Rendiconti di Matematica, Serie VII
Volume 26, Roma (2006), 121-188

Hierarchy of almost-periodic function spaces

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Abstract: *The various types of definitions of almost-periodic functions are examined and compared in order to clarify the hierarchy of almost-periodic function spaces. Apart from the standard definitions, we introduce also new classes and comment some other, less traditional, definitions, to make a picture as much as possible complete. Several new results concerning horizontal hierarchies are proved. Illustrating examples and counter-examples are shown.*

1 – Introduction

Ever since their introduction by H. Bohr in the mid-twenties, almost-periodic (a.p.) functions have played a role in various branches of mathematics. Also, in the course of time, various variants and extensions of Bohr's concept have been introduced, most notably by A. S. Besicovitch, V. V. Stepanov and H. Weyl. Accordingly, there are a number of monographs and papers covering a wide spectrum of notions of almost-periodicity and applications (see the large list of references).

An extension of Bohr's original (scalar) concept of a different kind are the generalizations to vector-valued almost-periodic functions, starting with Bochner's work in the thirties. Here, too, are a number of monographs on the subject, most notably by L. Amerio and G. Prouse [4] and by B. M. Levitan and

Key Words and Phrases: *Almost-periodic functions – Stepanov, Weyl and Besicovitch classes – Bochner transform – Approximation by trigonometric polynomials – Bohr compactification – Marcinkiewicz spaces – Hierarchy.*
A.M.S. Classification: 42A32 – 42A75 – 43A60

V. V. Zhikov [100]. This vector-valued (Banach space valued) case is particularly important for applications to (the asymptotic behavior of solutions to) differential equations and dynamical systems.

In recent years, this branch of the field has led to a kind of revival of the almost-periodicity field. One of the basic breakthroughs in this context was M. I. Kadets' solution of the "integration problem" in 1969 (see [74]), showing that the scalar result (the integral of a uniformly a.p. function is uniformly a.p., provided it is bounded) carries over to exactly the class of Banach spaces not containing an isomorphic copy of c_0, the scalar null sequences.

Starting from there, the more general question of which kind of (ordinary or partial) "almost-periodic differential equations" (of various types) has almost-periodic solutions (of the same or related type), has vividly been taken up, both in the linear and in the nonlinear case. In the nonlinear case, positive results are sparse, and hard to come by; mostly, because of the absence of the machinery of spectral theory. In the linear case, though, there has been tremendous progress within the past ten years, both in breadth and depth. A fairly complete account of this development is to be found in parts B and C of a recent monograph by W. Arendt, C. Batty, M. Hieber and F. Neubrander [10].

Hence, in the theory of almost-periodic (a.p.) functions, there are used many various definitions, mostly related to the names of H. Bohr, S. Bochner, V. V. Stepanov, H. Weyl and A. S. Besicovitch ([4], [16], [17], [22], [24], [25], [26], [27], [31], [32], [33], [34], [35], [39], [41], [55], [67], [99], [100], [105], [106], [107]).

On the other hand, it is sometimes difficult to recognize whether these definitions are equivalent or if one follows from another. It is well-known that, for example, the definitions of uniformly a.p. (u.a.p. or Bohr-type a.p.) functions, done in terms of a relative density of the set of almost-periods (the Bohr-type criterion), a compactness of the set of translates (the Bochner-type criterion, sometimes called *normality*), the closure of the set of trigonometric polynomials in the sup-norm metric, are equivalent (see, for example, [22], [41]).

The same is true for the Stepanov class of a.p. functions ([22], [35], [67], [107]), but if we would like to make some analogy for, e.g., the Besicovitch class of a.p. functions, the equivalence is no longer true.

For the Weyl class, the situation seems to be even more complicated, because in the standard (Bohr-type) definition, the Stepanov-type metric is used, curiously, instead of the Weyl one.

Moreover, the space of the Weyl a.p. functions is well-known [35], unlike the other classes, to be incomplete in the Weyl metric.

In [57], E. Fœlner already pointed out these considerations, without arriving at a clarification of the hierarchies.

Besides these definitions, there exists a lot of further characterizations, done, e.g., by J.-P. Bertrandias [20], R. Doss [48], [50], [51], A. S. Kovanko [86], [87], [88], [89], [91], [92], [93], B. M. Levitan [99], A. A. Pankov [107], and the references therein], A. C. Zaanen [135], which are, sometimes, difficult to compare with more standard ones.

Therefore, our main aim is to clarify the hierarchy of such classes, in a "horizontal", as well as a "vertical", way.

More precisely, we would like to fulfill at least the following table and to indicate the related relationships.

	a. periods	*normal*	*approx.*
Bohr	Def. 2.2	Def. 2.6	Def. 2.8
Stepanov	Def. 3.1	Def. 3.3	Def. 3.4
equi-Weyl	Def. 4.1	Def. 4.2	Def. 4.3
Weyl	Def. 4.4	Def. 4.5	Def. 4.6
Besicovitch	Def. 5.16	Def. 5.17	Def. 5.5

Table 1

Furthermore, we would like to collect and comment all the equivalent definitions in the literature to those in Table 1.

This goal is stimulated by our interest to apply elsewhere these notions to the theory of nonlinear a.p. oscillations (cf. [5], [6], [7], [8]). It occurs that the most suitable definitions w.r.t. applications to differential equations are those by means of almost-periods (the first column in Table 1). On the other hand, the obtained implications in Table 2 in Section 6 allow us to assume a bit more, when e.g. using the definitions by means of approximations (the third column in Table 1), but to get a bit less in terms of almost-periods. Moreover, because of the regularity of solutions, we can get in fact normal oscillations, even when considering a.p. equations in terms of almost-periods. Thus, imposing some additional restrictions on a.p. equations, and subsequently normal solutions, one might come back to almost-periodic forced oscillations, defined by means of approximations. We would like to follow this idea in one of our forthcoming papers. The present classification can be regarded as the first step of this aim. The next one should contain the integrals of a.p. functions from all given classes, playing the fundamental role in representing solutions of linear a.p. equations. The Bohr-Neugebauer type results (i.e. boundedness implies almost-periodicity) for linear a.p. systems with constant coefficients was recently investigated in this frame by our PhD student L. Radova in [109]. The final desired step is to build the theory of a.p. oscillations (for linear as well as nonlinear differential equations and inclusions), just on the basis of the indicated implications. This is, however, still a rather long way to go, and so we are not quoting here papers concerning a.p. solutions of differential equations, apart from those, where new classes of a.p. functions were introduced.

The paper is organized appropriately to Table 1.

In Sections 2 and 3, uniformly and the Stepanov a.p. classes are defined and compared.

In Sections 4 and 5, the same is done for the equi-Weyl, the Weyl and the Besicovitch a.p. classes. Some new results concerning the horizontal hierarchies are proved.

In Section 6, the most important properties, common to all the spaces, are illustrated. The desired hierarchy is clarified, fulfilling the Table 1 and showing several counter-examples, demonstrating non-equivalence.

All the theorems already presented in the literature are quoted with (sometimes only partial) references. The theorems without references are intended to be (as far as the authors know) original.

Sometimes, we will not distinguish between a function f and the values $f(x)$ it assumes; it will be clear from the context whether we mean the functions or their values. Furthermore, speaking about Weyl or Besicovitch metrics implicitly means to deal with the related quotient spaces, because otherwise we should rather speak about Weyl or Besicovitch pseudo-metrics.

Many further definitions of generalized spaces of almost periodic functions are present in literature (see the large list of the references). Some of them will be briefly introduced and discussed at the end of Sections 2–5 and in Section 7.

Finally, in Section 7, some concluding remarks and open problems will be pointed out.

2 – Uniformly almost-periodicity definitions and horizontal hierarchies

The theory of a.p. functions was created by H. Bohr in the Twenties, but it was restricted to the class of uniformly continuous functions.

Let us consider the space $C^0(\mathbb{R};\mathbb{R})$ of all continuous functions, defined on $\mathbb{R}$ and with the values in $\mathbb{R}$.

In this section, the definitions of almost-periodicity will be based on the topology of uniform convergence.

Definition 2.1. A set $X \subseteq \mathbb{R}$ is said to be *relatively dense* (r.d.) if there exists a number $l > 0$ (called the *inclusion interval*), s.t. every interval $[\, a, a+l\,]$ contains at least one point of X.

Definition 2.2. (see, for example, [4, p. 3], [22, p. 2], [67, p. 170]) [Bohr-type definition] A function $f \in C^0(\mathbb{R};\mathbb{R})$ is said to be *uniformly almost-periodic* (u.a.p.) if, for every $\epsilon > 0$, there corresponds a r.d. set $\{\tau\}_\epsilon$ s.t.

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \epsilon \qquad \forall \tau \in \{\tau\}_\epsilon \, . \tag{2.1}$$

Each number $\tau \in \{\tau\}_\epsilon$ is called an ϵ-*uniformly almost-period* (or a *uniformly ϵ-translation number*) of f.

Proposition 2.3([4, p. 5], [22, p. 2], [76, p. 155]) *Every u.a.p. function is uniformly continuous.*

Proposition 2.4([4, p. 5], [22, p. 2], [76, p. 155]) *Every u.a.p. function is uniformly bounded.*

Proposition 2.5([4, p. 6], [22, p. 3]) *If a sequence of u.a.p. functions f_n converges uniformly in $\mathbb{R}$ to a function f, then f is u.a.p., too.*

In other words, the set of u.a.p. functions is closed w.r.t. the uniform convergence. Since it is a closed subset of the Banach space $C_b := C^0 \cap L^\infty$ (i.e. the space of bounded continuous functions, endowed with the sup-norm), it is Banach, too.

Actually, it is easy to show that the space is a commutative Banach algebra, w.r.t. the usual product of functions (see, for example, [113, pp. 186–188]).

Definition 2.6. ([22, p. 10], [41, p. 14], [100, p. 4]) [normality or Bochner-type definition] A function $f \in C^0(\mathbb{R};\mathbb{R})$ is called *uniformly normal* if, for every sequence $\{h_i\}$ of real numbers, there corresponds a subsequence $\{h_{n_i}\}$ s.t. the sequence of functions $\{f(x+h_{n_i})\}$ is uniformly convergent.

The numbers h_i are called *translation numbers* and the functions $f^{h_i}(x) := f(x+h_i)$ are called *translates*.

In other words, f is uniformly normal if the set of translates is precompact in C_b (see [4], [107]).

Let us recall that a metric space X is *compact* (*precompact*) if every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements belonging to X contains a convergent (fundamental) subsequence.

Obviously, if X is a complete space, it is equivalent to say that X is precompact, relatively compact (i.e. the closure is compact) or compact.

The necessary (and, in a complete metric space, also sufficient) condition for the relative compactness, or an equivalent condition for the precompactness, of a set X can be characterized by means of (see, for example, [78], [102]):

the total boundedness (*Hausdorff theorem*): for every $\epsilon > 0$ there exists a finite number of points $\{x_k\}_{k=1,\dots,n}$ s.t.

$$X \subset \bigcup_{k=1}^{n} (x_k, \epsilon) \ ,$$

where (x_k, ϵ) denotes a spherical neighbourhood of x_k with radius ϵ; the set $\{x_k\}_{k=1,\dots,n}$ is called an ϵ-*net* for X.

Remark 2.7. Every trigonometric polynomial

$$P(x) \; = \; \sum_{k=1}^{n} a_k e^{i\lambda_k x} \;\; ; \quad (a_k \in \mathbb{R} \; ; \; \lambda_k \in \mathbb{R})$$

is u.a.p. Then, according to Proposition (2.5), every function f, obtained as the limit of a uniformly convergent sequence of trigonometric polynomials, is u.a.p.

It is then natural to introduce the third definition:

Definition 2.8. ([24, p. 224], [35, p. 36], [41, p. 9]) [approximation] We call $C^0_{ap}(\mathbb{R};\mathbb{R})$ the (Banach) space obtained as the closure of the space $\mathcal{P}(\mathbb{R};\mathbb{R})$ of all trigonometric polynomials in the space C_b, endowed with the sup-norm.

Remark 2.9. Definition (2.8) may be expressed in other words: a function f belongs to $C^0_{ap}(\mathbb{R};\mathbb{R})$ if, for any $\epsilon > 0$, there exists a trigonometric polynomial T_ϵ, s.t.

$$\sup_{x\in\mathbb{R}} |f(x) - T_\epsilon(x)| < \epsilon \; .$$

It is easy to show that C^0_{ap}, like C^0, is invariant under translations, that is C^0_{ap} contains, together with f, the functions $f^t(x) := f(x+t) \quad \forall t \in \mathbb{R}$ (see, for example, [100, p. 4]).

The three main definitions, (2.2), (2.6) and (2.8), are shown to be equivalent:

Theorem 2.10([4, p. 8], [22, pp. 11-12], [99, pp. 23-27], [100, p. 4], [107, pp. 7-8]) [Bochner criterion] *A continuous function f is u.a.p. iff it is uniformly normal.*

Theorem 2.11([24, p. 226], [107, p. 9]) *A continuous function f is u.a.p. iff it belongs to $C^0_{ap}(\mathbb{R};\mathbb{R})$.*

Remark 2.12. To show the equivalence among Definitions (2.2), (2.6) and (2.8), in his book [41, pp. 15-23], C. Corduneanu follows another way that will be very useful in the following sections: he shows that

$$(2.8) \quad \Longrightarrow \quad (2.6) \quad \Longrightarrow \quad (2.2) \quad \Longrightarrow \quad (2.8).$$

In order to satisfy the S. Bochner criterion, L. A. Lusternik has proved an Ascoli–Arzelà-type theorem, introducing the notion of equi-almost-periodicity.

Theorem 2.13([41, p. 143], [100, p. 7], [102, pp. 72-74]) [Lusternik] *The necessary and sufficient condition for a family $\mathcal{F}$ of u.a.p. functions to be precompact is that*

1) $\mathcal{F}$ *is* equi-continuous, *i.e. for any* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *s.t.*

$$|f(x_1) - f(x_2)| < \epsilon \qquad \text{if} \quad |x_1 - x_2| < \delta(\epsilon) \qquad \forall f \in \mathcal{F} \ ;$$

2) $\mathcal{F}$ *is* equi-almost-periodic, *i.e. for any* $\epsilon > 0$ *there exists* $l(\epsilon) > 0$ *s.t., every interval whose length is* $l(\epsilon)$, *contains a common* ϵ*-almost-period* ξ *for all* $f \in \mathcal{F}$, *i.e.*

$$|f(x+\xi) - f(x)| < \epsilon \quad \forall f \in \mathcal{F} \ ; \quad x \in \mathbb{R} \ ;$$

3) *for any* $x \in \mathbb{R}$, *the set of values* $f(x)$ *of all the functions in* $\mathcal{F}$ *is precompact.*

Remark 2.14. As already seen, for numerical almost-periodic functions, condition 3) in the Lusternik theorem coincides with the following:

3′) for any $x \in \mathbb{R}$, the set of values $f(x)$ of all the functions in $\mathcal{F}$ is uniformly bounded.

The u.a.p. functions, like the periodic ones, can be represented by their Fourier series.

Definition 2.15. For every function f, we will call as the *mean value of* f the number

$$M[f] \ = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\ dx =: \fint f(x)\ dx \ . \tag{2.2}$$

Theorem 2.16 ([22, pp. 12-15], [35, p. 45], [100, pp. 22-23]) [Mean value theorem] *The mean value of every u.a.p. function* f *exists and*

$$\begin{aligned} &\mathbf{a})\ M[f] = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)\ dx \ = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^0 f(x)\ dx \ ; \\ &\mathbf{b})\ M[f] = \lim_{T\to\infty} \frac{1}{2T} \int_{a-T}^{a+T} f(x)\ dx \ ; \qquad \text{uniformly w.r.t. } a \in \mathbb{R}. \end{aligned} \tag{2.3}$$

Remark 2.17. Every even function satisfies (2.3), while necessary condition for an odd function to be u.a.p. is that $M[f] = 0$. Furthermore, since, for every u.a.p. function f and for every real number λ, the function $f(x)e^{-i\lambda x}$ is still a u.a.p. function, the number

$$a(\lambda, f) := M[f(x)e^{-i\lambda x}] \tag{2.4}$$

always exists.

Theorem 2.18 ([22, p. 18], [100, pp. 23-24]) *For every u.a.p. function* f, *there always exists at most a countable infinite set of values* λ (*called the* Bohr–Fourier exponents or frequencies) *for which* $a(\lambda) \neq 0$.

The numbers $a(\lambda, f)$ are called the *Bohr–Fourier coefficients* and the set

$$\sigma(f) := \{\lambda_n \mid a(\lambda_n, f) \neq 0\}$$

is called the *spectrum of* f.

The formal series $\sum_n a(\lambda_n, f)e^{-i\lambda x}$ is called the *Bohr–Fourier series of* f and we write

$$f(x) \sim \sum_n a(\lambda_n, f)e^{-i\lambda x} \ . \tag{2.5}$$

Let us now consider the connection between the Bohr–Fourier exponents and the almost-periods. To this aim, we recall the so-called *Kronecker theorem* on the diophantine approximation (see, for example, [4, pp. 30-38], [22, p. 35], [41, pp. 146-150]).

LEMMA 2.19([41, pp. 146-147]) *Let*

$$f(x) \ \sim \ \sum_{k=1}^{+\infty} a(\lambda_k, f)e^{i\lambda_k x}$$

be a u.a.p. function. For every $\epsilon > 0$, *there correspond* $n \in \mathbb{N}$ *and* $\delta \in \mathbb{R}$, $0 < \delta < \pi$, *s.t. any real number* τ *which is a solution of the system of diophantine (or congruencial) inequalities*

$$|\lambda_k \tau| \ < \ \delta \ (\mathrm{mod} \ \ 2\pi) \ ; \quad k = 1, \dots , n$$

is an ϵ*-almost-period for* $f(x)$.

THEOREM 2.20([4, pp. 31-33], [22, p. 35], [41, pp. 147-149]) [Kronecker theorem] *Let* λ_k, θ_k $(k = 1, \dots , n)$ *be arbitrary real numbers. The system of diophantine inequalities*

$$|\lambda_k \tau - \theta_k| \ < \ \delta \ (\mathrm{mod} \ \ 2\pi) \ ; \quad k = 1, \dots , n$$

has solutions $\tau_\delta \in \mathbb{R}$, *for any* $\delta > 0$, *iff every relation*

$$\sum_{k=1}^{n} m_k \lambda_k \ = \ 0 \ ; \quad m_k \in \mathbb{N}$$

implies

$$\sum_{k=1}^{n} m_k \theta_k \ \equiv \ 0 \ (\mathrm{mod} \ \ 2\pi) \ ; \quad m_k \in \mathbb{N} \ .$$

Lemma 2.21([41, p. 149]) *Let*

$$f(x) \sim \sum_{k=1}^{+\infty} a(\lambda_k, f) e^{i\lambda_k x}$$

be a u.a.p. function and λ a real number which is rationally linearly independent of the Bohr–Fourier exponents λ_k. For every $\epsilon > 0$, there correspond a number $\delta \in \mathbb{R}$, $0 < \delta < \frac{\pi}{2}$ and $n \in \mathbb{N}$, s.t. there exists an ϵ-almost-period τ which satisfies the system of inequalities

$$|\lambda_k \tau| < \delta \pmod{2\pi} ; \quad |\lambda\tau - \pi| < \delta \pmod{2\pi} ; \quad k = 1, \ldots, n .$$

Proposition 2.22([22, p. 18], [100, pp. 31-33]) [Bohr Fundamental theorem] *The Parseval equation*

$$\sum_n |a(\lambda_n, f)|^2 = M\{|f(x)|^2\}$$

is true for every u.a.p. function.

Proposition 2.23([22, p. 27], [100, p. 24]) [Uniqueness theorem] *If two u.a.p. functions have the same Fourier series, then they are identical.*

In other words, two different elements belonging to C^0_{ap} cannot have the same Bohr–Fourier series.

