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A new approach to Bernoulli polynomials

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Dedicated to Professor Laura Gori on her 70th birthday

ABSTRACT: *Six approaches to the theory of Bernoulli polynomials are known; these are associated with the names of J. Bernoulli* [2]*, L. Euler* [4]*, E. Lucas* [8]*, P. E. Appell* [1]*, A. Hürwitz* [6] *and D. H. Lehmer* [7]*. In this note we deal with a new determinantal definition for Bernoulli polynomials recently proposed by F. Costabile [3]; in particular, we emphasize some consequent procedures for automatic calculation and recover the better known properties of these polynomials from this new definition. Finally, after we have observed the equivalence of all considered approaches, we conclude with a circular theorem that emphasizes the direct equivalence of three of previous approaches.*

1 – Short review of classical approaches

Bernoulli polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and in classical and numerical analysis. These polynomials can be defined by various methods depending on the applications. In particular, six approaches to the theory of Bernoulli polynomials are known; these are associated with the names of J. Bernoulli ([2], 1690), L. Euler ([4], 1738), P.E. Appell ([1], 1882), A. Hürwitz ([7], 1890), E. Lucas ([8], 1891) and D.H. Lehmer ([7], 1988). The term *Bernoulli polynomials* was used first in 1851 by Raabe [10] in connection with

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the following multiplication theorem

$$\frac{1}{m}\sum_{k=0}^{m-1} B_n\left(x+\frac{k}{m}\right) = m^{-n} B_n(mx). \tag{1}$$

In effect Jacob Bernoulli introduced the polynomials $B_n(m)$ already in 1690 (his work [2] was posthumously published in 1713) in parallel with the discovery of numbers B_n related to the calculation of the sum $S_n(m) = \sum\limits_{k=0}^{m-1} k^n$ of powers of the first natural numbers: introducing the remarkable formula

$$S_n(m) = \sum_{k=0}^{n} \frac{B_k}{k!}\frac{n!}{(n+1-k)!} m^{n+1-k}$$

he set

$$S_n(m) = \frac{1}{n+1}\left(B_{n+1}(m) - B_{n+1}(0)\right).$$

After the Jacob Bernoulli's discovery, Leonard Euler [4] proposed an approach to Bernoulli polynomials based on functional series expansion; in order to define the Bernoulli polynomials by this approach, known as *the generating function approach*, let us consider the function

$$F(x,t) = \begin{cases} e^{xt}\dfrac{t}{e^t-1} & \text{if} \quad t \neq 0, \\ 1 & \text{if} \quad t = 0, \end{cases}$$

with x a fixed complex number. The function $F(x,t)$ is, in particular, complex analytic in the disk $\{|t| < 2\pi\}$, therefore it can be expanded in a convergent power series of t centered at the origin, with coefficients that depend on complex number x:

$$e^{xt}\frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

After a number of calculations we can obtain from previous equation the relation

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

showing that $B_n(x)$ is a polynomial of degree n. A more general approach to Bernoulli polynomials can be obtained by using the so called Appell sequences [1], defined as follows: *a sequence of polynomials*

$$P_0(x), P_1(x), \ldots$$

is said to form an Appell sequence if

1. $\deg P_n(x) = n$ *for each* $n = 0, 1, \ldots$
2. $P'_n(x) = nP_{n-1}(x)$ *for each* $n = 0, 1, \ldots$

Usually such a sequence is normalized by setting $P_0(x) = 1$. Note that an Appell sequence can be obtained constructively producing $P_n(x)$ by an indefinite integration of $P_{n-1}(x)$:

$$P_n(x) = c_n + n \int_0^x P_{n-1}(t)\, dt \quad n = 1, 2, \ldots$$

and then choosing the constant of integration $c_n = P_n(0)$ in an appropriate way; in fact, an Appell sequence is completely determined by the numbers $P_n(0)$. So, for example, if we set $c_n = 0$ for each $n = 1, 2, \ldots$ we obtain $P_n(x) = x^n$, and, for this reason, the polynomials forming an Appell sequence are also called *generalized monomials*; the sequence of Bernoulli polynomials can be obtained by setting $c_n = B_n$. In 1890 A. Hürwitz gave the Fourier series expansions for $B_n(x)$

