Rendiconti di Matematica, Serie VII
Volume 26, Roma (2006), 267-290

Refinable functions from Blaschke products

MARIANTONIA COTRONEI – MARIA LAURA LO CASCIO

HONG OH KIM – CHARLES A. MICCHELLI – TOMAS SAUER

Dedicated to Laura Gori with friendship and esteem on the occasion of her 70th birthday.

Abstract: *In this paper we study the problem of constructing refinable orthonormal cardinal functions from Blaschke products. The digital filter perspective of our construction corresponds to what is called an infinite impulse response (IIR) filter. We show how to construct, at least numerically, stable filters of this type in contrast to the Butterworth filter which is the maximally flat filter in our class.*

1 – Introduction

The *sinc function*

$$S(x) = \frac{\sin \pi x}{\pi x}, \qquad x \in \mathbb{R},$$

has *three* remarkable properties that motivate us here. First, it is a *cardinal function*, that is,

$$S(k) = \delta_k := \begin{cases} 0, & k \in \mathbb{Z} \setminus \{0\}, \\ 1, & k = 0. \end{cases} \tag{1.1}$$

Key Words and Phrases: *Blaschke products – IIR filters – Refinable functions – Subdivision schemes.*
A.M.S. Classification: 42C40 – 65T60 – 94A12

Next, it has *orthonormal integer translates*, that is

$$(S, S(\cdot - k)) = \delta_k, \ k \in \mathbb{Z} \tag{1.2}$$

where $(\cdot,\cdot)$ is the standard inner product on $L^2(\mathbb{R})$. We simply say that S *is orthonormal*. The third property is that S is *refinable*.

The first property of the sinc function connects it to the Whittaker – Kotelnikov – Shannon series

$$f = \sum_{n\in\mathbb{Z}} f(n)\, S(\cdot - n), \tag{1.3}$$

which is valid for all functions $f \in L^2(\mathbb{R})$ whose Fourier transform is zero off the interval $I_\pi := [-\pi, \pi]$, cf. [10], [13], [14]. These functions are commonly referred to as *band–limited functions* with band width π. The totality of all such functions, denoted by B_0, is a reproducing kernel Hilbert space with reproducing kernel $K(x, y) := S(x - y)$, $x, y \in \mathbb{R}$, and from this property it follows that S is orthonormal.

The third property of the sinc function which is central to this paper rests upon the fact that the closed subspace

$$B_1 := \{f(2\cdot) \ : \ f \in B_0\} \tag{1.4}$$

of $L^2(\mathbb{R})$ contains B_0 as a closed subspace. This means that S is *refinable* in the sense of [2] and so it satisfies the *refinement equation*

$$\widehat{S}(2\cdot) = s\,\widehat{S}, \qquad \text{a.e. } \mathbb{R}, \tag{1.5}$$

where s is the 2π–periodic extension of the function $\chi_{I_{\pi/2}}$ restricted to I_π.

The study of refinable functions which are both orthonormal and cardinal was addressed in [6], [9]. In this paper, we provide a construction of such functions using *Blaschke products*. The method we propose is simple and yields refinable functions, which are different from those in [6], [9], with desirable properties not possessed by the sinc function.

In the following Section 2, we provide a necessary condition for the symbol of the refinement equation to combine cardinality, orthonormality and a zero of a prescribed order at -1. This condition is of purely algebraic nature and ignores the question of existence of a refinable functions. For that latter problem, we use two approaches, one based on subdivision in Section 3, and one based on a Fourier transform method in Section 4. Both approaches only yield *sufficient* conditions for the existence of the desired refinable functions and can be used for different purposes. In fact, the subdivision approach allows us to identify a whole one–parameter family of such functions, while the Fourier based method is more convenient to determine a class of refinable functions of arbitrary smoothness and to consider the limit process for a particular class when the smoothness parameter tends to infinity. The symbols corresponding to our orthonormal and cardinal refinable functions have to be Blaschke products, i.e., *rational* functions, and so the associated filters are IIR filters for which *stability* is an important issue that will be considered in Section 5.

2 – The form of the filter

To prepare ourselves, we let Δ be the open unit disc, $\overline{\Delta}$ its closure, $\partial\Delta$ its boundary and $\mathbb{C}$ the complex plane. Any complex-valued function a defined on $\partial\Delta$ determines a refinement equation

$$\widehat{A}(2\cdot) = \frac{1}{2}\, a\left(e^{-i\cdot}\right)\, \widehat{A} \tag{2.1}$$

for some function $A \in L^2(\mathbb{R})$.

