Rendiconti di Matematica, Serie VII
Volume 26, Roma (2006), 189-209

Some corrector results for composites with imperfect interface

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Abstract: *In this paper we give some corrector results for a problem modelling the stationary heat diffusion in a conductor with two components, a connected one* Ω_1^ε *and a disconnected one* Ω_2^ε*, consisting of* ε*-periodic connected components of size* ε*. The flow of heat is proportional, by mean of a function of order* ε^γ, $\gamma > -1$*, to the jump of the temperature field, due to a contact resistance on the interface. We prove a corrector result for the temperature in the component* Ω_1^ε*. Moreover, for* $-1 < \gamma \leq 1$ *we prove the strong convergence to zero of the gradient of the temperature in the component* Ω_2^ε*. Due to different a priori estimates, the case* $\gamma > 1$ *needs to be treated separately. These results complete the study of the asymptotic behaviour of this problem done in [10].*

1 – Introduction

In this paper we consider a domain Ω of $\mathbb{R}^n$, such that $\Omega = \Omega_1^\varepsilon \cup \overline{\Omega_2^\varepsilon}$, where Ω_1^ε is a connected domain and Ω_2^ε is a disconnected one, union of ε-periodic sets of size ε.

We prove some corrector results for the problem

$$(1.1) \qquad \begin{cases} -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) = f_1 & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) = f_2 & \text{in } \Omega_2^\varepsilon, \\ [A^\varepsilon \nabla u^\varepsilon] \cdot n = 0 & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n = -\varepsilon^\gamma h^\varepsilon [u^\varepsilon] & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

Key Words and Phrases: *Homogenization – Correctors – Perforated domains – Jump boundary conditions.*
A.M.S. Classification: 35B27 – 35J20 – 35J25

prescribing the continuity of the conormal derivatives on a contact surface $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ and a jump of the solution which is proportional to the conormal derivative by mean of a function of order ε^γ. Here, n denotes the unit outward normal to Ω_1^ε and $u_i^\varepsilon = u^\varepsilon|_{\Omega_i^\varepsilon}$ $i = 1, 2$.

This problem models the stationary diffusion in a two-component heat conductor with a contact resistance (see H. S. Carslaw and J. C. Jaeger [5] for a physical justification of the model). Therefore, its asymptotic behaviour describes the effective thermal conductivity of the homogenized composite and takes into account the influence of the contact barrier. The description of the limit problem has been studied in [10]. The corrector results presented here complete the asymptotic study therein.

Let us recall that in [10] is proved that, if $-1 < \gamma \leq 1$, then a suitable extension $P_1^\varepsilon u_1^\varepsilon$ of u_1^ε weakly converges to the solution u_1 in $H_0^1(\Omega)$ of the limit problem

$$(1.2) \qquad \begin{cases} -\text{div}\,(A^0\nabla u_1) = \theta_1 f_1 + \theta_2 f_2 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

with the convergences

$$(1.3) \qquad \begin{cases} A^\varepsilon\widetilde{\nabla u_1^\varepsilon} \rightharpoonup A^0\nabla u_1 & \text{weakly in } [L^2(\Omega)]^n, \\ A^\varepsilon\widetilde{\nabla u_2^\varepsilon} \rightharpoonup 0 & \text{weakly in } [L^2(\Omega)]^n, \end{cases}$$

where θ_i, for $i = 1, 2$, represents the proportion of the material in Ω_i^ε and $\sim$ denotes the zero extension to the whole of Ω. Moreover, $\widetilde{u_2^\varepsilon}$ weakly converges in $L^2(\Omega)$ to $\theta_2 u_1$ if $-1 < \gamma < 1$ and to $\theta_2(u_1 + c_h^{-1} f_2)$, with $c_h = \frac{1}{|Y_2|}\int_\Gamma h(y)d\sigma_y$, if $\gamma = 1$.

The constant matrix A^0 is the same as that obtained by D. Cioranescu and J. Saint Jean Paulin ([8], see also [9]) for the homogenization of the Laplace problem in the perforated domain Ω_1^ε with a Neumann condition on the boundary of the holes. Hence, the effective conductivity of the first conductor is the same as that obtained when there is no material occupying Ω_2^ε, with in the limit problem $\theta_1 f_1 + \theta_2 f_2$ instead of $\theta_1 f_1$. The flux related to u_2^ε asymptotically vanishes, thus the whole homogenized material behaves as if the composite Ω_2^ε does not contribute in the heat propagation.

The first corrector result of this paper states that, if $-1 < \gamma \leq 1$, the following convergence holds:

$$(1.4) \qquad \lim_{\varepsilon\to 0} ||\nabla u_1^\varepsilon - C^\varepsilon\nabla u_1||_{L^1(\Omega_1^\varepsilon)^n} = 0,$$

where C^ε is the same corrector matrix as that of the Laplace problem in the perforated domain Ω_1^ε with a Neumann condition on the holes. We also prove that

$$(1.5) \qquad \lim_{\varepsilon\to 0} ||\nabla u_2^\varepsilon||_{L^1(\Omega_2^\varepsilon)^n} = 0.$$

This strong convergence implies, in particular, that the weak convergence to zero of $A^\varepsilon \widetilde{\nabla u_2^\varepsilon}$, stated in [10], is actually strong.

The main difficulty for proving these convergences is to describe the asymptotic behaviour of the energy. In general, the convergence of the energy to that of the homogenized problem is straightforward. Here the situation is more complicated, due to the presence of the boundary term in the variational formulation.

For $-1 < \gamma < 1$, we prove that the energy, which includes a boundary term, converge to that of problem (1.2).

For the case $\gamma = 1$, we only can prove that the limit superior of the energy is lower than that of the homogenized problem (1.2). Nevertheless, this result is sufficient to prove convergences (1.4) and (1.5). Its proof is quite technical and mainly makes use of two lemmas. The first one (Lemma 3.3) is a variant of a lemma proved in [6] and transforms integral on the boundary Γ^ε into volume integrals on Ω_2^ε. The second one (Lemma 3.4), proved in [11], provides for a weakly converging sequences vanishing in Ω_2^ε a better inequality than that given by the lower semi-continuity.

In the case $\gamma > 1$ where, as shows a counterexample of [14], one cannot expect bounded a priori estimates for the solution, we replace as in [10] the function f_2 in problem (1.1) by $\varepsilon^{\frac{\gamma-1}{2}} f_2$. We prove that in this case we still have convergence (1.4). The question if (1.5) still holds for $\gamma = 1$ remains open.

The first homogenization results for this kind of boundary conditions was done, for some values of the parameter γ, by J. L. Auriault and H. I. Ene [1] by the multiple scales method. We refer to R. Lipton [15] for the study of the limit problem when $\gamma = 0$, to S. Monsurrò [17] for the case $\gamma \leq -1$ and to [10] for the case $\gamma > -1$. For similar homogenization problems we also refer to J. N. Pernin [18], E. Canon and J. N. Pernin [3], [4], H. Ene [12], H. Ene and D. Polisevski [13], H. K. Hummel [14] and to R. Lipton and B. Vernescu [16] (for other related references see also the bibliography of [10]).

In Section 2 we state the correctors results (Theorems 2.5 and 2.9). They are proved in Section 4. The asymptotic behaviour of the energy according to the different values of γ is studied in Section 3.

2 – Formulation of the problem and main results

In the following, Ω will be an open bounded subset of $\mathbb{R}^n$ and $\{\varepsilon\}$ a positive sequence converging to zero.

We denote by $Y =]0, l_1[\times \ldots]0, l_n[$ the reference cell and by Y_1 and Y_2 two non empty open subsets such that $Y = Y_1 \cup \overline{Y_2}$, with Y_1 connected and $\Gamma \doteq \partial Y_2$ Lipschitz continuous.