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k=-\infty}^{+\infty} k^{-n} e^{2\pi i k x} \quad 0 < x < 1$$

and used the *Fourier series approach* to Bernoulli polynomials in his lectures, as Lehmer report in [7]. In 1881 Lucas [8] derived the Bernoulli polynomial sequence using the *umbral calculus*: in the identity

$$B_n(x) = (B + x)^n$$

he claims that k at exponent of B^k in the power expansion of the right member of previous equation is replaced by the index of the Bernoulli number B_k to obtain

$$(B + x)^n = \sum_{k=0}^{n} \binom{n}{k} B^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Recently, Lehmer [7] proposed a new approach to Bernoulli polynomials based on the Raabe multiplication theorem (1) and derived from this approach the other definitions. In particular, Lehmer proved the following assertion:

1. for a given integer n there exists only one monic polynomial of degree n in x satisfying the functional equation
$$\frac{1}{m} \sum_{k=0}^{m-1} f\left(x + \frac{k}{m}\right) = m^{-n} f(mx)$$
for each $m > 1$;
2. for each n let us denote the solution of previous equation by $B_n(x)$; then the sequence $\{B_n(x)\}$ is an Appell sequence.

2 – A determinantal approach

More recently, a new definition for Bernoulli polynomials using a determinantal approach has been proposed by F. Costabile ([3], 1999). This definition requires only the knowledge of basic linear algebra.

Definition 1. The Bernoulli polynomial of degree $n = 0, 1, 2, \ldots$, it is denoted by $B_n(x)$ and is defined by

$$B_0(x) = 1$$

and

$$B_n(x) = \frac{(-1)^n}{(n-1)!} \begin{vmatrix} 1 & x & x^2 & x^3 & \ldots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 2 & 3 & \ldots & n-1 & n \\ 0 & 0 & 0 & \binom{3}{2} & \ldots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix} \tag{2}$$

for each $n = 1, 2, \ldots$

Remark 2. If we set in (2) $(-1)! := 1$ then the entry (i, j) is equal to $\binom{j-1}{i-3}$ for each $i = 2, \ldots, n+1$, $j = 1, \ldots, n+1$, $i - j \geq 1$.

Despite previous definition of $B_n(x)$ involves the calculation of a $(n+1)$-order determinant, its particular form, known as upper-Hessember, allows us to simplify the computational procedure. In fact, it is known that the algorithm of Gaussian elimination without pivoting for computing the determinant of an upper Hessemberg matrix is stable [5, p.27]; then a stable algorithm for numerical calculation of $B_n(x)$ can be obtained simply by applying the algorithm of Gaussian elimination without pivoting for computing the determinant (2). The following procedure allows us to recover a well-known formula for symbolic computation of Bernoulli polynomials

Lemma 3. *For the determinant H_n of an upper Hessemberg matrix of order n, with entries $h_{i,j}$, $h_{i,j} = 0$, $i - j \geq 2$*

$$H_n = \begin{vmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & \cdots & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & h_{2,3} & \ddots & & \ddots & h_{2,n} \\ 0 & h_{3,2} & h_{3,3} & \ddots & \ddots & \ddots & h_{3,n} \\ \vdots & 0 & h_{4,3} & h_{4,4} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & h_{n,n-1} & h_{n,n} \end{vmatrix} \tag{3}$$

the following recursive relation holds

$$H_n = \sum_{k=0}^{n-1} (-1)^{n-k-1}\, q_k(n) h_{k+1,n} H_k \tag{4}$$

with the following settings

$$q_{n-1}(n) = 1, \quad q_k(n) = \prod_{j=k+2}^{n} h_{j,j-1}, \quad k = 0, 1, \dots , n-2$$

or equivalently

$$q_{n-1}(n) = 1, \quad q_k(n) = h_{k+2,k+1} q_{k+1}, \quad k = 0, 1, \dots , n-2.$$

