

## Manifolds with multiplication on the tangent sheaf

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ABSTRACT: *This talk reviews the current state of the theory of  $F$ -(super)manifolds  $(M, \circ)$ , first defined in [6] and further developed in [5], [11], [12]. Here  $\circ$  is an  $\mathcal{O}_M$ -bilinear multiplication on the tangent sheaf  $\mathcal{T}_M$ , satisfying an integrability condition.  $F$ -manifolds and compatible flat structures on them furnish a useful weakening of Dubrovin's Frobenius structure which naturally arises in the quantum  $K$ -theory, theory of extended moduli spaces, and unfolding spaces of singularities.*

### 1 – Generalities: $F$ -structure vs Poisson structure

**1.1. Manifolds.** Manifolds considered in this talk can be  $C^\infty$ , analytic, or formal, eventually with even and odd coordinates (supermanifolds). The ground field  $K$  of characteristic zero is most often  $\mathbf{C}$  or  $\mathbf{R}$ . Each manifold is endowed with the structure sheaf  $\mathcal{O}_M$  which is a sheaf of commutative  $K$ -algebras, and the tangent sheaf  $\mathcal{T}_M$  which is a locally free  $\mathcal{O}_M$ -module of (super)rank equal to the (super)dimension of  $M$ .  $\mathcal{T}_M$  acts on  $\mathcal{O}_M$  by derivations, and is a sheaf of Lie (super)algebras with an intrinsically defined Lie bracket  $[\cdot, \cdot]$ .

There is a classical notion of *Poisson structure* on  $M$  which endows  $\mathcal{O}_M$  as well with a Lie bracket  $\{, \}$  satisfying a certain identity. Similarly, an  *$F$ -structure* on  $M$  endows  $\mathcal{T}_M$  with an extra operation: (super)commutative and

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associative  $\mathcal{O}_M$ -bilinear multiplication  $\circ$  with identity  $e$ .

In order to describe the structure identities imposed on these operations, we recall the notion of the Poisson tensor. Let generally  $A$  be a  $K$ -linear superspace (or a sheaf of superspaces) endowed with a  $K$ -bilinear multiplication and a  $K$ -bilinear Lie bracket  $[, ]$ . Then for any  $a, b, c \in A$  put

$$(1.1) \quad P_a(b, c) := [a, bc] - [a, b]c - (-1)^{ab}b[a, c].$$

(From here on,  $(-1)^{ab}$  and similar notation refers to the sign occurring in super-algebra when two neighboring elements get permuted.)

This tensor will be written for  $A = (\mathcal{O}_M, \cdot, \{, \})$  in case of the Poisson structure, and for  $A = (\mathcal{T}_M, \circ, [, ])$  in case of an  $F$ -structure.

We will now present parallel lists of basic properties of Poisson, resp.  $F$ -manifolds.

**1.2. Poisson (super)manifolds.** (i) <sub>$P$</sub> . *Structure identity:* for all local functions  $f, g, h$  on  $M$

$$(1.2) \quad P_f(g, h) \equiv 0.$$

(ii) <sub>$P$</sub> . Equivalently, each local function  $f$  on  $M$  becomes a local vector field  $X_f$  of the same parity on  $M$  via  $X_f(g) := \{f, g\}$

(iii) <sub>$P$</sub> . *Maximally nondegenerate case: symplectic structure.* There exist local canonical coordinates  $(q_i, p_i)$  such that for any  $f, g$

$$\{f, g\} = \sum_{i=1}^n (\partial_{q_i} f \partial_{p_i} g - \partial_{q_i} g \partial_{p_i} f).$$

Thus, locally all symplectic manifolds of the same dimension are isomorphic. Local group of symplectomorphisms is, however, infinite dimensional.

**1.3.  $F$ -manifolds.** (i) <sub>$F$</sub> . *Structure identity:* for all local vector fields  $X, Y, Z, U$

$$(1.3) \quad P_{X \circ Y}(Z, U) = X \circ P_Y(Z, U) + (-1)^{XY} Y \circ P_X(Z, U).$$

(ii) <sub>$F$</sub> . Since  $(\mathcal{T}_M, \circ)$  admits an identity  $e$ , each local vector field on  $M$  becomes a local vector function on the spectral cover  $\widetilde{M}$  of  $M$  where

$$\widetilde{M} := \text{Spec}_{\mathcal{O}_M}(\mathcal{T}_M, \circ)$$

The last notation is meaningful in each standard category of ringed spaces, because the tangent sheaf is free as  $\mathcal{O}_M$ -module. In particular, the structure covering  $\widetilde{M} \rightarrow M$  is flat.

However, generally  $\widetilde{M}$  is not a manifold because of nilpotents and/or singularities. Hertling's Theorem 3.6.4 below describes certain important cases when  $\widetilde{M}$  is a manifold.

(iii)<sub>F</sub>. *Maximally nondegenerate case: semisimple F-manifolds.*  $\widetilde{M}$  will be a manifold and even an unramified covering of  $M$  in the appropriate "maximally nondegenerate case", namely, when  $M$  is pure even, and locally  $(\mathcal{T}_M, \circ)$  is isomorphic to  $(\mathcal{O}_M^d)$  as algebra,  $d = \dim M$ .

In this case there exist local canonical coordinates  $(u_a)$  (Dubrovin's coordinates) such that the respective vector fields  $\partial_a := \partial/\partial u_a$  are orthogonal idempotents:

$$\partial_a \circ \partial_a = \delta_{ab} \partial_a.$$

Thus, locally all semisimple  $F$ -manifolds of the same dimension are isomorphic. Local automorphisms of an  $F$ -semisimple structure are generated by renumberings and shifts of canonical coordinates:

$$u_a \mapsto u_{\sigma(a)} + c_a$$

so that this structure is more rigid than the symplectic one.