It is worthwhile to introduce a further definition of the u.a.p. functions, which may be useful, by a heuristic way, to understand more deeply the structure of the space C^0_{ap} (see [107, pp. 5-9]).

Definition 2.24. The *Bohr compactification*, or the *compact hull*, of $\mathbb{R}$ is a pair $(\mathbb{R}_B, i_B)$, where $\mathbb{R}_B$ is a compact group and $i_B : \mathbb{R} \to \mathbb{R}_B$ is a group homomorphism, s.t. for any homomorphism $\Phi : \mathbb{R} \to \Gamma$ into a compact group Γ there exists a unique homomorphism $\Phi_B : \mathbb{R}_B \to \Gamma$ s.t. $\Phi = \Phi_B \circ i_B$.

The Bohr compactification of a given group is always uniquely determined up to isomorphisms. Since $\mathbb{R}$ is a locally compact abelian group, its Bohr compactification can be constructed by means of the group $\mathbb{R}'$ of the characters of $\mathbb{R}$ (that is, the group of all the homomorphisms χ from $\mathbb{R}$ into the circumference $T = \{z \in \mathbb{C} \mid |z| = 1\}$), that can be written as

$$\chi(x) = e^{i\xi x} ; \quad x \in \mathbb{R} ; \ \xi \in \mathbb{R} .$$

Since the map $\xi \to e^{i\xi x}$ defines an isomorphism between $\mathbb{R}$ and $\mathbb{R}'$, we can identify $\mathbb{R}'$ with $\mathbb{R}$.

In other words, the Bohr compactification may be interpreted as an isomorphism between $\mathbb{R}$ and a subgroup of the cartesian product (with the power of continuum) of the circumference T: if $T_\lambda \equiv T \ \forall \lambda \in \mathbb{R}$,

$$T^C = \prod_{\lambda \in \mathbb{R}} T_\lambda$$

endowed with an appropriate topology (for further information, see, e.g., [3], [9], [11], [16], [17], [68], [69], [70], [75], [112], [132]).

Theorem 2.25([107 , p. 7]) *$f \in C^0_{ap}$ iff there exists a function $\tilde{f} \in C^0(\mathbb{R}_B \ ; \ \mathbb{R})$ s.t.*

$$f \ = \ \tilde{f} \circ i_B \ =: i_B^* \tilde{f}$$

(i.e. f can be extended to a continuous function on $\mathbb{R}_B$).

Remark 2.26. The extension $\tilde{f}$ is unique and it satisfies

$$\sup_{x \in \mathbb{R}} |f(x)| \ = \ \sup_{y \in \mathbb{R}_B} |\tilde{f}(y)| \ .$$

Thus, we can establish an isometric isomorphism

$$i_B^* \ : \ C^0(\mathbb{R}_B \ ; \ \mathbb{R}) \ \sim \ C^0_{ap}(\mathbb{R} \ ; \mathbb{R})$$

and every u.a.p. function can be identified with a *continuous* function defined on $\mathbb{R}_B$. This isometry allows us to deduce many properties of C^0_{ap} by means of the properties of C^0 (see [52], [69], [107]).

The importance of the Bohr compactification will be more clear, when we study the Besicovitch-like a.p. functions, in Section 5.

The possibility to generalize the notion of almost-periodicity in the framework of continuous functions was studied by B. M. Levitan, who introduced the notion of N-almost-periodicity (see [67], [99], [100]), in terms of a diophantine approximation.

Definition 2.27. A number $\tau = \tau(\epsilon, N)$ is said to be an (ϵ, N)-*almost-period* of a function $f \in C^0(\mathbb{R}, \mathbb{R})$ if, for every x s.t. $|x| < N$,

$$|f(x+\tau) - f(x)| < \epsilon \ . \tag{2.6}$$

Definition 2.28. A function $f \in C^0(\mathbb{R}, \mathbb{R})$ is said to be an N-*almost-periodic* (N-a.p.) if we can find a countable set of real numbers $\{\Lambda_n\}_{n \in \mathbb{N}}$, depending on f and possessing the property that, for every choice of ϵ and N, we can find two positive numbers $n = n(\epsilon, N)$ and $\delta = \delta(\epsilon, N)$ s.t. each real number τ, satisfying the system of inequalities

$$|\Lambda_k \tau| < \delta \pmod{2\pi} \ ; \quad k = 1, 2, \dots, n \ ,$$

is an (ϵ, N)-almost-period of the function f, i.e. satisfies inequality (2.4).

Although every u.a.p. function is N-a.p., the converse is not true.

Example 2.29. ([67, p. 185], [100, pp. 58-59]) Given the function

$$p(x) = 2 + \cos x + \cos(\sqrt{2}x) \ ,$$

we have $\inf\limits_{x \in \mathbb{R}} p(x) = 0$; then the function $q(x) = \frac{1}{p(x)}$ is unbounded, and consequently it is not u.a.p. On the other hand, the function q is N-a.p.

Although this class of functions preserves many properties of the u.a.p. functions, many other properties do not hold anymore. For example, the mean value (2.2), in general, does not exist, even for bounded functions. Furthermore, we can associate, to every N-a.p. function, different Fourier series (see [99, pp. 150-153], [100, p. 62]). Nevertheless, this space is very useful to obtain generalizations of classical results in the theory of ordinary differential equations with almost-periodic coefficients (see [99]).

In [29], [130] (for a more recent reference, see also [106]), a further class of almost-periodic functions, called *almost-automorphic*, was introduced. It can be shown that this class is a subset of the space of N-almost-periodic functions. This class was furtherly generalized in [36], [110], where it is shown that this more general space of almost-automorphic functions coincides with the class of N-almost-periodic functions.

The theory of u.a.p. functions can be generalized to spaces of functions defined in $\mathbb{R}^n$ or, more generally, on groups (see, for example, [3], [16], [17], [30], [60], [101], [112], [131], [132]), and with the values in $\mathbb{R}^n$, in $\mathbb{C}$ or, more generally, in a metric, in a Banach or in a Hilbert space (see, for example, [4], [28], [41], [100], [107]).

These generalizations can be very useful to introduce and to study the Stepanov-like a.p. functions, described in the next section, based on the necessity to generalize the notion of almost-periodicity to discontinuous functions which must be, in any case, locally integrable.

One of the most important goals of this first generalization to discontinuous functions, as much as of the other spaces studied in the next sections, is to find a Parseval-like relation for the coefficients of the Bohr–Fourier series related to the functions belonging to these spaces and, consequently, to find approximation theorems for these spaces, which generalize Theorem (2.1).

3 – Stepanov almost-periodicity definitions and horizontal hierarchies

Since all the various extensions of the definition of a.p. functions will involve also discontinuous functions, by means of integrals on bounded intervals, it is natural to work with locally integrable functions, i.e. $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$.

First of all, let us introduce the following Stepanov norms and distances:

$$\| f \|_{S^p_L} = \sup_{x\in\mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} |f(t)|^p \, dt \right]^{\frac{1}{p}} ; \tag{3.1}$$

$$D_{S^p_L}(f \, , \, g) = \| f - g \|_{S^p_L} = \sup_{x\in\mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} |f(t) - g(t)|^p \, dt \right]^{\frac{1}{p}} . \tag{3.2}$$

Since L is a fixed positive number, we might expect infinite Stepanov norms; but it can be trivially shown that, for every $L_1, L_2 \in \mathbb{R}_+$, there exist $k_1, k_2 \in \mathbb{R}_+$ s.t.

$$k_1 \| f \|_{S^p_{L_1}} \leq \| f \|_{S^p_{L_2}} \leq k_2 \| f \|_{S^p_{L_1}}$$

i.e. all the Stepanov norms are equivalent.

Due to this equivalence, we can replace in formula (3.1) L by an arbitrary positive number. In particular, we can consider the norm, where $L = 1$.

Definition 3.1. ([4, pp. 76-77], [22, p. 77], [41, p. 156], [67, p. 189], [99, p. 200], [100, p. 33]) [Bohr-type definition] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be almost-periodic in the sense of Stepanov (S^p_{ap}) if, for every $\epsilon > 0$, there corresponds a r.d. set $\{\tau\}_\epsilon$ s.t.

$$\sup_{x\in\mathbb{R}} \left[\int_x^{x+1} |f(t+\tau) - f(t)|^p \, dt \right]^{\frac{1}{p}} < \epsilon \, ; \quad \forall \tau \in \{\tau\}_\epsilon \, . \tag{3.3}$$

Each number $\tau \in \{\tau\}_\epsilon$ is called an *ϵ-Stepanov almost-period* (or *Stepanov ϵ-translation number* of f).

Originally, V. V. Stepanov [114], [115] called the spaces S^1_{ap} and S^2_{ap} respectively "the class of almost-periodic functions of the second and the third type". N. Wiener [134] called the space S^2_{ap} "the space of pseudoperiodic functions". P. Franklin [66] called the spaces S^1_{ap} and S^2_{ap} respectively *apS* (almost-periodic summable functions) and *apSsq* (almost-periodic functions with a summable square).

The space S^1_{ap} will be shortly indicated as S_{ap}.

Theorem 3.2([67, p. 189], [99, Th. 5.2.3., p. 201]) *Every S^p_{ap}-function is*

a) *S^p-bounded*
and
b) *S^p-uniformly continuous, i.e.*

$$\forall \epsilon > 0 \ \ \exists\ \delta = \delta(\epsilon) \quad \text{s.t.} \quad \text{if } |h| < \delta, \quad \text{then} \quad D_{S^p}\left[f(x+h)\ ,\ f(x)\right] < \epsilon\ .$$

Definition 3.3. ([67, p. 189], [87], [129]) [S^p-normality] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said *S^p-normal* if the family of functions $\{f(x+h)\}$ (h is an arbitrary real number) is S^p-precompact, i.e. if for each sequence $f(x+h_1), f(x+h_2), \dots$, we can choose an S^p-convergent sequence.

Let us define the Banach space

$$BS^p \ := \ \{f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}) \mid \ \| f \|_{S^p} < +\infty\}\ .$$

Definition 3.4. ([24, p. 224], [35, p. 36]) [approximation] We will call $S^p(\mathbb{R}\ ;\ \mathbb{R})$ the space obtained as the closure in BS^p of the space $\mathcal{P}(\mathbb{R}\ ;\ \mathbb{R})$ of all trigonometric polynomials w.r.t. the norm (3.1).

We have, by virtue of Theorem (3.2) and by Definition (3.4), $S^p_{ap} \subset BS^p$, $S^p \subset BS^p$.

Using the appropriate implications (see [35, Th. 1, p. 47], [67, Th. 7, p. 190 and Th. 4, p. 191] or, analogously, [22, pp. 88-91], [107, pp. 26-27]), we can show the main

Theorem 3.5. *The three spaces, defined by the definitions* (3.1)*,* (3.3)*,* (3.4)*, are equivalent.*

Theorem 3.6([35, pp. 51-53], [67, Th. 6, p. 189]) *The spaces S^p_{ap} are complete w.r.t. the norm* (3.1)*.*

An important contribution to the study of the equivalence of the different definitions for the spaces of the Stepanov, the (equi-)Weyl and the Besicovitch type, came from A. S. Kovanko. Unfortunately, many of his papers (written in Russian or in Ukrainian) were published in rather obscure journals; furthermore, many of his results were written without any proof. It is however useful to quote these results, in order to clarify the several hierarchies. Since the notion of normality is related to precompactness, A. S. Kovanko ([86], [87]) studied the necessary and sufficient conditions to guarantee the precompactness of some subclasses of the spaces S^p_{ap}, by means of a Lusternik-type theorem, introducing the notion of S^p-equi-almost-periodicity.

Definition 3.7. Let $E \in \mathbb{R}$ be a measurable set and, for every closed interval $[a,b]$, let $E(a,b) := E \cap [a,b]$. Given two measurable functions f, g, let us define, for every $a \geq 0$,

$$\begin{aligned} E_a &:= \{x \in \mathbb{R} \quad \text{s.t.} \quad |f(x) - g(x)| \geq a\} \ ; \\ \delta_{[a,b]}(E) &:= \frac{\mu[E(a,b)]}{b-a} \qquad \text{(density of } E \text{ w.r.t. } [a,b]) \ ; \\ \delta_S^L(E) &:= \sup_{x \in \mathbb{R}} \frac{\mu[E(x, x+L)]}{L} \ , \end{aligned} \tag{3.4}$$

(where $\mu(X)$ represents the usual Lebesgue measure of a set X);

$$D^E_{S^p_L}[f,g] := \sup_{x \in \mathbb{R}} \left[\frac{1}{L} \int_{E(x,x+L)} |f(t) - g(t)|^p \ dt \right]^{\frac{1}{p}} \ ; \tag{3.5}$$

$$\mathcal{D}_S^L[f,g] := \inf_{0 < a < +\infty} [a + \delta_S^L(E_a)] \ ; \tag{3.6}$$

$$\overline{f_L(x)} := \frac{1}{L} \int_x^{x+L} f(t) \ dt \ .$$

Theorem 3.8([86], [87]) *The necessary and sufficient condition for a family $\mathcal{F}$ of S^p_{ap}-functions to be S^p*-precompact *for every value L is that, for every $\epsilon > 0, L > 0$,*

1) *there exists $\sigma = \sigma(\epsilon, L) > 0$ s.t.*

$$D^E_{S^p_L}[f, 0] < \epsilon \quad if \quad \delta_S^L(E) < \sigma \ , \qquad \forall f \in \mathcal{F};$$

2) *there exists $\rho = \rho(\epsilon, L)$ s.t.*

$$D_{S^p_L}[f, \overline{f_h}] < \epsilon \qquad \forall \ 0 < h < \rho \ , \qquad \forall f \in \mathcal{F};$$

3) *there exists a r.d. set of S^p-almost-periods $\{\tau(\epsilon, L)\}$, common to all the elements of $\mathcal{F}$, i.e.*

$$D_{S^p_L}[f^\tau, f] < \epsilon \ , \qquad \forall f \in \mathcal{F}.$$

Remark 3.9. In Theorem (3.8), conditions 2) and 3) can be respectively replaced by the conditions:

2′) for every $\epsilon > 0, L > 0$, there exists $\delta = \delta(\epsilon, L)$, s.t.

$$\mathcal{D}_S^L[f^h, f] < \epsilon \qquad \forall \quad 0 < h < \delta \ , \qquad \forall f \in \mathcal{F};$$

3′) there exists a r.d. set (w.r.t. the distance (3.6)) of almost-periods $\{\tau(\epsilon, L)\}$, common to every $f \in \mathcal{F}$, s.t.

$$\mathcal{D}_S^L[f^\tau, f] < \epsilon \qquad \forall\ f \in \mathcal{F}\ .$$

Remark 3.10. Since the spaces S^p_{ap} are subspaces of $L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$, they must be regarded as quotient spaces, where each element is an equivalence class w.r.t. the relation

$$f \sim g \iff D_{S^p}[f, g] = 0\ .$$

Consequently, two different functions belong to the same class iff they differ from each other by a function with S^p-norm equal to 0. This fact occurs when the two functions differ only on a set of the zero Lebesgue measure.

The theory of the S^p_{ap}-spaces can be included in the theory of C^0_{ap}-spaces with the values in a Banach space (see [4, pp. 7, 76-78], [7], [41, p. 137], [100, pp. 33-34], [107, pp. 24-28]), by means of the so-called *Bochner transform*, that will be briefly recalled here.

The Bochner-transform

$$f^b(x) = f(x + \eta)\ , \quad \eta \in [0, 1], \quad x \in \mathbb{R}\ ,$$

associates, to each $x \in \mathbb{R}$, a function defined on $[0, 1]$.

Thus, if $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$, then $f^b \in L^p_{loc}(\mathbb{R}\ ,\ L^p([0, 1]))$.

Consequently,

$$BS^p = \{f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}) \mid f^b \in L^\infty(\mathbb{R}\ ,\ L^p([0, 1]))\}\ ,$$

because $\|f\|^p_{S^p} = \|f^b\|_{L^\infty}$:

$$\| f^b \|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \| f^b \|_{L^p([0,1])} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left[\int_0^1 |f(x + \eta)|^p \, d\eta\right]^{\frac{1}{p}}\ .$$

Moreover, since for every $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$, $f^b \in C^0(\mathbb{R}, L^p([0, 1]))$, then

$$BS^p = \{f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}) \mid f^b \in C_b(\mathbb{R}\ ,\ L^p([0, 1]))\}\ ,$$

where C_b denotes the space of bounded continuous functions.

S. Bochner has shown (see [4, pp. 76-78]) that

$$S^p = \{f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R}) \mid f^b \in C^0_{ap}(\mathbb{R}, L^p([0,1]))\} \ .$$

Remark 3.11. Since

$$\| f \|^p_{S^p} = \| f^b \|_{C_b(\mathbb{R},L^p([0,1]))} \ ,$$

we have

$$f_n \to \ f \text{ in } S^p_{ap} \quad \Longleftrightarrow \quad f^b_n \to f^b \text{ in } C^0_{ap}(\mathbb{R} \ , \ L^p([0,1])) \ .$$

The possibility to relate the spaces S^p_{ap} to the space $C^0_{ap}(\mathbb{R}, L^p([0,1]))$ enables us to explain the similarity of the results obtained for S^p_{ap} and C^0_{ap}, in particular, for the equivalence of the three definitions of almost-periodicity.

In [66], [114] and [115], a very wide generalization of the spaces S^p_{ap} to measurable functions is shown.

Definition 3.12. A measurable function is said *measurable almost-periodic* (M_{ap}) if, for every $\epsilon > 0$, there exists a r.d. set $\{\tau_\epsilon\}$ s.t., for a fixed number d,

$$|f(x+\tau) - f(x)| \ < \ \epsilon$$

for every x except a set whose Lebesgue exterior measure in every interval of length d is less than $d\epsilon$, or whose density on every interval of length d is less than ϵ.

Originally, V. V. Stepanov [114], [115] called this space "class of almost periodic of the first type".

As remarked by V. V. Stepanov [115] and P. Franklin [66], the definition remains essentially unchanged if d is not fixed, but may be arbitrary; in this case the length L, related to the definition of relative density, depends on both d and ϵ.

Theorem 3.13([66], [115]) *For every $\epsilon > 0$, any function $f \in M_{ap}$ is bounded except a set of density less than ϵ in every interval of length d.*

The space M_{ap} can be also defined by means of an approximation theorem.

Theorem 3.14([66]) *A measurable function f belongs to M_{ap} iff there exists a sequence of trigonometric polynomials $\{P_\epsilon\}$ s.t., for every $\epsilon > 0$, $|f(x) - P_\epsilon(x)| < \epsilon$, for every x except a set of density less than ϵ in every interval of length d.*

It is important to underline that, while changing values of every u.a.p. function in every non-empty bounded set gives a function which cannot be u.a.p., for the functions belonging to S^p_{ap} or to M_{ap} an analogous property holds if we modify a function in a set with a nonzero Lebesgue measure.

On the other hand, V. V. Stepanov [115] has shown that, since the following inclusions hold (see Formula (6.4))

$$C^0_{ap} \subset S^{p_1}_{ap} \subset S^{p_2}_{ap} \subset S^1_{ap} \subset M_{ap} \qquad \forall p_1 > p_2 > 1 \,, \tag{3.7}$$

if a function belonging to one of the last four spaces in the sequence (3.7) is respectively uniformly continuous, p_1-integrable, p_2-integrable, uniformly integrable, then it belongs to the space of the corresponding earlier type.

Let us recall that (see, for example, [81], [115]) a measurable function f is said to be *uniformly integrable* if, for every $\epsilon > 0$ and $d > 0$, there corresponds a number $\eta > 0$ s.t.

$$\int_E |f(x)| \, dx \; < \; \epsilon \,,$$

for every set E s.t. $\mu(E) < \eta$ and $\mathrm{diam}(E) \le d$.