Proof. A proof of previous result can be accomplished by using the Laplace formula to calculate the determinant H_n. □

By applying previous result to determinant (2), we find:

$$B_n(x) = x^n - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k(x) \tag{5}$$

that can be used independently to define the Bernoulli polynomials [11]. In addition, previous formulas can be used to calculate the coefficients of the polynomial $B_n(x)$; in fact, by setting for each $k = 0, 1, 2, ..., n$

$$B_k(x) = \sum_{j=0}^{k} b_{kj} x^j$$

and by substituting previous relations in (5) we obtain

$$\begin{aligned} \sum_{j=0}^{n} b_{nj} x^j &= x^n - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} \sum_{j=0}^{k} b_{kj} x^j \\ &= x^n - \frac{1}{n+1} \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \tbinom{n+1}{k} b_{kj} x^j; \end{aligned}$$

comparing term by term the first member polynomial with the last member we finally obtain the relations

$$\begin{cases} b_{nj} = -\dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{n+1}{k} b_{kj}, & j = 0, 1, ..., n-1, \\ b_{nn} = 1. \end{cases}$$

3 – Properties of Bernoulli polynomials

Some of well-known properties of Bernoulli polynomials can be easily recovered from the determinantal definition (2) with some calculation and the knowledge of basic notions related to the theory of determinant; let us consider here some example.

Property 1 (Differentiation):*For the differentiation of Bernoulli polynomials can be used the relations*

$$B_n'(x) = nB_{n-1}(x), \quad n = 1, 2, ...$$

Proof. In order to recover this property starting from the determinantal approach, one can differentiate the determinant (2) using the properties of linearity, expand the resulting determinant with respect to the first column and recognize the factor $B_{n-1}(x)$ after multiplication of the i-th row by $i-2 \quad i = 3, ..., n$ and j-th column by $1/j \quad j = 1, ..., n$. ∎

Property 2 (Integral means conditions): *For each* $n \geq 1$*there is*

$$\int_0^1 B_n(x)\,dx = 0.$$

Proof. The proof of this property consists in a direct calculation. In fact, after the definite integration the first two line of determinant $\int_0^1 B_n(x)\,dx$ shall coincide. ∎

Property 3 (Differences): *For each* $n \geq 1$*there is*

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

Proof. A proof of previous property based on determinantal definition (2) can be accomplished with some calculation by using the linearity property of a determinant with respect to each row and the following well-known identity

$$(x+1)^i - x^i = \sum_{k=0}^{i-1} \binom{i}{k} x^k. \qquad ∎$$

In force of the primary connection between Bernoulli numbers B_n and Bernoulli polynomials, namely

$$B_n(0) = B_n$$

we obtain from determinantal definition of Bernoulli polynomials (2) a determinantal definition for the Bernoulli numbers as well, by the setting

$$B_0 = 1$$

(6)
$$B_n = \frac{(-1)^n}{(n-1)!} \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 2 & 3 & \cdots & n-1 & n \\ 0 & 0 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix} \qquad n = 1, 2, \ldots$$

Property 4 (A series representation in terms of Bernoulli numbers): *For Bernoulli polynomials we have*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n = 0, 1, \ldots$$

PROOF. In this case, we can start the proof argumentations by expanding the determinant (2) with respect to the first row; then, working on the cofactor of the power x^k, $k = 0, \ldots, n$, after some calculation by using the property of linearity with respect to each row or column one recognize that this cofactor is exactly $\binom{n}{k} B_{n-k}$. $\square$

Property 5 (The value at x=1): *For each* $n \geq 2$

$$B_n(1) = B_n.$$

PROOF. Even in this case the proof consists in a direct calculation: in fact, evaluating the determinant (2) at $x = 1$ previous equality results by expanding the evaluated determinant with respect to the first column.

