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The rational analogue of the Beckman-Quarles Theorem and the rational realization of some sets in E^d

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ABSTRACT: *We describe the recent developments concerning the rational analogues of the Beckman-Quarles Theorem, and discuss a related result concerning isometric embeddings in Q^m of subsets of E^n.*

1 – Let E^d denote the Euclidean d-space, and let Q^d denote the Euclidean rational d-space. A mapping $f : E^d \to E^d$ is called ρ-distance preserving if $\|x - y\| = \rho$ implies that $\|f(x) - f(y)\| = \rho$. The Beckman Quarles Theorem [1] asserts that every mapping $f : E^d \to E^d$ which preserves unit distance is an isometry, provided $d \geq 2$; for a discrete version, see Tyszka [9].

W. Benz [2, 3] and H. Lenz [7] noticed that if $d = 2, 3$ or 4, a unit-distance preserving mapping from Q^d into Q^d needs not be an isometry. A Tyszka [10] showed that every unit distance preserving mapping $f : Q^8 \to Q^8$ is an isometry. In a sequence of papers [12,13] we extended these results to all even dimensions d of the form $d = 4k(k+1)$ and all the odd dimensions d of the form $d = 2m^2 - 1$. W. Benz [2, 3] had shown that every mapping $f : Q^d \to Q^d$ which preserves the distances 1 and 2 (or, equivalently, 1 and n, $n \geq 2$) is an isometry, provided $d \geq 5$. We [14] had shown that every mapping $f : Q^d \to Q^d$ which preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$. R. Connelly and J. Zaks [5]

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showed that for all even d, $d \geq 6$, every unit distance preserving mapping $f : Q^d \to Q^d$ is an isometry. W. Hibi, my Ph.D. student, has recently proved [6] that for every $d \geq 5$, every unit-preserving mapping $f : Q^d \to Q^d$ is an isometry.

Let $Q(d, \rho)$ denote the graph whose vertices are the rational points of E^d and its edges are pairs of points (x, y) for which $\|x - y\| = \rho$. Denote by $\omega(G)$ the clique number of a graph G, and by $\omega(d)$ the clique number of $Q(d, 1)$. The values of $\omega(d)$ were given by Chilakamarry [4].

The main idea of W. Hibi [6] is the following lemma.

Lemma. *If $d \geq 5$, if $m = \omega(d) \leq d$ and if $A = \{A_1, \ldots, A_m, B, C\}$ is an $(m+2)$-points set in Q^d for which $\|A_i - A_j\| = \|A_j - B\| = \|B - C\| = 1, 1 \leq i < j \leq m$, then every unit preserving mapping $f : A \to Q^d$ has the property that $f(C) \neq f(A_1)$.*

To prove the lemma, observe that if $f : A \to Q^d$ is a unit preserving mapping for which $f(C) = f(A_1)$, then $\{f(A_1), \ldots, f(A_m), f(B)\}$ forms an $(m+1)$-clique in Q^d, contrary to the assumption that $\omega(d) = m$.

2 – Let A be a subset of E^n and let $d = \dim(\mathrm{Aff}(A))$. If there exists a rational space Q^m, for some m, which contains a congruent copy of A, then *the rational dimension* $\rho(A)$ of A is defined as the least m for which Q^m contains a congruent copy of A; otherwise $\rho(A)$ does not exist.

A set A is said to satisfy condition (ρ) if the following holds:

(ρ) $\|x - y\|^2$ is a rational number for all x, y in A.

Obviously, if a set A can be rationally embedded in E^n, then A must satisfy the condition (ρ).

We will show that condition (ρ) is sufficient for a set A in E^n to have a rational embedding in some E^m. We will establish the following theorems.

Theorem 1. *If the vertex set S of a d-simplex Δ^d in E^n satisfies the condition (ρ), then $\rho(S) \leq 4d$.*

Theorem 1 leads to our main result, which is the following.

Theorem 2. *If a set A in E^n satisfies the condition (ρ), and if $\dim(\mathrm{Aff}(A)) = d$, then $\rho(A) \leq 4d$.*

Moreover, if $\{V_0, \ldots, V_d\}$ is a subset of A which has affine dimension d, then there exists a set $\{V_0^*, \ldots, V_d^*\}$ of $d+1$ points in Q^{4d} and there exists an isometric embedding f of $\mathrm{Aff}\{V_0, \ldots, V_d\}$ onto a d-dimensional affine flat in E^{4d}, such that $f(V_i) = V_i^*$ holds for all i, $0 \leq i \leq d$, and $f(A)$ is contained in Q^{4d}; i.e., the mapping f isometrically embeds all the points of A into rational points in $\mathrm{Aff}\{V_0^*, \ldots, V_d^*\}$ in E^{4d}.