We let $R(\Delta)$ be the set of all rational functions of the form f/g where f and g are polynomials. A *Blaschke product* b of degree n is a function of the form

$$b(z) = c \prod_{k\in\mathbb{Z}_n} \frac{z - z_k}{1 - \overline{z}_k z}, \qquad z \in \overline{\Delta},$$

where $\{z_k : k \in \mathbb{Z}_n\} \subseteq \Delta$, $c \in \partial\Delta$ and $\mathbb{Z}_n := \{0, 1, \cdots, n-1\}$. The constant c can be chosen so that $b(1) = 1$. The totality of Blaschke products which satisfy this additional condition shall be denoted by $B_n(\Delta)$ (by convention, for $n = 0$ $B_n(\Delta)$ consists of the function which is identically one).

Our starting point is the following fact.

Lemma 2.1. *If $a \in R(\Delta)$ and A is an orthonormal cardinal function such that $A \in L^2(\mathbb{R})$, $\widehat{A} \in L^1(\mathbb{R})$ and*

$$\widehat{A}(2\cdot) = \frac{1}{2} a\left(e^{-i\cdot}\right)\, \widehat{A},$$

then there exist two nonnegative integers, $n_1, n_2 \in \mathbb{N}$ and Blaschke products $b_k \in B_{n_k}(\Delta)$, $k \in \mathbb{Z}_2$, with no common zeros such that

$$a(z) = z\, \frac{b_1\left(z^2\right)}{b_2\left(z^2\right)} + 1, \qquad z \in \overline{\Delta}. \tag{2.2}$$

PROOF. Our hypotheses on A and a imply that

$$a(1) = 2, \tag{2.3}$$
$$a(z) + a(-z) = 2, \tag{2.4}$$

and

$$|a(z)|^2 + |a(-z)|^2 = 4, \tag{2.5}$$

for all $z \in \partial\Delta$, cf. [6], [9].

Equations (2.4) and (2.5) imply that $|a-1|^2 = 1$ on $\partial\Delta$. Since $a-1 \in R(\Delta)$ and $a(1) = 2$ we get that $a - 1 = b_+/b_-$ where b_+ and b_- are Blaschke products without common zeros which have the value one at one. Substituting this form of a into equation (2.4) yields the desired result. ∎

It has been known for some time that regularity (in the sense of existence of derivatives) of a refinable function A implies that a must have a zero at -1 of an order which reflects the regularity of A. This important fact was proved in [2] in great generality (and even in [11] in the *non–stationary* case).

Michael Stessin reminded one of us that any Blaschke product $b \in B_n(\Delta)$ has the property that $b'(1) > 0$, and so, if the functions a and A satisfy the hypotheses of Lemma 2.1, then $a'(1) = 1 + b'(1) > 1$ and, consequently, $\widehat{A} \notin L^1(\mathbb{R})$. For this reason, we write equation (2.2) in the form

$$a(z) = z^{-1}\,\frac{b_1\left(z^2\right)}{b_2\left(z^2\right)} + 1 \tag{2.6}$$

for some pair b_1, b_2 of Blaschke products without common zeros. This can be achieved by either factoring a zero at the origin from b_2 obtaining

$$a(z) = z\frac{b_1\left(z^2\right)}{z^2\,\widetilde{b}_2\left(z^2\right)} + 1 = z^{-1}\,\frac{b_1\left(z^2\right)}{\widetilde{b}_2\left(z^2\right)} + 1$$

or, if $b_2(0) \neq 0$, by adding another zero at the origin to b_1 which gives

$$a(z) = z\frac{z^{-2}\,b_1\left(z^2\right)}{b_2\left(z^2\right)} + 1 = z^{-1}\,\frac{b_1\left(z^2\right)}{b_2\left(z^2\right)} + 1.$$

Indeed, this way any odd power of z could appear in (2.6), but the form given there is the most convenient one for our purpose here.

Next, we explain how to prescribe a zero at $z = -1$ of a given order for a in (2.6) by expressing the right hand side of this equation in an alternative form.