**1.4. Structure embedding of the spectral cover.** The canonical surjective morphism of sheaves of  $\mathcal{O}_M$ -algebras  $\text{Symm}_{\mathcal{O}_M}(\mathcal{T}_M) \rightarrow (\mathcal{T}_M, \circ)$  induces a closed embedding  $\widetilde{M} \rightarrow T^*M$ . Its image is Lagrangian (with respect to the canonical symplectic structure on  $T^*M$ .)

As a partial converse, an embedded submanifold  $N \subset T^*M$  is the spectral cover of some generically semisimple  $F$ -structure iff  $N$  is Lagrangian.

**1.5. Local decomposition theorem.** For any point  $x$  of a pure even  $F$ -manifold  $M$ , the tangent space  $T_x M$  is endowed with the structure of a  $K$ -algebra. This  $K$ -algebra can be represented as a direct sum of local  $K$ -algebras. The decomposition is unique in the following sense: the set of pairwise orthogonal idempotent tangent vectors determining it is well defined.

C. Hertling has shown that this result extends to a neighborhood of  $x$ .

Generally, define the sum of two  $F$ -manifolds:

$$(M_1, \circ_1, e_1) \oplus (M_2, \circ_2, e_2) := (M_1 \times M_2, \circ_1 \boxplus \circ_2, e_1 \boxplus e_2)$$

A manifold is called *indecomposable* if it cannot be represented as a sum in a nontrivial way.

**THEOREM 1.5.1.** *Every germ  $(M, x)$  of a complex analytic  $F$ -manifold decomposes into a direct sum of indecomposable germs such that for each summand, the tangent algebra at  $x$  is a local algebra.*

*This decomposition is unique in the following sense: the set of pairwise orthogonal idempotent vector fields determining it is well defined.*

For a proof, see [5], Theorem 2.11. It is based on the following reinterpretation of (1.3) as an integrability condition: for any two local vector fields  $X, Y$ ,

$$\text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + (-1)^{XY} Y \circ \text{Lie}_X(\circ).$$

Here  $\circ$  is understood as a tensor, to which one can apply a Lie derivative.

If  $(T_x M, \circ)$  is semisimple, this theorem shows that  $M$  is semisimple in a neighborhood of  $x$ , and implies the existence and uniqueness (up to reindexing and shifts) of Dubrovin's canonical coordinates.

For  $F$ -manifolds with a compatible flat structure, there exists a considerably more sophisticated operation of *tensor product*. This will be explained in the last section.

## 2 – Compatible flat structures and Euler fields

**2.1. Flat structures.** Let  $M$  be a (super)manifold. An (affine) flat structure on  $M$  is given by any of the following data:

- (i) A torsionless flat connection  $\nabla_0 : \mathcal{T}_M \rightarrow \Omega_M^1 \otimes_{\mathcal{O}_M} \mathcal{T}_M$ .
- (ii) A local system  $\mathcal{T}_M^f \subset \mathcal{T}_M$  of flat vector fields, which forms a sheaf of supercommutative Lie algebras of rank  $\dim M$  such that  $\mathcal{T}_M = \mathcal{O}_M \otimes_K \mathcal{T}_M^f$ .
- (iii) An atlas whose transition functions are affine linear.

**2.2. Compatibility of a flat structure with a multiplication.** Assume that  $\mathcal{T}_M$  is endowed with an  $\mathcal{O}_M$ -bilinear (super)commutative and associative multiplication  $\circ$ , eventually with unit  $e$ . At this stage, we do not assume that (1.3) holds.

DEFINITION 2.2.1. a) A flat structure  $\mathcal{T}_M^f$  on  $M$  is called compatible with  $\circ$ , if in a neighborhood of any point there exists a vector field  $C$  such that for arbitrary local flat vector fields  $X, Y$  we have

$$(2.1) \quad X \circ Y = [X, [Y, C]].$$

$C$  is called a local vector potential for  $\circ$ .

- b)  $\mathcal{T}_M^f$  is called compatible with  $(\circ, e)$ , if a) holds and moreover,  $e$  is flat.

PROPOSITION 2.2.2. *If  $\circ$  admits a compatible flat structure, then it satisfies the structure identity (1.3) so that  $(M, \circ)$  is an  $F$ -manifold.*

For a proof, see [11], Prop. 2.4.

**2.3. Pencils of flat connections.** Consider the following input data:

- (i) A flat structure  $\nabla_0 : \mathcal{T}_M \rightarrow \Omega_M^1 \otimes_{\mathcal{O}_M} \mathcal{T}_M$  on  $M$ .
- (ii) an (odd) global section  $A \in \Omega_M^1 \otimes_{\mathcal{O}_M} \text{End}(\mathcal{T}_M)$  (Higgs field).

Produce from it the following output data:

- (iii) A pencil of connections  $\nabla_\lambda = \nabla_\lambda^A$  on  $\mathcal{T}_M$ :

$$\nabla_\lambda := \nabla_0 + \lambda A.$$

- (iv) An  $\mathcal{O}_M$ -bilinear composition law  $\circ = \circ^A$  on  $\mathcal{T}_M$ :

$$X \circ^A Y := i_X(A)(Y), \quad i_X(df \otimes G) := Xf \cdot G$$

**PROPOSITION 2.3.1.**  *$(M, \circ^A, \nabla_0)$  is an  $F$ -manifold with compatible flat structure if and only if  $\nabla_\lambda^A$  is a pencil of torsionless flat connections.*

*In this case,  $(M, \circ^A, \nabla_\lambda^A)$  is an  $F$ -manifold with compatible flat structure for any  $\lambda$  as well.*

**2.4. Euler fields.** Consider the following problem:

*Given  $(M, \circ, \nabla, e)$  as in 2.2.1 and the respective pencil of torsionless flat connections  $\nabla_\lambda$ , extend it to a flat connection on  $pr_M^*(\mathcal{T}_M)$  over  $\widehat{M} := M \times (\lambda$ -space).*

The missing data is an extra covariant derivative in the  $\lambda$ -direction

$$\widehat{\nabla}_{\partial_\lambda}(Y) = \partial_\lambda Y + H(Y)$$

where  $H$  is an even endomorphism of  $pr_M^*(\mathcal{T}_M)$  depending on  $\lambda$ .