Proof of Theorem 1. Let $\{V_0, \ldots, V_d\}$ be the vertex set of a d-simplex in E^n for which $\|V_i - V_j\|^2 = d_{ij}^2$ is a rational number for all i and for all j, $0 \le i \le j \le d$.

For each i, $1 \le i \le d$, define W_i to be a point in Q^{4d} of the form

$$W_i = (0, \ldots, 0, w_{4i-3}, w_{4i-2}, w_{4i-1}, w_{4i}, 0, \ldots, 0),$$

in which the four coordinates $w_{4i-3}, \ldots, w_{4i}$ will be defined later. We inductively define the points $V_0^*, \ldots, V_d^*$ as follows. The point V_0^* is taken as the origin; assume that all the points $V_1^*, \ldots, V_{m-1}^*$ have been defined and they are of the form

$$V_1^* = W_1 \quad \text{and} \quad V_k^* = \sum_{j=1}^{k-1} b_{k,j} W_j + W_k \in Q^{4d}, \qquad 2 \le k \le m-1,$$

in which all the coefficients $b_{k,j}$ are rational numbers, W_k are rational points and for which $\|V_i^* - V_j^*\| = \|V_i - V_j\| = d_{ij}$ holds for all i and for all j, $1 \le i < j \le m-1$.

Define the point V_m^* to be of the form

$$V_m^* = \sum_{i=1}^{m-1} b_{m,i} W_i + W_m \in Q^{4d},$$

in which all the $b_{m,j}$ are rational numbers for which $\|V_i^* - V_j^*\| = \|V_i - Vj\| = d_{ij}$ will hold for all i and for all j, $1 \le i < j \le m$.

In the case $d = 1$, two points V_0 and V_1 are given in E^n, such that $\|V_0 - V_1\|^2 = d_{12}^2$ is a positive rational number. By Lagrange's Four Squares Theorem [8], there exist four rational numbers α, β, γ and δ such that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \|V_0 - V_1\|^2$. The two points V_0^* and V_1^* in Q^4 are defined by $V_0^* = (0, 0, 0, 0)$ and $V_1^* = (\alpha, \beta, \gamma, \delta)$.

The case $d = 2$ deals with triangles, and it had been treated in ([6], Lemma 2, see also [14], Lemma 4 and Lemma 5); it will be included here as well.

We will determine the rational coefficients $b_{m,1}, b_{m,2}, \ldots, b_{m,m-1}$ and rational coordinates $w_{4m-3}, w_{4m-2}, w_{4m-1}$ and w_{4m} of W_m as follows.

The following m equations, $0 \le j \le m-1$, are required to hold:

$$\|V_m^* - V_j^*\|^2 = d_{m,j}^2$$

in particular,

$$\|V_m^* - V_0^*\|^2 = d_{m,0}^2 = \sum_{i=1}^{m-1} b_{m,i}^2 \|W_i\|^2 + \|W_m\|^2;$$

$$\|V_m^* - V_1^*\|^2 = d_{m,1}^2 = (b_{m,1} - 1)^2 \|W_1\|^2 + \sum_{i=2}^{m-1} b_{m,i}^2 \|W_i\|^2 + \|W_m\|^2.$$

Therefore,

$$\|V_m^* - V_1^*\|^2 - \|V_m^* - V_0^*\|^2 = d_{m,1}^2 - d_{m,0}^2 = (1 - 2b_{m,1}) \|W_1\|^2.$$

Thus $b_{m,1}$ is a rational number, since

$$b_{m,1} = \frac{\|W_1\|^2 - d_{m,1}^2 + d_{m,0}^2}{2\|W_1\|^2} = \frac{d_{0,1}^2 - d_{m,1}^2 + d_{m,0}^2}{2d_{0,1}^2}.$$

Next,

$$\begin{aligned}
&\|V_m^* - V_2^*\|^2 - \|V_m^* - V_1^*\|^2 = d_{m,2}^2 - d_{m,1}^2 = \\
&= (b_{m,1} - b_{2,1})^2\|W_1\|^2 + (b_{m,2} - 1)^2\|W_2\|^2 + \sum_{i=3}^{m-1} b_{m,i}^2\|W_i\|^2 + \\
&\quad - (b_{m,1} - 1)^2\|W_1\|^2 - \sum_{i=2}^{m-1} b_{m,i}^2\|W_i\|^2 = \\
&= \left[(b_{m,1} - b_{2,1})^2 - (b_{m,1} - 1)^2\right]\|W_1\|^2 + (1 - 2b_{m,2})\,\|W_2\|^2.
\end{aligned}$$

It follows that $b_{m,2}$ is a rational number, and so on.

