can be easily verified.

It is worth mentioning explicitly the important particular case of interval quadrature formula of Gauss-Christoffel type.

COROLLARY 4.4. *Let μ be any integrable function on $[a, b]$ which is continuous and positive on (a, b). Then, for every given set of non-negative numbers* $\mathbf{h}$ *satisfying the condition*

$$h_1 + \ldots + h_n \le b - a\,,$$

there exists a unique set of nodes $\mathbf{x} \in D(\mathbf{h})$ such that

$$\int_a^b \mu(t)f(t)\,dt = \sum_{k=1}^n a_k \frac{1}{m_k} \int_{x_k}^{x_k+h_k} \mu(t)f(t)\,dt$$

for every algebraic polynomial of degree less than or equal to $2n-1$.

Acknowledgements

The author is grateful to Irina Georgieva for her help with the figures.

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Lavoro pervenuto alla redazione il 27 novembre 2004
ed accettato per la pubblicazione il 1 marzo 2005.
Bozze licenziate il 1 settembre 2005

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This research was supported by the Sofia University Science Foundation, Grant No. 160/2004.