The difficulty related to the space M_{ap} consists in the definition of frequencies. In fact, if a measurable function is not integrable, then the quantities (2.4) need not exist, in general. The problem can be overcome by considering a sequence of *cut-off functions*

$$g_n(x) = \begin{cases} f(x), & \text{for } |f(x)| \le n \\ n\dfrac{f(x)}{|f(x)|}, & \text{for } |f(x)| > n \,. \end{cases}$$

In fact, the functions g_n are uniformly integrable, and consequently S^1_{ap} (see [115]). So, by virtue of Theorem (6.5), we have a countable set of frequencies, given by the union of all the frequencies $a(\lambda, g_n)$ of the functions g_n. Rejecting all the frequencies s.t. $\lim_{n\to+\infty} a(\lambda, g_n) = 0$, this set can be interpreted as the spectrum of the measurable function f, even if the limit of some sequence of frequencies is not finite or does not exist at all. It can be shown [66] that the spectrum of f does not depend on the choice of the sequence of cut-off functions (instead of $a_n = \{n\}$, we could consider another increasing sequence $a_k = \{n_k\}$, s.t. $\lim_{k\to+\infty} n_k = +\infty$ and s.t. there exists $K > 0$, for which $n_{k+1} - n_k < K$).

Let us remark that, when we restrict ourselves to uniformly integrable functions, this definition of the spectrum coincides with the classical one for the S^1_{ap}-functions (see [66]).

4 – Weyl and equi-Weyl almost-periodicity definitions and horizontal hierarchies

Although the three definitions of the C^0_{ap} and S^p_{ap}-spaces are related to the same norms (respectively, the sup-norm and (3.1)), the classical definitions of Weyl spaces are using two different norms: (3.1) and the Weyl norm

$$\| f \|_{W^p} = \lim_{L\to\infty} \sup_{x\in\mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} |f(t)|^p \, dt \right]^{\frac{1}{p}} = \lim_{L\to\infty} \| f \|_{S^p_L} \ , \tag{4.1}$$

induced by the distance

$$D_{W^p}(f,g) = \lim_{L\to\infty} \sup_{x\in\mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} |f(t)-g(t)|^p \, dt \right]^{\frac{1}{p}} = \lim_{L\to\infty} D_{S^p_L}[f,g] \ . \tag{4.2}$$

It can be easily shown that these limits always exist (see [22, pp. 72-73], [99, pp. 221-222]).

In order to clarify the reason of the usage of two different norms, let us introduce in a “naive” way six definitions.

Definition 4.1. ([8], [22, p. 77], [24, pp. 226-227], [67, p. 190], [99, p. 200]) [Bohr-type definition] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be *equi-almost-periodic in the sense of Weyl* ($e - W^p_{ap}$) if, for every $\epsilon > 0$, there correspond a r.d. set $\{\tau\}_\epsilon$ and a number $L_0 = L_0(\epsilon)$ s.t.

$$\sup_{x\in\mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} |f(t+\tau)-f(t)|^p \, dt \right]^{\frac{1}{p}} < \epsilon \ ; \quad \forall \tau \in \{\tau\}_\epsilon \ ; \quad \forall L \geq L_0(\epsilon) \ . \tag{4.3}$$

Each number $\tau \in \{\tau\}_\epsilon$ is called an *ϵ-equi-Weyl almost-period* (or *equi-Weyl ϵ-translation number* of f).

Definition 4.2. [equi-W^p-normality] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be *equi-W^p-normal* if the family of functions $\{f^h\}$ (h is an arbitrary real number) is S^p_L-precompact for sufficiently large L, i.e. if, for each sequence $f^{h_1}, f^{h_2}, \ldots$, we can choose an S^p_L-fundamental subsequence, for a sufficiently large L.

Definition 4.3. [approximation] We will denote by *equi-$W^p(\mathbb{R}\ ;\ \mathbb{R})$* the space obtained as the closure in BS^p of the space $\mathcal{P}(\mathbb{R}\ ;\ \mathbb{R})$ of all trigonometric polynomials w.r.t. the norm (3.1) for sufficiently large L, i.e. for every $f \in e-W^p$ and for every $\epsilon > 0$ there exist $L_0 = L_0(\epsilon)$ and a trigonometric polynomial T_ϵ s.t.

$$D_{S^p_{ap}}[f\ ,\ T_\epsilon] < \epsilon \qquad \forall\, L \geq L_0(\epsilon) \ .$$

Definition 4.4. ([8], [88]) [Bohr-type definition] A function $f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ is said to be *almost-periodic in the sense of Weyl (W^p_{ap})* if, for every $\epsilon > 0$, there corresponds a r.d. set $\{\tau\}_\epsilon$ s.t.

$$(4.4) \qquad \lim_{L\to\infty} \sup_{x\in\mathbb{R}} \left[\frac{1}{L}\int_x^{x+L} |f(t+\tau)-f(t)|^p \, dt\right]^{\frac{1}{p}} < \epsilon\,; \qquad \forall \tau \in \{\tau\}_\epsilon\ .$$

Each number $\tau \in \{\tau\}_\epsilon$ is called an *ϵ-Weyl almost-period* (or *a Weyl ϵ-translation number* of f).

Definition 4.5. [W^p-normality] A function $f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ is said to be *W^p-normal* if the family of functions $\{f^h\}$ (h is arbitrary real number) is W^p-precompact, i.e. if for each sequence $f^{h_1}, f^{h_2}, \ldots,$ we can choose a W^p-fundamental subsequence.

Analogously to the Stepanov spaces, we can introduce the space

$$BW^p \ := \ \{f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R}) \quad \text{s.t.} \quad \| f \|_{W^p} < \ +\infty\} \ .$$

Definition 4.6. ([22, pp. 74-75], [24, p. 225], [35, pp. 35-36]) [approximation] We denote by $W^p(\mathbb{R}\ ;\ \mathbb{R})$ the space obtained as the closure in BW^p of the space $\mathcal{P}(\mathbb{R}\ ;\ \mathbb{R})$ of all trigonometric polynomials w.r.t. the norm (4.1).

The spaces $e - W^1_{ap}$ and W^1_{ap} will be shortly indicated as $e - W_{ap}$ and W_{ap}.
Definition (4.4) has been used in [8], but, as already pointed out by the authors, it was introduced by A. S. Kovanko in the paper without proofs [88].
Due to the equivalence of all the S^p_L-norms, to find a number L_1 s.t., by means of Definition (4.3), a sequence of polynomials converges in the norm $S^p_{L_1}$ implies that the sequence converges in every S^p_L-norm; it follows that the spaces given by Definition (4.3) coincide with the spaces S^p_{ap}.
On the other hand, the following theorem holds.

Theorem 4.7. ([22, pp. 82-83], [35, Th. 2, p. 48]) *A function $f \in W^p$ satisfies Definition* (4.1).

Consequently, we cannot expect the equivalence of the definitions for each type of spaces. As shown in [8], the space defined by means of Definition (4.1) is an intermediate space between S^p_{ap} and W^p_{ap} and the inclusion is strict (see [8, Example 1]).

Theorem 4.8. ([22, p. 83], [24, pp. 232-233], [67, p. 190], [99, pp. 222-223]) *A function $f \in e - W^p_{ap}$ belongs to BW^p.*

REMARK 4.9. It can be easily shown (see [35, p. 37]) that the sets BS^p and BW^p coincide, but the different norms imply a big difference between the two spaces. In fact, although BS^p is complete w.r.t. the Stepanov norm (see [35, pp. 51-53]), the space BW^p is incomplete w.r.t. the Weyl norm (see [35, pp. 58-61]).

On the other hand, since the set of S^p-bounded functions coincides with the set of W^p-bounded functions, every $e - W^p_{ap}$-function is also $e - W^p$-bounded and S^p-bounded.

THEOREM 4.10. ([22, p. 84], [24, pp. 233-234], [67, p. 190], [99, pp. 223-224]) *A function $f \in e - W^p_{ap}$ is equi-W^p-uniformly continuous, i.e. for any $\epsilon > 0$ there exist two positive numbers $L_0 = L_0(\epsilon)$ and $\delta = \delta(\epsilon)$ s.t., if $|h| < \delta$, then*

$$D_{S^p_L} \{f^h, f\} < \epsilon \quad , \quad \forall L \geq L_0(\epsilon) \ .$$

THEOREM 4.11. ([67, p. 191]) *For every function $f \in e - W^p_{ap}$ and every $\epsilon > 0$, we can find a trigonometric polynomial P_ϵ, satisfying the inequality*

$$D_{W^p}(f, P_\epsilon) \ < \ \epsilon \ .$$

The meaning of the last theorem is that Definition (4.1) $\Longrightarrow$ Definition (4.6).

Consequently, by Theorems (4.7) and (4.11), we have shown that Definition (4.1) is equivalent to Definition (4.6). The same result has been obtained in [22, pp. 82-91], [24, pp. 231-241], [66].

THEOREM 4.12. *The space of $e - W^p$-normal functions is equivalent to $e - W^p_{ap}$.*

PROOF. The proof is based partly on [8], [99] and [123].

SUFFICIENCY: fix $\epsilon > 0$. Since $\{f^h \ s.t. \ h \in \mathbb{R}\}$ is $e - W^p$-precompact, there exists $L_0 = L_0(\epsilon)$ s.t.

$$\begin{gathered} \forall L \geq L_0(\epsilon) \quad , \quad \forall h \in \mathbb{R} \quad \exists\, j = 1, \ldots , n \quad s.t. \\ D_{S^p_L}[f^{h-h_j}, f] \ = \ D_{S^p_L}[f^{h_j}, f^h] \ < \ \epsilon \ . \end{gathered} \tag{4.5}$$

Thus, the numbers $\tau = h - h_j$ are $S^p_L - \epsilon$-almost periods. Take

$$k \ = \ \max_{j=1,\ldots,n} \ |h_j| \tag{4.6}$$

and let $a \in \mathbb{R}$ be arbitrary. If $h = a + k$ and h_j satisfy (4.5), we obtain, due to (4.6), that $h - h_j \in [a, a + 2k]$. Thus, each interval of length $2k$ contains an

$e - W^p - \epsilon$-almost period of f and the number $2k$ is a constant of relative density to the set

$$\left\{ \tau \quad s.t. \quad \tau = h - h_j \ , \ h \in \mathbb{R} \ , \ j = 1, \dots, n \ ; \ D_{S_L^p}[f^{h_j}, f^h] \ < \ \epsilon \right\} \ ,$$

which is consequently r.d.

Necessity: assume that f is an $e - W^p_{ap}$ function and fix $\epsilon > 0$. By virtue of Theorem (4.10), the function is $e - W^p$-uniformly continuous, i.e.

$$\exists L_0 = L_0(\epsilon) \ s.t. \ \forall L \geq L_0 \quad \exists \delta > 0 s.t. \forall \ |w| < \delta \quad D_{S_L^p} \ [f, f^w] < \frac{\epsilon}{2} \ . \tag{4.7}$$

Let k be a constant of relative density to the set $\{\tau \ s.t. D_{S^p_{L_1}} \ [f, f^\tau] \ < \frac{\epsilon}{2}\}$, (i.e., for every interval I of length k there exist $\tau \in I$ and $L_1 > 0$ s.t. $D_{S_L^p}[f, f^\tau] \ < \frac{\epsilon}{2}$, for every $L \geq L_1$). To these numbers k and δ we associate a positive integer n s.t.

$$n\delta \ \leq \ k \ < \ (n+1)\delta \tag{4.8}$$

and put $h_j = j \cdot \delta$ $(j = 1, \dots, n)$. For any $h \in \mathbb{R}$, in the interval $[-h, -h+k]$ of length k we find some $S^p_{L_1} - \frac{\epsilon}{2}$-almost period τ, s.t.

$$D_{S_L^p} \ [f^\tau, f] \ < \frac{\epsilon}{2} \quad , \quad \forall L \geq L_1 \ . \tag{4.9}$$

Futhermore, we choose h and τ in such a way that

$$|h + \tau - h_j| \ < \ \delta \tag{4.10}$$

(this is possible because of (4.8) and the fact that $\tau \in [-h, -h+k]$). Take $L_2 = \max\{L_0, L_1\}$. By means of (4.7), (4.9) and (4.10), we write, for every $L \geq L_2$,

$$\begin{aligned} D_{S_L^p} \ [f^h, f^{h_j}] &\leq D_{S_L^p} \ [f^h, f^{h+\tau}] \ + \ D_{S_L^p} \ [f^{h+\tau}, f^{h_j}] = \\ &= D_{S_L^p} \ [f^\tau, f] \ + \ D_{S_L^p} \ [f^{h+\tau-h_j}, f] \ < \ \frac{\epsilon}{2} + \frac{\epsilon}{2} \ = \ \epsilon \ . \end{aligned}$$

This shows that $\{f^{h_j} \ ; \ j = 1, \dots, n\}$ is a finite ϵ-net to $\{f^h \ ; \ h \in \mathbb{R}\}$, w.r.t. the equi-Weyl metric. ∎

Due to the fact that the spaces C^0_{ap} and S^p_{ap} are complete, it is possible to state the Bochner criterion in terms of compactness instead of pre-compactness (see [35, pp. 51-53], [99, pp. 23-27, 199-200, 216-220], [137, pp. 10-11, 38]). Surprisingly, the spaces BW^p and W^p are not complete w.r.t. the Weyl norm (see [8], [35, pp. 58-61], [88], [99, pp. 242-247]).

As for the Stepanov spaces, A. S. Kovanko ([88], [91], [92]) studied the necessary and sufficient conditions to guarantee the compactness of some subclasses of the spaces $e - W^p_{ap}$ and W^p_{ap}, by means of a Lusternik-type theorem.

In order to find necessary and sufficient conditions for an $e - W^p_{ap}$ function to be $e - W^p$-normal, let us introduce another definition.

Definition 4.13. ([91], [92]) A sequence of W^p-bounded functions $\{f_n\}$ is called

1) $e - W^p$*-uniformly fundamental* if, for every $\epsilon > 0$, there exists $L_0(\epsilon)$ s.t.
$$\limsup_{m,n\to\infty} D_{S^p_L}[f_m, f_n] < \epsilon \qquad \forall L \geq L_0 \ ;$$
2) $e - W^p$*-uniformly convergent* if there exists a function $f \in BW^p$ s.t., for every $\epsilon > 0$, there exists $L_0(\epsilon)$ s.t.
$$\limsup_{n\to\infty} D_{S^p_L}[f, f_n] < \epsilon \qquad \forall L \geq L_0 \ .$$

Theorem 4.14([91]) *A sequence of functions belonging to BW^p is $e-W^p$-uniformly fundamental iff it is $e - W^p$-uniformly convergent. It means that the space BW^p, endowed with the norm of $e-W^p$-uniform convergence, is complete.*

Theorem 4.15([92]) *A set $\mathcal{M}$ of $e - W^p_{ap}$ functions is compact, w.r.t. the $e - W^p$-uniformly convergence, if for every $\epsilon > 0$,*

i) $\exists \sigma > 0 \ , \ T_1 > 0$ *s.t.*
$$D^E_{S^p_T}[f, 0] < \epsilon \qquad \text{if} \qquad \delta^T_S(E) < \sigma \quad , \quad \forall T \geq T_1 \quad ; \quad \forall f \in \mathcal{M} \ ;$$
ii) *($e - W^p$-equi-continuity)* $\exists \eta > 0 \ , \ T_2 > 0$ *s.t.*
$$D_{S^p_T}[f^h, f] \ < \ \epsilon \qquad \text{if} \qquad |h| < \eta \quad ; \quad \forall T \geq T_2 \quad ; \quad \forall f \in \mathcal{M} \ ;$$
iii) *($e-W^p$-equi-almost-periodicity)* $\exists \ T_3 > 0$ *and a r.d. set* $\{\tau_\epsilon\}$ *of real numbers s.t.*
$$D_{S^p_T}[f^\tau, f] \ < \ \epsilon \qquad \text{if} \quad \tau \in \{\tau_\epsilon\} \ ; \quad \forall T \geq T_3 \ ; \ \forall \ f \in \mathcal{M} \ ,$$
where $D_{S^p_T}$, $D^E_{S^p_T}$ and $\delta^T_S(E)$ are respectively given by (3.2), (3.4), (3.1).

Theorem 4.16([92]) *The necessary and sufficient condition in order to have $f \in e - W^p_{ap}$ is that the set of all the translates $\{f^\tau\}$ be relatively compact in the sense of $e - W^p$-uniform convergence.*

To show the second theorem about normality, we need some introductory definitions, too.

Definition 4.17. Given a Lebesgue-measurable set $E \subseteq \mathbb{R}$, let $E(a,b) := E \cap (a,b)$, for every interval (a,b), and $|E(a,b)|$ its Lebesgue measure. Let us denote

$$\delta_W(E) = \lim_{T\to\infty} \left[\sup_{a\in\mathbb{R}} \frac{|E(a-T,a+T)|}{2T} \right] .$$

Definition 4.18. For every $f,\phi \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, let us introduce the distance

$$D^E_{W^p}(f,\phi) \ := \ \lim_{T\to\infty} \left\{ \sup_{a\in\mathbb{R}} \left[\frac{1}{2T} \int_{E(a-T,a+T)} |f-\phi|^p \ dx \right]^{\frac{1}{p}} \right\} .$$

This distance, when $E \equiv \mathbb{R}$, coincides with the Weyl distance $D_{W^p}(f,\phi)$.

In order to avoid any confusion about the concept of compactness, A. S. Kovanko introduced the so-called *ideal limits* of every Cauchy sequence of W^p_{ap} functions. The distance between two ideal limits f and g is defined in the following way:

$$D_{W^p}[f,g] \ := \ \lim_{m,n\to+\infty} D_{W^p}[f_m, g_n] \ ,$$

where the sequences $\{f_m\}, \{g_n\}$ are two Cauchy sequences whose ideal limits are respectively f and g.

We are now ready to state the following Lusternik-type theorem:

Theorem 4.19([88]) *The necessary and sufficient condition for the relative compactness in the Weyl norm of a class $\mathcal{M}$ of functions $f \in W^p_{ap}$ is that, for every $\epsilon > 0$,*

i) *there exists a number $\sigma > 0$ s.t. $D^E_{W^p}(f,0) < \epsilon$ if $\delta_W(E) < \sigma$, for every function $f \in \mathcal{M}$;*

ii) *(W^p-equi-continuity) there exists a number $\eta > 0$ s.t.*

$$D_{W^p}[f^h, f] < \epsilon \quad \text{if} \quad |h| < \eta \ ,$$

for every function $f \in \mathcal{M}$;

iii) *(W^p-equi-almost-periodicity) there exists a r.d. set of almost-periods $\{\tau_\epsilon\}$ s.t.*

$$D_{W^p}\left\{f^\tau, f\right\} < \epsilon$$

for every function $f \in \mathcal{M}$.

Remark 4.20. In both Theorems (4.15) and (4.19), conditions ii) and iii) are the integral versions of the corresponding hypotheses in the Lusternik theorem for C^0_{ap}-functions; on the other hand, in the original Lusternik theorem the first condition is related to the Ascoli-Arzelà theorem; in Theorems (4.15) and (4.19) it is substituted by a condition that recalls the L^p-version of the Ascoli-Arzelà theorem, given by M. Riesz, M. Fréchet and A. N. Kolmogorov (see, for example, [37, Theorem IV.25 and Corollary IV.26, pp. 72-74]).

Theorem 4.21([88]) *The spaces of W^p-normal functions in the sense of Kovanko and W^p_{ap} are equivalent.*

If we weaken the hypothesis on compactness and ask only the pre-compactness for the set of translates $\{f^h\}$, we need an auxiliary condition:

Hypothesis ([8]). Let $f \in L^1_{loc}(\mathbb{R}, \mathbb{R})$, with $D_{W^p}(f) < +\infty$, be uniformly continuous in the mean, i.e.

$$\forall \, \frac{\epsilon}{3} > 0 \quad \exists \, \delta > 0 \quad \forall |h| < \delta \; : \; \frac{1}{l} \int_0^l \; |f^h(t) - f(t)| \, dt \; < \; \frac{\epsilon}{3} \; ,$$

uniformly w.r.t. $l \in (0, +\infty)$.