**PROPOSITION 2.4.1.** *Assume that  $\nabla e = 0$  and  $E := H(e)$  does not depend on  $\lambda$ . Then  $\widehat{\nabla}_{\partial_\lambda}$  defines a flat extension of  $\nabla_\lambda$  if and only if  $H$  can be given by the formula*

$$H(X) = X \circ E + \lambda^{-1}(\nabla_X E - X),$$

*and  $E$  is a vector field on  $M$  satisfying the following conditions:*

$$P_E(X, Y) = X \circ Y, \quad [E, \text{Ker } \nabla] \subset \text{Ker } \nabla.$$

This means that  $E$  is an Euler field of weight one in the following sense:

DEFINITION 2.4.2. An Euler field  $E$  of weight  $d_0$  for  $(M, \circ, \nabla, e)$  is defined by the conditions:  $E$  is even, and

$$P_E(X, Y) = d_0 X \circ Y, \quad [E, \text{Ker } \nabla] \subset \text{Ker } \nabla$$

One easily checks that local Euler fields form a sheaf of linear spaces and Lie algebras. Commutator of two Euler fields is an Euler field of weight zero. Identity is an Euler field of weight zero.

For proofs and further details, see [11].

### 3 – Frobenius manifolds, moduli spaces, mirrors

DEFINITION 3.1 A Frobenius manifold is an  $F$ -manifold endowed with a compatible flat structure  $\nabla$  and a pseudo-Riemannian metric  $g : S^2(\mathcal{T}_M) \rightarrow \mathcal{O}_M$  such that

- (i)  $g$  is flat, and  $\nabla =$  the Levi-Civita connection of  $g$ .
- (ii)  $g(X \circ Y, Z) = g(X, Y \circ Z)$ .

An Euler field  $E$  is called compatible with the Frobenius structure if

- (iii)  $\text{Lie}_E g = Dg$  for a constant  $D$ .

#### 3.2. Associativity equations for the $F$ - and Frobenius structures.

Let us start with an  $F$ -manifold endowed with a compatible flat structure  $\nabla$ . If we choose a local flat coordinate system  $(x^a)$  and write the local vector potential  $C$  as  $C = \sum_c C^c \partial_c$ ,  $X = \partial_a$ ,  $Y = \partial_b$ , then (2.1) becomes

$$(3.1) \quad \partial_a \circ \partial_b = \sum_c C_{ab}{}^c \partial_c, \quad C_{ab}{}^c = \partial_a \partial_b C^c.$$

If moreover  $e$  is flat, we may choose local flat coordinates so that  $e = \partial_0$ , and the conditions  $e \circ \partial_b = \partial_b$  reduce to  $C_{0b}{}^c = \delta_b^c$ .

If we now choose an arbitrary  $C$  and define a composition  $\circ : \mathcal{T}_M^f \otimes \mathcal{T}_M^f \rightarrow \mathcal{T}_M$  by the formula (2.1), it will be automatically supercommutative in view of the Jacobi identity. Associativity, however, is a quadratic differential constraint on  $C$  which was called the *oriented associativity equations* in [9]: for any  $a, b, c, f$

$$(3.2) \quad \sum_e C_{ab}{}^e C_{ec}{}^f = (-1)^{a(b+c)} \sum_e C_{bc}{}^e C_{ea}{}^f.$$

The choice of a flat invariant metric  $(g_{ab})$  as in 3.1 allows us to reduce  $d$  functions  $C^a$  to one potential  $\Phi$  such that  $C^a = \sum_b \partial_b \Phi g^{ba}$ . Associativity equations for  $\Phi$  are also called the WDVV equations:

$$(3.3) \quad \forall a, b, c, d: \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{a(b+c)} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad}.$$

Here  $\Phi_{abc} := \partial_a \partial_b \partial_c \Phi$ . The flat identity  $e = \partial_0$  equation reduces to  $\Phi_{0ab} = g_{ab}$ . The Euler field equations take the form

$$E\Phi = (d_0 + D)\Phi + (\leq \text{quadratic terms}), \quad E g = D g.$$

**3.3. Semisimple case.** Assume now that  $M$  is endowed with a structure of semisimple Frobenius manifold. Then the multiplication  $\circ$  takes a simple form in canonical local coordinates  $(u^i)$ ,  $\partial_i := \partial/\partial u^i$ ,  $\partial_i \circ \partial_j = \delta_{ij}$ . Put  $e_j = \partial_j$  so that  $e = \sum e_i$ .

Instead, the flat metric becomes an unknown functional variable. It diagonalizes in the canonical coordinates, and its choice can be reduced to that of a *local metric potential*  $\eta = \eta(u)$  such that  $g(\partial_i, \partial_j) = \delta_{ij} \eta_i$ ,  $\eta_i := \partial_i \eta$ . Flatness and existence of  $\Phi$  reduce to the Darboux–Egoroff equations on  $\eta$ :

$$\gamma_{ij} := \frac{1}{2} \frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}}, \quad e_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad e \gamma_{ij} = 0.$$

The flat identity equation becomes

$$e \eta = \text{const}.$$

Finally, the Euler field equation takes form

$$E \eta = (D - d_0) \eta + \text{const}.$$

**3.4. Examples: quantum cohomology.** The simplest examples of (formal) Frobenius manifolds coming from algebraic geometry are furnished by the quantum cohomology of projective spaces.