For any $k \in Z^n$, we denote

$$Y_i^k := k_l + Y_i, \qquad \Gamma_k := k_l + \Gamma,$$

where $k_l = (k_1 l_1, \dots, k_n l_n)$ and $i = 1, 2$. We assume that

$$\partial\Omega \cap \big(\cup_{k \in Z^n} (\varepsilon \Gamma_k) \big) = \emptyset \tag{2.1}$$

and, for any fixed ε, we set

$$K_\varepsilon := \{ k \in Z^n |\ \varepsilon Y_2^k \subset \Omega \neq \emptyset \}.$$

Then, we define the two components of Ω and the interface (see fig. 1 below) respectively by

$$\Omega_i^\varepsilon := \Omega \cap \{ \cup_{k \in K_\varepsilon} \varepsilon Y_i^k \},\ i = 1, 2, \qquad \Gamma^\varepsilon = \partial \Omega_2^\varepsilon.$$

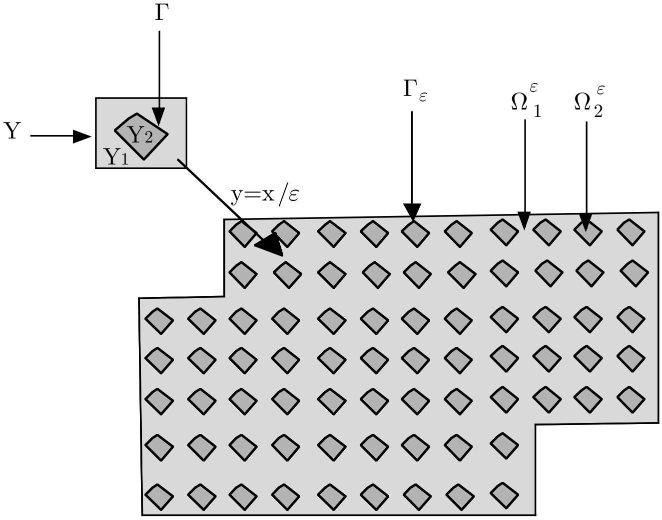

Figure 1.

Observe that (2.1) implies the fact that $\partial\Omega \cap \Gamma^\varepsilon = \emptyset$, so that the component Ω_1^ε is connected and the component Ω_2^ε is union of disjoint translated sets of εY_2, whose number is of order ε^{-n}.

In what follows, we denote by

- $\sim$ the zero extension to the whole of Ω of functions defined on Ω_1^ε or Ω_2^ε,
- χ_ω the characteristic function of any open set $\omega \subset \mathbb{R}^n$,
- $m_\omega(v) = \frac{1}{|\omega|} \int_\omega f\, dx$ the average on Y of any function $v \in L^1(\omega)$.

We recall that

$$\chi_{\Omega_i^\varepsilon} \rightharpoonup \theta_i := \frac{|Y_i|}{|Y|}, \quad \text{weakly in } L^2(\Omega). \tag{2.2}$$

For α, β such that $0 < \alpha < \beta$, let A be a Y-periodic matrix field satisfying

$$\begin{cases} (A(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ |A(x)\lambda| \leq \beta\lambda, \end{cases} \tag{2.3}$$

for any $l \in \mathbb{R}^n$ and a.e. in Y and set, for any $\varepsilon > 0$,

$$A^\varepsilon(x) = A(x/\varepsilon). \tag{2.4}$$

Let h be an Y-periodic function such that

$$h \in L^\infty(\Gamma), \text{ and } 0 < h_0 < h(y), \ \ y \text{ a.e. in } \Gamma, \tag{2.5}$$

for some $h_0 \in \mathbb{R}_+^*$ and set

$$h^\varepsilon(x) = h\Big(\frac{x}{\varepsilon}\Big). \tag{2.6}$$

We introduce the space V^ε defined by

$$V^\varepsilon := \{u_1^\varepsilon \in H^1(\Omega_1^\varepsilon) |\ u_1^\varepsilon = 0 \text{ on } \partial\Omega\},$$

equipped with the norm $\|u\|_{V^\varepsilon} := \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}$ and the space H_0^ε defined by

$$H_0^\varepsilon := \{u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \mid u_1^\varepsilon \in V^\varepsilon \quad \text{and} \quad u_2^\varepsilon \in H^1(\Omega_2^\varepsilon)\},$$

equipped with the norm

$$\|u^\varepsilon\|^2_{H_0^\varepsilon} := \|\nabla u_1^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon\|u_1^\varepsilon - u_2^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)}.$$

Let us recall the following extension results in V^ε, due to D. Cioranescu and J. Saint Jean Paulin:

Lemma 2.1 ([8]). i) *There exists a linear continuous extension operator P_1 belonging to $\mathcal{L}(H^1(Y_1); H^1(Y)) \cap \mathcal{L}(L^2(Y_1); L^2(Y))$ such that, for some positive constant C*

$$\begin{cases} \|P_1 v_1\|_{L^2(Y)} \leq C\|v_1\|_{L^2(Y_1)}, \\ \|\nabla P_1 v_1\|_{L^2(Y)} \leq C\|\nabla v_1\|_{L^2(Y_1)}, \end{cases}$$

for every $v_1 \in H^1(Y_1)$.

ii) *There exists an extension operator P_1^ε belonging to $\mathcal{L}(L^2(\Omega_1^\varepsilon); L^2(\Omega)) \cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ such that, for some positive constant C (independent of ε)*

$$\begin{cases} \|P_1^\varepsilon v_1\|_{L^2(\Omega)} \leq C\|v_1\|_{L^2(\Omega_1^\varepsilon)}, \\ \|\nabla P_1^\varepsilon v_1\|_{L^2(\Omega)} \leq C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}, \end{cases}$$

for every $v_1 \in V^\varepsilon$.

Observe that this lemma provides a Poincaré inequality in V^ε independent of ε, i.e. there exists a positive constant $C > 0$ (independent of ε) satisfying

$$\|v_1\|_{L^2(\Omega_1^\varepsilon)} \leq C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}, \qquad \forall\, v_1 \in V^\varepsilon.$$

Let $f_1^\varepsilon \in L^2(\Omega_1^\varepsilon)$, $f_2^\varepsilon \in L^2(\Omega_2^\varepsilon)$ and g be given in $H^{-1}(\Omega)$. Our aim is to study the correctors for the following problem

$$\text{(2.7)} \qquad \begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_1^\varepsilon) = f_1^\varepsilon + P_1^{\varepsilon\,*}(g) & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}(A^\varepsilon \nabla u_2^\varepsilon) = f_2^\varepsilon & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n_1^\varepsilon = -A^\varepsilon \nabla u_2^\varepsilon \cdot n_2^\varepsilon & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n_1^\varepsilon = -\varepsilon^\gamma h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where n_i^ε is the unitary outward normal to Ω_i^ε, $i = 1, 2$ and $P_1^{\varepsilon\,*}$ is the adjoint operator of the extension operator P_1^ε given by Lemma 2.1. By definition, $P_1^{\varepsilon\,*}$ is in $\mathcal{L}(H^{-1}(\Omega)\,; V_\varepsilon')$ and for $g \in H^{-1}(\Omega)$, $P_1^{\varepsilon\,*}(g)$ is given by

$$P_1^{\varepsilon\,*} g \,:\, v \in V_\varepsilon \longrightarrow < g, P_1^\varepsilon v >_{H^{-1}(\Omega), H_0^1(\Omega)} .$$