Theorem 4.22([8]) *If a W^p_{ap} function satisfies the Hypothesis, then it is W^p-uniformly continuous, i.e. for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ s.t., if $|h| < \delta$, then*

$$D_{W^p} \; \{f^h, f\} \; < \epsilon \; . \tag{4.11}$$

Theorem 4.23([8]) *Let $f \in L^1_{loc}(\mathbb{R}, \mathbb{R})$ be a W^p-function satisfying the Hypothesis. Then $f \in W^p_{ap}$ iff it is W^p-normal.*

Corollary 4.24. *Every W^p-normal function is W^p_{ap}.*

Following analogous proofs to u.a.p. functions (see [22, pp. 11-12] or [41, p. 16, Theorem 10]), it is possible to show the following

Theorem 4.25. *Every W^p-function is W^p-normal.*

PROOF. Let us consider an arbitrary W^p-function f and a sequence of trigonometric polynomials $\{T_n\}, W^p$- converging to f. Let us take a sequence of real numbers $\{h_n\}$ and a subsequence $\{h_{1n}\}$ s.t. $\{T_1(x+h_{1n})\}$ is W^p-convergent. Then, we can extract from $\{h_{1n}\}$ a subsequence $\{h_{2n}\}$ s.t. $\{T_2(x+h_{2n})\}$ is W^p-convergent, too, and so on. In this way, we construct a subsequence $\{h_{rn}\}$, for every $r \in \mathbb{N}$ s.t. $\{T_q(x+h_{rn})\}$ is W^p-convergent, for every $q \leq r$. Let us take the subsequence $\{h_{rr}\}$, which is a subsequence of every sequence $\{h_{qn}\}$, with the exception of at most a finite number of terms. Consequently, the sequence $\{T_n(x+h_{rr})\}$ is W^p-convergent, for every $n \in \mathbb{N}$. Given $\epsilon > 0$, let $n \in \mathbb{N}$ be sufficiently large so that

$$D_{W^p}[f, T_n] \ < \ \frac{\epsilon}{3} \ . \tag{4.12}$$

There exists $N(\epsilon) > 0$ s.t.

$$\begin{aligned} &D_{W^p}[f(x+h_{rr}), f(x+h_{qq})] \leq D_{W^p}[f(x+h_{rr}), T_n(x+h_{rr})] + \\ &\quad + D_{W^p}[T_n(x+h_{rr}), T_n(x+h_{qq})] + D_{W^p}[T_n(x+h_{qq}), f(x+h_{qq})] < \epsilon \\ &\quad \forall q, r \geq N(\epsilon) \ . \end{aligned} \tag{4.13}$$

Thus, the sequence $\{f(x+h_{rr})\}$ is W^p-fundamental, and consequently the function f is W^p-normal. □

REMARK 4.26. The analogy of Theorem (4.25) for $e - W^p$ spaces is guaranteed by the fact that

- **a)** the spaces $e - W^p$ coincide with the spaces S^p;
- **b)** the spaces S^p coincide with the spaces S^p-normal (see Theorem (3.5));
- **c)** the spaces S^p-normal are included in the spaces $e - W^p$-normal (see Formula (6.4)).

The converse of Corollary (4.24) or Theorem (4.25) is, in general, not true.

EXAMPLE 4.27. (cf. [127, pp. 20-21]) (Example of an equi-W^1-normal function which is not an equi-W^1-function) Let us consider the function, defined on $\mathbb{R}$,

$$f(x) = \begin{cases} 1 \ , & \text{for } 0 < x < \dfrac{1}{2} \ ; \\ 0 \ , & \text{elsewhere } . \end{cases}$$

For every $L, \tau \in \mathbb{R}$, $L \geq 1$, we have

$$\int_x^{x+L} |f(t+\tau) - f(t)| \ dt \ \leq \ 1 \ .$$

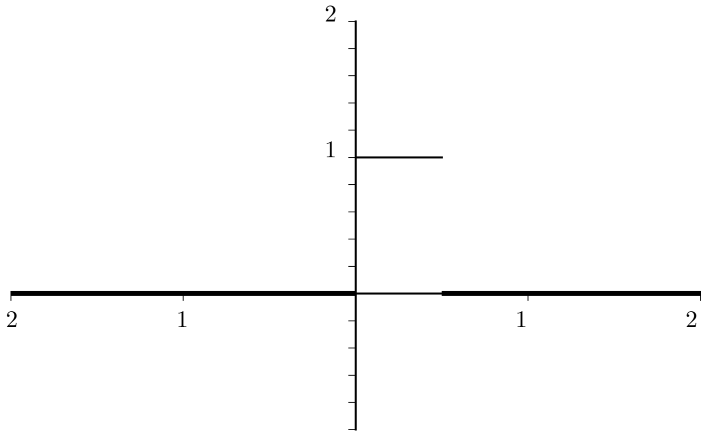

Fig. 1

Thus,

$$D_{S_L}[f^\tau, f] \;=\; \sup_{x\in \mathbb{R}} \left\{ \frac{1}{L} \int_x^{x+L} |f(t+\tau) - f(t)|\, dt \right\} \;\leq\; \frac{1}{L} \;. \tag{4.14}$$

For every $\epsilon > 0$, there exists $L \geq 1$, s.t.

$$D_{S_L}[f^\tau, f] \;\leq\; \epsilon \qquad \forall \tau \in \mathbb{R} \;.$$

Consequently, the function belongs to $e - W_{ap}$.

From Theorem (4.12), we conclude that the function is $e - W$-normal.

On the other hand, there always exists $x \in \mathbb{R}$ such that, for every $\tau > \epsilon$, where $0 < \epsilon < \frac{1}{2}$, we have $(L = 1)$

$$\int_x^{x+1} |f(t+\tau) - f(t)|\, dt \;>\; \epsilon \;.$$

Therefore, if $\epsilon < \frac{1}{2}$, then $(L = 1)$

$$D_{S_1}[f^\tau, f] \;>\; \epsilon \qquad \forall \tau > \epsilon \;.$$

For $\tau \geq L - \frac{1}{2}$, we get even

$$D_{S_L}[f^\tau, f] \;\geq\; \frac{1}{2L} \;.$$

So, the function is not S_{ap}. Since the sets S_{ap} and $e - W^1$ coincide, we have the claim.

EXAMPLE 4.28. (Example of a W^1_{ap}-function which is not a W^1-normal function) The example is partly based on [127, pp. 42-47].

Let us consider the function

$$f(x) = \begin{cases} 0\ , & \text{if } x \in (-\infty, 0]\ ; \\ \sqrt{\dfrac{n}{2}}, & \text{if } x \in (n-2; n-1]\ ,\ n = 2, 4, 6 \dots\ ; \\ -\sqrt{\dfrac{n}{2}}, & \text{if } x \in (n-1; n]\ ,\ n = 2, 4, 6 \dots \end{cases}$$

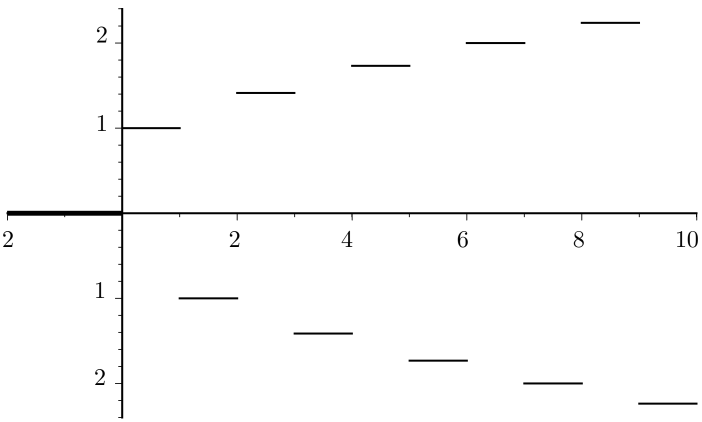

Fig. 2

Let us show that this function is a W_{ap}-function. To this aim, let us consider the set $\{x + 2k\ ,\ k \in \boldsymbol{Z}\}$ and let us show that

$$D_W[f^{2k}, f] = 0 \qquad \forall k \in \boldsymbol{Z}\ .$$

If $k = 0$, the proof is trivial. Furthermore, if $k < 0,\ k = -m$, then

$$D_W[f^{2k}, f] = D_W[f, f^{-2k}] = D_W[f^{2m}, f]\ .$$

It will be then sufficient to study the case $k > 0$. Since we will consider the limit for $L \to +\infty$, let us take $L > 2k$. There exists an integer i such that

$$2k \le 2i \le L < 2(i+1)\ . \tag{4.15}$$

Let us compute

$$D_{S_L}[f^{2k}, f] = \sup_{x \in \mathbb{R}} \left\{ \frac{1}{L} \int_x^{x+L} |f(t+2k) - f(t)|\ dt \right\}\ . \tag{4.16}$$

Since, in the interval $(-\infty, -2i)$, we have $|f(x+2k) - f(x)| = 0$; in the interval $(-2i, 0)$, the function $|f(x+2k) - f(x)|$ is increasing; in the interval $(0, +\infty)$, the function $|f(x+2k) - f(x)|$ is decreasing, the maximum value for the integral in (4.14) is obtained in an interval including 0. Considering (4.15), we can write

$$D_{S_L}[f^{2k}, f] = \sup_{x \in \mathbb{R}} \left\{ \frac{1}{L} \int_x^{x+L} |f(t+2k) - f(t)| \, dt \right\} \leq$$
$$\leq \frac{1}{2i} \int_{-2(i+1)}^{2(i+1)} |f(t+2k) - f(t)| dt = \frac{1}{2i} \int_{-2(i+1)}^{0} |f(t+2k) - f(t)| dt +$$
$$+ \frac{1}{2i} \int_0^{2(i+1)} |f(t+2k) - f(t)| dt = \frac{1}{i} \sum_{j=1}^{k} \sqrt{j} + \frac{1}{i} \sum_{j=1}^{i+1} \left[\sqrt{j+k} - \sqrt{j} \right] =$$
$$= \frac{1}{i} \sum_{j=1}^{k} \sqrt{j} + \frac{1}{i} \sum_{j=1}^{i+1} \left[\frac{k}{\sqrt{j+k} + \sqrt{j}} \right] \leq \frac{1}{i} \sum_{j=1}^{k} \sqrt{j} + \frac{1}{i} \sum_{j=1}^{i+1} \frac{k}{\sqrt{j}} \, .$$

By virtue of the Cauchy integral criterion for positive series, or by induction, it can be shown that

$$\sum_{j=1}^{l} \frac{1}{\sqrt{j}} \leq 2\sqrt{l} \qquad \forall l \in \mathbb{N} \, .$$

Consequently,

$$D_{S_L}[f^{2k}, f] \leq \left[\frac{1}{i} \sum_{j=1}^{k} \sqrt{j} \right] + \frac{2k\sqrt{i+1}}{i} \, .$$

Passing to the limit for $L \to +\infty$, we obtain

$$D_W[f^{2k}, f] = \lim_{L \to +\infty} D_{S_L}[f^{2k}, f] \leq$$
$$\leq \lim_{L \to +\infty} \left[\frac{1}{i} \sum_{j=1}^{k} \sqrt{j} + \frac{2k\sqrt{i+1}}{i} \right] = 0 \, .$$

Then the set $\{2k; k \in \boldsymbol{Z}\}$ represents, for every $\epsilon > 0$, a set of $W^1_{ap} - \epsilon$-almost-periods for the function f, which is, consequently, W^1-almost-periodic. In Example (5.37), it will be shown that this function is not B^1-normal. Furthermore, in Section 6 we will show that the space of W^1-normal functions is included in the space of B^1-normal functions. Consequently, this function is not W^1-normal.

Let us observe that this function does not satisfy both the conditions of the Hypothesis. In fact,

$$\|f\|_{S_L} = \sup_{x \in \mathbb{R}} \frac{1}{L} \int_x^{x+L} |f(t)| \, dt \geq \frac{1}{L} \int_0^{L} |f(t)| \, dt \, .$$

For every L, there exists $k > 0$ s.t. $2k \leq L < 2k+1$. Then

$$\|f\|_{S_L} \geq \frac{1}{2k+1} \int_0^{2k} |f(t)|\, dt = \frac{1}{2k+1} \left[2 \sum_{j=1}^{k} \sqrt{j} \right] \geq \frac{4}{3(2k+1)} k^{\frac{3}{2}} ,$$

where the last inequality is obtained by virtue of the Cauchy integral criterion of convergence. Consequently, letting $L \to +\infty$, we obtain that the function is unbounded in the W^1-norm. Furthermore, the function is not uniformly continuous in the mean. In fact,

$$\frac{1}{L} \int_0^L |f(t+h) - f(t)| dt \geq \frac{1}{2k+1} \times$$
$$\times \left\{ \sum_{j=1}^{k} \left[\int_{2j-1-h}^{2j-1} 2\sqrt{j} dt + \int_{2j-h}^{2j} |\sqrt{j+1} + \sqrt{j}| dt \right] \right\} =$$
$$= \frac{h}{2k} \left[3 + \sqrt{k+1} + \sum_{j=2}^{k} 4\sqrt{j} \right] \geq \frac{h}{2k} \left[3 + \sqrt{k+1} + \frac{8}{3}(k^{\frac{3}{2}} - 1) \right] \geq \frac{4}{3} h\sqrt{k} ,$$

where we have again used the Cauchy integral criterion.

The nonuniformity follows immediately. Furthermore, the function is not W^1-continuous, since

$$\|f^h - f\|_{S^1} \geq \sup_{x \in \mathbb{R}} \left[\frac{4}{3} h\sqrt{k} \right]$$

and

$$\|f^h - f\|_{W^1} \geq \lim_{k \to +\infty} \sup_{x \in \mathbb{R}} \left[\frac{4}{3} h\sqrt{k} \right] = +\infty .$$

The previous example shows that Theorems (4.8) and (4.10) cannot be extended to W^p_{ap}-functions because, in general, a W^p_{ap}-function is neither BW^p nor W^1-continuous.

Example 4.29. ([8], [127, p. 48]) (Example of a W^1-normal function which is not a W^1-function) In [8], the Heaviside step function

$$H(x) = \begin{cases} 0 , & \text{if } x < 0 \\ 1 , & \text{if } x \geq 0 \end{cases}$$

is shown to be W_{ap}, but not $e - W_{ap}$, that is, by virtue of Theorems (4.7) and (4.11), not W^1. In fact, a relative density of the set $\{\tau \; / \; D_{S_l}(f, f^\tau) < \varepsilon\}$, for some l, requires arbitrary large values of τ's in this set (we can extract some sequences of $\tau_n \longrightarrow \infty$ with $n \to \infty$). If we demand that l is (perhaps large but) constant for all τ's (we fix ε), then for most of them we get $\tau > l$ and subsequently $D_{S_l}(f^\tau, f) = \frac{1}{l} \cdot l = 1 =$ const. So, $D_{S_l}(f^\tau, f) < \varepsilon$ is impossible (for all τ's, simultaneously), by which f is not $e - W_{ap}$. On the other hand, we have $D_W(f^\tau, f) = \lim_{l\to\infty} \frac{1}{l} \cdot \tau = 0$ (we can assume that $\tau < l$, since $l \to \infty$), by which f is W_{ap}. Therefore, $e - W_{ap} \subset W_{ap}$.

Furthermore, J. Stryja [127, p. 48] has shown that the function is W-normal.

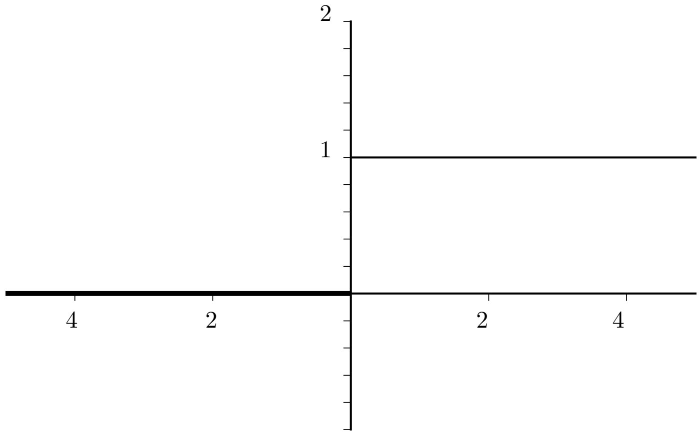

Fig. 3

In fact, since, from the almost-periodicity of H, for any $\tau \in \mathbb{R}$,

$$D_W[H^\tau, H] \;=\; 0 \;,$$

then we have that, for every $\epsilon > 0$ and for every set of translates $\{H(x+a); a \in \mathbb{R}\}$, there exists a finite ϵ-net w.r.t. the distance D_W, given by the only value $H(x)$. The W-normality follows immediately.

Let us finally observe that, as can be easily seen, the Heaviside function is BW^1 and uniformly continuous in the mean, which are the two sufficient conditions to guarantee the W-normality of a W_{ap}-function.

As already observed in [8], since the spaces BW^p and W^p are incomplete in the Weyl norm, the between lying spaces $W^p_{ap} \cap BW^p$ and W^p-normal $\cap BW^p$ are incomplete, too. Furthermore, the spaces $e - W^p$, equivalent to S^p_{ap}, are complete in the equi-Weyl norm. However, it is an open question, whether the spaces $e - W^p_{ap} \cap BS^p$ are complete or not.

A. S. Kovanko has generalized the definition of almost-periodic functions in the sense of Weyl, by means of a Bohr-like definition.

Definition 4.30. ([81]) A measurable function is said *asymptotically almost-periodic* (a.a.p.) if, for every $\epsilon > 0$, there correspond two positive numbers $l = l(\epsilon), T_0 = T_0(\epsilon)$ s.t., in every interval of length $l(\epsilon)$, there exists an

ϵ-asymptotic almost-period $\tau(\epsilon)$ s.t. the inequality

$$|f(x+\tau) - f(x)| \ < \ \epsilon \ ,$$

holds for every $x \in \mathbb{R}$, except a set whose density, w.r.t. every interval of length greater than T_0, is less than ϵ.

THEOREM 4.31([81]) *Every a.a.p. function f s.t. f^p is uniformly integrable is W^p.*

For the a.a.p. function, it is possible to state an approximation theorem.

THEOREM 4.32([81]) *An integrable function f is a.a.p. iff, for every $\epsilon > 0$, there exist a trigonometric polynomial P_ϵ and a positive number $T = T_\epsilon$ s.t.*

$$|f(x) - P_\epsilon(x)| \ < \ \epsilon \ ,$$

holds for every $x \in \mathbb{R}$, except a set whose density, w.r.t. every interval of length greater than T_0, is less than ϵ.

In [129], H. D. Ursell introduced four new definitions in terms of normality and almost-periods. He called the first two classes respectively W-normal and W_{ap}. Here, in order to avoid any confusion with Definitions (4.1), (4.2), (4.4) and (4.5), we will call these classes respectively $\overline{W}$-normal and $\overline{W}_{ap}$.

DEFINITION 4.33. ([129]) A function $f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be $\overline{W}$*-normal* if, for every sequence $\{f(x+h_n)\}$, there exists a subsequence $\{f(x+h_{n_k})\}$ s.t.

$$\lim_{j,k\to+\infty} \left[\lim_{T\to+\infty} \frac{1}{T} \int_0^T \ |f(x+h_{n_j}) - f(x+h_{n_k})| \ dx \right] \ = \ 0 \ .$$

This definition is evidently more general than Definition (4.5) of the W^p-normality, for $p = 1$ (put $x = 0$ in (4.4)). The limit is namely not made uniformly w.r.t. every interval $[a, a+T]$, but only on the interval $[0, T]$. However, we do not know whether or not H. D. Ursell's $\overline{W}$-normal space is complete.