Put  $H := H^*(\mathbf{P}^r, \mathbf{C}) = \bigoplus_{a=0}^r \mathbf{C} \Delta_a$ ,  $\Delta_a :=$  dual class of  $\mathbf{P}^{r-a}$ .

Denote by  $(x^a)$  linear coordinates on  $H$  dual to  $(\Delta_a)$ . Take the Poincaré pairing for the flat metric:  $g_{ab} = \delta_{a+b,r}$ . Obviously,  $x^a$  are flat coordinates.

One can construct the following formal solution to the WDVV equations:

$$\Phi^{\mathbf{P}^r}(x) := \frac{1}{6} \left( \sum_a x^a \Delta_a \right)^3 + \sum_{d, n_a \geq 0} N(d; n_2, \dots, n_r) \frac{(x^2)^{n_2} \dots (x^r)^{n_r}}{n_2! \dots n_r!} e^{dx^1}$$

Here  $N(d; n_2, \dots, n_r)$  is the number of rational curves of degree  $d$  in  $\mathbf{P}^r$  intersecting  $n_a$  hyperplanes of codimension  $a$  in general position. Purely formally, one can prove the following statement:

**PROPOSITION 3.4.1.** (i) *There exists a unique solution to the WDVV equations of this form satisfying  $N(1; 0, \dots, 0, 2) = 1$ .*

(ii)  *$e := \partial/\partial x^0$  is the flat identity.*

(iii)  *$E = \sum_a (1-a)x^a \partial_a + (r+1)\partial_1$  is an Euler field of weight one,  $Eg = (2-r)g$ .*

In much greater generality, the theory of Gromov–Witten invariants endows the cohomology space of any smooth projective or compact symplectic manifold with the canonical structure of formal Frobenius manifold. For algebro-geometric aspects of this construction, see [10].

Recently A. Givental defined and developed quantum  $K$ -theory: cf. [4] and [7]. His construction produces  $F$ -manifolds with a flat invariant metric which are not quite Frobenius because  $e$  is not flat: see [7]. Dubrovin's almost Frobenius manifolds ([3], sec. 3, Definition 9) are  $F$ -manifolds with nonflat identity as well.

**3.5. Examples: unfolding spaces of isolated singularities.** Another important class of examples comes from K. Saito's theory of unfolding singularities. Again, the simplest example, that of singularity  $z^{n+1}$  at  $z = 0$ , can be described directly and explicitly.

The unfolding space can be identified with the space of polynomials

$$\{p(z) = z^{n+1} + a_1 z^{n-1} + \dots + a_n\} = \{(a_i) \in \mathbf{C}^n\}.$$

Consider the following space  $M$ : a point of  $M$  is an ordering of the roots  $(\rho_i)$  of  $p'(z) = 0$ ,  $i = 1, \dots, n$ , supplemented by the value of  $a_n$ . (We exclude the case of multiple roots.) The functions  $u^i := p(\rho_i)$  are local coordinates on  $M$ .

**PROPOSITION 3.5.1.** *Put  $e_i = \partial/\partial u^i$  and define  $\circ$  by  $e_i \circ e_j = \delta_{ij} e_i$ . Put*

$$g = \sum_i \frac{(du^i)^2}{p''(\rho_i)}.$$

*Then  $(M, g, \circ)$  is a semisimple Frobenius manifold with metric potential*

$$\eta = -\frac{1}{2(n-1)} \sum_i \rho_i^2,$$



flat identity  $e = \sum e_i = \frac{\partial}{\partial a_n}$ , and Euler field

$$E = \sum u^i e_i = \frac{1}{n+1} \sum_{i=1}^n (i+1) a_i \frac{\partial}{\partial a_i}.$$

K. Saito's theory endows any unfolding space of an isolated hypersurface singularity with a canonical structure of an irreducible generically semisimple  $F$ -manifold with an Euler field, admitting many structures of a Frobenius manifold which are determined by a choice of the so called *primitive form*. Existence of primitive forms is a difficult fact. See [5] for a comprehensive exposition of K. Saito's theory in the context of  $F$ -manifolds.

The  $F$ -structure itself, and the construction of an Euler field, is furnished by an elementary construction which can be easily generalized to include geometric situations essential for mirror constructions. Cf. [10], III.8, where this generalization was called *K. Saito's framework*.

**3.6. Mirror isomorphisms.** The simplest manifestations of the famous mirror phenomenon, discovered by physicists, are highly nontrivial isomorphisms between certain Frobenius manifolds representing quantum cohomology ( $A$ -side) and other Frobenius structures carried by generalized, or extended, moduli spaces ( $B$ -side) and their appropriate subspaces, for example, cut out by the condition of integrality of the spectrum of the operator  $\text{ad } E$  acting upon flat vector fields. In this subsection, I will briefly review some of the aspects of this fascinating story directly involving  $F$ -manifolds.

**3.6.1. Extended moduli spaces.** Roughly speaking, an unobstructed deformation problem (like that of a Calabi Yau complex structure  $X$ ) classically is governed by a certain cohomology space  $H^d$  of a fixed weight associated to  $X$  which becomes the tangent space to the moduli space at the point corresponding to  $X$ .

Any known construction of what can be considered as an extended moduli space produces the tangent space which is the full cohomology algebra  $H^*$ .

The existing definitions of extended moduli spaces of, say, complex structures, are not really satisfactory in at least two respects. First, generally they furnish only a formal neighborhood of the classical moduli space. Second, it is not at all clear what kind of structures are parametrized by their nonclassical points. Any progress in this problem necessarily involves enlarging the geometric universe so as to include dg-spaces (to treat obstructed cases as well), noncommutative spaces etc. For a review, see [12], §4, and references therein.

