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On a problem of Lesniak, Polimeni, and Vanderjagt

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Abstract: *We give almost complete solution to the following problem: for a fixed* S*, what is the minimum value* $p = \mu_H(S)$ *such that a pair* (S, p) *has a Hamiltonian realization? We give a criterion for a pair* (S, p) *to have a Hamiltonian realization.*

1 – Introduction

The *degree set* of a graph G, D_G, is the set of all distinct vertex degrees of G. The graph G is called a *realization* of D_G. Let S be a set of non-negative integers, and let p be a positive integer. A *realization of a pair* (S, p) is a graph G such that $D_G = S$ and $|V(G)| = p$. We consider the following problem.

Problem 1 (Lesniak, Polimeni, and Vanderjagt [4]). For a fixed S, what is the minimum value $p = \mu_H(S)$ such that the pair (S, p) has a Hamiltonian realization?

Note that necessary and sufficient conditions for realizability a set S and a pair (S, p) by an arbitrary graph were found by Kapoor, Polimeni, and Wall [3] and Sipka [5], respectively.

We shall consider a set of integers S in following form:

$$S = \{k_1, k_2, \ldots, k_n\}, \quad \text{where} \quad \leq k_1 < k_2 < \cdots < k_n.$$

Theorem 1 (Lesniak, Polimeni, and Vanderjagt [4]). $\mu_H(S)$ *exists if and only if* $k_1 \geq 2$.

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Another result of Lesniak, Polimeni, and Vanderjagt [4] is that

$$k_n + 1 \leq \mu_H(S) \leq \sum_{i=1}^{n}(k_i + 1).$$

Some particular results on $\mu_H(S)$ were obtained by Chernyak [1], but they are far from the exact value.

Theorem 2.

(i) *If $k_1 \geq 3$ then $\mu_H(S) = k_n + 1$.*

(ii) *If $k_1 \geq 2$ then $k_n + 1 \leq \mu_H(S) \leq k_n + 2$.*

(iii) *If $k_1 = 2$ then $\mu_H(S) = k_n + 1$ for each odd n, and $\mu_H(S)$ may take the both values $k_n + 1$ or $k_n + 2$ for each even n.*

Concerning Hamiltonian realizations of pairs, we have the following result.

Theorem 3. *A pair (S, p) has a Hamiltonian realization if and only if $k_1 \geq 2$, $p \geq \mu_H(S)$, the number $pk_1k_2 \cdots k_n$ is even, and $(S, p) \neq (\{3, 2t\}, 2t + 2)$ for all integers $t \geq 4$.*

This result can be applied to Eulerian realizations of pairs.

Corollary 1. *A pair (S, p) has a Eulerian realization if and only if $p \geq k_n + 1$ and all $k_1, k_2, \ldots, k_n$ are even.*

Note also that a criterion for realizability a pair (S, p) of Sipka [5] easily follows from Theorem 3.

2 – Proofs

We shall use the following well-known facts, see Harary [2].

Fact 1. The complete graph K_{2n+1} can be represented as disjoint union of n Hamiltonian cycles.

Fact 2. The complete graph K_{2n} can be represented as disjoint union of $n - 1$ Hamiltonian cycles and a perfect matching.

Fact 1 and Fact 2 imply that there exists a regular Hamiltonian graph of degree r and order p for $2 \leq r < p$ such that rp is even. We denote any such graph by $H_{r,p}$. Also $H(S, p)$ (respectively, $H(S)$) will denote a Hamiltonian realization of a pair (S, p) (respectively, a set S). First we prove the statements (i) and (ii) of Theorem 2.

Proof. (i) Let $k_1 \geq 3$ in S. Clearly, $\mu_H(S) \geq k_n + 1$. We use induction on n to show that there exists a graph L_n of type $H(S, k_n + 1)$. For $n = 1$ and $n = 2$, we may choose $L_1 = K_{k_1+1}$ and $L_2 = K_{k_1-2} + C_{k_2-k_1+3}$. Here and below

$$G + H = \overline{\overline{G} \cup \overline{H}}.$$

Let $n \geq 3$. We construct an auxiliary set $S' = \{k'_1, k'_2, \dots, k'_{n-2}\}$, where $k'_i = k_{i+1} - k_1 + 2$, $i = 1, 2, \dots, n-2$. By the inductive hypothesis, S' has a Hamiltonian realization, say H, of order $k'_{n-1} + 1$.