DEFINITION 4.34. ([129]) A function $f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be $\overline{W}_{ap}$ if, for every $\epsilon > 0$, there exist a r.d. set of numbers τ and a number $T_0 = T_0(\epsilon)$ s.t.

$$\frac{1}{T} \int_0^T \ |f(x+\tau) - f(x)| \ dx \quad \forall T > T_0 \ .$$

H. D. Ursell shows that Definition (4.33) implies Definition (4.34). He also claims that the converse is true, but he does not prove the statement.

Furthermore, he introduces the space W_{*-ap}, which is equivalent to the $e - W_{ap}$ space, and the W_*-normal space.

Definition 4.35. ([129]) A function $f \in L^1_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be W_*-*normal* if, for every sequence $\{f(x+h_n)\}$, there exists a subsequence $\{f(x+h_{n_k})\}$ s.t.

$$\lim_{j,k\to+\infty} \left\{ \lim_{T\to+\infty} \sup_{a\in\mathbb{R}} \left[\frac{1}{T} \int_a^{a+T} |f(x+h_{n_j}) - f(x+h_{n_k})| \, dx \right] \right\} = 0 .$$

This space is evidently the same as the W-normal one. H. D. Ursell shows that the $e - W_{ap}$ space is contained in the W_*-normal one and he concludes that the $\overline{W}_{ap}$, $\overline{W}$-normal, $e - W_{ap}$ and W_*-normal spaces are equivalent. This last statement is again not proved and is a bit surprising.

It seems to us that H. D. Ursell's statement is false, because it is easy to show that the Heaviside step function, which is not $e - W_{ap}$, is $\overline{W}$-normal.

As already observed, due to their continuity, the elements of C^0_{ap} are in fact real functions; furthermore, since every space S^p_L is a subspace of $L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$, it is obtained as a quotient space, w.r.t. the usual equivalence relation for (Bochner-) Lebesgue integrable functions:

$$f \sim g \in S^p_{ap} \quad \Longleftrightarrow \quad \mu\{x \in \mathbb{R} | f(x) \neq g(x)\} = 0 ,$$

where μ is the usual Lebesgue measure.

On the other hand, the elements of the space W^p_{ap} are more general classes of equivalence. In fact, two different functions, belonging to the same class, may differ even on a set with Lebesgue measure greater than 0 (even infinite), provided

$$f - g \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}) .$$

Consequently, to handle elements in W^p_{ap} (and, *a fortiori*, as will be shown in the next section, in B^p_{ap}), is less convenient than to work with S^p_{ap}-functions.

5 – Besicovitch almost-periodicity definitions and horizontal hierarchies

As already pointed out, the structure of the Weyl spaces is more intricated than S^p_{ap} and C^0_{ap}, because every element of the space is a class of $L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ functions, which may differ from each other even on a set of an infinite Lebesgue measure. We have not deepened the question in the previous section, but it is necessary to talk about this fact for the Besicovitch spaces.

Following [107], let us consider the *Marcinkiewicz spaces*

$$\begin{aligned}\mathcal{M}^p(\mathbb{R}) &= \\ &= \Big\{ f:\mathbb{R}\to\mathbb{R}, f\in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}), \text{s.t.} \|f\|_p = \\ &= \limsup_{T\to\infty} \left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^p\ dx\right)^{\frac{1}{p}} < +\infty \Big\}\ \forall p\geq 1\ .\end{aligned}$$

For the case $p=+\infty$, we have

$$\mathcal{M}^\infty(\mathbb{R}) = \left\{ f:\mathbb{R}\to\mathbb{R}\ ,\ f\in L^1_{loc},\ \text{s.t.}\ \|\ f\ \|_\infty = \|\ f\ \|_{L^\infty} < +\infty \right\}\ .$$

$\mathcal{M}^p$, endowed with the seminorm

$$\|\ f\ \|_p = \begin{cases} \limsup_{T\to\infty}\left(\dfrac{1}{2T}\ \displaystyle\int_{-T}^{T}|f(x)|^p dx\right)^{\frac{1}{p}}, & \text{if } 1\leq p<+\infty \\ \|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}}\ |f(x)|\ , & \text{if } p=+\infty \end{cases} \tag{5.1}$$

is a seminormed space.

Sometimes it is convenient to use the seminorm ([23], [35, p. 42])

$$\|f\|_p^* = \max\left\{ \limsup_{T\to+\infty}\left[\frac{1}{T}\int_0^T |f(x)|^p\ dx\right]^{\frac{1}{p}}, \limsup_{T\to+\infty}\left[\frac{1}{T}\int_{-T}^0 |f(x)|^p\ dx\right]^{\frac{1}{p}} \right\}\ ,$$

which is equivalent to the seminorm (5.1), because

$$\left(\frac{1}{2}\right)^{\frac{1}{p}}\ \|f\|_p^*\ \leq \|\ f\ \|_p \leq\ \|f\|_p^*\ .$$

Theorem 5.1([104]) [Marcinkiewicz] *The space $\mathcal{M}^p$ is a Fréchet space, i.e. a topological seminormed complete space.*

The proof is essentially based on the following

Lemma 5.2. *For a seminormed space $(X,\|\cdot\|)$, the following conditions are equivalent*:

i) *X is complete;*
ii) *every absolutely convergent series is convergent* (*i.e.* $\forall\{x_n\}_{n\in\mathbb{N}}\subset X$ *s.t.* $\sum_{n=1}^\infty \|x_n\|\ <+\infty$, *there exists* $x\in X$ *s.t.* $\lim_{N\to+\infty}\|x-\sum_{n=1}^N\ x_n\|\ =0$).

Let us note that the limits in the Marcinkiewicz space are not uniquely determined. In fact, two different functions, differing from each other (even on an infinite set) by a function belonging to L^p, can be the limits of the same Cauchy sequence of elements in $\mathcal{M}^p$. Following the standard procedure, let us consider the kernel of the seminorm (5.1)

$$K_p = \{f \in \mathcal{M}^p \ s.t. \ \| f \|_p = 0\} \ .$$

Let us consider the equivalence relation

$$f \sim g \quad \Longleftrightarrow \quad \| f - g \|_p = 0 \quad ; \quad f, g \in \mathcal{M}^p \tag{5.2}$$

and the quotient space

$$M^p(\mathbb{R}) = \mathcal{M}^p / K_p \ ,$$

denoting by $\hat{f}$ the element belonging to M^p, corresponding to the function f.

Since $\mathcal{M}^p$ is a seminormed space and K_p is a subspace, then (5.1) represents a norm on M^p. Since $\mathcal{M}^p$ is complete, then M^p is a Banach space. This fact follows from the well-known

LEMMA 5.3. *Let $(X, \| \cdot \|)$ be a seminormed space. Then*

i) *the kernel $K = \{x \in X \ s.t. \ \|x\| = 0\}$ is a linear subspace of X;*
ii) *if $[x]$ is an equivalence class, then $\|[x]\| := \|x\|$ defines a norm on the quotient space X/K;*
iii) *if X is complete, then X/K is a Banach space.*

Let us now consider the class

$$\mathcal{W}^p = \left\{ f \colon \mathbb{R} \to \mathbb{R}, f \in M^p \text{ s.t. } \exists \lim_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x)|^p \ dx \right)^{\frac{1}{p}} \right\} \subset M^p \ .$$

It is possible to show that this class is not a linear space, because it is not closed w.r.t. the summation.

EXAMPLE 5.4. Let us consider the functions

$$f_1(x) = \begin{cases} 0 \ , & \text{for } x \leq 1 \\ x + \sqrt{2 + \sin \log x + \cos \log x} \ , & \text{for } x > 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 \ , & \text{for } x \leq 1 \\ -x \ , & \text{for } x > 1 \ . \end{cases}$$

We have

$$\frac{1}{2T}\int_{-T}^{T} f_1^2(x)dx = \frac{1}{2T}\Big[\frac{T^3-1}{3} + T(2+\sin\log T) - 2+$$
$$+\int_1^T 2x\sqrt{2+\sin\log x+\cos\log x}\; dx\Big].$$

Then

$$⨍ f_1^2(x)dx = \lim_{T\to+\infty}\left[\frac{T^2}{6} - \frac{1}{6T} + \frac{2+\sin\log T}{2} - \frac{1}{T}\right] +$$
$$+\lim_{T\to+\infty}\frac{\displaystyle\int_1^T x\sqrt{2+\sin\log x+\cos\log x}\; dx}{T} =$$

(applying de L'Hôspital's rule)

$$= \lim_{T\to+\infty}\frac{T^2}{6} + \lim_{T\to+\infty}\frac{2+\sin\log T}{2} + \lim_{T\to+\infty} T\sqrt{2+\sin\log T+\cos\log T} \geq$$

$$\geq \lim_{T\to+\infty}\left[\frac{T^2}{6}+\frac{1}{2}\right] = +\infty\ .$$

On the other hand,

$$⨍ f_2^2(x)\; dx = \lim_{T\to+\infty}\frac{1}{2T}\int_1^T x^2\; dx = \lim_{T\to+\infty}\frac{1}{2T}\left[\frac{x^3}{3}\right]_1^T = +\infty\ .$$

However, the function

$$g(x) = f_1(x)+f_2(x) = \begin{cases} 0\ , & \text{for } x\leq 1 \\ \sqrt{2+\sin\log x+\cos\log x}\ , & \text{for } x>1 \end{cases}$$

is such that

$$\int_{-T}^{T} g^2(x)\; dx = \int_1^T [2+\sin\log x+\cos\log x]\; dx = T(2+\sin\log T) - 2\ ,$$

and consequently

$$⨍ g^2(x)\; dx = \lim_{T\to+\infty}\left[\frac{2+\sin\log T}{2} - \frac{1}{T}\right] = \lim_{T\to+\infty}\left[1+\frac{\sin\log T}{2}\right]$$

does not exist.

Definition 5.5. ([35, p. 36], [67, p. 192], [135, pp. 103-108]) [approximation] We will denote by $B^p(\mathbb{R})$ the Besicovitch space obtained as the closure in M^p of the space $\mathcal{P}(\mathbb{R}, \mathbb{R})$ of all trigonometric polynomials.

In other words, an element in B^p can be represented by a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ s.t., for every $\epsilon > 0$, there exists $P_\epsilon \in \mathcal{P}$ s.t.

$$\limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x) - P_\epsilon(x)|^p \, dx \right)^{\frac{1}{p}} < \epsilon \; .$$

Proposition 5.6. ([35, p. 45], [97]) *The space B^p is a closed subspace of $\mathcal{W}^p$.*

Consequently, since B^p is a closed subset of the complete space M^p, it is complete, too.

It is possible to introduce another space as the completion of the space $\mathcal{P}$.

Definition 5.7. ([12]) We will denote by $\mathcal{B}^p$ the space obtained as the *abstract* completion of the space $\mathcal{P}$ w.r.t. the norm

$$|||P|||_p = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |P(x)|^p \, dx \right)^{\frac{1}{p}} \; ; \; P \in \mathcal{P} \; . \tag{5.3}$$

By the definition, $\mathcal{B}^p$ is a Banach space and its elements are classes of Cauchy sequences of trigonometric polynomials w.r.t. the norm (5.3). Thus, according to this definition, it is rather difficult to understand the meaning of an element of $\mathcal{B}^p$. The following theorem allows us to assign to every element of this space a real function.

Theorem 5.8. $B^p \equiv \mathcal{B}^p$.

Proof. First of all, let us remark that, for every element of the space $\mathcal{P}$, the norms (5.1) and (5.3) coincide.

Both B^p and $\mathcal{B}^p$ contain a subspace isomorphic to the space $\mathcal{P}$. Let us identify these subspaces. Let $\hat{P} \in \mathcal{B}^p$ be an equivalence class of Cauchy sequences of trigonometric polynomials w.r.t. the norm (5.3). Then, every sequence $\{P_n\}_{n\in\mathbb{N}} \in \hat{P}$ is such that $\|P_n - P_m\|_p \to 0$, for $m, n \to \infty$. It follows that $\{P_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in B^p, and consequently there exists $\hat{f} \in B^p$ such that $\|P_n - \hat{f}\|_p \to 0$, for $n \to \infty$. In other words, the class $\hat{P}$ uniquely determines a class $\hat{f}$ of functions belonging to $L^p_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ w.r.t. the equivalence relation

$$f, g \in \hat{f} \iff \|f - g\|_p = 0 \; .$$

In fact, if the sequence $\{Q_n\}_{n\in\mathbb{N}} \in \hat{P}$ were such that $\|Q_n - g\|_p \to 0$, then we should have

$$\|f-g\|_p \le \|f-P_n\|_p + \|P_n-Q_n\|_p + \|Q_n-g\|_p \to 0 , \qquad \text{whenever } n\to\infty ,$$

because, if $\{P_n\}_{n\in\mathbb{N}}, \{Q_n\}_{n\in\mathbb{N}} \in \hat{P}$, then $\|P_n - Q_n\|_p \to 0$, whenever $n \to \infty$.

On the other hand, a class $\hat{f} \in B^p$ uniquely determines a class of trigonometric polynomials $\hat{P} \in \mathcal{B}^p$. In fact, let $\{P_n\}_{n\in\mathbb{N}}, \{Q_n\}_{n\in\mathbb{N}} \subset B^p(\mathbb{R})$ be such that $\|P_n - \hat{f}\|_p \to 0$, $\|Q_n - \hat{f}\|_p \to 0$, whenever $n\to\infty$ (i.e. there exist $f, g \in \hat{f}$ such that $\|f - P_n\|_p \to 0$ and $\|Q_n - g\|_p \to 0$). Let us show that there exists $\hat{P} \in \mathcal{B}^p$ such that $\{P_n\}_{n\in\mathbb{N}}$, $\{Q_n\}_{n\in\mathbb{N}} \in \hat{P}$. In fact,

$$\|P_n - Q_n\|_p \le \|f - P_n\|_p + \|Q_n - f\|_p \to 0 , \qquad \text{whenever } n\to\infty .$$

Then the two sequences are equivalent and belong to the same class of the space B^p_{ap}. It is easy to show that the equivalence between the two spaces is an isometry. □

Remark 5.9. Considering the closure of the space $\mathcal{P}$ w.r.t. the norm (5.1) in the space $\mathcal{M}^p$, we obtain a space $\tilde{B}^p$, which is still a seminormed, complete space, whose elements are still functions; its quotient space, w.r.t. the equivalence relation (5.2), is B^p.

Definition (5.5) is obtained as an approximation definition. It is possible to show that this definition is equivalent to a *Bohr-like* one, provided we introduce a new property of numerical sets.

Definition 5.10. ([22, pp. 77-78], [24, p. 227], [57]) A set $X \subset \mathbb{R}$ is said to be *satisfactorily uniform* (s.u.) if there exists a positive number l such that the ratio r of the maximum number of elements of X included in an interval of length l to the minimum number is less than 2.

Every s.u. set is r.d. The converse is, in general, not true.

Although, for example, the set $\boldsymbol{Z}$ is r.d. and s.u. in $\mathbb{R}$, the set $X = \boldsymbol{Z} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ is r.d., but it is not s.u.: in fact, due to the presence of the accumulation point 0, $r = +\infty$ $\forall l > 0$. Thus, a r.d. set, in order to be s.u., cannot have any finite accumulation point.

Definition 5.11. ([22, p. 78], [24, p. 227], [57, p. 6]) A function $f \in L^p_{\rm loc}(\mathbb{R};\mathbb{R})$ is said to be *almost-periodic in the sense of Besicovitch* ($\mathcal{B}^p_{ap}$) if, for every $\epsilon > 0$, there corresponds a s.u. set $\{\tau_k\}_{k\in\boldsymbol{Z}}$ ($\tau_j < \tau_i$ if $j < i$) s.t., for each i,

$$\text{(5.4)} \qquad \limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x+\tau_i) - f(x)|^p \, dx \right)^{\frac{1}{p}} < \epsilon ,$$

and, for every $c > 0$,

$$(5.5)\quad \limsup_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}\left[\limsup_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n}\frac{1}{c}\int_{x}^{x+c}|f(t+\tau_i)-f(t)|^p dt\right]dx\right)^{\frac{1}{p}}<\epsilon.$$

The space $\mathcal{B}^1_{ap}$ will be shortly indicated by $\mathcal{B}_{ap}$.

THEOREM 5.12([22, pp. 95-97, 100-101], [24, pp. 247-257]) *The spaces $\mathcal{B}^p_{ap}$ and B^p are equivalent.*

It can be readily checked that Definition (5.11) is rather cumbersome, even in its simplified form, obtained substituting conditions (5.4) and (5.5) with the simplest one [23]

$$(5.6)\quad \limsup_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}\left[\limsup_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n}|f(x+\tau_i)-f(x)|^p\right]dx\right)^{\frac{1}{p}}<\epsilon.$$

It can be be shown [22], [24] that the spaces given by these two different definitions are equivalent.

A. S. Besicovitch introduced even a simpler definition, which permits us to introduce another space.

DEFINITION 5.13. ([22, p. 112], [24, p. 267]) A function $f \in L^1_{\text{loc}}$ is said to be $\overline{B}^1_{ap}$ if

$$\liminf_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|\,dx < +\infty$$

and, for every $\epsilon > 0$, there corresponds a s.u. set of numbers τ_i s.t., for each i, (5.4) and (5.5) are satisfied.

THEOREM 5.14([22, pp. 113-123], [24, pp. 268-269]) $\overline{B}^1_{ap} \subset \mathcal{B}_{ap}$.

The inclusion is strict, as shown in [22, pp. 126-129], [24, pp. 286-291].

It is worthwhile to observe that, although $\overline{B}^1_{ap}$ is strictly contained in $\mathcal{B}_{ap}$, to every function in $\overline{B}^1_{ap}$ there corresponds a function $\mathcal{B}_{ap}$ with the same Bohr–Fourier series. This property is related to the following

THEOREM 5.15([22, pp. 123], [24, pp. 281-282]) *To every function $f \in \mathcal{B}_{ap}$, there corresponds a $\overline{B}^1_{ap}$-function differing from f by a function the mean value of whose modulus is zero.*

Due to the difficulty of the original definition, several authors have studied alternative (and simpler) definitions of the Besicovitch spaces, each of them based on Bohr-like or Bochner-like properties in the Besicovitch norm. It is then convenient to consider the norm (5.1) rather than the norm given by Definition (5.11). To this aim, we need some preliminary definitions in terms of (5.1).

Definition 5.16. ([8], [20, p. 69], [48]) [Bohr-type definition] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is said to be *almost-periodic in the sense of Doss* (B^p_{ap}) if, for every $\epsilon > 0$, there corresponds a r.d. set $\{\tau\}_\epsilon$ s.t.

$$\limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x+\tau) - f(x)|^p \, dx \right)^{\frac{1}{p}} < \epsilon \qquad \forall \tau \in \{\tau\}_\epsilon \ .$$

Each number $\tau \in \{\tau\}_\epsilon$ is called an ϵ-B^p *almost-period* (or a $B^p$$-$$\epsilon$-*translation number*) of f.

Definition 5.17. ([46], [50]) [normality or Bochner-type definition] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is called B^p-*normal* if, for every sequence $\{h_i\}$ of real numbers, there corresponds a subsequence $\{h_{n_i}\}$ s.t. the sequence of functions $\{f(x+h_{n_i})\}$ is B^p-convergent, i.e.

$$\lim_{m,n\to+\infty} \limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+h_n) - f(x+h_m)|^p \, dx = 0 \ .$$

Definition 5.18. ([20, p. 15], [48], [50]) [continuity] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is called B^p-*continuous* if

$$\lim_{\tau\to 0} \limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+\tau) - f(x)|^p \, dx = 0 \ .$$

The space of all the B^p-continuous functions will be indicated with B^p_c. Clearly, it is a (complete) subspace of $\mathcal{M}^p$.