In [12], S. Merkulov works with dg-manifolds, and defines for them the notion of *cohomology  $F$ -structure*. He then proves the following two results which show the ubiquity of  $F$ -structures:

**THEOREM 3.6.2 .** (i) *The smooth part of the extended moduli space of deformations of complex or symplectic structure on a compact manifold is endowed with a canonical  $F$ -structure.*

(ii) *If the  $\mathcal{G}_\infty$  operad acts upon a dg vector space  $(V, d)$ , then the formal graded manifold which is the completion of  $H^*(V, d)$  has a canonical structure of the cohomological  $F$ -manifold.*

**3.6.3. Unfolding spaces vs extended moduli spaces.** Let now  $f(x_0, \dots, x_n)$  be a *homogeneous polynomial* with an isolated singularity at zero. Denote by  $X$  the projective hypersurface  $f = 0$ , i.e. the base at infinity of the affine cone  $f = 0$ . An unfolding of  $f$  can be written in the form  $f + \sum_i t_i g_i$  where  $g_i$  span  $\mathbf{C}[x]$  modulo the Jacobian ideal. The unfolding contains a subspace consisting of the homogeneous polynomials of the same degree as  $f$ . Geometrically, it parametrizes unfoldings which correspond to the projective deformations of  $X$  and therefore can be associated with at least a part of the classical moduli space of deformations of  $X$ . Other points deform  $f$  and break the affine cone structure of  $f = 0$ .

Thus, this picture is parallel to the one we would like to see in the extended deformation theory. From the general perspective of deformation theory, it looks somewhat naive and ad hoc. On the other hand, it does have some desirable features: the unfolding space is global (or at least, not just formal in certain directions), and it is quite clear what geometric objects correspond to the nonclassical points.

Furthermore, K. Saito's construction provides the  $F$ -structure and many Frobenius structures in this context as well.

For applications to the mirror isomorphisms it is thus important to have an intrinsic characterization of  $F$ -manifolds which can be obtained in this way. At least locally, it is furnished by the following beautiful theorem due to C. Hertling ([5], Theorems 5.3 and 5.6). All spaces below are germs of complex analytic spaces.

**THEOREM 3.6.4.** (i) *The spectral cover space  $\widetilde{M}$  of the  $F$ -structure on the germ of the unfolding space of an isolated hypersurface singularity is smooth.*

(ii) *Conversely, Let  $M$  be an irreducible germ of a generically semisimple  $F$ -manifold with the smooth spectral cover  $\widetilde{M}$ . Then it is (isomorphic to) the germ of the unfolding space of an isolated hypersurface singularity. Moreover, any isomorphism of germs of such unfolding spaces compatible with their  $F$ -structure comes from a stable right equivalence of the germs of the respective singularities.*

Recall that the stable right equivalence is generated by adding sums of squares of coordinates and making invertible analytic coordinate changes.

In view of this result, it would be interesting to understand the following

**3.6.5. Problem.** *Which quantum cohomology Frobenius spaces have smooth spectral covers?*

For a discussion of generically simple quantum cohomology and Dubrovin's conjectures, see [1].

#### 4 – Pencils of flat connections on an external bundle and Dubrovin's twisting

**4.1. Pencils of flat connections on an external bundle.** In Proposition 2.3.1, we have characterized  $F$ -manifolds with a compatible flat connection in terms of pencils of flat connections on the tangent sheaf.

Let us now weaken the input data: consider a locally free sheaf  $\mathcal{F}$  on a manifold  $M$  and a pencil (affine line)  $\mathcal{P}$  of connections  $\nabla : \mathcal{F} \rightarrow \Omega_M^1 \otimes_{\mathcal{O}_M} \mathcal{F}$ . The difference of any two connections is an odd global section  $\nabla_1 - \nabla_0 = A \in \Omega_M^1 \otimes_{\mathcal{O}_M} \text{End} \mathcal{F}$ . Any two differences are proportional with a constant coefficient; we will often choose one arbitrarily. Any  $\nabla$  can be extended to an odd derivation (denoted again  $\nabla$ ) of  $\Omega_M^* \otimes_{\mathcal{O}_M} T(\mathcal{F})$  in the standard way, where  $T(\mathcal{F})$  is the total tensor algebra of  $\mathcal{F}$ .

Assume that all connections in  $\mathcal{P}$  are flat, that is for any  $\nabla \in \mathcal{P}$ ,  $\nabla A = 0$  and  $A \wedge A = 0$ . Again, in view of the De Rham lemma,  $A$  can be everywhere locally written as  $\nabla B$  where  $B = B_{\nabla}$  (it generally depends on  $\nabla$ ) is an even section of  $\text{End} \mathcal{F}$ . In this setting, we cannot assume  $\mathcal{P}$  to be torsionless: this makes no sense for an external bundle. To produce an  $F$ -structure, we will have instead to fix an additional piece of data.

Choose a coordinate neighborhood  $U$  in  $M$  over which the linear superspace  $F$  of local  $\nabla$ -flat sections of  $\mathcal{F}$  trivializes  $\mathcal{F}$ . Denote by  $\tilde{F}$  the affine supermanifold associated with  $F$ . Denote by  $q : \tilde{\mathcal{F}} \rightarrow M$  be the fibration of supermanifolds which is “the total space” of  $\mathcal{F}$  as a vector bundle: for example, in algebraic geometry this is the relative affine spectrum of  $\text{Symm}_{\mathcal{O}_M}(\mathcal{F}^*)$ ,  $\mathcal{F}^*$  being the dual sheaf. Hence local sections of  $\mathcal{F}$  become local sections of  $q$ . Clearly,  $\nabla$  trivializes  $q$  over  $U$ : we have a well defined isomorphism  $q^{-1}(U) = \tilde{F} \times U$  turning  $q$  into projection.