We end up with the following.

$$\begin{aligned}
&\|V_m^* - V_{m-1}^*\|^2 - \|V_m^* - V_{m-2}^*\|^2 = d_{m,m-1}^2 - d_{m,m-2}^2 = \\
&= \sum_{i=1}^{m-1-1} (b_{m,i} - b_{m-1,i})^2\|W_i\|^2 + (b_{m,m-1} - 1)^2\|W_{m-1}\|^2 + \\
&\quad - \sum_{i=1}^{m-2-1} (b_{m,i} - b_{m-2,i})^2\|W_i\|^2 - (b_{m,m-2} - 1)^2\|W_{m-2}\|^2 - b_{m,m-1}^2\|W_{m-1}\|^2 = \\
&= (b_{m,m-2} - b_{m-1,m-2})^2\|W_{m-2}\|^2 + (1 - 2b_{m,m-1})\|W_{m-1}\|^2 + \\
&\quad - (b_{m,m-2} - 1)^2\|W_{m-2}\|^2.
\end{aligned}$$

It follows that $b_{m,m-1}$ is a rational number.

As a consequence, it is possible to find rational coefficients $b_{m,i}$ for all i, $1 \le i \le m-1$, for which V_m^* has the required form (except possibly for the part of W_m) and for which

$$\|V_m^* - V_{k-1}^*\|^2 - \|V_m^* - V_{k-2}^*\|^2 = d_{m,k-1}^2 - d_{m,k-2}^2$$

holds for all k, $1 \le k \le m-1$.

Finally, from the equation

$$\|V_m^* - V_0^*\|^2 = d_{m,0}^2 = \sum_{i=1}^{m-1} b_{m,i}^2\|W_i\|^2 + \|W_m\|^2$$

and the rationality of all the coefficients $b_{m,i}$ we conclude that $\|W_m\|^2$ is a rational number.

Next, we want to show that the expression one get for $\|W_m\|^2$ is non-negative.

In fact, it follows easily from the form of the points V_i^* that for all k, $1 \leq k \leq m$, $\mathrm{Aff}\{V_0^*, V_1^*, \ldots, V_k^*\} = \mathrm{Aff}\{0, W_1, \ldots, W_k\}$, and also that $\|W_k\|$ is the height of the k-simplex $\mathrm{conv}\{0, W_1, \ldots, W_k\}$ with respect to its facet $\mathrm{conv}\{0, W_1, \ldots, W_{k-1}\}$; It is also the distance from the point V_k^* to $\mathrm{Aff}\{0, W_1, \ldots, W_{k-1}\}$. Thus, a priori, $\|W_k\| \geq 0$, with equality to zero holding if, and only if, W_k is already in $\mathrm{Aff}\{0, W_1, \ldots, W_{k-1}\}$.

It follows that the possibly non-zero coordinates $w_{4m-3}, w_{4m-2}, w_{4m-1}$ and w_{4m} of W_m can be chosen, using Lagrange's Four Square Theorem, as rational numbers. It follows that W_m can be chosen as a rational point, which implies that V_m^* is a rational point in Q^{4d}. Thus, $\|V_m^* - V_0^*\|^2 = d_{m,0}^2$, hence $\|V_m^* - V_1^*\|^2 = d_{m,1}^2$, which implies that $\|V_m^* - V_k^*\|^2 = d_{m,k}^2$ for all k, $1 \leq k \leq m-1$.

This completes the proof of Theorem 1. $\square$

Corollary 1. *If the vertex set S of a d-simplex Δ^d in E^n satisfies the condition (ρ) and if one of the edges of Δ^d has, in addition, a rational length, then $\rho(S) \leq 4d-3$.*

To prove Corollary 1, observe that if we take the two vertices of Δ^d which are at a rational distance α as the points V_0 and V_1, then we can choose $V_0^* = (0)$ and $V_1^* = (\alpha)$, thus save three dimensions at the beginning, and continue as in the proof of Theorem 1. This result has been used previously ([14], Lemma 4, [6], Corollary 1), showing that certain triangles can be rationally embedded in Q^5.