We will suppose that

$$\text{(2.8)} \qquad \begin{cases} \text{i)}\ \ \widetilde{f_1^\varepsilon} \rightharpoonup \theta_1 f_1 & \text{weakly in } L^2(\Omega), \\ \text{ii)}\ \ \widetilde{f_2^\varepsilon} \rightharpoonup \theta_2 f_2 & \text{weakly in } L^2(\Omega). \end{cases}$$

Then, the variational formulation of problem (2.7) is:

$$\text{(2.9)} \qquad \begin{cases} \text{Find } u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \text{ in } H_0^\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla v_1\, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla v_2\, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)(v_1 - v_2) d\sigma \\ \displaystyle = \int_{\Omega_1^\varepsilon} f_1^\varepsilon\, v_1\, dx + < g, P_1^\varepsilon v_1 >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega_2^\varepsilon} f_2^\varepsilon\, v_2\, dx,\ \forall\, (v_1, v_2) \in H_0^\varepsilon. \end{cases}$$

The existence and uniqueness of the solution u^ε of (2.9) for every $\varepsilon > 0$ is a consequence of the Lax–Milgram theorem and of the following proposition (see [17]):

PROPOSITION 2.2 ([17]). *The norm of* H_0^ε *is equivalent to the norm of* $V^\varepsilon \times H^1(\Omega_2^\varepsilon)$. *Moreover, there exist two positive constant* C_1, C_2, *independent of* ε, *such that*

$$C_1 \|v\|_{H_0^\varepsilon} \leq \|v\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)} \leq C_2 \|v\|_{H_0^\varepsilon}, \quad \forall\, v \in H_0^\varepsilon.$$

Let us recall the following homogenization result given in [10], concerning the case $-1 < \gamma \leq 1$:

Theorem 2.3 ([10]). *Let* A^ε *and* h^ε *be defined by* (2.3)-(2.6)*. Suppose that* f_1^ε *and* f_2^ε *satisfy* (2.8) *and let* g *be given in* $L^2(\Omega)$*. Let* $-1 < \gamma \leq 1$ *and* u^ε *be the solution of problem* (2.7)*. Then, there exists a positive constant* C *is independent of* ε *and an extension operator* $P_1^\varepsilon \in \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ *such that*

$$\text{(2.10)} \qquad \begin{cases} \text{i)}\ \ P_1^\varepsilon u_1^\varepsilon \rightharpoonup u_1 & \textit{weakly in } H_0^1(\Omega), \\ \text{ii)}\ \ A^\varepsilon \widetilde{\nabla u_1^\varepsilon} \rightharpoonup A^0 \nabla u_1 & \textit{weakly in } [L^2(\Omega)]^n, \\ \text{iii)}\ \ \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} < C\varepsilon^{-\gamma/2}, & \end{cases}$$

and the following convergences hold:

$$\text{(2.11)} \qquad \begin{cases} \text{i)}\ \ \widetilde{u_2^\varepsilon} \rightharpoonup u_2 & \textit{weakly in } L^2(\Omega), \\ \text{ii)}\ \ A^\varepsilon \widetilde{\nabla u_2^\varepsilon} \rightharpoonup 0 & \textit{weakly in } [L^2(\Omega)]^n. \end{cases}$$

The function u_1 *is the unique solution in* $H_0^1(\Omega)$ *of the problem*

$$\text{(2.12)} \qquad \begin{cases} -\text{div}\,(A^0 \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g & \textit{in } \Omega, \\ u_1 = 0 & \textit{on } \partial\Omega, \end{cases}$$

with θ_i*,* $i = 1, 2$*, given by* (2.2)*. The homogenized matrix* A^0 *is defined by*

$$\text{(2.13)} \qquad A^0 l = \frac{1}{|Y|} \int_{Y_1} A \nabla \widehat{w}_\lambda \, dy,$$

where $\widehat{w}_\lambda \in H^1(Y_1)$ *is the solution, for any* $l \in \mathbb{R}^n$*, of*

$$\text{(2.14)} \qquad \begin{cases} -\text{div}\,(A \nabla \widehat{w}_\lambda) = 0 & \textit{in } Y_1, \\ (A \nabla \widehat{w}_\lambda) \cdot n_1 = 0 & \textit{on } \Gamma, \\ \widehat{w}_\lambda - \lambda \cdot y & Y\textit{-periodic}, \\ m_{Y_1}(\widehat{w}_\lambda - \lambda \cdot y) = 0. & \end{cases}$$

Moreover, for $-1 < \gamma < 1$*, one has*

$$\text{(2.15)} \qquad \begin{cases} \text{i)}\ \ u_2 = \theta_2 u_1, \\ \text{ii)}\ \ \|P_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} \to 0, \end{cases}$$

while, for $\gamma = 1$*,*

$$\text{(2.16)} \qquad u_2 = \theta_2 (u_1 + c_h^{-1} f_2),$$

where $c_h = \frac{1}{|Y_2|} \int_\Gamma h(y) d\sigma_y$*.*

REMARK 2.4. In [10] this result has been proved in the case where $g = 0$ and $f_1^\varepsilon = f_{|\Omega_1^\varepsilon}$, $f_2^\varepsilon = f_{|\Omega_2^\varepsilon}$ for some $f \in L^2(\Omega)$, so that $\theta_1 f_1 + \theta_2 f_2 = f$. Nevertheless, it easily seen that the results is still valid if the data are chosen as in Theorem 2.3. Indeed, the proof in this case follows exactly the same outlines of that given in [10]. One only needs to use, in the terms where g appears, the fact that for any sequence $\{v^\varepsilon\}$ in $H_0^1(\Omega)$ (see Lemma 2.1 of [2]) the following implication holds:

$$\left(v^\varepsilon \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)\right) \Longrightarrow \left(P_\varepsilon^1(v^\varepsilon|_{\Omega_\varepsilon}) \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)\right). \tag{2.17}$$

The main result of this paper is the following corrector result, which completes the convergence results given by Theorem 2.3.

THEOREM 2.5 (correctors for the case $-1 < \gamma \leq 1$). *Let $(e_i)_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$ and $\widehat{w}_i \in H^1(Y_1)$ the solution of problem (2.14) for $\lambda = e_i$, $i = 1, \ldots, n$. Define the corrector matrix $C^\varepsilon = (C_{ij}^\varepsilon)_{1 \leq i,j \leq n}$ by*

$$\begin{cases} C_{ij}^\varepsilon(x) = \widetilde{C_{ij}}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \Omega, \\[2ex] C_{ij}(y) = \dfrac{\partial \widehat{w}_j}{\partial y_i}(y) & \text{a.e. on } Y_1, \end{cases} \tag{2.18}$$

where here $\sim$ denotes the zero extension to the whole of Y.

Let us suppose that the assumptions of Theorem 2.3 are satisfied. If $\gamma = 1$, we also suppose ∂Y_2 of class C^2 and $f_2^\varepsilon = f_{2|\Omega_2^\varepsilon}$, with f_2 given in $L^2(\Omega)$.

Then, one has the following convergences:

$$\begin{cases} \text{i) } \lim\limits_{\varepsilon \to 0} ||\nabla u_1^\varepsilon - C^\varepsilon \nabla u_1||_{L^1(\Omega_1^\varepsilon)^n} = 0, \\[2ex] \text{ii) } \lim\limits_{\varepsilon \to 0} ||\nabla u_2^\varepsilon||_{L^2(\Omega_2^\varepsilon)^n} = 0, \\[2ex] \text{iii) } A^\varepsilon \widetilde{\nabla u_2^\varepsilon} \to 0 \quad \text{strongly in } [L^2(\Omega)]^n. \end{cases} \tag{2.19}$$

Moreover, if $C \in (L^r(Y_1))^{n \times n}$ for some r such that $2 \leq r \leq \infty$ and $\nabla u_1 \in (L^s(\Omega))^n$ for some s such that $2 \leq s < \infty$, then

$$\lim_{\varepsilon \to 0} ||\nabla u_1^\varepsilon - C^\varepsilon \nabla u_1||_{L^t(\Omega_1^\varepsilon)^n} = 0,$$

where $t = \min\left\{2, \frac{r\,s}{r+s}\right\}$.