Let now  $u \in F$  be a  $\nabla$ -flat section of  $\mathcal{F}$  over  $U$ ,  $\nabla B = A$ ,  $B \in \text{End} \mathcal{F}$ . Then  $Bu$  is a section of  $\mathcal{F}$ ; we will identify it with a section of  $q$  as above. Projecting to  $\tilde{F}$ , we finally get a morphism  $Bu : U \rightarrow \tilde{F}$ . Thus  $Bu$  denotes several different although closely related objects. If we choose a basis of flat sections in  $F$  and a system of local coordinates  $(x^a)$  on  $M$ ,  $B$  becomes a local even matrix function  $B(x)$  acting from the left on columns of local functions. Then  $Bu$  becomes the

map  $U \rightarrow \tilde{F} : x \mapsto B(x)u$ . Since  $B$  is defined up to  $\text{Ker } \nabla$ , this map is defined up to a constant shift.

DEFINITION 4.1.1 A section  $u$  of  $\mathcal{F}$  is called a primitive section with respect to  $\nabla \in \mathcal{P}$ , if it is  $\nabla$ -flat, and  $Bu$  is a local isomorphism of  $U$  with a subdomain of  $\tilde{F}$ .

Since  $F$  is linear,  $\tilde{F}$  has a canonical flat structure  $\nabla^F : \mathcal{T}_{\tilde{F}} \rightarrow \Omega_{\tilde{F}}^1 \otimes \mathcal{T}_{\tilde{F}}$ . If  $u$  is primitive,  $Bu$  is a local isomorphism allowing us to identify locally  $(Bu)^*(\mathcal{T}_{\tilde{F}}) = \mathcal{T}_M$ . Moreover, we can consider the pullback of  $\nabla^F$  with respect to  $Bu$ :

$$(4.1) \quad \nabla^* := (Bu)^*(\nabla^F) : \mathcal{T}_M \rightarrow \Omega_M^1 \otimes \mathcal{T}_M.$$

Clearly, the local flat structures induced by the maps  $Bu$  on  $M$  do not depend on local choices of  $B$  and glue together to a flat structure on all of  $M$  determined by  $\nabla$  and  $u$ . It follows also that  $\nabla^*$  is flat and torsionless.

There is another important isomorphism produced by this embedding. Namely, restricting  $(Bu)^*$  to  $F \subset \mathcal{T}_{\tilde{F}}$  we can identify  $(Bu)^*(F) \subset \mathcal{T}_M$  with  $F \subset \mathcal{C}$  and then by  $\mathcal{O}_M$ -linearity construct an isomorphism  $\beta^* : \mathcal{F} \rightarrow \mathcal{T}_M$ . The connection  $\nabla^*$  identifies with  $\nabla$  under this isomorphism. The inverse isomorphism  $(\beta^*)^{-1}$  can be described as  $X \mapsto i_X(\nabla Bu)$ .

Denote  $\mathcal{P}^*$  the pencil of flat connections  $\beta^*(\mathcal{P})$ .

**4.1.2. Example.** Assume that  $(M, \circ, e, \nabla_0)$  is an  $F$ -manifold endowed with a compatible flat structure and a  $\nabla_0$ -flat identity. Put  $\mathcal{F} := \mathcal{T}_M$  and construct  $A$  so that  $X \circ Y = i_X(A)(Y)$ . Then  $e$  is a primitive section which induces exactly the initial flat structure  $\nabla_0^* = \nabla_0$ .

In fact, let  $(x^a)$  be a  $\nabla_0$ -flat local coordinate system on  $M$  such that  $e = \partial_0$ , and  $(\partial_0, \dots, \partial_m)$  the dual basis of flat vector fields. Then an easy argument shows that as a vector field,  $Be = \sum_a (x^a + c^a) \partial_a$  where  $c^a$  are constants.

A converse statement holds as well.

**THEOREM 4.2.** *Let  $(M, \mathcal{F}, \nabla, A, u)$  be a pencil of flat connections on an external bundle endowed with a primitive section. Then  $\mathcal{P}^*$  is a pencil of torsionless flat connections, and  $e := \beta^*(u)$  is an identity for one of the associated  $F$ -manifold structures  $\circ$ .*

For a proof, see [9], §5, and [11], §4.

**4.3. The tangent bundle considered as an external bundle.** Let now  $(M, \mathcal{P})$  be a manifold with a unital pencil of torsionless flat connections as in sec. 2.3. Assume moreover that one (hence each) identity  $e_A$  is flat with respect to some  $\nabla_0 \in \mathcal{P}$ . Choose as origin  $\nabla_0$  and a coordinate  $\lambda$  on  $\mathcal{P}$  so that  $(M, \mathcal{P})$  determines an  $F$ -manifold structure  $\circ$  with  $\nabla_0$ -flat identity  $e$  and multiplication tensor  $A$ .

It is natural to study the family of all  $F$ -manifold structures that can be obtained from this one by treating  $\mathcal{T}_M$  as an external bundle with the pencil  $\mathcal{P}$ , choosing different  $\nabla \in \mathcal{P}$  and different  $\nabla$ -flat primitive sections, and applying Theorem 4.2.

Let us call an even global vector field  $\varepsilon$  a *virtual identity*, if it is invertible with respect to  $\circ$  and its inverse  $u := \varepsilon^{-1}$  belongs to  $\text{Ker } \nabla$  for some  $\nabla \in \mathcal{P}$ .