Suppose that we have a Hamiltonian graph F. If we delete all edges of a Hamiltonian cycle from F, the resulting graph will be denoted by F^o. We define

$$L_n = \overline{(\overline{H^o \cup O_j})^o \cup O_t}, \tag{1}$$

$l = k_1 - 2$, $j = k_n - k_{n-1}$, and O_n denotes the edgeless graph of order n.

(ii) It is sufficient to consider the case of $k_1 = 2$. If $S = \{2\}$ then $\mu_H(S) = 3$, and the result follows. Suppose that $n \geq 2$. We denote $S_1 = S \setminus \{2\}$. By (i), there exists a graph R of type $H(S_1, k_n + 1)$. It remains to subdivide an edge of a Hamiltonian cycle in R. □

The statement (iii) of Theorem 2 will be proved below.

Fact 3. If $k_1 = 2$ in S and $p \geq \mu_H(S)$, then the pair (S, p) has a Hamiltonian realization.

Proof. It is sufficient to do the subdivision described above exactly $p - k_n - 1$ times. □

A graph G is called *traceable* if G has a Hamiltonian path P. We say that P *traces* G.

Fact 4. If $k_1 = 2$ in S and $p \geq \mu_H(S)$, then the pair (S, p) has a Hamiltonian realization.

Proof. It is sufficient to consider the case of $i = m$. We use induction on n as in the proof of Theorem 2(i). The graphs L_1 and L_2 are uniquely defined. For $n \geq 3$, the graph L_n [see (1)] is not uniquely defined: it depends on choice of Hamiltonian cycles. Suppose that a Hamiltonian realization H of S' satisfies Fact 4, i.e., H has a complete subgraph M of order k'_{m-1} and a Hamiltonian cycle C^1 that traces M. We delete C^1 when constructing H^o. Let us fix an edge $e = uv \in E(C^1) \setminus E(M)$. The vertices of O_j will be denoted by $w_1, w_2, \dots, w_j$. A Hamiltonian cycle C^2 in the graph $\overline{H^o \cup O_j}$ [see (1)] we choose in the following way:

$$C^2 = (E(C^1) \setminus e) \cup \{u, w_1\} \cup \left(\bigcup_{i=1}^{j-1} w_t w_{t+1} \right) \cup \{w_j v\}.$$

Deleting C^2 produces $(\overline{H^o \cup O_j})^o$. Then the graph L_n can be constructed. The graph L_n contains a complete subgraph $G_m = M \cup K_l$ of order k_m. Also, there is a Hamiltonian cycle in L_n that traces G_m. □

LEMMA 1. *Let $a \geq 3$ be an odd integer, and let $p \geq a+2$. The set $\{a, a+2\}$ has a traceable realization H such that*

- *$|V(H)| = p$,*
- *exactly $a+1$ vertices of H have degree a, and*
- *the end-vertices of a Hamiltonian path have degrees a.*

PROOF. A graph G *admits a switching* $d = (u, v, w, x)$, where vertices $u, v, w, x \in V(G)$ are pairwise distinct, if $uv, wx \in E(G)$ and $vw, xu \notin E(G)$. A *switching* d is deleting the edges uv, wx, and creating edges vw, xu.

It is sufficient to prove the lemma for $a+2 \leq p \leq 2a+3$. Indeed, let G satisfies Lemma 1, and let uv be an edge of a Hamiltonian path in G. We construct G' by applying switching $d = (u, v, w, x)$, where wx is an edge of K_{a+2}. The graph G' also satisfies Lemma 1, and $|V(G')| = |V(G)| + a + 2$.

Now we consider five possible cases.

CASE 1. $p = 2a + 3$.

We construct the graphs G' described above for $G = K_{a+1}$, uv being an arbitrary edge of G.

CASE 2. $p = 2a + 2$.

Let $H_1 = H_2 = K_{a+1}$ and $V(H_i) = \{v^i_j : i = 1, 2 \text{ and } j = 1, 2, \ldots, a+1\}$. We delete a perfect matching from H_1 [see Fact 2], and we add edges $v^1_j v^2_j$ for all $j = 1, 2, \ldots, a+1$.

CASE 3. $p = 2a + 1$.

Let $H_1 = K_{a+1}$, $H_2 = K_a$, $V(H_1) = \{v_1, v_2, \ldots, v_{a+1}$, and $V(H_2) = \{u_1, u_2, \ldots, u_a$. We delete the edges $v_i v_{i+1}$, $i = 1, 2, \ldots, a$, from H_1, and we add edges $u_j v_j$ and $u_j v_{j+1}$, $j = 1, 2, \ldots, a$.