This result will be proved in Section 4. Its proof makes use of the asymptotic behaviour of the energy associated to problem (2.7), which here is not immediate, since one has to take into account the boundary term in the energy. The three cases $-1 < \gamma < 1$, $\gamma = 1$ and $\gamma > 1$ need to be treated separately, the more delicate one being the case $\gamma = 1$. They are studied in Section 3.

Remark 2.6. The corrector result for the component u_1^ε is the same as that obtained by D. Cioranescu and J. Saint Jean Paulin ([8], see also [9]) for the homogenization of the Laplace problem in the perforated domain Ω_1^ε, with a Neumann condition on the boundary of the holes. Convergence ii) shows that $\widetilde{\nabla u_2^\varepsilon}$ is strongly converging to zero in $L^2(\Omega)$. This easily implies that actually convergence (2.11)ii) is also strong, that is (2.19)iii) holds.

Let us recall that in the case $\gamma > 1$ (see [14]) one cannot expect boundedness of the solutions. To overcome this difficulty and in order to have a non-trivial limit behaviour, one can consider the following problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_1^\varepsilon) = f_1^\varepsilon + P_1^{\varepsilon\,*}(g) & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}(A^\varepsilon \nabla u_2^\varepsilon) = \varepsilon^{\frac{\gamma-1}{2}} f_2^\varepsilon & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n_1^\varepsilon = -A^\varepsilon \nabla u_2^\varepsilon \cdot n_2^\varepsilon & \text{on } \Gamma^\varepsilon, \\ -A^\varepsilon \nabla u_1^\varepsilon \cdot n_1^\varepsilon = \varepsilon^\gamma h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

where, as before, $f_1^\varepsilon \in L^2(\Omega_1^\varepsilon)$ and $f_2^\varepsilon \in L^2(\Omega_2^\varepsilon)$, g is given in $H^{-1}(\Omega)$ and $P_1^{\varepsilon\,*}$ is the adjoint operator of the extension operator P_1^ε given by Lemma 2.1. The variational formulation of (2.20) is then

$$\begin{cases} \text{Find } u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \text{ in } H_0^\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla v_1 \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla v_2 \, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)(v_1 - v_2)\, d\sigma \\ \displaystyle = \int_{\Omega_1^\varepsilon} f_1^\varepsilon \, v_1 dx + < g, P_1^\varepsilon v_1 >_{H^{-1}(\Omega), H_0^1(\Omega)} + \varepsilon^{\frac{\gamma-1}{2}} \int_{\Omega_2^\varepsilon} f_2^\varepsilon \, v_2 \, dx, \\ \forall\, (v_1, v_2) \in H_0^\varepsilon. \end{cases} \tag{2.21}$$

The asymptotic behaviour of this system in given by the following result, proved in [10]:

Theorem 2.7 ([10]). *Let* A^ε *and* h^ε *be defined by* (2.3)-(2.6). *Suppose that* f_1^ε *and* f_2^ε *satisfy* (2.8) *and let* g *be given in* $L^2(\Omega)$. *Let* $\gamma > 1$ *and* u^ε *be the solution of problem* (2.20). *Then, there exists an extension operator* $P_1^\varepsilon \in \mathcal{L}(V^\varepsilon, H_0^1(\Omega))$ *such that*

$$\begin{cases} \text{i) } P_1^\varepsilon u_1^\varepsilon \rightharpoonup u_1 & \textit{weakly in } H_0^1(\Omega), \\ \text{ii) } A^\varepsilon \widetilde{\nabla u_1^\varepsilon} \rightharpoonup A^0 \nabla u_1 & \textit{weakly in } [L^2(\Omega)]^n, \\ \text{iii) } \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} < C\varepsilon^{-\gamma/2}, \end{cases} \tag{2.22}$$

where C is independent of ε and u_1 is the unique solution of the problem

$$\begin{cases} -\mathrm{div}\,(A^0\nabla u_1) = \theta_1 f_1 + g & \textit{in } \Omega, \\ u_1 = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.23}$$

with A^0 given by (2.13) *and* (2.14).

Moreover,

$$A^\varepsilon \widetilde{\nabla u_2^\varepsilon} \rightharpoonup 0 \quad \textit{weakly in } [L^2(\Omega)]^n. \tag{2.24}$$

Remark 2.8. In [10] this result has been proved in the case where $g = 0$ and $f_1^\varepsilon = f_{|\Omega_1^\varepsilon}$, $f_2^\varepsilon = f_{|\Omega_2^\varepsilon}$ for some $f \in L^2(\Omega)$. Nevertheless, the results is still valid under the above assumptions on the data (see also Remark 2.4 above).

Theorem 2.9 (Corrector for the case $\gamma > 1$). *Let $C^\varepsilon = (C_{ij}^\varepsilon)_{1\le i,j\le n}$ be the corrector matrix defined by* (2.14) *and* (2.18).

Under the assumptions of Theorem 2.7, *one has the following convergence*:

$$\lim_{\varepsilon\to 0} ||\nabla u_1^\varepsilon - C^\varepsilon \nabla u_1||_{L^1(\Omega_1^\varepsilon)^n} = 0.$$

Moreover, if $C \in (L^r(Y_1))^{n\times n}$ for some r such that $2 \le r \le \infty$ and $\nabla u_1 \in (L^s(\Omega))^n$ for some s such that $2 \le s < \infty$, then

$$\lim_{\varepsilon\to 0} ||\nabla u_1^\varepsilon - C^\varepsilon \nabla u_1||_{L^t(\Omega_1^\varepsilon)^n} = 0,$$

where $t = \min\left\{2, \frac{r\,s}{r+s}\right\}$.

This result will be proved in Section 4. Its proof makes use of the asymptotic behaviour of the energy of the first component u_1^ε of the solution of problem (2.20), which will be studied in Section 3.

Remark 2.10. The possible strong convergence to zero in $L^2(\Omega)$ is here an open question. This is related to the fact that we do not know the weak limit behaviour in $L^2(\Omega)$ of the (bounded) sequence $\varepsilon^{\frac{\gamma-1}{2}}\,\widetilde{u_2^\varepsilon}$ (see also Remark 3.6 below).

3 – The asymptotic behaviour of the energies

In this section we study the limit behaviour of the energies associated to problems (2.7) and (2.20). The three cases $-1 < \gamma < 1$, $\gamma = 1$ and $\gamma > 1$ need to be treated separately.

Proposition 3.1 (case $-1 < \gamma < 1$). *Let* $-1 < \gamma < 1$. *Under the assumptions of Theorem* 2.5 *one has the following convergence of the energy*:

$$\text{(3.1)} \quad \begin{cases} \lim\limits_{\varepsilon \to 0} \Big(\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) \\ = \displaystyle\int_\Omega A^0 \nabla u_1 \nabla u_1 \, dx, \end{cases}$$

where A^0 *and* u_1 *are given by Theorem* 2.3.