THEOREM 2.2. *If a and A satisfy the hypotheses of Lemma* 2.1 *then there is a polynomial p of exact degree n such that*

$$a(z) := z^{-1}\frac{p\left(z^2\right)}{\tilde{p}\left(z^2\right)} + 1, \qquad z \in \Delta. \tag{2.7}$$

Moreover, a has a zero of order k at -1 if and only if $k \leq 2n+1$ and in that case there is a symmetric polynomial q of exact degree $2n+1-k$ such that

$$p\left(z^2\right) = \frac{1}{2}\left[(1+z)^k q(z) + (1-z)^k q(-z)\right], \qquad z \in \mathbb{C}. \tag{2.8}$$

Conversely, if p is given by (2.8) *where q is a symmetric polynomial of exact degree $2n+1-k$ then a defined by* (2.7) *satisfies* (2.3)-(2.5) *and has a zero of order k at -1.*

We begin our discussion with a given *monic* polynomial p of degree n. We identify the coefficients of p relative to the monomial basis, take their complex conjugate and write them in reverse order to form the polynomial

$$\tilde{p}(z) := z^n \overline{p\left(\frac{1}{\overline{z}}\right)}, \qquad z \in \mathbb{C}.$$

When $\tilde{p} = p$ we say p is *symmetric.*

The rational function $p/\tilde{p}$ has modulus one on $\partial\Delta$ and therefore can be written in the form $c\,b_0/b_1$ where $c \in \partial\Delta$, $b_k \in B_{n_k}(\Delta)$, $k \in \mathbb{Z}_2$ and $n := n_0 + n_1$ is the number of zeros of p in $\mathbb{C} \setminus \partial\Delta$. Indeed, if $p = p_0 p_1 p_2$ where p_0 has zeros in Δ, p_1 has zeros in $\partial\Delta$ and p_2 has zeros in $\mathbb{C} \setminus \overline{\Delta}$ then $p_0/\tilde{p}_0 = b_0$, $p_1/\tilde{p}_1 = c$ and $p_2/\tilde{p}_2 = b_1^{-1}$. Conversely, any Blaschke product $b \in B_n(\Delta)$ can easily be written in the form $p/\tilde{p}$ where the degree of p is n.

Let us now consider the question of when the function

$$a(z) := z^{-1}\frac{p(z^2)}{\tilde{p}(z^2)} + 1, \qquad z \in \Delta,$$

has a zero of *exact* order $k \in \mathbb{N}$ at $z = -1$, that is, $a^{(j)}(-1) = 0$, $j \in \mathbb{Z}_k$, and $a^{(k)}(-1) \neq 0$. To this end, we write a as

$$a(z) = \frac{z\tilde{p}(z^2) + p(z^2)}{z\tilde{p}(z^2)}, \qquad z \in \Delta. \tag{2.9}$$

The denominator of this rational function does not vanish at -1 and so for the function a to have a $k-$fold zero at -1 means there exists a polynomial q such that

$$z^{2n+1}\overline{p\left(\frac{1}{\overline{z}^2}\right)} + p\left(z^2\right) = (1+z)^k q(z). \tag{2.10}$$

The left hand side is a polynomial of exact degree $2n+1$, since p is of exact degree n. Hence the degree of q is $2n+1-k$ which implies that $k \le 2n+1$.

Let us solve (2.10) for the polynomial p. This can be accomplished by replacing z by $-z$ in equation (2.10) and adding the resulting equation to equation (2.10) to obtain the formula

$$p\left(z^2\right) = \frac{1}{2}\left[(1+z)^k q(z) + (1-z)^k q(-z)\right].$$

We now substitute this expression for p into (2.10) and conclude that the polynomial

$$v(z) := (1+z)^k \left[z^{2n+1-k} \overline{q\left(\frac{1}{\overline{z}}\right)} - q(z) \right] \tag{2.11}$$

is *even*. Since v has a $k-$fold zero at -1 it also must have a $k-$fold zero at 1 and so

$$v(z) = (1-z^2)^k w(z), \tag{2.12}$$

where w is a polynomial of exact degree $2n-2k$. Therefore, when $k > n$ we conclude that $v = 0$ and consequently q is a symmetric polynomial. Conversely, when q is symmetric and p is given by (2.10) then the rational function a defined by (2.9) does indeed have a $k-$fold zero in -1. This gives *complete characterization* of this property of a which we sought, at least for $k > n$.