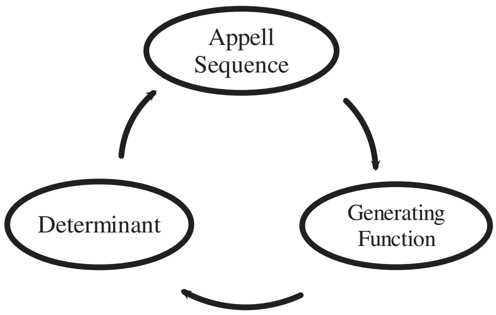
4 – Equivalence of definitions

It is possible to prove that all previous approaches lead to at least one of

$$(7)\quad \begin{cases} B_n'(x)=nB_{n-1}(x)\,, \\ B_n(0)=B_n,\ n\geq 1, \end{cases} \begin{cases} B_n'(x)=nB_{n-1}(x)\,, \\ B_1(0)=-B_1(1)\,, \\ B_n(0)=B_n(1)\,,\ n>1, \end{cases} \begin{cases} B_n'(x)=nB_{n-1}(x)\,, \\ \int_0^1 B_n(t)\,dt=0,\ n\geq 1, \end{cases}$$

that yield, jointly with condition $B_0(x) = 1$, the same sequence of polynomials, i.e. the sequence of Bernoulli polynomials. In this sense previous definitions are equivalent. In addition, we prove the following

THEOREM 4. *The following circular diagram holds, were the arrows mean that the pointed approaches can be derived from the previous one as theorems:*



PROOF. Determinantal approach $\Rightarrow$Appell's approach. As we saw, from determinantal definition (2) of Bernoulli polynomials easily follows that these polynomials form the following Appell sequence

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x)\,, \\ \int_0^1 B_n(t)\,dt = 0, \quad n \geq 1. \end{cases}$$

Appell's approach $\Rightarrow$ Euler's approach. On the other hand, it is known that the polynomials forming an Appell sequence $\{P_n(x)\}$ can be introduced by means of closed form formulas by using the generating function of the sequence, i.e. the function

$$e^{xt} f(t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

with the following setting

$$f(t) = \sum_{n=0}^{\infty} P_n(0) \frac{t^n}{n!}$$

Euler's approach $\Rightarrow$ Determinantal approach. Finally a known algorithm for calculating the quotient of two power series [9] can be used to derive the determinant form (2) for Bernoulli polynomials from the Euler approach. In fact, in the equation

$$e^{xt} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$$

let us replace functions e^{xt} and $e^t - 1$ with their Taylor series expansions (in xt and in t respectively) at the origin; so we have

$$\frac{1 + \frac{xt}{1!} + \frac{x^2t^2}{2!} + ... + \frac{x^nt^n}{n!} + ...}{1 + \frac{t}{2!} + \frac{t^2}{3!} + ... + \frac{t^n}{(n+1)!} + ...} = \frac{B_0(x)}{0!} + \frac{B_1(x)}{1!} t + ... + \frac{B_n(x)}{n!} t^n + ...$$

In order to write the Taylor series expansion of the function $e^{xt} \frac{t}{e^t - 1}$ with respect to t at the origin, we can compare the left member of previous equation with the right member, and multiplying the right member by the denominator of the fraction on the left member we obtain

$$\begin{aligned} 1 + \frac{xt}{1!} + \frac{x^2t^2}{2!} + ... + \frac{x^nt^n}{n!} + ... &= \left(\frac{B_0(x)}{0!} + \frac{B_1(x)}{1!} t + ... + \frac{B_n(x)}{n!} t^n + ... \right) \cdot \\ &\quad \cdot \left(1 + \frac{t}{2!} + ... + \frac{t^n}{(n+1)!} + ... \right). \end{aligned}$$