CASE 4. $p = 2a$.

In the complete bipartite graph $K_{a-1,a+1}$ one of the parts has exactly $a+1$ vertices. We construct a 1-regular graph in this part.

CASE 5. $a + 2 \leq p \leq 2a - 1$.

In the complete graph K_{a+1} there are $(a-1)/2$ disjoint Hamiltonian cycles and a perfect matching. We consider each of the $(a-1)/2$ Hamiltonian cycles as a disjoint union of two perfect matchings. Thus, we have represented K_{a+1} as a disjoint union of a Hamiltonian cycle and $a-2$ perfect matchings. Now we construct a Hamiltonian graph H by deleting i perfect matchings from K_{a+1}, $i = 1, 2, \ldots, a-2$. As a required graph, we take $H + O_i$. □

Lemma 2. *If $k_1 \geq 2$ in S, at least one of the numbers $k_1, k_2, \dots, k_m$ is even, where $m = \lfloor (n+1)/2 \rfloor$, and $p \geq \mu_H(S)$, then the pair (S,p) has a Hamiltonian realization.*

Proof. For $k_1 = 2$, the statement follows from Fact 2. Let $k_1 \geq 3$. It is sufficient to consider the case of $p \geq k_n + 2$. If k_i is odd for some $i \in \{1, 2, \dots m\}$, then we consider the graph described in Fact 4. We denote it by G_i, $V(G_i) = \{v_1, v_2, \dots, v_{k_i}\}$. Let v_1 and v_2 be the end-vertices of a Hamiltonian path in G_i. We delete edges from G_i. By Lemma 1, we can realize the set $\{k_i - 1, k_i\}$ on $p - k_n + k_i - 1$ vertices. Now we identify the vertices of degree $k_i - 1$ in this realization with the vertices $v_1, v_2, \dots, v_{k_i}$ in such a way that the end-vertices of the Hamiltonian path coincide with v_1 and v_2. □

We write $(S,p) \in \mathcal{A}$ if $|S| = n = 2m$, $k_1 \geq 3$, all $k_1, k_2, \dots, k_m$ are odd, all $k_{m+1}, k_{m+2}, \dots, k_n$ are even, and $p \geq k_n + 2$.

Lemma 3. *Let $(S,p) \notin \mathcal{A}$, and let $k_1 \geq 2$ in S. The pair (S,p) has a Hamiltonian realization if and only if $p \geq \mu_H(S)$ and the number $pk_1k_2 \cdots k_n$ is even.*

Proof. Necessity is obvious.

Sufficiency. We may assume that $p \geq k_n + 2$, $k_1, k_2, \dots, k_m$ are odd, and $k_1 \geq 3$. We set

$$k'_i = p - k_{n-i+1} + 1, i = 1, 2, \dots, n,$$

and $S' = \{k'_1, k'_2, \dots, k'_n\}$. Clearly, $k'_1 \geq 3$ and $p \geq k'_n + 2$. We show that at least one of $k'_1, k'_2, \dots, k'_m$ is even for $(S,p) \notin \mathcal{A}$.

If p is odd then there is an even integer among $k_{m+1}, k_{m+2}, \dots, k_n$. Therefore at least one of $k'_1, k'_2, \dots k'_m$ is even. If p is even and n is odd, then $k'_m = p - k_m + 1$ is even. If both p and n are odd, then there is an odd integer among $k_{m+1}, k_{m+2}, \dots, k_n$ [otherwise $(S,p) \in \mathcal{A}$]. We denote such a number by k_r, $m + 1 \leq r \leq n$. Then $k'_{n-r+1} = p - k_r + 1$ is even, and $1 \leq n - r + 1 \leq m$.

According to Lemma 2, (S',p) has a Hamiltonian realization H. It is easy to check that $\overline{H^o}$ is a Hamiltonian realization for the pair (S,p). □

It remains to consider the case of $(S,p) \in \mathcal{A}$. Note that Lemma 3 does not hold for $(S,p) \in \mathcal{A}$.

Lemma 4. *Let $(S,p) \in \mathcal{A}$, and let $k_1 \geq 2$ in S. The pair (S,p) has a Hamiltonian realization if and only if $(S,p) \neq (\{3, 2t\}, 2t+2)$ for all $t \geq 4$.*

PROOF. We consider eight possible cases.

CASE 1. $p \geq k_n + k_1 + 3$.