Let us now show that this fact still holds even when $k \le n$. In this case, combining (2.11) and (2.12) gives us the equation

$$(1-z)^k w(z) = z^{2n+1-k} \overline{q\left(\frac{1}{\overline{z}}\right)} - q(z). \tag{2.13}$$

Replace z by $1/\overline{z}$ in this equation to obtain the formula

$$\overline{z}^{2n+1-k} \left(1 - \frac{1}{\overline{z}}\right)^k w\left(\frac{1}{\overline{z}}\right) = \overline{q(z)} - \overline{z}^{2n+1-k} q\left(\frac{1}{\overline{z}}\right), \tag{2.14}$$

which yields the identity

$$z^{2n+1-k} \left(1 - \frac{1}{z}\right) \overline{w\left(\frac{1}{\overline{z}}\right)} = -(1-z)^k w(z)$$

or equivalently

$$z^{2n+1-2k} (-1)^{k+1} \overline{w\left(\frac{1}{\overline{z}}\right)} = w(z).$$

However, we have already pointed out that w is a polynomial of at most degree $2n-2k$. Therefore, by the above equation it is identically zero and we conclude even in the case $k \le n$ that q is a symmetric polynomial.

Combining these observations completes the proof of Theorem 2.2.

In view of (2.7) and (2.8), the rational function a also has the form

$$a(z) = \frac{2(1+z)^k q(z)}{(z+1)^k q(z) - (1-z)^k q(-z)}, \tag{2.15}$$

where q is a symmetric polynomial of exact degree $N = 2n+1-k$ and can yield refinable functions which are real–valued, orthonormal and cardinal. From now on, we consider only the rational function a of the form (2.15) where q is a *real* symmetric polynomial. We note that the maximum order of the zero of a at $z=-1$ is $2n+1$ and it is attained by taking $q=1$, so that

$$a_n(z) := \frac{2(1+z)^{2n+1}}{(z+1)^{2n+1} - (1-z)^{2n+1}}, \qquad z \in \mathbb{C}. \tag{2.16}$$

This is the autocorrelation symbol of the Butterworth filter of degree $2n+1$, see [7, (12.6-1), p. 246], which we discovered after we derived Theorem 2.2. Recall also from [7] that the Butterworth filter is essentially a “square root” of a collecting the poles inside the unit circle and thus yielding stability.

3 – Stationary subdivision and an example

Associated with a bi–infinite sequence $a = (a_k : k \in \mathbb{Z}) \in \ell_1(\mathbb{Z})$ is the subdivision operator, defined for all $\lambda = (\lambda_k : k \in \mathbb{Z}) \in \ell_\infty(\mathbb{Z})$ as

$$S_a\lambda = \left(\sum_{k\in\mathbb{Z}} a_{j-2k}\,\lambda_k \;:\; j \in \mathbb{Z}\right).$$

The subdivision scheme is said to be convergent if for any $\lambda \in \ell_\infty(\mathbb{Z})$ there exists a continuous function $f \in C(\mathbb{R})$ such that

$$\lim_{r\to\infty} \sup_{k\in\mathbb{Z}} \left|(S_a^r\lambda)_k - f\left(2^{-r}k\right)\right| = 0$$

and for at least one $\lambda \in \ell_\infty(\mathbb{Z})$ the function f is not identically zero. It follows that whenever the subdivision scheme converges there exists a refinable function A in $C(\mathbb{R})$ which satisfies the refinement equation

$$A = \sum_{k\in\mathbb{Z}} a_k\, A\,(2\cdot -k) \tag{3.1}$$

and that

$$f = \sum_{k \in \mathbb{Z}} A\,(\cdot - k)\ \lambda_k, \tag{3.2}$$

see [2]. We denote the right hand side of (3.2) by $A * \lambda$. We refer to A as the *refinable function associated to the mask* a. Note that here we express the refinement equation in the spatial domain while in (2.1) the frequency domain version is used.

This section is motivated by the study of the convergence of the subdivision scheme for the coefficient vector of the Laurent expansion of the special family of rational functions corresponding to

$$a(z) = 1 + z^{-1} \frac{z^2 - t}{1 - tz^2}, \qquad z \in \mathbb{C}, \tag{3.3}$$

where $t \in \mathbb{R}$, $|t| < 1$. This is the simplest case of the rational masks given in (2.6). Here, $n = 1$ and $k = 1$ unless $t = -\frac{1}{3}$ in which case (3.3) becomes the Butterworth autocorrelation with $k = 3$.