Proof. Let us choose $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ in the variational formulation (2.9). One has

$$\text{(3.2)} \quad \begin{cases} \displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \\ = \displaystyle\int_{\Omega_1^\varepsilon} f_1^\varepsilon \, u_1^\varepsilon \, dx + < g, P_1^\varepsilon u_1^\varepsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega_2^\varepsilon} f_2^\varepsilon \, u_2^\varepsilon \, dx. \end{cases}$$

Observe now that from (2.8), (2.10) and (2.15) one has

$$\lim_{\varepsilon \to 0} \int_{\Omega_2^\varepsilon} f_2^\varepsilon \, u_2^\varepsilon \, dx = \int_\Omega \widetilde{f_2^\varepsilon} P_1^\varepsilon u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} f_2^\varepsilon (P_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon) \, dx = \int_\Omega \theta_2 \, f_2 \, u_1 \, dx.$$

Hence, again from (2.8) and (2.10) and in view of the limit equation (2.12) we obtain

$$\begin{cases} \lim\limits_{\varepsilon \to 0} \Big(\displaystyle\int_{\Omega_1^\varepsilon} f_1 \, u_1^\varepsilon \, dx + < g, P_1^\varepsilon u_1^\varepsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega_2^\varepsilon} f_2 \, u_2^\varepsilon \, dx \Big) \\ = \lim\limits_{\varepsilon \to 0} \Big(\displaystyle\int_\Omega \widetilde{f_1^\varepsilon} \, P_1^\varepsilon u_1^\varepsilon \, dx + < g, P_1^\varepsilon u_1^\varepsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} \Big) + \int_\Omega \theta_2 f_2 u_1 \, dx \\ = \displaystyle\int_\Omega \theta_1 \, f_1 u_1 + \theta_2 f_2 u_1 \, dx + < g, u_1 >_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega A^0 \nabla u_1 \nabla u_1 \, dx. \end{cases}$$

This, together with (3.2), gives convergence (3.1). □

The case $\gamma = 1$ is more delicate and we can only prove the following inequality:

PROPOSITION 3.2 (case $\gamma = 1$). *Let $\gamma = 1$. Under the assumptions of Theorem 2.5 one has the following asymptotic behaviour of the energy*:

$$\begin{cases} \limsup\limits_{\varepsilon\to 0}\Big(\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx \le \\ \le \int_\Omega A^0 \nabla u_1 \nabla u_1 \, dx, \end{cases} \tag{3.3}$$

where A^0 and u_1 are given by Theorem 2.2.

To prove Proposition 3.2 we need to use two technical lemmas. The first one is an adaptation to the case of a disconnected set of Lemma 3.1 of [6] (see also [10], Lemma 3.1 for $p = 2$).

LEMMA 3.3. *Suppose that Γ is of class C^2. Let $g \in L^\infty(\Gamma)$ and set $c_g := \frac{1}{|Y_2|}\int_\Gamma g(y) d\sigma_y$. Let v_ε, for every ε, be a function in $W^{1,1}(\Omega_2^\varepsilon)$ such that for some positive constant c one has*

$$||v_\varepsilon||_{W^{1,1}(\Omega_2^\varepsilon)} \le c. \tag{3.4}$$

Then,

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{\Gamma^\varepsilon} g(x/\varepsilon) v_\varepsilon(x)\, d\sigma = \liminf_{\varepsilon\to 0} c_g \int_{\Omega_2^\varepsilon} v_\varepsilon(x)\, dx.$$

PROOF. We adapt to our case the proof of Lemma 3.1 of [6]. Let $\psi_g \in W^{1,\infty}(Y_2)$ be the unique solution of the problem

$$\begin{cases} -\triangle \psi_g = -c_g & \text{in } Y_2, \\ \nabla \psi_g \cdot n_2 = g & \text{on } \Gamma, \\ m_Y(\psi_g) = 0, \end{cases}$$

where n_2 denotes the unit outward normal to Y_2. Observe that the regularity of Γ implies that ψ_g exists and is in $W^{1,\infty}(Y_2)$. Then, still denoting ψ_g the extension by periodicity of ψ_g to $\bigcup_{k\,\in\, Z^n} Y_2^k$, by a change of scale one has

$$\varepsilon \int_{\Gamma^\varepsilon} g(x/\varepsilon) v(x) d\sigma = \varepsilon \int_{\Omega_2^\varepsilon} \nabla_y \psi_g(x/\varepsilon) \nabla_x v(x)\, dx + c_g \int_{\Omega_2^\varepsilon} v(x) dx, \tag{3.5}$$

for every $v \in W^{1,1}(\Omega_2^\varepsilon)$. On the other hands, from (3.4) one has

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Omega_2^\varepsilon} \nabla_y \psi_g(x/\varepsilon) \nabla_x v_\varepsilon(x)\, dx = 0.$$

Then, choosing $v = v_\varepsilon$ in (3.5) and passing to the limit inferior as $\varepsilon \to 0$, one has the result. □

The second lemma has been proved in [11].

Lemma 3.4 ([11]). *Let $\mathcal{O}$ be an open set of $\mathbb{R}^n$ and $\{\mathcal{O}_\varepsilon\}_\varepsilon \subset \mathcal{O}$ a sequence of open subsets of $\mathcal{O}$. Suppose that $\{v_\varepsilon\}_\varepsilon \subset L^p(\mathcal{O}_\varepsilon)$, $p > 1$, is such that, as $\varepsilon \to 0$,*

$$\begin{cases} \chi_{\mathcal{O}_\varepsilon} \rightharpoonup \chi_0, & \text{in } L^\infty(\mathcal{O}) \text{ weakly } *, \\ \widetilde{v_\varepsilon} \rightharpoonup \chi_0 v & \text{weakly in } L^p(\mathcal{O}). \end{cases}$$

Then

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}_\varepsilon} |v_\varepsilon|^p \, dx \geq \int_{\mathcal{O}} \chi_0 |v|^p \, dx.$$

Proof of Proposition 3.2. Let us choose $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ in the variational formulation (2.9). Using (2.8)i) and the fact that $f_2^\varepsilon = f_{2|\Omega_2^\varepsilon}$, together with (2.16), (2.10), (2.11) and the limit equation (2.12), one has

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \Big(\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx \Big) \\
&= \limsup_{\varepsilon\to 0} \Big(-\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) \\
&\quad + \lim_{\varepsilon\to 0} \Big(\int_{\Omega_1^\varepsilon} f_1^\varepsilon \, u_1^\varepsilon \, dx + < g, P_1^\varepsilon u_1^\varepsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega_2^\varepsilon} f_2 \, u_2^\varepsilon \, dx \Big) \\
&= \limsup_{\varepsilon\to 0} \Big(-\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) \\
&\quad + \lim_{\varepsilon\to 0} \Big(\int_{\Omega} \widetilde{f_1^\varepsilon} \, P_1^\varepsilon u_1^\varepsilon \, dx + < g, P_1^\varepsilon u_1^\varepsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} f_2 \, \widetilde{u_2^\varepsilon} \, dx \Big) \\
&= -\liminf_{\varepsilon\to 0} \Big(\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) + \int_{\Omega} \theta_1 f_1 u_1 \, dx \\
&\quad + < g, u_1 >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} f_2 \theta_2 (u_1 + c_h^{-1} f_2) \, dx \\
&= -\liminf_{\varepsilon\to 0} \Big(\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) + \int_{\Omega} A^0 \nabla u_1 \nabla u_1 \, dx \\
&\quad + \int_{\Omega} \theta_2 c_h^{-1} f_2^2 \, dx.
\end{aligned}$$

Hence, to prove (3.3), it will be sufficient to show that

$$\liminf_{\varepsilon\to 0} \Big(\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) \geq \int_{\Omega} \theta_2 c_h^{-1} f_2^2 \, dx. \tag{3.6}$$