We put

$$S' = S \setminus \{k_1\}, p' = p - k_1 - 1,$$

and we construct a graph of type $H(S', p')$ by Lemma 3. We choose an edge uv in this graph, and an edge wx in the complete graph K_{k_1+1}. It remains to do the switching (u, v, w, x).

CASE 2. $p = k_n + k_1 + 1$ and $k_2 - k_1 \geq 3$.

We put

$$S' = \{k_3 - k_1 + 1, k_4 - k_1 + 1, \dots, k_n - k_1 + 1, \}, p' = k_n - 2,$$

and we construct a graph of type $H(S', p')$.

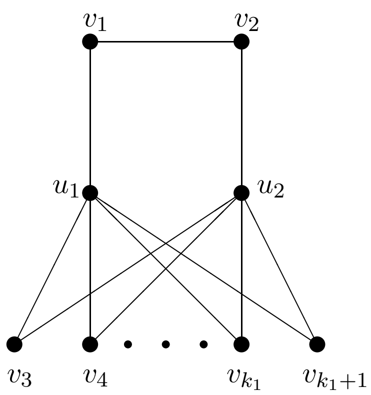

Fig. 1: The graph B.

Finally, we connect all vertices of this graph with $v_1, v_2, \dots, v_{k_1+1}$ of the graph B shown in fig. 1.

CASE 3. $p = k_n + k_1 + 1$ and $k_2 - k_1 = 2$.

It is possible if $n \geq 4$ only. We put

$$S' = \{k_3 - k_1 + 1, k_4 - k_1 + 1, \dots, k_n - k_1 + 1\}, p' = k_n - 2,$$

and we construct a graph of type $H(S', p')$. Finally, we connect all vertices of this graph with the vertices $v_1, v_2, \dots, v_{k_1+1}$ of the graph B' that is obtained from B [see fig. 1] by deleting the edge u_1v_1 and adding edges u_1u_2 and u_2v_1.

Case 4. $p = k_n + k_1 + 1$ and $k_2 - k_1 = 1$.

It is possible if $S = \{t, t+1\}$, $p = 2t + 2$, and $t \geq 3$ is odd only. We take adjacent vertices u and v. One half of vertices of $H_{t-1,2t}$ we connect with u, and the other half is connected with v.

Case 5. $p = k_n + k_1 + 1$ and $k_1 \geq 5$.

We put

$$S' = \{k_2 - k_1 + 1, k_3 - k_1 + 1, \dots k_n - k_1 + 1, \}, p' = k_n - 2.$$

We connect two vertices of degree $k_1 - 1$ in the graph K_{2,k_1-1}, and we connected all vertices of degree two with all vertices of a graph of type $H(S', p')$.

Case 6. $k_n + 2 \leq p \leq k_n + k_1 - 3$ and $k_1 \geq 5$.

We put

$$i = p - k_n, S' = \{k_1 - i, k_2 - i, \dots, k_{n-1} - i\}, p_i = k_n.$$

We construct a graph of type $H(S_i, p_i) + O_i$.

Case 7. $p = k_n + 2$, $k_1 = 3$, and $n \geq 4$.

It is possible if $S = \{3, 2t\}$, $p = 2t + 2$, and $t \geq 2$ only. It is straightforward to check that a required realization exists if and only if $t = 2$ or $t = 3$.

Case 8. $p = k_n + 2$, $k_1 = 3$, and $n \geq 4$.

We put

$$S' = \{k_2 - 2, k_3 - 2, \dots, k_{n-1} - 2\}, p' = k_n - 2.$$

Clearly, $(S', p') \in \mathcal{A}$. Excluding

$$k_2 = 5, k_n = k_{n-1} + 2, \tag{2}$$

we can construct a graph of type $H(S', p')$ using Cases 1 – 7. Now we delete an edge uv from K_4, and connect u, v with all vertices of a graph of type $H(S', p')$.

If the condition (2) holds, then this construction gives inductive step. Base of induction: the set $S = \{3, 5, t, t + 2\}$ has a Hamiltonian realization of order $t+4$ for all even $t \geq 6$. Indeed, we connect all vertices of a graph of type $H_{t-4,t-2}$ with the vertices u, v, w, x of the graph C shown in fig. 2. ∎

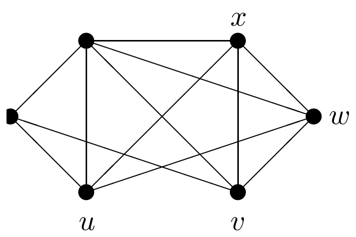

Fig. 2: The graph C.