Theorem 3.1. *Let a be defined as in* (3.3)*. For $t \in \left(-\frac{1}{2}, 0\right)$ the subdivision scheme determined by the coefficients of the Laurent expansion of a in* (3.3) *converges and the associated refinable function is continuous, orthonormal and cardinal.*

Later, by using a Fourier analysis approach to convergence, we shall show that for $t = -\frac{1}{3}$ the associated refinable function is even continuously differentiable. In general, subdivision and the Fourier approach are two methods of estimating regularity or proving the existence of refinable functions, and they are not mutually exclusive. In fact, for the proof of Theorem 3.1 we find the subdivision approach more convenient, while in other situations it will be easier to check the conditions of Theorem 4.2. In either case, *characterizations* would be based on considering some sort of spectral radius, i.e., an unbounded number of iterations.

To prove this theorem, we provide a sufficient condition for the convergence of the subdivision scheme. To this end, we introduce the forward difference operator Δ, defined for $\lambda \in \ell_\infty(\mathbb{Z})$ as

$$(\Delta\lambda)_k = \lambda_{k+1} - \lambda_k, \qquad k \in \mathbb{Z},$$

and set $u = (1\ :\ k \in \mathbb{Z})$.

Theorem 3.2. *If the bi–infinite sequence $a = (a_j\ :\ j \in \mathbb{Z})$ satisfies the conditions that $(|j\, a_j|\ :\ j \in \mathbb{Z}) \in \ell_1(\mathbb{Z})$, $S_a u = u$ and there exists a number $r \in \mathbb{N}$ such that $\|S_b^r\|_\infty < 1$, where $\Delta S_a = S_b \Delta$, then the subdivision scheme S_a converges.*

The proof of this fact is based on the following lemma which extends an idea in [12].

Lemma 3.3. *If the bi–infinite sequence* $a = (a_j \ : \ j \in \mathbb{Z})$ *satisfies the condition* $(|j\, a_j| \ : \ j \in \mathbb{Z}) \in \ell_1(\mathbb{Z})$ *then there exists a positive constant* M *such that for all* $\lambda \in \ell_\infty(\mathbb{Z})$ *we have that* $\|S_a \lambda\|_\infty \leq M \, \|\Delta\lambda\|_\infty$.

Proof. Note that for any subdivision operator S_a with $S_a u = 0$ we have that $S_a \lambda = X \Delta \lambda$ where X is the bi–infinite matrix whose elements are defined as

$$X_{jk} = \sum_{\ell=k+1}^{\infty} a_{j-2\ell}, \qquad j,k \in \mathbb{Z}.$$

Alternatively, we have for $j,k \in \mathbb{Z}$ the representation

$$X_{jk} = -\sum_{\ell=-\infty}^{k} a_{j-2\ell}.$$

Now, we can estimate the quantity $\|X\|_\infty$ by first noting that

$$\|X\|_\infty = \sup_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |X_{jk}| \,,$$

then choosing $j \in \mathbb{Z}$ and observing that

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} |X_{jk}| &= \sum_{k \geq j/2} \left| \sum_{\ell=k+1}^{\infty} a_{j-2\ell} \right| + \sum_{k < j/2} \left| \sum_{\ell=-\infty}^{k} a_{j-2\ell} \right| \leq \\
&\leq \sum_{k \geq j/2} \sum_{\ell > k} |a_{j-2\ell}| + \sum_{k < j/2} \sum_{\ell \leq k} |a_{j-2\ell}| = \\
&= \sum_{\ell > j/2} \sum_{j/2 \leq k < \ell} |a_{j-2\ell}| + \sum_{\ell < j/2} \sum_{\ell \leq k < j/2} |a_{j-2\ell}| \leq \\
&\leq \sum_{\ell > j/2} (\ell - j/2) \, |a_{j-2\ell}| + \sum_{\ell < j/2} (j/2 - \ell) \, |a_{j-2\ell}| \leq \sum_{\ell \in \mathbb{Z}} \left| \ell - \frac{j}{2} \right| \, |a_{j-2\ell}| = \\
&= \begin{cases} \displaystyle\sum_{\ell \in \mathbb{Z}} |\ell| \; |a_{2\ell}|, & j \in 2\mathbb{Z}, \\ \displaystyle\sum_{\ell \in \mathbb{Z}} \left(|\ell| + \frac{1}{2} \right) |a_{2\ell+1}|, & j \in 2\mathbb{Z}+1, \end{cases}
\end{aligned}$$

which is independent of $j \in \mathbb{Z}$. ∎

With this lemma in hand, we can now return to the proof of Theorem 3.2.