By multiplying the series on the right hand side of previous equation according to the Cauchy-product rules, this equation leads to the following system of infinite equations in the unknown $c_i(x) = \frac{B_i(x)}{i!}$, $i = 0, 1, ...$

$$\begin{cases} c_0(x) = 1 \\ c_0(x) \frac{1}{2!} + c_1(x) = \frac{x}{1!} \\ c_0(x) \frac{1}{3!} + c_1(x) \frac{1}{2!} + c_2(x) = \frac{x^2}{2!} \\ \vdots \\ c_0(x) \frac{1}{(n+1)!} + c_1(x) \frac{1}{n!} + ... + c_n(x) = \frac{x^n}{n!} \\ \vdots \end{cases}$$

The special form of the previous system (lower triangular) allows us to work out the unknown $c_n(x)$ operating with the first $n+1$ equations only, by applying the Cramer method:

$$c_n(x) = \frac{\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ \frac{1}{2!} & 1 & 0 & \dots & 0 & \frac{x}{1!} \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 & \frac{x^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \dots & 1 & \frac{x^{n-1}}{(n-1)!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} & \frac{x^n}{n!} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2!} & 1 & 0 & \dots & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \dots & 1 & 0 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} & 1 \end{vmatrix}} =$$

$$= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ \frac{1}{2!} & 1 & 0 & \dots & 0 & \frac{x}{1!} \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 & \frac{x^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \dots & 1 & \frac{x^{n-1}}{(n-1)!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} & \frac{x^n}{n!} \end{vmatrix} \qquad n = 1, 2, \dots$$

From the above steps it follows that

$$\text{(8)} \qquad B_n(x) = n! \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ \frac{1}{2!} & 1 & 0 & \dots & 0 & \frac{x}{1!} \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 & \frac{x^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \dots & 1 & \frac{x^{n-1}}{(n-1)!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} & \frac{x^n}{n!} \end{vmatrix} \qquad n = 1, 2, \dots$$

Finally, the determinant (2) can be obtained from previous determinant by means of a transposition and elementary row and column operations. In fact the transposition of (8) is

$$B_n(x) = n!\begin{vmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{n!} & \frac{1}{(n+1)!} \\ 0 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(n-1)!} & \frac{1}{n!} \\ 0 & 0 & 1 & \cdots & \frac{1}{(n-2)!} & \frac{1}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2!} \\ 1 & \frac{x}{1!} & \frac{x^2}{1!} & \cdots & \frac{(x-1)^n}{(n-1)!} & \frac{x^n}{n!} \end{vmatrix} \qquad n = 1, 2, \ldots$$

and multiplying the i-th row $i = 2, .., n$ by $\frac{1}{(i-2)!}$ and the j-th column $j = 2, .., n+1$ by $(j-1)!$ we obtain

$$B_n(x) = \frac{1! \cdots (n-2)! n!}{1! 2! \cdots n!}\begin{vmatrix} 1 & \frac{1!}{2!} & \frac{2!}{3!} & \cdots & \frac{(n-1)!}{n!} & \frac{n!}{(n+1)!} \\ 0 & \frac{1}{0!1!} & \frac{2!}{0!2!} & \cdots & \frac{(n-1)!}{0!(n-1)!} & \frac{n!}{0!n!} \\ 0 & 0 & \frac{2!}{1!1!} & \cdots & \frac{(n-1)!}{1!(n-2)!} & \frac{n!}{1!(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(n-1)!}{(n-2)!1!} & \frac{n!}{(n-2)!2!} \\ 1 & \frac{x}{1!} & \frac{2!x^2}{2!} & \cdots & \frac{(n-1)!(x-1)^n}{(n-1)!} & \frac{n!x^n}{n!} \end{vmatrix} \qquad n = 1, 2, \ldots$$

that is exactly (2) after the exchange of the first row with the last one. □

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