Corollary 2. *If the vertex set $S = \{V_0, \ldots, V_d\}$ of a d-simplex Δ^d in E^n satisfies the condition (ρ) and if for every $m, 1 \leq m \leq d$, the distance from V_m to $\mathrm{Aff}\{V_0, \ldots, V_{m-1}\}$ is a rational number, then $\rho(S) = d$, i.e., S can be rationally embedded in Q^d.*

To prove Corollary 2, we repeat the proof of Theorem 1, and whenever there is a need to fix W_m, it turns out that $\|W_m\|$ is a rational number, hence only one non zero rational coordinate suffices for W_m, therefore d non zero rational coordinates will suffice for the entire $\{W_1, W_2, \ldots, W_d\}$.

Lemma 1. *If the vertex set $S = \{V_0, \ldots, V_d\}$ of a d-simplex Δ^d is contained in E^n, if U is a point of $\mathrm{Aff}\{V_0, \ldots, V_d\}$ and if $\{V_0, \ldots, V_d, U\}$ satisfies the condition (ρ), then the barycentric coordinates of U with respect to $\{V_0, \ldots, V_d\}$ are all rational numbers and $\rho(V_0, \ldots, V_d, U) = \rho(V_0, \ldots, V_d) \leq 4d$.*

Proof of Lemma 1. Let $S = \{V_0, \dots, V_d\}$ be the vertex set of a d-simplex Δ^d in E^n, let U be a point of $\mathrm{Aff}\{V_0, \dots, V_d\}$ and let $\{V_0, \dots, V_d, U\}$ satisfy the condition (ρ). The procedure of the proof of Theorem 1 to the set $\{V_0, \dots, V_d, U\}$ in E^n yields a congruent set of rational points $\{V_0^*, \dots, V_d^*, U^*\}$, in which V_0^* is the origin and U^* is a point in $\mathrm{Aff}\{V_0^* = 0, V_1^*, \dots, V_d^*\}$. Therefore there exist rational coefficients $b_{k,j}$ for which

$$V_k^* = \sum_{j=1}^{k-1} b_{k,j} W_j + W_k \in Q^{4d}, \qquad 1 \le k \le d, \quad \text{and}$$

$$U^* = \sum_{j=1}^{d} b_{d+1,j} W_j \in Q^{4d}.$$

We wish to emphasize that there is no need for a W_{d+1}, in the expression for U^*, since the point U^* is in $\mathrm{Aff}\{V_0^* = 0, V_1^*, \dots, V_d^*\}$, because the point U is in $\mathrm{Aff}\{V_0, V_1, \dots, V_d\}$. The sets $\{V_0, V_1, \dots, V_d, U\}$ and $\{V_0^* = 0, V_1^*, \dots, V_d^*, U^*\}$ are congruent, and from the expressions for $V_0^*, V_1^*, \dots, V_d^*$ and U^* it follows that there exist rational numbers $\lambda_1, \dots, \lambda_d$, for which $U^* = \sum_i \lambda_i V_i^* = (1 - \sum_i \lambda_i)V_0^* + \sum_i \lambda_i V_i^*$, in which all the coefficients add up to 1. It follows that the barycentric coordinates of U^* with respect to $\{V_0^*, V_1^*, \dots, V_d^*\}$ are $((1 - \sum_i \lambda_i), \lambda_1, \dots, \lambda_d)$, which are all rational numbers. By congruency, the barycentric coordinates of U with respect to $\{V_0, V_1, \dots, V_d\}$ are $((1-\sum_i \lambda_i), \lambda_1, \dots, \lambda_d)$, which are all rational numbers. This completes the proof of Lemma 1. ∎

Lemma 1 yields the following consequence.

Corollary 3. *If a set $\{V_0, \dots, V_d\}$ consists of $d+1$ affinely independent rational points in E^n, then the following sets are equal.*

(1) *The set A of all the rational points of* $\mathrm{Aff}(V_0, \dots, V_d)$.