Definition 5.19. ([20, p. 15]) [regularity] A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is called B^p-*regular* if, for every $l \in \mathbb{R}$,

$$\limsup_{T\to\pm\infty} \frac{1}{2T} \int_{T-l}^{T} |f(x)|^p \, dx = 0 \ . \tag{5.7}$$

This condition implies that a B^p-regular function cannot assume too large values in finite intervals. The space of all the B^p-regular functions will be indicated by B^p_r. Clearly, it is a (complete) subspace of $\mathcal{M}^p$ (see [20, p. 16]). Besides, since

$$\limsup_{T\to\pm\infty} \frac{1}{2T} \int_{T-l}^{T} |f(x)|^p \, dx \le \|f\|_p^p \ ,$$

it follows that every null function in the Besicovitch norm is regular.

THEOREM 5.20([50]) *A function $f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ belongs to the space B^p iff*

1) *f is B^p-bounded, i.e. it belongs to $\mathcal{M}^p$;*
2) *f is B^p-continuous;*
3) *f is B^p-normal;*
4) *for every $\lambda \in \mathbb{R}$,*

$$\lim_{L\to+\infty} \limsup_{T\to+\infty} \int_{-T}^{T} \left| \frac{1}{L} \int_{x}^{x+L} f(t)e^{i\lambda t}\, dt - \frac{1}{L} \int_{0}^{L} f(t)e^{i\lambda t}\, dt \right| dx \;=\; 0\,.$$

REMARK 5.21. Condition 4) is, actually, formed by infinite conditions, each one for each value of λ. Each of them is independent of the others. For example, it can be proved (see [50]) that, for every λ_0, the functions $f(x) = e^{i\lambda_0 x}\operatorname{sign} x$ satisfy conditions 1), 2), 3) and condition 4), for every value $\lambda \neq \lambda_0$.

Condition 4) can be replaced by the following condition ([50]):

4′) to every $\lambda \in \mathbb{R}$, there corresponds a number $a(\lambda)$ s.t.

$$\lim_{L\to+\infty} \limsup_{T\to+\infty} \int_{-T}^{T} \left| \frac{1}{L} \int_{x}^{x+L} f(t)e^{i\lambda t}\, dt - a(\lambda) \right| dx \;=\; 0\,;$$

or by (see [50])

4″) for every $a \in \mathbb{R}$, there exists a function $f^{(a)} \in L^p$, a-periodic and s.t.

$$\lim_{n\to+\infty} \left\{ \limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left[\frac{1}{n} \sum_{k=0}^{n-1} |f(x+ka) - f^{(a)}(x)|^p\, dx \right] \right\} \;=\; 0\,. \tag{5.8}$$

Moreover, condition 3) can be replaced by a Bohr-like condition ([48]):

3′) for every ϵ, the set of numbers τ for which

$$\limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+\tau) - f(x)|\, dx \;<\; \epsilon$$

is r.d.

It follows that, under conditions 1), 2) and 4), a function is B^p-normal iff it is almost-periodic in the sense of R. Doss.

Comparing Theorems (5.12) and (5.20), let us note that introducing a Bohr-like definition as in condition 3′) represents a weaker structural characterization than Definition (5.11).

J.-P. Bertrandias has restricted his analysis to B_c^p-functions (see Definition 5.18), showing the equivalence of the different definitions.

DEFINITION 5.22. ([20, p. 69]) A function $f \in B_c^p$ is called $\mathcal{M}^p$-*almost-periodic* ($\mathcal{M}_{ap}^p$) if, for every $\epsilon > 0$, there exists a r.d. set $\{\tau_\epsilon\}$ s.t.

$$\limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(x+\tau) - f(x)|^p \, dx \right)^{\frac{1}{p}} < \epsilon \qquad \forall \tau \in \{\tau\}_\epsilon \ .$$

DEFINITION 5.23. ([20, p. 69]) [normality] A function $f \in B_c^p$ is called $\mathcal{M}^p$-normal if the set $\{f^\tau\}$ of its translates is B^p-precompact.

THEOREM 5.24([20, p. 69]) *Definitions* (5.22) *and* (5.23) *are equivalent.*

In order to show the equivalence with the third type of definition, we need a preliminary definition.

DEFINITION 5.25. ([20, p. 50]) Given a family $\{k_\lambda(x)\}_{\lambda\in\mathbb{R}}$ of B^p-constant functions (see Definition (6.9)), we will call by a *generalized trigonometric polynomial*, or by a *trigonometric polynomial with B^p-constant coefficients*, the function

$$\sum_{\lambda\in\mathbb{R}} k_\lambda(x) e^{i\lambda x} \ .$$

The class of generalized trigonometric polynomials will be denoted by P^p. It is easy to show that this is a linear subspace of B_c^p. Obviously, the space $\mathcal{P}$ is a subspace of P^p.

THEOREM 5.26([20, p. 71]) *A function $f \in B_c^p$ is $\mathcal{M}_{ap}^p$ iff it is the B^p-limit of a sequence of generalized trigonometric polynomials. In other words, $\mathcal{M}_{ap}^p$ is the closure, w.r.t. the Besicovitch norm, of the space P^p.*

Since $\mathcal{M}_{ap}^p$ is a closed subspace of the complete space B_c^p, it is complete, too. The space B^p is a complete subspace of $\mathcal{M}_{ap}^p$.

THEOREM 5.27([20, p. 72]) *The space $\mathcal{M}_{ap}^p$ is a complete subspace of B_r^p.*

THEOREM 5.28([20, p. 72]) [Uniqueness theorem] *If two functions belonging to $\mathcal{M}_{ap}^p$ have the same generalized Bohr–Fourier coefficients, they are equivalent in the Besicovitch norm.*

Comparing Theorem (5.20), Remark (5.21) and Definitions (5.22), (5.23), it is clear that Condition 4) in Theorem (5.20), or the equivalent $4'$), is the necessary and sufficient condition in order for a $\mathcal{M}^p_{ap}$-function to be a B^p-function. For example, the functions $f_\lambda(x) = e^{i\lambda x}\ \mathrm{sign}\, x$, are $\mathcal{M}^p_{ap}$-functions (since $\mathrm{sign}\, x$ is a B^p-constant), but they are not B^p-functions, as already shown in Remark (5.21).

J.-P. Bertrandias has also proved a further characterization of the Besicovitch functions, in terms of *correlation functions*, whose discussion would bring us far from the goal of this paper. For more information, see [20, pp. 70-71].

On the other hand, A. S. Kovanko has introduced a new class of functions and has proved its equivalence with the space B^p_{ap}.

DEFINITION 5.29. ([93]) Given a function $f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ and a set $E \subset \mathbb{R}$, let us define

$$\overline{M}^E\{|f|^p\} \ = \ \{D^E_{B^p}\ [f,0]\}^p \ := \ \limsup_{T\to+\infty} \left[\frac{1}{2T}\int_{E\cap(-T,T)} |f(x)|^p\ dx\right]$$

and

$$\overline{\delta}E \ := \ \limsup_{T\to+\infty} \frac{|E(-T,T)|}{2T} \ ,$$

where

$$|E(-T,T)| \ = \ \mu[E\cap(-T,T)] \ .$$

Observe that that, if $E = \mathbb{R}$,

$$\overline{M}^E\{|f|^p\} \ = \ \|f\|^p_p \ .$$

$f(x)$ is said to be B^p-*uniformly integrable* ($f \in B^p_{u.i.}$) if $\forall \epsilon > 0 \quad \exists\ \eta(\epsilon) > 0$ s.t.

$$\overline{M}^E\{|f|^p\} \ < \ \epsilon \ , \quad \text{whenever} \quad \overline{\delta}E \ < \ \eta \ .$$

DEFINITION 5.30. ([93], [94]) A function f is said to belong to the *class* A_p if

1) $f \in B^p_{u.i.}$;
2) $\forall \epsilon > 0 \ \ \exists \eta > 0$ and a r.d. set of ϵ-almost periods τ s.t.
$$|f(x+t) - f(x)| \ < \ \epsilon \quad \text{for} \quad \tau - \eta \ < \ t \ < \ \tau + \eta \ ,$$
for arbitrary $x \in \mathbb{R}$, possibly with an exception of a set E_t, s.t. $\overline{\delta}E_t < \eta$;
3) for every $a > 0$ $\exists$ a-periodic function $f^{(a)}(x)$ which is a.e. bounded and s.t. (5.8) holds.

REMARK 5.31. In [93], A. S. Kovanko shows that, if a function f belongs to A_p, then it belongs to L^p. Thus, condition $4''$) in Remark (5.21) is a consequence of condition 3) in Definition (5.30).

THEOREM 5.32([93]) $B^p \ \equiv \ A_p$.

G. Bruno and F. R. Grande proved a Lusternik-type theorem, very similar to the corresponding theorem for C^0_{ap}-functions.

Theorem 5.33([38]) *Let $\mathcal{F}$ be a family of elements belonging to B^p_{ap}, $1 \leq p < +\infty$, closed and bounded. Then the following statements are equivalent*:

1) *$\mathcal{F}$ is compact in the B^p-norm;*
2) *$\mathcal{F}$ is B^p-equi-continuous, i.e. for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ s.t., if $|h| < \delta$, then*
$$D_{B^p}\ [f^h, f]\ < \epsilon\ ,\quad \forall f \in \mathcal{F}$$
and B^p-equi-almost-periodic, i.e., for any $\epsilon > 0$, there exists $l(\epsilon) > 0$ s.t. every interval whose lenghth is $l(\epsilon)$contains a common ϵ-almost-period ξ for all $f \in \mathcal{F}$, i.e.
$$D_{B^p}\ [f^\xi, f]\ < \epsilon \qquad \forall f \in \mathcal{F}\ .$$

Theorem 5.34([38]) *Every B^p-function is B^p-normal.*

Remark 5.35. Theorem (5.34) is also a corollary of Theorem (5.20), by means of which we also prove that every B^p-function is B^p_{ap}, B^p-bounded and B^p-continuous.

Theorem 5.36([38]) *Every B^p-normal function is B^p_{ap}.*

For both Theorems (5.34) and (5.36), the converse is not true.

Example 5.37. (Example of a function which is B^p_{ap}, but not B^p-normal) The example is based partly on [127, pp. 42-47]. In Example (4.28), it has been shown that the function

$$f(x) = \begin{cases} 0\ , & \text{if } x \in (-\infty, 0]\ ; \\ \sqrt{\dfrac{n}{2}}\ , & \text{if } x \in (n-2; n-1]\ ,\ n = 2, 4, 6 \dots\ ; \\ -\sqrt{\dfrac{n}{2}}\ , & \text{if } x \in (n-1; n]\ ,\ n = 2, 4, 6 \dots\ ; \end{cases}$$

is a W^1_{ap}-function. Furthermore, in Section 6, it will be shown that every W^p_{ap}-function is B^p_{ap}. Consequently, since the function is W^1_{ap}, it is B^1_{ap}. Now, we want to show that it is not B^1-normal. Let us take $c \in \mathbb{R}$, $c \neq 2k$; $k \in \boldsymbol{Z}$. Without any loss of generality, we can suppose $c > 0$. In fact, if $c < 0$, $(c = -d)$, then
$$D_B[f^d, f] = D_B[f, f^{-d}] = D_B[f^c, f]\ .$$

Since we will have to consider the limit for $T \to \infty$, let us take $T > c$. Then there exist $i, l \in \mathbb{N}$ s.t.

$$2i \leq c < 2(i+1) \ ; \qquad 2l \leq T < 2(l+1) \ .$$

Put $\delta := c - 2i$. We distinguish two cases:

a) $0 \leq \delta \leq 1$;
b) $1 < \delta < 2$.

Since we have to evaluate

$$\frac{1}{2T} \int_{-T}^{T} |f(t+c) - f(t)| \ dt \ ,$$

let us compute the difference $|f(t+c) - f(t)|$ in intervals whose union is strictly included in the interval $[-T, T]$. For the case a), we will take the intervals $(j-1-\delta, j-1)$, $j = 2, 4, 6, \ldots$, for the case b), we will take the intervals $(j, j+\delta)$, $j = 2, 4, 6, \ldots$ We have respectively

$$|f(t+c) - f(t)| = \sqrt{\frac{j}{2}} + \sqrt{\frac{j+2i}{2}} \ ;$$
$$|f(t+c) - f(t)| = \sqrt{\frac{j+2}{2}} + \sqrt{\frac{j+2+2i}{2}} \ .$$

Since the second equality can be obtained from the first one by means of a variable shifting, let us consider only the case a). We have

$$\frac{1}{2T} \int_{-T}^{T} |f(t+c) - f(t)| \ dt \ \geq \ \frac{1}{2l} \sum_{n=1}^{l} \int_{2n-1-\delta}^{2n-1} |f(t+c) - f(t)| \ dt \ =$$
$$= \ \frac{1}{2l} \sum_{n=1}^{l} \delta \left(\sqrt{n+i} + \sqrt{n} \right) \ \geq \ \frac{1}{l} \sum_{n=1}^{l} \delta \sqrt{n} \ \geq \ \frac{\delta}{l} \ \frac{2}{3} l^{\frac{3}{2}} \ ,$$

where the last inequality is obtained by virtue of the Cauchy integral criterion of convergence. Thus, we get

$$\frac{1}{2T} \int_{-T}^{T} |f(t+c) - f(t)| \ dt \ \geq \ \frac{2\delta}{3} \sqrt{l} \ .$$

Passing to the limit for $T \to \infty$, i.e. for $l \to \infty$, provided $c \neq 2k$ ($k \in \boldsymbol{Z}$), we obtain

$$D_B[f^c, f] = \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t+c) - f(t)| \ dt \ \geq \ \frac{\delta}{3} \lim_{l \to \infty} \sqrt{l} = +\infty \ .$$

If we fix a number $a \in \mathbb{R}$ and a sequence $\{a_i\}_{i \in \mathbb{N}}$ s.t. $a_i - a \neq 2k$; $k \in \boldsymbol{Z}$, then

$$D_B[f^a, f^{a_i}] \;=\; D_B[f^{a-a_i}, f] = \;+\infty\;.$$

We conclude that the sequence of translates $\{f(x+a_i)\}$ is not relatively compact and consequently the function is not B^1-normal.

Let us note that the function cannot satisfy all the conditions 1), 2), 3) in Theorem (5.20) (otherwise, according to Remark (5.21), it would be B^1-normal). Let us show that f is not B^1-continuous. In fact,

$$|f(x+\alpha) - f(x)| \;=\; \begin{cases} \sqrt{2n}\;, & \text{if } x \in (n-1-\alpha, n-1] \\ \sqrt{\dfrac{n}{2}} + \sqrt{\dfrac{n-2}{2}}\;, & \text{if } x \in (n-2-\alpha, n-2]\;, \end{cases}$$

for every $n = 2, 4, 6, \ldots$

Let us take $T = 2l$. Then

$$\int_{-T}^{T} |f(x+\alpha) - f(x)| dx = \sum_{k=1}^{l} \left[\sqrt{k} + \sqrt{k-1}\right] \alpha + \sum_{k=1}^{l} 2\sqrt{k}\alpha = \left[3\sqrt{l} + \sum_{k=1}^{l-1} 4\sqrt{k}\right]\alpha.$$

Passing to the limit,

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+\alpha) - f(x)|\; dx \;\geq\; \lim_{l\to+\infty} \frac{\alpha}{4l} \left[3\sqrt{l} + \sum_{k=1}^{l-1} 4\sqrt{k}\right] \;\geq$$

$$\geq\; \alpha \lim_{l\to+\infty} \left[\frac{3}{4\sqrt{l}} \;+\; \frac{2}{3l}(l-1)^{\frac{3}{2}}\right] \;=\; +\infty\;,$$

and the claim follows.

Example 5.38. (Example of a function which is B^p-normal, but not B^p) The example is partly based on [35, p. 107] and [57, p. 5]. Let us consider the function

$$\operatorname{sign}(x) \;=\; \begin{cases} -1\;, & \text{if } x < 0 \\ 0\;, & \text{if } x = 0 \\ 1\;, & \text{if } x > 0\;. \end{cases}$$

As shown in [57, p. 5], this function is not B^p: it is sufficient to recall that, by virtue of Theorem (6.2), for every B^p-function, there exists the mean value (2.2) and

$$M[f] \;=\; \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)\; dx \;=\; \lim_{T\to\infty} \frac{1}{T} \int_{-T}^0 f(x)\; dx\;.$$

However,

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \operatorname{sign} x\; dx = \;1\;;$$

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T}^0 \operatorname{sign} x\; dx = \;-1\;.$$

Thus, Theorem (6.2) is not fulfilled. On the other hand, $\operatorname{sign} x$ is a B^p_{ap}-function, because

$$|\operatorname{sign}(x+\tau) - \operatorname{sign}(x)| \;=\; \begin{cases} 2\,, & \text{if } x \in [-\tau, 0) \\ 0\,, & \text{elsewhere}\,, \end{cases}$$

and

$$\|\operatorname{sign}(x+\tau) - \operatorname{sign}(x)\|_p \;=\; 0 \qquad \forall \tau \;\in \mathbb{R}\,.$$

Let us show that it is B^p-normal, too.

In fact, for every choice of h_m, h_n (let us choose, without any loss of generality, $h_m > h_n > 0$), we have

$$\int_{-T}^{T} |\operatorname{sign}(x+h_m) - \operatorname{sign}(x+h_n)|^p \; dx \;=\; \int_{-h_m}^{-h_n} dx \;=\; h_m - h_n\,,$$

and consequently

$$\|\operatorname{sign}(x+h_m) - \operatorname{sign}(x+h_n)\|_p \;=\; 0$$

as well as the B^p-normality.

More generally, we can consider a function $f(x) = e^{-i\lambda x}\operatorname{sign} x$, which is not B^p, for every $\lambda \in \mathbb{R}$ (see Remark (5.21)).

On the other hand, each of the functions $e^{-i\lambda x}\operatorname{sign} x$ is B^p-normal. In fact, for every choice of h_m, h_n ($h_m > h_n$), we have

$$\int_{-T}^{T} |f(x+h_m) - f(x+h_n)| \; dx \;=\; \int_{-h_m}^{-h_n} 1 \; dx \;=\; h_m - h_n\,,$$

and consequently

$$⨍ \; |e^{i\lambda(x+h_m)} - e^{i\lambda(x+h_n)}| \; dx \;=\; 0 \qquad \forall\; h_m, h_n \in \mathbb{R}\,.$$

So, the claim follows.

However, for these functions, Formula (2.3) holds. In fact

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T e^{-i\lambda x}\operatorname{sign} x\; dx = \lim_{T\to+\infty} \frac{i}{\lambda T}\left[e^{-i\lambda T} - 1\right] \;=\; 0\,;$$
$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T}^0 e^{-i\lambda x}\operatorname{sign} x\; dx = \lim_{T\to+\infty} \frac{i}{\lambda T}\left[e^{i\lambda T} - 1\right] \;=\; 0\,.$$

This example shows that Theorem (6.2) in Section 6, characterizing B^p-functions, is, in general, not satisfied by B^p_{ap} and B^p-normal functions.

The importance and the properties of the spaces B^p_{ap} can be better understood applying again the Bohr compactification. Let us introduce on $\mathbb{R}_B$ the normalized *Haar measure* μ (i.e. the positive regular Borel measure s.t. $\mu(U) = \mu(U+s)$, for every Borel subset $U \subset \mathbb{R}_B$ and for every $s \in \mathbb{R}_B$ (invariance property) and s.t. $\mu(\mathbb{R}_B) = 1$ (normality property)).