PROOF (THEOREM 3.2). Choose any finitely supported mask $g \in \ell_1(\mathbb{Z})$ such that the subdivision scheme S_g converges with an associated compactly supported refinable function $G \in C(\mathbb{R})$ which is nonnegative on $\mathbb{R}$ and has the property that

$$\sum_{j \in \mathbb{Z}} G(\cdot - j) = 1.$$

For $\lambda \in \ell_\infty(\mathbb{R})$, we consider the sequence of functions

$$f^n := G * (S_a^n \lambda)(2^n \cdot), \qquad n \in \mathbb{Z}_\infty.$$

Consequently, we have that

$$f^{n+1} - f^n = G * ((S_a - S_g) S_a^n \lambda) \left(2^{n+1} \cdot\right).$$

Because $(S_a - S_g) u = 0$ and $a - g$ satisfies the condition of Lemma 3.3, we conclude that there exists a constant M such that

$$\left\|f^{n+1} - f^n\right\|_\infty \leq \|(S_a - S_g) S_a^n \lambda\|_\infty \leq M \|\Delta S_a^n \lambda\|_\infty = M \|S_b^n \Delta \lambda\|_\infty .$$

Hence, $\{f^n : n \in \mathbb{Z}_\infty\}$ is a Cauchy sequence and thus converges uniformly to a continuous function. □

We now are ready to use Theorem 3.2 to prove Theorem 3.1. To this end, we define the rational function q by setting, for $z \in \mathbb{C}$, $q(z) := a(z)/(z+1)$. We see that the coefficients of q are given by

$$q_j = \begin{cases} 0, & j < -1 \\ -t, & j = -1 \\ t^k (1+t) & j = 2k, \quad k \in \mathbb{Z}_\infty, \\ -t^{k+1}(1+t) & j = 2k+1, \quad k \in \mathbb{Z}_\infty. \end{cases} \tag{3.4}$$

Our goal is to show that $\left\|S_q^2\right\|_\infty < 1$ for $t \in \left(-\frac{1}{2}, 0\right)$. We begin by recalling from [2] that

$$\left\|S_q^2\right\|_\infty = \max_{\epsilon \in \mathbb{Z}_4} \sum_{k \in \mathbb{Z}} \left|q^1_{4k-\epsilon}\right|,$$

where $q^1 = \left(q^1_j : j \in \mathbb{Z}\right)$ is the bi–infinite vector whose coordinates are given for $j \in \mathbb{Z}$ as

$$q^1_j = \sum_{k \in \mathbb{Z}} q_{j-2k} q_k.$$

Therefore, we obtain for $k \in \mathbb{Z}$ that

$$\begin{aligned} q^1_{4k} &= (1+t)\, t^k \left(1+t-t^{k+2}\right), \\ q^1_{4k-1} &= -(1+t)t^{k+1}\left(1-t^{k+1}\right), \\ q^1_{4k-2} &= -t^{2k+1}(1+t), \\ q^1_{4k-3} &= -t^k(1+t)\left(1-t^{k+1}\right). \end{aligned}$$

Since $1+t-t^{k+2} \geq 0$ for all $k \in \mathbb{Z}_\infty$ if and only if $\frac{1-\sqrt{5}}{2} \leq t < 0$, we conclude for any such t that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left|q^1_{4k}\right| &\leq \frac{1+t}{1+t^2}, \\ \sum_{k\in\mathbb{Z}} \left|q^1_{4k-1}\right| &\leq 1-\frac{1+t}{1+t^2}, \\ \sum_{k\in\mathbb{Z}} \left|q^1_{4k-2}\right| &\leq \frac{-t}{1-t}. \end{aligned}$$

Consequently, we observe that the upper bounds are less than 1 for any t such that $\frac{1-\sqrt{5}}{2} \leq t < 0$. We also get that

$$\sum_{k\in\mathbb{Z}} \left|q^1_{4k-3}\right| \leq 2t^2 - \frac{t(1+t)}{1+t^2}.$$

Now, observe that, as a function of t, the upper bound is decreasing for $-\frac{1}{2} \leq t < 0$ and is less than 1 for $t = -\frac{1}{2}$. Hence, by appealing to Theorem 3.2, the proof of convergence is established.