To do that, we make use of Lemmas 3.3 and 3.4. First, we apply Lemma 3.3 with $p = 1$, $g = h$ and $v_\varepsilon = (P_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon)^2$. We obtain

$$
(3.7)\qquad \begin{cases} \liminf\limits_{\varepsilon\to 0}\Big(\varepsilon \displaystyle\int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma\Big) = \liminf\limits_{\varepsilon\to 0}\Big(\varepsilon \displaystyle\int_{\Gamma^\varepsilon} h^\varepsilon (P_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma\Big) \\ = \liminf\limits_{\varepsilon\to 0} c_h \displaystyle\int_{\Omega_2^\varepsilon} (P_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon)^2 dx. \end{cases}
$$

Observe now that, thanks to (2.10), (2.11) and (2.16), we have

$$
\widetilde{(P_1^\varepsilon u_1^\varepsilon|_{\Omega_2^\varepsilon})} - \widetilde{u_2^\varepsilon} = \chi_{\Omega_2^\varepsilon} P_1^\varepsilon u_1^\varepsilon - \widetilde{u_2^\varepsilon} \rightharpoonup -\theta_2 c_h^{-1} f_2, \qquad \text{weakly in } L^2(\Omega).
$$

Consequently, we can apply Lemma 3.4 with $p = 2$, $\mathcal{O}_\varepsilon = \Omega_2^\varepsilon$, $\chi_0 = \theta_2$,$v_\varepsilon = P_1^\varepsilon u_1^\varepsilon{}_{|_{\Omega_2^\varepsilon}} - u_2^\varepsilon$ and $v = -c_h^{-1} f_2$. We have

$$
\liminf_{\varepsilon\to 0} c_h \int_{\Omega_2^\varepsilon} (P_1^\varepsilon u_1^\varepsilon|_{\Omega_2^\varepsilon} - u_2^\varepsilon)^2 dx \geq c_h \int_\Omega \theta_2 (-c_h^{-1} f_2)^2\, dx = \int_\Omega \theta_2 c_h^{-1} f_2^2\, dx.
$$

This, together with (3.7), gives (3.6) and concludes the proof. □

Proposition 3.5 (case $\gamma > 1$). *Let* $\gamma > 1$. *Under the assumptions of Theorem* 2.9 *one has the following convergence*:

$$
(3.8)\qquad \begin{cases} \lim\limits_{\varepsilon\to 0}\Big(\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon\, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) u_2^\varepsilon\, d\sigma\Big) \\ = \displaystyle\int_\Omega A^0 \nabla u_1 \nabla u_1\, dx, \end{cases}
$$

where A^0 *and* u_1 *are given by Theorem* 2.7.

Proof. Choosing $u^\varepsilon = (u_1^\varepsilon, 0)$ in the variational formulation (2.21) gives

$$
\begin{aligned} &\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon\, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) u_2^\varepsilon\, d\sigma \\ &\quad = \int_{\Omega_1^\varepsilon} f_1^\varepsilon\, u_1^\varepsilon\, dx + < g, P_1^\varepsilon u_1^\varepsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} . \end{aligned}
$$

Then, using (2.8) and (2.22), one gets

$$
\begin{cases} \lim\limits_{\varepsilon\to 0}\Big(\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon\, dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) u_2^\varepsilon\, d\sigma\Big) \\ = \displaystyle\int_{\Omega_1^\varepsilon} \theta_1 f_1\, u_1\, dx + < g, u_1 >_{H^{-1}(\Omega), H_0^1(\Omega)} . \end{cases}
$$

This, using u_1 as test function in the limit problem (2.23), gives the result. □

Remark 3.6. Here we can only study the behaviour of the first component of the energy. Indeed, since we do not know the weak limit behaviour in $L^2(\Omega)$ of the (bounded) sequence $\{\varepsilon^{\frac{\gamma-1}{2}}\widetilde{u_2^\varepsilon}\}$, we cannot compute the limit of $\int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx$.

4 – Proof of the corrector results

The proof of Theorem 2.5 is based on the convergence of the energies given in Section 3 and on the following main result:

Proposition 4.1. *Under the assumptions of Theorem* 2.5, *there exists a positive constant c independent of* ε *such that for any* $\Phi \in (\mathcal{D}(\Omega))^n$, *one has*

$$\limsup_{\varepsilon\to 0} \left(\|\nabla u_1^\varepsilon - C^\varepsilon \Phi\|_{L^2(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \right) \le c \|\nabla u_1 - \Phi\|_{L^2(\Omega)}.$$

Proof. Let $\Phi = (\Phi_1, \dots \Phi_n) \in (\mathcal{D}(\Omega))^n$. From (2.3) and (2.4) one gets

$$\begin{aligned} &\alpha \|\nabla u_1^\varepsilon - C^\varepsilon \Phi\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} \\ &\le \int_{\Omega_1^\varepsilon} A^\varepsilon \left(\nabla u_1^\varepsilon - C^\varepsilon \Phi\right)\left(\nabla u_1^\varepsilon - C^\varepsilon \Phi\right) dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx \\ &= \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx - \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \left(C^\varepsilon \Phi\right) dx - \int_{\Omega_1^\varepsilon} A^\varepsilon \left(C^\varepsilon \Phi\right) \nabla u_1^\varepsilon \, dx \\ &\quad + \int_{\Omega_1^\varepsilon} A^\varepsilon \left(C^\varepsilon \Phi\right)\left(C^\varepsilon \Phi\right) dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon. \end{aligned} \tag{4.1}$$

We want to pass to the limit for $\varepsilon \to 0$ in each term. Concerning the energy terms, we use Proposition 3.1 for $-1 < \gamma < 1$ and Proposition 3.2 for $\gamma = 1$. In the first case this gives

$$\begin{cases} \limsup\limits_{\varepsilon\to 0} \Big(\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx \Big) \le \\ \le \lim\limits_{\varepsilon\to 0} \Big(\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \, dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \, dx + \\ + \varepsilon^\gamma \displaystyle\int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)^2 d\sigma \Big) = \int_\Omega A^0 \nabla u_1 \nabla u_1 \, dx, \end{cases} \tag{4.2}$$

the second limit superior being a limit since $\gamma < 1$.

Observe now that, from the definition (2.18) of C^ε, one can write

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \ (C^\varepsilon \Phi)\ dx = \sum_{i=1}^n \lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \ (\Phi_i \nabla \widehat{w}_i^\varepsilon)\ dx \\
&= \sum_{i=1}^n \Big(\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \ \nabla (\Phi_i\, \widehat{w}_i^\varepsilon)\ dx - \lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \ \nabla \Phi_i\, \widehat{w}_i^\varepsilon\ dx \Big).
\end{aligned} \tag{4.3}
$$

Set, for $i = 1, \dots, n$,

$$
\widehat{\chi}_i(y) = -\widehat{w}_i(y) + y_i, \qquad \text{in } Y_1 \tag{4.4}
$$

and

$$
\widehat{w}_i^\varepsilon(x) = x_i - \varepsilon(P_1(\widehat{\chi}_i^\varepsilon)(x/\varepsilon)), \qquad \text{in } \Omega, \tag{4.5}
$$

where the extension operator P_1 is defined in Lemma 2.1. By a change of scales one has

$$
\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla \widehat{w}_i^\varepsilon \nabla v\ dx = 0, \quad \forall v \in H_0^1(\Omega) \tag{4.6}
$$

and, by standard arguments

$$
\begin{cases}
\widehat{w}_i^\varepsilon \to x_i & \text{weakly in } H^1(\Omega), \\
\widehat{w}_i^\varepsilon \to x_i & \text{strongly in } L^2(\Omega), \\
\widehat{\eta}_i^\varepsilon \doteq \chi_{\Omega_i^\varepsilon} A^\varepsilon \nabla \widehat{w}_i^\varepsilon \rightharpoonup A^0 e_i & \text{weakly in } \ [L^2(\Omega)]^n.
\end{cases} \tag{4.7}
$$