It is possible [107] to show that the space B^p_{ap} is isomorphic to the space $L^p(\mathbb{R}_B, \mathbb{R})$, where L^p is taken w.r.t. the Haar measure defined on $\mathbb{R}_B$. It follows that

$$\|f\|^p_{B^p} = \begin{cases} \|\tilde{f}\|^p_{L^p(\mathbb{R}_B,\mu)} = \displaystyle\int_{\mathbb{R}_B} |\tilde{f}(x)|^p \, d\mu(x) \,, & \text{if } 1 \le p < +\infty \\ \text{ess } \sup_{x\in\mathbb{R}_B} |\tilde{f}(x)| \,, & \text{if } p = +\infty \,, \end{cases}$$

where $\tilde{f}$ is the extension by continuity of f from $\mathbb{R}$ to $\mathbb{R}_B$.

From this isomorphism, many properties for the spaces B^p_{ap} can be obtained. For example, two functions differing from each other even on the whole real axis can belong to the same Besicovitch class, because two functions belonging to the same $L^p(\mathbb{R}_B, \mathbb{R})$-class may differ from each other on a set of the Haar measure zero and the real numbers are embedded in the Bohr compactification as a dense set of the Haar measure zero. Furthermore, recalling the inclusions among the spaces L^p on compact sets, we have

$$B^\infty_{ap} \subset B^{p_1}_{ap} \subset B^{p_2}_{ap} \subset B^1_{ap} \qquad \forall\, p_1 > p_2 > 1 \,,$$

where

$$B^\infty_{ap} = \bigcap_{p\in\mathbb{N}} B^p_{ap} \,.$$

Furthermore ([64], [71], [72]), the spaces B^p_{ap}, $1 \le p < \infty$, are reflexive spaces and their duals are given by B^q_{ap}, where q is s.t. $\frac{1}{q} + \frac{1}{p} = 1$. The spaces B^p_{ap} are not separable (see, for example, [135, p. 108]). In particular, the space B^2_{ap} is a non-separable Hilbert space, in which the exponents $e^{i\lambda x}$ ($\lambda \in \mathbb{R}$) form an orthonormal basis. Other properties can be found in [12], [64], [71], [72], [107, pp. 11-12], [108].

6 – Vertical hierarchies. Properties. Examples and counter-examples

In the previous sections, we have shown that, although for the spaces C^0_{ap} and S^p_{ap}, the three definitions, in terms of relative density, normality and polynomial approximation, are equivalent, for the remaining spaces (e-W^p_{ap}, W^p_{ap}, B^p_{ap}) the equivalence does not hold anymore.

It is then important to check the relationships among every definition obtained w.r.t. one norm and the less restrictive definitions related to more general classes. Before studying these relations and the vertical hierarchies among the spaces up to now studied, let us recall the most important properties that are common to all these spaces.

In fact, many of the properties of the u.a.p. functions can be satisfied by the functions belonging to the spaces of generalized a.p. functions. For the sake of simplicity, let us indicate with G^p the either (generic) space S^p, e-W^p, W^p or B^p (similarly for the spaces G^p_{ap} and G^p-normal). If not otherwise stated, the following theorems will be valid for any of the spaces studied.

First of all, let us underline the connection of almost-periodic functions with the trigonometric series.

Theorem 6.1([22, p. 104], [24, p. 262], [35, p. 45], [67, pp. 191, 193]) *Every G^p-function can be represented by its Fourier series, given by formula* (2.5).

Theorem 6.2([22, p. 93], [24, p. 244-245], [35, p. 45], [67, p. 191] [Mean value theorem] *The mean value (2.2) of every G^p-function f exists and*

$$\text{(6.1)} \qquad \mathbf{a})\ M[f] = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx \ = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^0 f(x)\,dx$$

$$\text{(6.2)} \qquad \mathbf{b})\ M[f] = \lim_{T\to\infty} \frac{1}{2T}\int_{a-T}^{a+T} f(x)\,dx\ ;$$

where the last limit exists uniformly w.r.t. $a \in \mathbb{R}$, for every function in S^p, in $e - W^p$ and in W^p.

Theorem (6.2) is related to a property of the S^p-norm, stated by S. Koizumi.

Theorem 6.3([77]) *A function $f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ belongs to BS^p iff there exists a positive constant K' s.t.*

$$\limsup_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(x+t)|^p\,dt \ \le\ K'\ , \qquad \text{uniformly w.r.t. } x\in\mathbb{R}\ .$$

Remark 6.4. Repeating the considerations done in Remark (2.17), for every $f \in G^p$, the quantities $a(\lambda)$, given by (2.4), are finite, for every $\lambda \in \mathbb{R}$.

This fact is no longer true, in general, for G^p_{ap}-functions, as shown in Example (7.7).

Theorem 6.5([22, p. 104], [24, p. 262], [35, p. 45]) *For every G^p-function f, there always exists at most a countable infinite set of the Bohr–Fourier exponents λ, for which $a(\lambda) \neq 0$, where $a(\lambda)$ are given by* (2.4).

Theorem 6.6([22, p. 109], [35, p. 47]) [Bohr Fundamental Theorem] *The Parseval equation*

$$\sum_n |a(\lambda_n, f)|^2 \;=\; M\{|f|^2\}$$

is true for every G^2-function.

Theorem 6.7([22, p. 109], [24, p. 266], [35, p. 45]) [Uniqueness Theorem] *If two G^p-functions f, g have the same Fourier series, then they are identical, i.e.*

$$D_G\;[f, g] \;=\; 0\;.$$

In other words, two different elements belonging to G^p cannot have the same Bohr–Fourier series.

The functions whose G^p-norm is equal to zero are called G^p-zero functions ([35, p. 38]).

Proposition 6.8. *Every G^p-zero function is a G^p-function and belongs to the class of the function $f(x) \equiv 0$.*

Definition 6.9. ([20]) A G^p-bounded function is said G^p-*constant* if, for every real number τ,

$$\|f^\tau - f\|_{G^p} \;=\; 0\;.$$

Proposition 6.10. *Every G^p-zero function is G^p-constant.*

Proof. In fact, for every $\tau \in \mathbb{R}$, $\|f^\tau - f\|_{G^p} \leq \|f^\tau\|_{G^p} + \|f\|_{G^p} = 0$. □

The converse is, in general, not true. For example, the function $f(x) \equiv 1$ is, obviously, G^p-constant, but

$$\|f\|_p \;=\; \limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} 1\;dx \;=\; 1 \;\neq\; 0\;.$$

Proposition 6.11. *Every G^p-constant is G^p_{ap} and G^p-normal.*

The last property cannot be extended to the G^p-functions. For example, the function $f(x) = \operatorname{sign} x$ is B^p-constant. It is B^p_{ap}, but it is not B^p (see Example (5.38)).

Theorem 6.12([22, pp. 110-112], [35, p. 47]) [Riesz–Fischer theorem] *To any series $\sum a_n e^{i\lambda_n x}$, for which $\sum |a_n|^2$ converges, corresponds a G^2-function having this series as its Bohr–Fourier series.*

In order to establish the desired vertical hierarchies, let us recall the most important relationships among the norms (2.1), (3.1), (4.1), (5.1) (see, for example, [22, pp. 72-76], [24, pp. 220-224], [35, pp. 36-37]).

For every $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ and for every $p \geq 1$, the following inequalities hold:

$$\|f\|_{C^0} \;\geq\; \|f\|_{S^p_L} \;\geq\; \|f\|_{W^p} \;\geq\; \|f\|_p \, .$$

Consequently, we obtain, for every $p \geq 1$,

$$\begin{gathered} C^0_{ap} \subseteq S^p \subseteq W^p \subseteq B^p; \\ C^0_{ap} \subseteq S^p-\text{normal} \subseteq e-W^p-\text{normal} \subseteq W^p-\text{normal} \subseteq B^p-\text{normal}; \\ C^0_{ap} \subseteq S^p_{ap} \subseteq e-W^p_{ap} \subseteq W^p_{ap} \subseteq B^p_{ap} \, . \end{gathered} \tag{6.3}$$

Furthermore, it is easy to show, by virtue of the Hölder inequality, that, for every $1 \leq p_1 < p_2$,

$$\|f\|_{G^{p_1}} \;\leq\; \|f\|_{G^{p_2}} \; ,$$

and, consequently,

$$C^0_{ap} \;\subseteq\; G^{p_2} \;\subseteq\; G^{p_1} \;\subseteq\; G^1 \;\subseteq\; L^1_{\text{loc}} \, . \tag{6.4}$$

Defining the spaces

$$BG^p \;:=\; \{f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}) \quad s.t. \quad \| \, f \, \|_{G^p} < \; +\infty\} \;\; ,$$

from formula (6.4) and from Definition (2.8), the following theorem holds.

Theorem 6.13([22, pp. 75-76], [35, p. 38]) *The spaces G^p coincide with the spaces obtained as the closures of the space C^0_{ap} w.r.t. the norms (3.2), (4.1), (5.1).*

Furthermore, the following properties for bounded functions hold:

Theorem 6.14([35, p. 37]) *Every G^p-function is G^p-bounded.*

Remark 6.15. Theorem (6.14) does not hold, in general, for the spaces W^p_{ap} and B^p_{ap}, as shown by Example (4.28).

Theorem 6.16([35, pp. 62-63]) *Every bounded function belonging to G^1 belongs to every space G^p, $\forall p > 1$.*

Inclusions (6.3) can be improved, recalling some theorems and examples.

THEOREM 6.17([48]) *If there exists a number M s.t. the exponents of a S^p_{ap}-function f are less in modulus than M, then f is equivalent to a u.a.p. function.*

THEOREM 6.18([4, Th. VII, p. 78], [22, pp. 81-82], [41, p. 158, Th. 6.16]) [Bochner] *If $f \in S^p_{ap}$ is uniformly continuous, then f is u.a.p.*

H. D. URSELL [129] has shown an example of a continuous S^p_{ap}-function, which is not uniformly continuous and which is not C^0_{ap}.

On the other hand, it is not difficult to show that the space C^0_{ap} is strictly contained in S^p_{ap}, for every p.

In his book [99], B. M. Levitan shows two interesting examples.

EXAMPLE 6.19. ([99, pp. 209-210]) Let $f \in C^0_{ap}$. Then the function

$$F(x) = \operatorname{sign}(f(x)) = \begin{cases} 1\ , & \text{if } f(x) > 0 \\ 0\ , & \text{if } f(x) = 0 \\ -1\ , & \text{if } f(x) < 0 \end{cases}$$

is S^1_{ap}.

EXAMPLE 6.20. ([99, pp. 212-213]) Given the quasi-periodic (and, *a fortiori*, almost-periodic) function $\phi(x) = 2 + \cos x + \cos\sqrt{2}x$, the function

$$f(x) = \sin\left(\frac{1}{\phi(x)}\right)$$

is S^1_{ap}.

However, in order to have S^p_{ap}-functions which are not in C^0_{ap}, it would be sufficient to consider, for example, the functions obtained modifying the values of an $f \in C^0_{ap}$ on all the relative integers, because the elements of $L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ (and, consequently, of S^p_{ap}) are the classes of functions obtained by means of the equivalence relation $f \sim g$ if $f = g$, a.e. in $\mathbb{R}$.

The following example shows a function $f \in S^p_{ap}$ which is unbounded.

EXAMPLE 6.21. The function

$$f(x) = \begin{cases} \cos x\ , & \text{if } x \neq k\pi\ , \\ k\ , & \text{if } x = k\pi\ , \end{cases}$$

is not continuous and it is unbounded, but it belongs to the same class of $L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ as the function $g(x) = \cos x$, which is, obviously, u.a.p. Thus, it is S^p_{ap}, for every p.

THEOREM 6.22([22, p. 77], [67, p. 190], [99, p. 222]) *If, in the norm* (4.1), $\limsup_{\epsilon\to 0} L(\epsilon)$ *is finite, then a function $f \in e\text{-}W^p_{ap}$ is an S^p_{ap}-function.*

In other words, the spaces S^p_{ap} can be interpreted as uniform W^p-spaces. The spaces S^p_{ap} are strictly included in $e-W^p_{ap}$.

EXAMPLE 6.23. (Example of an $e-W^p$-normal function which does not belong to S^p_{ap}) The example is partly based on [127, pp. 20-21]. In Example (4.27), we have already proved that the function, defined on $\mathbb{R}$,

$$f(x)=\begin{cases} 1\ , & \text{for } \ 0<x<\dfrac{1}{2}\ , \\ 0\ , & \text{elsewhere}\ , \end{cases}$$

is $e-W$-normal, but not S_{ap}.

Let us note that, by Formula (4.14), for every $L\geq 1$,

$$D_{S_L}\ [f^\tau, f]\ <\ \epsilon \qquad \text{if} \quad L=L(\epsilon)\geq \frac{1}{\epsilon}\ .$$

Thus, $\limsup_{\epsilon\to 0}\ L(\epsilon)\ =\ +\infty$, and the hypothesis in Theorem (6.22) is not satisfied.

Another example can be found in [35] (Main Example 2, pp. 70-73 and Main Example II, pp. 115-116).

In Example (4.29), we have shown that the Heaviside step function

$$H(x)\ =\ \begin{cases} 0\ , & \text{if } x<0 \\ 1\ , & \text{if } x\geq 0\ , \end{cases}$$

is a W^1_{ap}-function, but not $e-W^1_{ap}$. Since the space $e-W^1_{ap}$ corresponds to the space W^1 (see Theorems (4.7), (4.11)) and the spaces W^p are included in the spaces W^p-normal, Example (4.28) shows a second example of a W^p_{ap}-function which is not an $e-W^p_{ap}$-function, other than [8, Example 1].

As already shown for the W^p_{ap} and $e-W^p_{ap}$ spaces, the three definitions for a more general space can be inserted among those of a more restrictive space. This situation is very clear when we compare the W^p and B^p-definitions.

EXAMPLE 6.24. (Example of a B^p-function which is not a W^p_{ap}-function) Let us take the function

$$f(x)=\begin{cases} n^{\frac{1}{2p}}\ , & \text{if } n^2\leq x<n^2+\sqrt{n}\ , \\ 0\ , & \text{elsewhere}\ , \end{cases}$$

where $p\in\mathbb{R}\ ,\ p>1$ and $n\in\mathbb{N}\ ,\ n\geq 1$.

As pointed out in [20, p. 42], this function is unbounded, B^p-bounded and B^p-constant. Consequently, by virtue of Proposition (6.11), it is B^p_{ap}.

Let us compute $\|f\|_q$, for every $q \geq 1$. It is sufficient to take $T = N^2 + \sqrt{N}$, $N \in \mathbb{N}$.

$$\|f\|_q^q = \lim \sup_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^q dx = \lim \sup_{N\to+\infty} \frac{1}{2(N^2+\sqrt{N})}\int_0^{N^2+\sqrt{N}} |f(x)|^q =$$

$$= \lim \sup_{N\to+\infty} \frac{1}{2(N^2+\sqrt{N})}\sum_{k=1}^{N} \sqrt{k}k^{\frac{q}{2p}} .$$

For $q = p$, we have

$$\|f\|_p^p = \lim_{N\to+\infty} \frac{1}{2(N^2+\sqrt{N})} \frac{N(N+1)}{2} = \frac{1}{4} .$$

For $q > p$, we have

$$\|f\|_q^q \geq \lim \sup_{N\to+\infty} \frac{1}{2(N^2+\sqrt{N})} \frac{2p}{3p+q} N^{\frac{1}{2}(3+\frac{q}{p})} = +\infty ,$$

because $\frac{1}{2}\ (3+\frac{q}{p})\ > 2$.

For $q < p$, we have

$$\|f\|_q^q \leq \lim \sup_{N\to+\infty} \frac{1}{2(N^2+\sqrt{N})} \frac{2p}{3p+q} (N+1)^{\frac{1}{2}(3+\frac{q}{p})} = 0 ,$$

because $\frac{1}{2}\ (3+\frac{q}{p})\ < 2$.

In the last two cases, we have used the Cauchy integral criterion.

It follows that $f \in B^q$ $,\forall q < p$, because it is a B^q-zero function.

Let us show that $f \notin W^q_{ap}$.

Without any loss of generality, we can take $\tau > 0$ and $T > \tau$. There exists a real number M s.t. $\sqrt{M} > T > \tau$. Thus,

$$\sup_{x\in\mathbb{R}} \frac{1}{T} \int_x^{x+T} |f(x+\tau) - f(x)|^q \, dx \geq \frac{1}{T} \int_{N^2+\sqrt{N}-\tau}^{N^2+\sqrt{N}} N^{\frac{q}{2p}} \, dx = \frac{\tau}{T} N^{\frac{q}{2p}} ,$$

for every $N \geq M$. Consequently, taking the limit for $N \to +\infty$, we arrive at

$$\|f^\tau - f\|_{W^q} = +\infty \qquad \forall q \geq 1 \,; \forall \tau > 0 \,; \forall T > \tau .$$

Let us observe that, since $B^q \subset B^q$-normal and W^q-normal $\subset W^q_{ap}$, this example shows simultaneously a function which is B^q, but not W^q-normal, and a function which is B^q-normal, but not W^q_{ap}.

Let us finally remark that this function does not satisfy Condition (6.2) in Theorem (6.2), uniformly w.r.t. a. In fact, for every $T > 0$, there exists $N \in \mathbb{N}$ s.t. $\sqrt{N} > 2T$. Consequently,

$$\left| \sup_{a \in \mathbb{R}} \frac{1}{2T} \int_{a-T}^{a+T} f(x) dx - M[f] \right| \geq \frac{1}{2T} \int_{N^2 + \frac{\sqrt{N}}{2} - T}^{N^2 + \frac{\sqrt{N}}{2} + T} N^{\frac{1}{2p}} dx = N^{\frac{1}{2p}} \to_{N \to +\infty} +\infty \,,$$

and the claim follows.

Example 6.25. (Example of a W^1-normal function which is not a B^1-function) The example is partly based on [127, p. 48]. As shown in Example (4.29), the function $H(x)$ is W^1-normal. On the other hand, it does not satisfy (6.1), because

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T H(x)\, dx = 1 \neq \lim_{T \to \infty} \frac{1}{T} \int_{-T}^0 H(x)\, dx = 0 \,.$$

Consequently, it is not B^1.

It means that $H(x)$ does not satisfy all the conditions of Theorem (5.20). In fact, while, for every $\lambda \neq 0$ and for L sufficiently large,

$$\begin{aligned} &\frac{1}{2TL} \left[\int_{-T}^{T} \Big| \int_x^{x+L} H(t) e^{i\lambda t} dt - \int_0^L H(t) e^{i\lambda t} dt \Big| dx \right] = \\ &= \frac{1}{2TL} \left[\int_{-T}^{0} \left| \int_0^{x+L} e^{i\lambda t} dt - \int_0^L e^{i\lambda t} dt \right| dx + \int_0^T \left| \int_x^{x+L} e^{i\lambda t} dt - \int_0^L e^{i\lambda t} dt \right| dx \right] = \\ &= \frac{1}{2TL|\lambda|} \left[\int_{-T}^{0} \left| e^{i\lambda x} - 1 \right| dx + \int_0^T \left| e^{i\lambda x} - 1 \right| \left| e^{i\lambda L} - 1 \right| dx \right] \leq \frac{3}{L|\lambda|} \,, \end{aligned}$$

and consequently,

$$\begin{aligned} &\lim_{L \to +\infty} \lim_{T \to +\infty} \frac{1}{2TL} \left[\int_{-T}^{T} \left| \int_x^{x+L} H(t) e^{i\lambda t} dt - \int_0^L H(t) e^{i\lambda t} dt \right| dx \right] = \\ &= \lim_{L \to +\infty} \frac{3}{L|\lambda|} = 0 \,, \end{aligned}$$

for $\lambda = 0$,

$$\begin{aligned} &\lim_{L \to +\infty} \lim_{T \to +\infty} \frac{1}{2TL} \left[\int_{-T}^{T} \Big| \int_x^{x+L} H(t) e^{i\lambda t}\, dt - \int_0^L H(t) e^{i\lambda t}\, dt \,\Big|\, dx \right] = \\ &= \lim_{L \to +\infty} \lim_{T \to +\infty} \frac{1}{2TL} \left[\int_{-T}^{0} |x|\, dx \right] = \lim_{L \to +\infty} \lim_{T \to +\infty} \frac{T}{4L} = +\infty \,, \end{aligned}$$

and the function does not satisfy condition 4), for every $\lambda \in \mathbb{R}$.