4 – Orthonormal cardinal functions with rational symbol

In this section, we use the rational symbol in (2.15) to construct an orthonormal cardinal refinable function A. To this end, we solve the refinement equation (2.1) for $\widehat{A}$, estimate its decay at $\pm\infty$ and then obtain A from the Fourier inversion formula. We begin by writing a in the form

$$a(z) = 2\left(\frac{1+z}{2}\right)^k b(z), \qquad z \in \mathbb{C}, \tag{4.1}$$

where

$$b(z) = \frac{2^k q(z)}{(1+z)^k q(z) - (1-z)^k q(-z)}, \qquad z \in \mathbb{C}, \tag{4.2}$$

and, assuming that $k = 2m+1$ is odd,

$$q(z) = \sum_{j \in \mathbb{Z}_{n-m+1}} \alpha_{2j} (1+z)^{N-2j} (1-z)^{2j}, \qquad z \in \mathbb{C}, \tag{4.3}$$

is a real symmetric polynomial of degree $N = 2n+1-k$.

We estimate the decay of the infinite product

$$G(w) = \prod_{\ell \in \mathbb{N}} \frac{1}{2} a(e^{-iw/2^\ell}) \tag{4.4}$$

using the methods in [4]. Since a has no poles on $\partial\Delta$ and $a(1) = 2$, the infinite product converges uniformly on compact subsets of the real line $\mathbb{R}$. In order to estimate the decay of G, we have to find an upper bound for the function b on $\partial\Delta$. We denote the denominator of b by d which is given for $z \in \mathbb{C}$ as

$$d(z) = (1+z)^k q(z) - (1-z)^k q(-z). \tag{4.5}$$

Also, we introduce the function Q, defined for $w \in \mathbb{R}$ as

$$Q(w) = \left(\cos \frac{w}{2}\right)^{2\lfloor k/2 \rfloor - k + 1} \sum_{j \in \mathbb{Z}_{n-\lfloor k/2 \rfloor + 1}} (-1)^j \alpha_{2j} \left(\cos^2 \frac{w}{2}\right)^{n - \lfloor k/2 \rfloor - j} \left(\sin^2 \frac{w}{2}\right)^j . \tag{4.6}$$

We record below the connection between these functions which is verified by direct computation.

Lemma 4.1. *For $w \in \mathbb{R}$ and an odd positive integer k we have that*

(a) $\left|q\left(e^{-iw}\right)\right| = 2^{2n+1-k} \left|Q(w)\right|$.

(b) $\left|d\left(e^{-iw}\right)\right| = 2^{2n+1} \left[\left(\cos^2 \frac{w}{2}\right)^k Q(w)^2 + \left(\sin^2 \frac{w}{2}\right)^k Q(w+\pi)^2\right]^{1/2}$.

(c) $\left|b\left(e^{-iw}\right)\right| = \left|Q(w)\right| / \left[\left(\cos^2 \frac{w}{2}\right)^k Q(w)^2 + \left(\sin^2 \frac{w}{2}\right)^k Q(w+\pi)^2\right]^{1/2}$.

Theorem 4.2. *Let a be the rational symbol in* (4.1) *where q is a real symmetric polynomial of degree $N = 2n+1-k$, k an odd positive integer and suppose that there are positive constants L and U such that for all $w \in \mathbb{R}$*

$$0 < L \le |Q(w)| \le U < \infty.$$

Then the infinite product in (4.4) *converges uniformly on compact subsets of $\mathbb{R}$ and there is a positive constant $c \in \mathbb{R}$ such that for all $w \in \mathbb{R}$ we have*

$$|G(w)| \le c\,(1+|w|)^{-\frac{k+1}{2} + \log_2 \frac{U}{L}} . \tag{4.7}$$

Furthermore, if $U/L < 2^k$, then A with $\widehat{A} = G$ is an orthonormal refinable function, and if $U/L < 2^{k-1/2}$, then A is an orthonormal cardinal function.

In particular, if $k = 2n+1$, then for all $w \in \mathbb{R}$

$$|G(w)| \le c\,(1+|w|)^{-(n+1)} \tag{4.8}$$

and A is an orthonormal cardinal function.