Then, choosing $v_1 = \Phi_i\, \widehat{w}_i^\varepsilon$ and $v_2 = \Phi_i\, x_i$ as test function in (2.9), one has

$$
\begin{cases}
\displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \ \nabla (\Phi_i\, \widehat{w}_i^\varepsilon)\ dx = -\int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \ \nabla (\Phi_i\, x_i)\ dx \\
\displaystyle -\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) \Phi_i (\widehat{w}_i^\varepsilon - x_i) d\sigma + \int_{\Omega_1^\varepsilon} f_1^\varepsilon\, \Phi_i\, \widehat{w}_i^\varepsilon\ dx \\
\displaystyle + < g, P_1^\varepsilon \big(\Phi_i\, \widehat{w}_i^\varepsilon|_{\Omega_1^\varepsilon}\big) >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega_2^\varepsilon} f_2^\varepsilon\, \Phi_i\, x_i\ dx.
\end{cases} \tag{4.8}
$$

Now, observe that from (2.10)iii), (4.5) and a change of scales it results

$$
\begin{aligned}
\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon) \Phi_i (\widehat{w}_i^\varepsilon - x_i) d\sigma &\le \varepsilon^\gamma \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \|\varepsilon \widehat{\chi}_i^\varepsilon (x/\varepsilon)\|_{L^2(\Gamma^\varepsilon)} \\
&\le c\,\varepsilon^{\gamma+1} \varepsilon^{-\gamma/2} \varepsilon^{-1/2} = c\,\varepsilon^{(\gamma+1)/2},
\end{aligned}
$$

where c is independent of ε. Consequently

$$\lim_{\varepsilon\to 0}\varepsilon^{\gamma}\int_{\Gamma^{\varepsilon}} h^{\varepsilon}(u_1^{\varepsilon}-u_2^{\varepsilon})\Phi_i(\widehat{w}_i^{\varepsilon}-x_i)d\sigma=0,$$

since $\gamma>-1$. Then, passing to the limit in (4.8) as $\varepsilon\to 0$ and using (2.11)ii), (2.17), (4.7), (2.8) and the fact that $f_2^{\varepsilon}=f_{2|\Omega_2^{\varepsilon}}$ if $\gamma=1$, one derives

$$\begin{cases}\displaystyle\lim_{\varepsilon\to 0}\int_{\Omega} A^{\varepsilon}\nabla u_1^{\varepsilon}\ \nabla\left(\Phi_i\,\widehat{w}_i^{\varepsilon}\right)\,dx=\int_{\Omega}\theta_1 f_1\,\Phi_i\,x_i\,dx\\ +<g,\Phi_i\,x_i>_{H^{-1}(\Omega),H_0^1(\Omega)}+\int_{\Omega}\theta_2 f_2\,\Phi_i\,x_i\,dx.\end{cases}$$

This, together with (4.3) and convergences (2.10)ii) and (4.7), gives

$$\begin{cases}\displaystyle\lim_{\varepsilon\to 0}\int_{\Omega_1^{\varepsilon}} A^{\varepsilon}\nabla u_1^{\varepsilon}\ \left(C^{\varepsilon}\Phi\right)\,dx\\ \displaystyle=\sum_{i=1}^{n}\Big(\int_{\Omega}\theta_1 f_1\,\Phi_i\,x_i\,dx+<g,\Phi_i\,x_i>_{H^{-1}(\Omega),H_0^1(\Omega)}\\ \displaystyle+\int_{\Omega}\theta_2 f_2\,\Phi_i\,x_i\,dx\ -\int_{\Omega}A^0\nabla u_1\nabla\Phi_i\,x_i\,dx\Big).\end{cases}\tag{4.9}$$

Since

$$\int_{\Omega}A^0\nabla u_1\nabla\Phi_i\,x_i\,dx=\int_{\Omega}A^0\nabla u_1\nabla(\Phi_i\,x_i)\,dx-\int_{\Omega}A^0\nabla u_1\nabla\Phi_i\,e_i\,dx,$$

using $\Phi_i\,x_i$ as test function in (2.12) we obtain from (4.9) that

$$\lim_{\varepsilon\to 0}\int_{\Omega_1^{\varepsilon}} A^{\varepsilon}\nabla u_1^{\varepsilon}\ \left(C^{\varepsilon}\Phi\right)\,dx=\int_{\Omega}A^0\nabla u_1\ \Phi\,dx.\tag{4.10}$$

To treat the third term of the right-hand side of (4.1), let us take $\Phi_i\,u_1^{\varepsilon}$ as test function in (4.6). We obtain, by taking into account (4.6), (2.10)i) and (4.7)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_1^{\varepsilon}} A^{\varepsilon}\left(C^{\varepsilon}\Phi\right)\ \nabla u_1^{\varepsilon}\,dx=\sum_{i=1}^{n}\lim_{\varepsilon\to 0}\int_{\Omega_1^{\varepsilon}}A^{\varepsilon}\nabla\widehat{w}_i^{\varepsilon}\ \nabla u_1^{\varepsilon}\ \Phi_i\,dx\\ &=\sum_{i=1}^{n}\lim_{\varepsilon\to 0}\Big(\int_{\Omega_1^{\varepsilon}}\widehat{\eta}_i^{\varepsilon}\nabla\left(\Phi_i u_1^{\varepsilon}\right)\,dx-\int_{\Omega_1^{\varepsilon}}\widehat{\eta}_i^{\varepsilon}\nabla\Phi_i\ u_1^{\varepsilon}\,dx\Big)\\ &=\sum_{i=1}^{n}\lim_{\varepsilon\to 0}\int_{\Omega_1^{\varepsilon}}\widehat{\eta}_i^{\varepsilon}\nabla\Phi_i\ u_1^{\varepsilon}\,dx=-\sum_{i=1}^{n}\int_{\Omega}A^0 e_i\nabla\Phi_i\ u_1\,dx\\ &=\int_{\Omega}A^0\Phi\nabla u_1\,dx.\end{aligned}\tag{4.11}$$

For the last term in (4.1) we now choose $\Phi_i\Phi_j\widehat{w}_j^\varepsilon$ as test function in (4.6). A standard computation gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon\,(C^\varepsilon\Phi)\,(C^\varepsilon\Phi)\;dx = \sum_{i,j=1}^{n}\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon\nabla\widehat{w}_i^\varepsilon\;\nabla\widehat{w}_j^\varepsilon\;\Phi_i\;\Phi_j\;dx\\
&= \sum_{i,j=1}^{n}\lim_{\varepsilon\to 0}\Big(\int_{\Omega_1^\varepsilon}\widehat{\eta}_i^\varepsilon\nabla(\Phi_i\;\Phi_j\;\widehat{w}_j^\varepsilon)\;dx - \int_{\Omega_1^\varepsilon}\widehat{\eta}_i^\varepsilon\nabla(\Phi_i\;\Phi_j)\;\widehat{w}_j^\varepsilon\;dx\Big)\\
&= -\sum_{i,j=1}^{n}\int_\Omega A^0 e_i\nabla(\Phi_i\Phi_j)\;x_j\;dx = \int_\Omega A^0\;\Phi\;\Phi\;dx.
\end{aligned}\tag{4.12}$$