**PROPOSITION 4.3.1.** *a) Inverted virtual identities in  $\text{Ker } \nabla$  are exactly primitive sections of  $(\mathcal{T}_M, \nabla)$  considered as an external bundle.*

*b) The map  $(\beta^*)^{-1}$  sends a local vector field  $X$  to  $X \circ u = X \circ \varepsilon^{-1}$ .*

**4.4. Dubrovin's twisting.** Let now  $(M, \mathcal{P}, e, \varepsilon)$  be a flat pencil on  $\mathcal{T}_M$  such that the respective  $F$ -manifold is endowed with a  $\nabla_0$ -flat identity  $e$ , and a  $\nabla$ -flat inverse virtual identity  $\varepsilon^{-1}$ .

**THEOREM 4.4.1.** *a) Denote by  $*$  the new multiplication on  $\mathcal{T}_M$  induced by  $(\beta^*)^{-1}$  from  $\circ$ :*

$$(4.2) \quad X * Y = \varepsilon^{-1} \circ X \circ Y.$$

*Then  $(M, *, \varepsilon)$  is an  $F$ -manifold with identity  $\varepsilon$  (hence our term "virtual identity").*

*b) With the same notation, assume moreover that  $\nabla_0 \neq \nabla$  and that  $A$  is normalized as  $A = \nabla - \nabla_0$ . Then  $e$  is an Euler field of weight one for  $(M, *, \varepsilon)$ .*

For a proof, see [11].

The relationship between  $(M, \circ, e)$  and  $(M, *, \varepsilon)$  is almost symmetric, but not quite. To explain this, we start with a part that admits a straightforward check:

$$(4.3) \quad X \circ Y = e^{*-1} * X * Y$$

which is the same as (4.2) with the roles of  $(\circ, e)$  and  $(*, \varepsilon)$  reversed. Here  $e^{*-1}$  denotes the solution  $v$  to the equation  $v * e = \varepsilon$ , that is  $\varepsilon^{-1} \circ v \circ e = \varepsilon$ , therefore  $v = e^{*-1} = \varepsilon^{\circ 2}$ . After this remark one sees that (4.3) follows from (4.2).

Slightly more generally, one easily checks that the map inverse to  $X \mapsto \varepsilon^{-1} \circ X$  reads  $Y \mapsto e^{*-1} * Y$  as expected.

If the symmetry were perfect, we would now expect  $\varepsilon$  to be an Euler field of weight one for  $(M, \circ, e)$ . However, one can check that this is not always true.

The reason of this is that the vector field  $e$  is not a virtual identity for  $(M, *, \varepsilon)$ :  $e^{*-1} = \varepsilon^2$  is not flat with respect to any  $\nabla^* \in \mathcal{P}^*$ , so that if we start with  $(M, *, \varepsilon)$  and construct  $\circ$  via (4.3), we cannot apply Theorem 4.2 anymore.

Dubrovin has introduced this twisting operation in the context of Frobenius manifolds and called it (almost) duality in [3].

### 5 – Toric compactifications and tensor products of $F$ -manifolds with compatible flat structure

**5.1. Setup.** Let  $M$  be the formal completion of a linear (super)space  $T$  at 0, that is formal spectrum of  $\mathcal{R} := \mathbf{C}[[x^a]]$ . Denote by  $\nabla$  the flat connection on  $M$  with  $\text{Ker } \nabla = T$ .

As we have already remarked, the classification of all formal  $F$ -manifold structures on  $M$  compatible with  $\nabla$  is equivalent to the classification of the solutions to the matrix differential equation

$$(5.1) \quad \nabla C \wedge \nabla C = 0, \quad C \in \text{End } T \widehat{\otimes}_{\mathbf{C}} m,$$

where  $m$  is the maximal ideal of  $\mathcal{R}$ . Namely, given  $C$ , put

$$X \circ Y := i_X(\nabla C)(Y) = (XC)Y$$

In this section we will explain the following result from [9]:

**THEOREM 5.2.** (i) *Solutions  $C$  to (5.1) bijectively correspond to the representations in  $T$  of a certain algebra  $H_{*T}$  constructed from homology spaces of permutohedral toric compactifications.*

(ii) *This allows one to define a tensor product operation on formal  $F$ -manifolds endowed with a compatible flat structure.*

The algebra  $H_{*T}$  is essentially motivic. Its possible relation to the motivic fundamental groups (of Hodge–Tate type) deserves further study.

**5.3. Permutohedral fans.** Let  $B$  be a finite set. An  $N + 1$ -partition  $\tau = \{\tau_1, \dots, \tau_{N+1}\}$  of  $B$ , by definition, is a *totally ordered set of  $N + 1$  pairwise disjoint non-empty subsets of  $B$  whose union is  $B$ .*

If  $N + 1 \geq 2$ ,  $\tau$  determines a well ordered family of  $N$  2-partitions  $\sigma^{(a)}$ :

$$\sigma_1^{(a)} := \tau_1 \cup \dots \cup \tau_a, \quad \sigma_2^{(a)} := \tau_{a+1} \cup \dots \cup \tau_{N+1}, \quad a = 1, \dots, N.$$

A sequence of  $N$  2-partitions  $(\sigma^{(i)})$  is called *good* if it can be obtained by such a construction.

Put  $N_B := \mathbf{Z}^B / \mathbf{Z}$ , the subgroup being embedded diagonally. Similarly,  $N_B \otimes \mathbf{R} = \mathbf{R}^B / \mathbf{R}$ . Vectors in this space (resp. lattice) are functions  $B \rightarrow \mathbf{R}$  (resp.  $B \rightarrow \mathbf{Z}$ ) considered modulo constant functions. For a subset  $\beta \subset B$ , the function  $\chi_\beta = 1$  on  $\beta$  and 0 elsewhere, determines such a vector.