EXAMPLE 6.26. ([127, pp. 42-47]) (Example of a W^1_{ap}-function which is not a B^1-normal function) In Example (4.28), we have shown that the function

$$f(x)=\begin{cases} 0\ , & \text{if } x\in(-\infty,0]\ ;\\ \sqrt{\dfrac{n}{2}}\ , & \text{if } x\in(n-2;n-1]\ ,\ n=2,4,6\dots\ ;\\ -\sqrt{\dfrac{n}{2}}\ , & \text{if } x\in(n-1;n]\ ,\ n=2,4,6\dots\ ;\end{cases}$$

belongs to $e-W^1_{ap}$. On the other hand, in Example (5.37), the function is shown not to be B^1-normal.

EXAMPLE 6.27. ([35, Example 3b, pp. 58-61, pp. 111-114], [127, pp. 34-38]) (Example of a B^p-function which is not a W^p-function) The function

$$f(x)=\begin{cases} 1, & \text{if } x\in\left[n-\dfrac12,n+\dfrac12\right], n\in\boldsymbol{Z}, n \bmod 2=0 \ \text{ but } n \bmod 2^2\neq 0\ ,\\ 2, & \text{if } x\in\left[n-\dfrac12,n+\dfrac12\right], n\in\boldsymbol{Z}, n \bmod 2^2=0 \ \text{ but } n \bmod 2^3\neq 0\ ,\\ 3, & \text{if } x\in\left[n-\dfrac12,n+\dfrac12\right], n\in\boldsymbol{Z}, n \bmod 2^3=0 \ \text{ but } n \bmod 2^4\neq 0\ ,\\ \vdots & \\ 0, & \text{elsewhere}\ ,\end{cases}$$

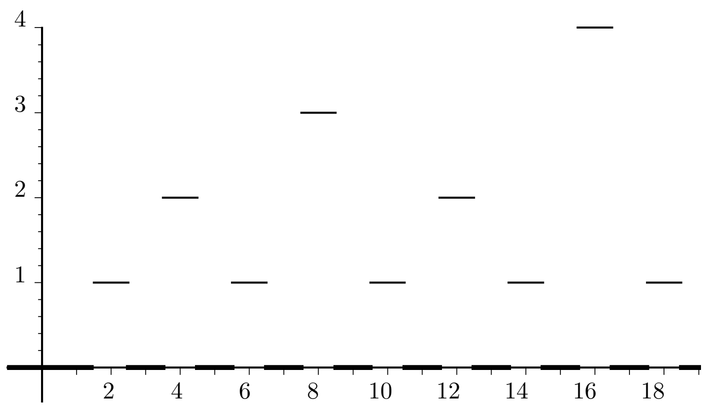

Fig. 4

is a B^p-function which is not a W^p-function.

Taking into account the last four examples, we can conclude that there does not exist any inclusive relation between the spaces W^p_{ap} and B^p.

The previous theorems and examples allow us to write down Table 2.

	a. periods		*normal*		*approx.*
Bohr	C^0_{ap}	$\Leftrightarrow$	u. normal	$\Leftrightarrow$	u.a.p.
	$\Downarrow\not\Uparrow$		$\Downarrow\not\Uparrow$		$\Downarrow\not\Uparrow$
Stepanov	S^p_{ap}	$\Leftrightarrow$	S^p-normal	$\Leftrightarrow$	S^p
	$\Downarrow\not\Uparrow$		$\Downarrow\not\Uparrow$		$\Updownarrow$
equi-Weyl	$e-W^p_{ap}$	$\Leftrightarrow$	$e-W^p$-normal	$\substack{\Leftarrow \\ \not\Rightarrow}$	$e-W^p$
	$\Downarrow\not\Uparrow$	$\Nwarrow$	.... $\Downarrow\not\Uparrow$	$\Searrow$	$\Downarrow\not\Uparrow$
Weyl	W^p_{ap}	$\substack{\Leftarrow \\ \not\Rightarrow}$	W^p-normal	$\substack{\Leftarrow \\ \not\Rightarrow}$	W^p
	$\Downarrow\not\Uparrow$	$\not\Swarrow\not\Nearrow$	.... $\Downarrow\not\Uparrow$	$\not\Swarrow\not\Nearrow$	$\Downarrow\not\Uparrow$
Besicovitch	B^p_{ap}	$\substack{\Leftarrow \\ \not\Rightarrow}$	B^p-normal	$\substack{\Leftarrow \\ \not\Rightarrow}$	B^p

Table 2

7 – Further generalizations. Open problems. Perspectives

As for the case of the C^0_{ap}-functions, it is possible to generalize the theory of the G^p-functions to spaces of functions defined on arbitrary groups (see, for example, [58], [63], [65]).

Many authors have furthermore generalized in different directions the notion of almost-periodicity; for example, R. Doss ([47], [49], in terms of diophantine approximations), S. Stoinski ([116], [117], [118], [119], [120], [121], [122], [123], [124], [125], [126], in terms of ϵ-almost-periods), K. Urbanik ([128], in terms of polynomial approximations), etc.

Besides that, the contribution by A. S. Kovanko to the theory of generalized a.p. functions is significant in this context: namely in [79], [80], he introduced ten different definitions of a.p. functions (of types $A, \overline{A}, B, B', \overline{B}, \overline{B}', C, C', \overline{C}, \overline{C}'$), in terms of ϵ-almost-periods, showing that the space of A-a.p. functions is the largest one and that the space of C-a.p. functions is an intermediate space between W^2 and B^2. See also [85], [90], [93], [96].

Furthermore, in [81], [82], [83], [84], [94], [95], he extended the theory of a.p. functions to non-integrable functions, in terms of polynomial approximations. He introduced the space of α-a.p. functions, which coincides with the space M_{ap}, by virtue of Definition (3.12) and Theorem (3.14), and the spaces of α_k-a.p. functions, which are the extensions of the spaces B^p to measurable functions. These spaces are included in the space of α-a.p. functions. Moreover, it is possible to prove a Bohr-like property for the spaces of α_k-a.p. functions and to show that, for every α_1-a.p. function, the mean value exists and the set of values $a(\lambda)$, defined by (2.4), for which $\alpha(\lambda) \neq 0$, is at most countable.

In [84], the author introduces the space of β-a.p. functions, in terms of a Bohr-like definition, where the set of almost-periods is satisfactorily uniform, like in Definition (5.10), and shows that this space coincides with the space of α-a.p. functions. Thus, he shows the equivalence of two definitions (Bohr-like and approximation) for these spaces. Moreover, he proves that the space B^1 is included in this space. Finally, he states a necessary and sufficient condition in order for a B^1-function to be α-a.p., in terms of the so-called asymptotic uniform integrability (see [84]). E. Fœlner [57] has specialized the study of these spaces, considering B^p-bounded functions and bounded functions, proving interesting relationships with the spaces B^p.

Generalizing the theory of weakly a.p. functions (see [53], [54]), J.-P. Bertrandias [20, pp. 64-68, 71] has introduced the spaces of B^p-weakly a.p. functions, showing that every B^p_{ap}-function is B^p-weakly a.p. (see [20, p. 71]).

In this section, we will only concentrate our attention to the generalizations given by C. Ryll-Nardzewski, S. Hartman, J. P. Kahane (see [75] and the references therein), which are related to the Bohr-Fourier coefficients, not considering the almost-periodicity of the functions, and C. Zhang [136] (cf. [2]). We will also mention possible multivalued extensions in [5], [6], [8], [40], [42], [52].

Definition 7.1. ([75]) A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is called *almost-periodic in the sense of Hartman* (shortly, H^1_{ap}) if, for every $\lambda \in \mathbb{R}$, the number

$$a_f(\lambda) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)e^{-i\lambda x}\, dx$$

exists and is finite.

Definition 7.2. ([75]) A function $f \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R})$ is called *almost-periodic in the sense of Ryll-Nardzewski* (shortly, R^1_{ap}) if, for every $\lambda \in \mathbb{R}$, the number

$$b_f(\lambda) = \lim_{T\to+\infty} \frac{1}{T} \int_{X}^{X+T} f(x)e^{-i\lambda x}\, dx, \quad \text{exists uniformly w.r.t. } x \in \mathbb{R}\ , \tag{7.1}$$

and is finite.

Every R^1_{ap}-function is, obviously, H^1_{ap} (and, for every λ and for every $f \in R^1_{ap}$, $a_f(\lambda) = b_f(\lambda)$). The converse, in general, is not true.

Example 7.3. (Example of a H^1_{ap}-function, neither belonging to R^1_{ap} nor to B^1) We have already shown in Example (5.38) that the function $f(x) = \operatorname{sign} x$ does not belong to B^1. Nevertheless, $f \in H^1_{ap}$ and its spectrum is empty. In fact, for every $\lambda \neq 0$,

$$\int_{-T}^{T} f(x)e^{-i\lambda x}\, dx = -\int_{-T}^{0} e^{-i\lambda x}\, dx + \int_{0}^{T} e^{-i\lambda x}\, dx = \frac{2}{i\lambda}[1-\cos(\lambda T)]\ .$$

Thus,

$$a_f(\lambda) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x} dx = \lim_{T\to+\infty} \frac{1}{i\lambda T}[1-\cos(\lambda T)] = 0 \quad \forall \lambda \neq 0.$$

If $\lambda = 0$,

$$a_f(0) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx = \lim_{T\to+\infty} \frac{1}{2T} \left[-\int_{-T}^{0} dx + \int_{0}^{T} dx \right] = 0 .$$

Finally, let us prove that $f \notin R^1_{ap}$. It is sufficient to show that property (7.1) does not hold for $\lambda = 0$. In fact,

$$\int_X^{X+T} \operatorname{sign} x\ dx = \begin{cases} -\displaystyle\int_X^0 dx + \int_0^{X+T} dx, & \text{for } X < 0 \\ \displaystyle\int_X^{X+T} dx\ , & \text{for } X \geq 0 \end{cases} =$$

$$= \begin{cases} 2X + T\ , & \text{for } X < 0\ , \\ T\ , & \text{for } X \geq 0\ . \end{cases}$$

Then

$$b_f(0) = \lim_{T\to+\infty} \frac{1}{T} \int_X^{X+T} \operatorname{sign} x\ dx = 1\ .$$

Since

$$\left| \frac{1}{T} \int_X^{X+T} \operatorname{sign} x\ dx - 1 \right| = \begin{cases} \left| \dfrac{2X}{T} \right|, & \text{for } X < 0\ , \\ 0\ , & \text{for } X \geq 0\ , \end{cases}$$

we have

$$\left| \frac{1}{T} \int_X^{X+T} \operatorname{sign} x\ dx - b_f(0) \right| < \epsilon\ ,$$

whenever $\left|\frac{2X}{T}\right| < \epsilon$, i.e. $\forall T > \frac{2|X|}{\epsilon}$. Consequently, the limit $b_f(0)$ is not uniform w.r.t. X and $f(x) = \operatorname{sign} x \notin R^1_{ap}$.

It can be observed that, in this case, $a_f(0) = 0 \neq b_f(0) = 1$. On the other hand, it can be easily shown that $b_f(0) = 0\ \forall \lambda \neq 0$, uniformly w.r.t. $X \in \mathbb{R}$.

As pointed out in [75] and from Theorem (6.2) and Remark (6.4), every S^p_{ap} and W^p_{ap}-function is R^1_{ap} and every B^p-function is H^1_{ap}, while there is no relation between the spaces B^p_{ap} and R^1_{ap} and between the spaces B^p_{ap} and H^1_{ap}.

Example 7.4. (Example of a function belonging to B^1_{ap}, but not to R^1_{ap}) In Example (6.24), we could see a function which is B^1_{ap}, but not W^1_{ap}, showing that (6.2) does not hold uniformly w.r.t. $a \in \mathbb{R}$. Consequently, the function does not belong to R^1_{ap}.

Analogously to the spaces G^p, we can introduce the *spectrum* $\sigma(f)$ for a H^1_{ap}-function $f(x)$ as

$$\sigma(f) \ = \ \{\lambda \in \mathbb{R} \quad s.t.\ a_f(\lambda) \ \neq \ 0\} \ .$$

THEOREM 7.5([75]) *The spectrum of every function $f \in H^1_{ap}$ is at most countable.*

REMARK 7.6. Every non negative H^1_{ap}-function is B^1-bounded, because, in this case, $a_f(0) = \|f\|_1$. This property is no longer true for general H^1_{ap}-functions.

EXAMPLE 7.7. (Example of a function which is B^p_{ap}, for every $p \geq 1$, but not H^1_{ap}) Let us take the function, with values in $\mathbb{C}$,

$$f(x) = e^{i \log |x|} \ .$$

Clearly,

$$\|f\|_p^p \ = ⨍ \ \left|e^{i \log |x|}\right|^p dx \ = \ 1 \ .$$

Besides that, J.-P. BERTRANDIAS [20, p. 42] has shown that f is B^p-constant and, consequently, by virtue of Proposition (6.11), it is B^p_{ap}, for every $p \geq 1$.

Let us show that this function does not belong to H^1_{ap}, in particular, that it does not have a mean value. In fact,

$$M[f] \ = ⨍ \ e^{i \log |x|} \ dx \ = \ \lim_{T \to +\infty} \ \frac{1}{T} \int_0^T \ e^{i \log x} \ dx \ = \ \frac{(1-i)}{2} \lim_{T \to +\infty} \ e^{i \log T} \ .$$

Since the limit in the last equality does not exist, the claim follows.

THEOREM 7.8([75]) *Every function $f \in H^1_{ap}$, belonging to some Marcinkiewicz space $\mathcal{M}^p$, $p > 1$, is the sum of a B^p_{ap}-function and of a H^1_{ap}-function whose spectrum is empty.*

J. BASS [13], [14], [15] and J.-P. BERTRANDIAS [18], [19], [20] introduced the spaces of pseudo-random functions, in terms of correlation functions, showing that these spaces are included in B^p_c and in the space of Hartman functions whose spectrum is empty (see [19]). There is no relation between the spaces B^p and the spaces of pseudo-random functions, but some theorems concerning operations involving B^p and pseudo-random functions can be proved (see, for example, [14, pp. 28-31]).

In the framework of the resolution of systems of ordinary differential equations, C. Zhang has introduced in [136] a new class of a.p. functions.

Definition 7.9. ([2], [136]) Set

$$PAP_0(\mathbb{R}) \; = \; \left\{ \phi \in \; C^0(\mathbb{R}) \; s.t. \; \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \; |\phi(x)| \; dx \; = \; 0 \right\} .$$

A function $f \in C^0(\mathbb{R})$ is called a *pseudo almost-periodic function* if it is the sum of a function $g \in C^0_{ap}$ and of a function $\phi \in PAP_0(\mathbb{R})$.

g is called the *almost-periodic component* of f and ϕ is the *ergodic perturbation.*

E. Ait Dads and O. Arino [2] have furtherly generalized these spaces to measurable functions, introducing the spaces $\tilde{P}AP$. As remarked in [2], the mean value, the Bohr-Fourier coefficients and the Bohr-Fourier exponents of every pseudo a.p. function are the same of its a.p. component.

As concerns almost-periodic multifunctions (considered in [5], [6], [8]; cf. also [40], [42], [52]), let us introduce the following metrics:

$$(\textit{Bohr}) \qquad D(\varphi, \psi) := \sup_{t \in \mathbb{R}} \; d_H(\varphi(t), \psi(t)),$$

$$(\textit{Stepanov}) \qquad D_{S^p_L}(\varphi, \psi) := \sup_{x \in \mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} d_H(\varphi(t), \psi(t))^p \, dt \right]^{\frac{1}{p}},$$

$$(\textit{Weyl}) \qquad D_{W^p}(\varphi, \psi) := \lim_{L \to \infty} \sup_{x \in \mathbb{R}} \left[\frac{1}{L} \int_x^{x+L} d_H(\varphi(t), \psi(t))^p \, dt \right]^{\frac{1}{p}} = \lim_{L \to \infty} D_{S^p_L}(\varphi, \psi),$$

$$(\textit{Besicovitch}) \qquad D_{B^p}(\varphi, \psi) := \limsup_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} d_H(\varphi(t), \psi(t))^p \, dt \right]^{\frac{1}{p}},$$

where $\varphi, \psi \; : \; \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ are measurable multifunctions with nonempty bounded, closed values and $d_H(\cdot, \cdot)$ stands for the Hausdorff metric.

Since every multifunction, say $P : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, is well-known (see e.g. [42], [52]) to be measurable if and only if there exists a sequence $\{p_n\}$ of measurable (single-valued) selections of P, i.e. $p_n \subset P \; \forall n \in \mathbb{N}$, such that P can be Castaing-like represented as follows

$$P(t) \; = \; \overline{\bigcup_{n \in \mathbb{N}} p_n(t)} \; ,$$

the standard (single-valued) measure-theoretic arguments make the distance $d_H(\varphi, \psi)$ to become a single-valued measurable function.

Therefore, replacing the metrics in definitions in Table 1 (cf. also Table 2) by the related ones above, we have correct definitions of almost-periodic multifunctions.

Definition 7.10. We say that a measurable multifunction $\varphi : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ with nonempty, bounded, closed values is *G-almost-periodic* if G means any of the respective classes defined in Table 1 (cf. also Table 2) with metrics replaced by the above ones, i.e. those involving the Hausdorff metric.

Remark 7.11. Although S^p_{ap}-multifunctions with nonempty (convex) compact values possess (single-valued) S^p_{ap}-selections (see [42], [52]), the same is not true for C^0_{ap}-multifunctions (see [40]). It is an open problem whether or not W^p_{ap} or B^p_{ap}-multifunctions possess the respective (single-valued) selections. For equi-W^p_{ap}-multifunctions, the problem was affirmatively answered quite recently by L. I. Danilov [43].

Instead of defining further classes of a.p. functions (see the large list of references), let us conclude with posing some further open problems.

Since, unfortunately, $W^p_{ap} \not\Rightarrow B^p$, it is a question under which additional assumptions, say A, we would have $W^p_{ap} \overset{A}{\Rightarrow} B^p$, because then the following linear sequence would take place: u.a.p. $\Leftrightarrow$ u. normal $\Leftrightarrow C^0_{ap} \Rightarrow S^p \Leftrightarrow S^p -$ normal $\Leftrightarrow S^p_{ap} \Leftrightarrow e - W^p \Rightarrow e - W^p -$ normal $\Leftrightarrow e - W^p_{ap} \Leftrightarrow W^p \Rightarrow W^p -$ normal $\Rightarrow W^p_{ap} \overset{A}{\Rightarrow} B^p \Rightarrow B^p -$ normal $\Rightarrow B^p_{ap}$.

For this is, in view of Theorem (5.20) and Remark (5.21), sufficient that the given functions are B^p-bounded, B^p-continuous and one of conditions 4) or 4$'$) or 4$''$) in Theorem (5.20) and Remark (5.21) takes place.

Because of possible applications to differential equations or inclusions, it would be also interesting to know what happens with the hierarchy in Table 2, provided a.p. functions are additionally uniformly continuous.

Acknowledgements

We are indebted to Prof. M. Amar (Rome), Prof. A. Avantaggiati (Rome) and Prof. A. Pankov (Vinnitsa) for fruitful discussions on this topic. We are also indebted to our former student Mgr. J. Stryja (Olomouc) who helped us to construct several counter-examples.

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Lavoro pervenuto alla redazione il 2 febbraio 2005
ed accettato per la pubblicazione il 20 settembre 2005.
Bozze licenziate il 10 maggio 2006

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Supported by the Council of Czech Government (J 14/98: 153100011) and by the Grant No. 201-00-0768 of the Grant Agency of Czech Republic.