We can now pass to the limit superior in (4.1). Thanks to (3.3), (4.2), (4.10)-(4.12) we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\big(\alpha\|\nabla u_1^\varepsilon - C^\varepsilon\Phi\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)}\big)\\
&\le \int_\Omega A^0\nabla u_1\nabla u_1\,dx - \int_\Omega A^0\nabla u_1\Phi\,dx - \int_\Omega A^0\Phi\nabla u_1\,dx\\
&\quad + \int_\Omega A^0\Phi\Phi\,dx = \int_\Omega A^0\,(\nabla u_1 - \Phi)\,(\nabla u_1 - \Phi)\;dx,
\end{aligned}$$

which gives the result, since A^0 is a constant matrix. ◻

Proof of Theorem 2.5. The result follows from a density argument and from Proposition 4.1. Let $\delta > 0$ and $\Phi_\delta \in (\mathcal{D}(\Omega))^n$ such that

$$\|\nabla u_1 - \Phi_\delta\|_{L^2(\Omega)} \le \delta.$$

From (2.18) and Proposition (4.1) one obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\big(\|\nabla u_1^\varepsilon - C^\varepsilon\nabla u_1\|_{L^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)}\big)\\
&\le \limsup_{\varepsilon\to 0}\big[\|\nabla u_1^\varepsilon - C^\varepsilon\Phi_\delta\|_{L^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} + \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u_1\|_{L^1(\Omega)}\big]\\
&\le \limsup_{\varepsilon\to 0}\big[c_1\|\nabla u_1^\varepsilon - C^\varepsilon\Phi_\delta\|_{L^2(\Omega)} + \|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)}\big] + c_2\|\nabla u_1 - \Phi_\delta\|_{L^2(\Omega)}\\
&\le c\,c_1\|\nabla u_1 - \Phi_\delta\|_{L^2(\Omega)} + c_2\,\delta \le c_3\,\delta.
\end{aligned}$$

This gives convergences i) and ii) in (2.19), while convergence iii) is an immediate consequence of ii) and (2.3)-(2.4). Finally, the last statement follows by a standard argument (see for instance [7, Chapter 8]), using a similar computation and the Hölder inequality. ◻

Proof of Theorem 2.9. The proof of Theorem 2.9 makes use of similar arguments as that used in the proof of Theorem 2.5, once one has proved that there exists a constant $c > 0$, independent of ε, such that for any $\Phi \in (\mathcal{D}(\Omega))^n$,

$$\limsup_{\varepsilon\to 0} \|\nabla u_1^\varepsilon - C^\varepsilon \Phi\|_{L^2(\Omega_1^\varepsilon)} \le c\|\nabla u_1 - \Phi\|_{L^2(\Omega)}. \tag{4.13}$$

To show that, let $\Phi = (\Phi_1, \dots \Phi_n) \in (\mathcal{D}(\Omega))^n$. From (2.3) and (2.4) one gets

$$\begin{aligned} \alpha\|\nabla u_1^\varepsilon - C^\varepsilon\Phi\|^2_{L^2(\Omega_1^\varepsilon)} &\le \int_{\Omega_1^\varepsilon} A^\varepsilon\left(\nabla u_1^\varepsilon - C^\varepsilon\Phi\right)\left(\nabla u_1^\varepsilon - C^\varepsilon\Phi\right) dx \\ &= \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon\, dx - \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \left(C^\varepsilon\Phi\right) dx \\ &\quad - \int_{\Omega_1^\varepsilon} A^\varepsilon\left(C^\varepsilon\Phi\right)\nabla u_1^\varepsilon\, dx + \int_{\Omega_1^\varepsilon} A^\varepsilon\left(C^\varepsilon\Phi\right)\left(C^\varepsilon\Phi\right) dx. \end{aligned} \tag{4.14}$$

From Proposition 3.5, Theorem 2.7 and convergence (4.12) one has

$$\begin{aligned} &\lim_{\varepsilon\to 0}\Big(\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon\, dx - \int_{\Omega_1^\varepsilon} A^\varepsilon\left(C^\varepsilon\Phi\right)\nabla u_1^\varepsilon\, dx + \int_{\Omega_1^\varepsilon} A^\varepsilon\left(C^\varepsilon\Phi\right)\left(C^\varepsilon\Phi\right) dx\Big) \\ &= \int_\Omega A^0 \nabla u_1 \nabla u_1\, dx - \int_\Omega A^0 \Phi \nabla u_1\, dx + \int_\Omega A^0 \Phi\Phi\, dx, \end{aligned} \tag{4.15}$$

where for passing to the limit in the second term of the left-hand side we used the same argument as that used to prove (4.11).

It remains now to pass to the limit in the second term on the right-hand side of (4.14). As in the proof of Theorem 2.5, we write this term as follows:

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \;\left(C^\varepsilon\Phi\right)\, dx = \sum_{i=1}^n \lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \;\left(\Phi_i \nabla \widehat{w}_i^\varepsilon\right) dx \\ \displaystyle = \sum_{i=1}^n \Big(\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \;\nabla\left(\Phi_i\, \widehat{w}_i^\varepsilon\right) dx - \lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \;\nabla\Phi_i\, \widehat{w}_i^\varepsilon\; dx\Big). \end{cases} \tag{4.16}$$

Let $\widehat{w}_i^\varepsilon$ be defined by (4.5). Choosing here $v_1 = \Phi_i\, \widehat{w}_i^\varepsilon$ and $v_2 = 0$ as test function in (2.21), one has

$$\begin{cases} \displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \;\nabla\left(\Phi_i\, \widehat{w}_i^\varepsilon\right) dx = -\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)\Phi_i \widehat{w}_i^\varepsilon\; d\sigma + \int_{\Omega_1^\varepsilon} f_1^\varepsilon\, \Phi_i\, \widehat{w}_i^\varepsilon\; dx \\ + < g, P_1^\varepsilon\left(\Phi_i\, \widehat{w}_i^\varepsilon|_{\Omega_1^\varepsilon}\right) >_{H^{-1}(\Omega), H_0^1(\Omega)} . \end{cases} \tag{4.17}$$

Now, observe that from (2.22)iii), (4.5) and a change of scales one has

$$\begin{aligned} \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)\Phi_i \widehat{w}_i^\varepsilon\; d\sigma &\le \varepsilon^\gamma \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \|\widehat{w}_i^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \\ &\le c\,\varepsilon^\gamma \varepsilon^{-\gamma/2}\varepsilon^{-1/2} = c\,\varepsilon^{(\gamma-1)/2}, \end{aligned}$$

where c is independent of ε. Consequently, we have

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)\Phi_i(\widehat{w}_i^\varepsilon - x_i)\, d\sigma = 0,$$

since here $\gamma > 1$. Then, passing to the limit in (4.17) as $\varepsilon \to 0$ and using (2.22)ii), (2.17) and (4.7) one derives

$$\lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \nabla u_1^\varepsilon \,\nabla\left(\Phi_i\, \widehat{w}_i^\varepsilon\right)\, dx = \int_\Omega \theta_1 f_1\, \Phi_i\, x_i\, dx + < g, \Phi_i\, x_i >_{H^{-1}(\Omega), H_0^1(\Omega)} .$$

Thanks to (4.16) and convergences (2.22)ii) and (4.7), this gives

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0} \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon\ (C^\varepsilon \Phi)\ dx \\ \displaystyle = \sum_{i=1}^n \Big(\int_\Omega \theta_1 f_1\, \Phi_i\, x_i\, dx + < g, \Phi_i\, x_i >_{H^{-1}(\Omega), H_0^1(\Omega)} \\ \displaystyle - \int_\Omega A^0 \nabla u_1 \nabla \Phi_i\, x_i\, dx \Big). \end{cases}$$

From this equality, using $\Phi_i\, x_i$ as test function in (2.23), we have

$$\lim_{\varepsilon\to 0} \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon\ (C^\varepsilon \Phi)\ dx = \int_\Omega A^0 \nabla u_1\ \Phi\, dx. \tag{4.18}$$

This, together with (4.14) and (4.15) gives (4.13) and concludes the proof. □

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Lavoro pervenuto alla redazione il 22 febbraio 2006
ed accettato per la pubblicazione il 1 marzo 2006.
Bozze licenziate il 10 maggio 2006

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