The fan  $\Phi_B$  in  $N_B \otimes \mathbf{R}$ , by definition, consists of the following  $l$ -dimensional cones  $C(\tau)$  labeled by all  $(l + 1)$ -partitions  $\tau$  of  $B$ .

If  $\tau$  is the trivial 1-partition,  $C(\tau) = \{0\}$ .

If  $\sigma$  is a 2-partition,  $C(\sigma)$  is generated by  $\chi_{\sigma_1}$ , or, equivalently,  $-\chi_{\sigma_2}$ , modulo constants.

Generally, let  $\tau$  be an  $(l+1)$ -partition, and  $\sigma^{(i)}$ ,  $i = 1, \dots, l$ , the respective good family of 2-partitions. Then  $C(\tau)$  is defined as a cone generated by all  $C(\sigma^{(i)})$ .

**5.4. Permutohedral toric varieties.** Denote by  $L_B$  the compactification of the torus  $(\mathbf{G}_m)^B/\mathbf{G}_m$  associated with the fan  $\Phi_B$ . The permutation group of  $B$  acts upon it.

**5.5. The algebra  $H_*$ .** It is a graded algebra whose  $n$ -th component of  $H_*$  is the homology space  $H_{*n} := H_*(L_n)$  where  $L_n := L_{\{1, \dots, n\}}$ . The homology space is spanned by cycles  $\mu(\tau)$  indexed, as well as cones of  $\Phi_n$ , by partitions  $\tau$  of  $\{1, \dots, n\}$ .

The multiplication law is given in terms of these generators by the following prescription. Let  $\tau^{(1)}$  (resp.  $\tau^{(2)}$ ) be a partition of  $\{1, \dots, m\}$  (resp. of  $\{1, \dots, n\}$ ), then

$$\mu(\tau^{(1)})\mu(\tau^{(2)}) = \mu(\tau^{(1)} \cup \tau^{(2)})$$

where the concatenated partition of  $\{1, \dots, m, m+1, \dots, m+n\}$  is defined in an obvious way, shifting all the components of  $\tau^{(2)}$  by  $m$ .

**5.6. The algebra  $H_*T$ .** As above, let  $T$  be a linear superspace. Define  $H_*T$  as the algebra of symmetric coinvariants of the diagonal tensor product

$$H_*T := \left( \bigoplus_{n=1}^{\infty} H_{*n} \otimes T^{\otimes n} \right)_{\mathbf{S}}.$$

The symmetric group  $\mathbf{S}_n$  acts upon the  $n$ -th graded component.

Given a linear representation  $\rho : H_*T \rightarrow \text{End } T$ , its *matrix correlators* are defined by

$$\tau^{(n)} \langle \Delta_{a_1} \dots \Delta_{a_n} \rangle_{\rho} := \rho(\mu(\tau^{(n)}) \otimes \Delta_{a_1} \otimes \dots \otimes \Delta_{a_n}).$$

Here  $\tau^{(n)}$  runs over all partitions of  $\{1, \dots, n\}$  whereas  $(a_1, \dots, a_n)$  runs over all maps  $\{1, \dots, n\} \mapsto I : i \mapsto a_i$ .

*Top matrix correlators* of  $\rho$  correspond to the identical partitions  $\varepsilon^{(n)}$  of  $\{1, \dots, n\}$ :

$$\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle_{\rho} := \varepsilon^{(n)} \langle \Delta_{a_1} \dots \Delta_{a_n} \rangle_{\rho}.$$

Given  $\rho$ , construct the series

$$C_{\rho} = \sum_{n=1}^{\infty} \sum_{(a_1, \dots, a_n)} \frac{x^{a_n} \dots x^{a_1}}{n!} \langle \Delta_{a_1} \dots \Delta_{a_n} \rangle_{\rho} \in \mathbf{C}[[x]] \otimes \text{End } T.$$

THEOREM 5.7. (i) We have

$$\nabla C_\rho \wedge \nabla C_\rho = 0.$$

(ii) Conversely, let  $\Delta(a_1, \dots, a_n) \in \text{End } T$  be a family of linear operators defined for all  $n \geq 1$  and all maps  $\{1, \dots, n\} \rightarrow I : i \mapsto a_i$ . Assume that the parity of  $\Delta(a_1, \dots, a_n)$  coincides with the sum of the parities of  $\Delta_{a_i}$  and that  $\Delta(a_1, \dots, a_n)$  is (super)symmetric with respect to permutations of  $a_i$ 's. Finally, assume that the formal series

$$C = \sum_{n=1}^{\infty} \sum_{(a_1, \dots, a_n)} \frac{x^{a_n} \dots x^{a_1}}{n!} \Delta(a_1, \dots, a_n) \in \mathbf{C}[[x]] \otimes \text{End } T$$

satisfies  $\nabla C \wedge \nabla C = 0$ . Then there exists a well defined representation  $\rho : H_* T \rightarrow \text{End } T$  such that  $\Delta(a_1, \dots, a_n)$  are the top correlators  $\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle_\rho$  of this representation.

**5.8. Comultiplication and tensor product.** Since components of  $H_*$  are homology groups of compact manifolds, they admit a natural comultiplication. This allows one to define ring homomorphisms

$$\Delta_{T_1, T_2} : H_{*T_1 \otimes T_2} \rightarrow H_{*T_1} \otimes H_{*T_2}.$$

Hence, if  $\rho_i$  is a representation of  $H_{*T_i}$ ,  $i = 1, 2$ , the tensor product  $\rho_1 \otimes \rho_2$  induces a representation of  $H_{*T_1 \otimes T_2}$ .

Using Theorem 5.7, we can translate this operation into a tensor product of formal  $F$ -manifolds with a compatible flat structure.

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