Lemma 3 and Lemma 4 imply Theorem 3.

Now we prove Corollary 1.

Proof. Lesniak, Polimeni, and Vanderjagt [4] proved that a set S has a Eulerian realization if and only if all k_i are even. Moreover, the minimum order of a Eulerian realization is k_n+1. Thus, it is sufficient to note that if all k_i are even, then the pair (S,p) can be realized by a connected graph for all $p \geq k_n+2$. It follows immediately from Theorem 3, since $k_n+2 \geq \mu_H(S)$. ∎

It remains to prove Theorem 2(iii).

Proof. (iii) Let $S=\{k_1=2,k_2,\ldots,k_n\}$. If $n=1$ then $\mu_H(S)=3$. For $n=2$, Lesniak, Polimeni, and Vanderjagt [4] showed that $\mu_H(\{2,t\})=t+2$ if $t\geq 4$, and $\mu_H(\{2,4\})=4$.

We use induction on n to show that $\mu_H(S)=k_n+1$ for each odd n. The statement holds for $n=1$ and $n=2$. Let $n\geq 3$. We consider the following two constructions.

$(2,2,k_n)$-Construction. We choose a Hamiltonian realization H of order $p'=k_n-2$ of the set $S'=\{k_1=2,k_2,\ldots,k_{n-1}-1\}$. Then we do a switching (u_1,u_2,v_1,v_2), where u_1u_2 is an edge of K_3 and v_1v_2 is an edge of a Hamiltonian cycle of H. We connect the third vertex of the K_3, u_3, with all the vertices of H. As a result, we obtain a graph of type $H(S,k_n+1)$.

The $(2,2,k_n)$-Construction cannot be implemented in the following two situation only:

$$k_n = k_{n-1}+1, \tag{3}$$

and

$$k_2,k_3,\ldots,k_{n-1} \quad \text{are even, but } k_n \text{ is odd.} \tag{4}$$

$(2,k_n,k_n)$-Construction. We choose a Hamiltonian realization H of order $p'=k_n-2$ of the set $S'=\{k_2-2,k_3-2,\ldots,k_{n-1}-2\}$. Then we connect all the vertices of H with two vertices of K_3. As a result, we obtain a graph of type $H(S,k_n+1)$.

The $(2,k_n,k_n)$-Construction cannot be implemented in the following two situation only:

$$k_2 = 3, \tag{5}$$

and

$$k_2,k_3,\ldots,k_n \text{ are odd.} \tag{6}$$

As it follows from (3), (4), (5), and (6), there is the following unconsidered case, namely $S = \{2, 3, k_3, \dots, k_{n-2}, k_n - 1, k_n\}$, where $n \geq 3$ is odd. For $n = 3$, we have $S = \{2, 3, 4\}$ and $\mu_H(S) = 5$. If $n \geq 5$, we propose additional constructions that give induction step.

1) Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$. We delete the edge u_1u_2. Then we construct a Hamiltonian realization of order $p' = k_n - 3$ of $S' = \{k_3 - 2, k_4 - 2, \dots, k_{n-2} - 2\}$ [using either Theorem 3 or the inductive hypothesis]. Finally we connect all vertices of the graph with u_2 and u_3. This construction cannot be done only if

(7) $$k_3, k_4, \dots, k_{n-2} \text{ are odd, but } k_n \text{ is even.}$$

2) Let $V(K_5) = \{u_1, u_2, u_3, u_4, u_5\}$. We delete the edges u_1u_2 and u_1u_3. Then we construct a Hamiltonian realization of order $p' = k_n - 4$ of $S' = \{k_3 - 3, k_4 - 3, \dots, k_{n-2} - 3\}$ [using either Theorem 3 or the inductive hypothesis]. Finally we connect all vertices of the graph with u_3, u_4 and u_5. This construction cannot be implemented only if

(8) $$k_3 = 4$$

or

(9) $$k_3, k_4, \dots k_{n-2} \text{ are even, but } k_n \text{ is odd.}$$

Since the family of sets defined by (7) is disjoint from the family of sets defined by (8) and (9), the proof for even n is complete.

Similar constructions show that $\mu_H(S) = k_n + 1$ if $k_1 = 2$, $k_2 \geq 5$ and n is even.

Finally, we note that $\mu_H(S) = k_n + 2$ for $S = \{2, 3, \dots, j, 2j + 2, 2j + 3, \dots, 3j\}$. It is easy to see that such a set has no a Hamiltonian realization of order $3j + 1$. ∎

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