(2) *The set B of all the points W of* $\mathrm{Aff}(V_0, \dots, V_d)$ *for which $\|W - V_i\|^2$ is a rational number for all i, $0 \le i \le d$.*

(3) *The set C of all the points of the form $\sum_i \lambda_i V_i$ for which λ_i are rational numbers and $\sum_i \lambda_i = 1$.*

Recall that the n-dimensional volume V of $\mathrm{conv}\{W_1, W_2, \dots, W_{n+1}\}$, for points $W_1, W_2, \dots, W_{n+1}$ in E^k, can be determined by using all the mutual distances. The volume V is given by the famous Euler-Cayley-Menger formula (see [11])

$$V^2 = \frac{(-1)^{n+1}}{2^n(n!)^2}\begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{1,2}^2 & d_{1,3}^2 & \dots & d_{1,n+1}^2 \\ 1 & d_{2,1}^2 & 0 & d_{2,3}^2 & \dots & d_{2,n+1}^2 \\ 1 & d_{3,1}^2 & d_{3,2}^2 & 0 & \dots & d_{3,n+1}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & d_{n+1,1}^2 & d_{n+1,2}^2 & d_{n+1,3}^2 & \dots & 0 \end{vmatrix}$$

where $d_{i,j} = \|W_i - W_j\|$. In particular, *the dimension* of the affine hull of a set A in E^k is determined in terms of the mutual distances of points of A. Thus, $\dim(\mathrm{Aff}(A))$ is equal to the maximum of the value of d for which A contains a set of $d+1$ points whose convex hull has a positive d-dimensional volume. In addition, if $\dim(\mathrm{Aff}(A)) = d$, then the $(d+1)$-volume of the convex hull of any $d+2$ points of A is equal to zero.

Proof of Theorem 2. Let A be a set of points in E^n which satisfies condition (ρ), and let $\dim(\mathrm{Aff}(A)) = d$. It follows that the set A contains the vertex set $\{V_0, \dots, V_d\}$ of a d-simplex Δ^d. Based on the proof of Theorem 1, there exist a set $\{V_0^*, \dots, V_d^*\}$, congruent to $\{V_0, \dots, V_d\}$ in Q^{4d}.

Let U be any point of $A \backslash \{V_0, \dots, V_d\}$, and apply the construction of the proof of Theorem 1 to $\{V_0, \dots, V_d, U\}$. The dimension $\dim(\mathrm{Aff}(\{V_0, \dots, V_d\}))$ is equal to d, which is also the dimension of $\mathrm{Aff}\{V_0, \dots, V_d, U\}$. Thus, the attempt to find a suitable point U^*, as described in the proof of Theorem 1, will yield a rational point U^* which is already in $\mathrm{Aff}\{V_0^*, \dots, V_d^*\}$, i.e., it has by Lemma 1 rational barycentric coordinates, with respect to $\{V_0^*, \dots, V_d^*\}$. Thus, U^* is already in Q^{4d}. Consider the two congruent sets of points $\{V_0, \dots, V_d\}$ in E^n and $\{V_0^*, \dots, V_d^*\}$ in Q^{4d}; there exists an isometric embedding f of $\{V_0, \dots, V_d\}$ into Q^{4d} for which $f(V_i) = (V_i^*)$ holds for all i, $0 \le i \le d$. Using barycentric coordinates, based on $\{V_0, \dots, V_d\}$ for $\mathrm{Aff}(\{V_0, \dots, V_d\})$, and on $\{V_0^*, \dots, V_d^*\}$ for $\mathrm{Aff}(\{V_0^*, \dots, V_d^*\})$, it follows that there exists an embedding F of A into $\mathrm{Aff}(\{V_0^*, \dots, V_d^*\})$, which is essentially a mapping of the set A into $\mathrm{Aff}(\{V_0^*, \dots, V_d^*\}) \cap Q^{4d}$.

Therefore $\rho(A) \le 4d$, which completes the proof of Theorem 2. ☐

We close by the following remark. If we replace in the proof of Theorem 1 $\{W_1, ..., W_{d-1}\}$ by the usual basis $\{e_1, ..., e_{d-1}\}$ of E^{d-1} and replace W_d by $b_{d,d}e_d$, we get that $b_{i,j}$ is the j-th coordinate of V_i^*. These $b_{i,j}$ can be constructed by a ruler and a compass, given all the d_{ij}. Therefore we have the following theorem.

Theorem 3. *Given all the mutual distances d_{ij} of a d-simplex in E^d, it is possible to construct with a ruler and a compass (in the plane) all the coordinates in E^d of the vertices $V_0, \dots, V_d$ of a d-simplex, for which $\|V_i - V_j\| = d_{ij}$ holds for all i and j, $1 \le i < j \le d$.*

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