Proof. Our hypothesis on Q implies for all $w \in \mathbb{R}$ that

$$\frac{L}{U} \leq \left|b\left(e^{-iw}\right)\right| \leq \frac{U}{L} 2^{(k-1)/2}.$$

Therefore, the infinite product in (4.4) converges uniformly on compact subsets of $\mathbb{R}$. Since

$$B_1 := \max\left\{\left|b\left(e^{-iw}\right)\right| \ : \ w \in I_\pi\right\} \leq \frac{U}{L} 2^{(k-1)/2},$$

we conclude that there exists a positive constant c such that for all $w \in \mathbb{R}$ we have

$$|G(w)| \leq c\,(1+|w|)^{-\frac{k+1}{2}+\log_2 \frac{U}{L}},$$

cf. [4]. The other assertions are consequences of the above decay. $\square$

We remark that the decay rate in (4.8) can be improved by considering the quantity $B_2 := \max\left\{\left|b(z)b\left(z^2\right)\right| \ : \ z \in \partial\Delta\right\}$. For example, if $n = 1$ and $k = 3$ then $B_1 = 2$ but $B_2 = 18/7$. Therefore, we get for all $w \in \mathbb{R}$ that

$$|G(w)| \leq c\,(1+|w|)^{-3+\frac{1}{2}\log_2 B_2} = c\,(1+|w|)^{-2.26\cdots},$$

an improvement over the estimate $|G(w)| \leq c\,(1+|w|)^{-2}$, $w \in \mathbb{R}$, obtained by using B_1. From this decay rate we see that A is continuously differentiable on $\mathbb{R}$. This case corresponds to the parameter $t = -\frac{1}{3}$ in (3.3) which is discussed after Theorem 3.1. In general, the computation of B_j for general $j \geq 2$ is beyond us.

The regularity estimate for G given in (4.7) associated with the rational function a given by (4.1) is also discussed in [3], [8]. However, in both instances the authors are only concerned with orthonormal refinable functions. The rational filter (4.1) which we study here also appear in [1], [15] where their relationship to discrete splines is pointed out.

Our final remarks in this section concern the limiting behavior as $n \to \infty$ for the refinable function A_n corresponding to the symbol

$$a_n(z) = 2\frac{(1+z)^{2n+1}}{(1+z)^{2n+1} - (1-z)^{2n+1}}.$$

Recall that

$$\widehat{A}_n(w) = \prod_{\ell \in \mathbb{N}} \frac{1}{2}\, a_n(e^{-iw/2^\ell}), \qquad w \in \mathbb{R}.$$

Since we have that

$$\frac{1}{2} a_n(e^{-iw}) = \frac{1}{1 - i(-1)^n \left(\tan\dfrac{w}{2}\right)^{2n+1}}$$

it follows that

$$\lim_{n\to\infty} \frac{1}{2} a_n \left(e^{-iw}\right) = s(w) \tag{4.9}$$

uniformly for any compact subsets of $\mathbb{R} \setminus \left\{\frac{2k+1}{2}\pi \ : \ k \in \mathbb{Z}\right\}$. Recall that the Fourier transform of the sinc function S is given by the equation $\widehat{S} = \chi_{I_\pi}$ and that the function s appears in the refinement equation (1.5). These comments motivate the following result.

THEOREM 4.3. *There holds*

$$\lim_{n\to\infty} \widehat{A}_n = \widehat{S}$$

where convergence takes place a.e. in $\mathbb{R}$, *in* $L^1(\mathbb{R})$ *and in* $L^2(\mathbb{R})$. *Moreover, we have that*

$$\lim_{n\to\infty} A_n = S$$

in $L^2(\mathbb{R})$ *and uniformly on* $\mathbb{R}$.

For the proof we define three functions H, P and R for $w \in \mathbb{R}$ as

$$P(w) := \begin{cases} \dfrac{1}{\cos^6(w/2)}, & |w| \le \dfrac{\pi}{2} \\ \dfrac{2}{\cos^6(w/2) + \sin^6(w/2)}, & \dfrac{\pi}{2} \le |w| \le \pi, \end{cases}$$

$R(w) = \cos^6 \frac{w}{2}\, P(w)$ and

$$H(w) := \prod_{\ell \in \mathbb{N}} R\left(2^{-\ell} w\right).$$

To proceed further, one should verify that

$$\max\left\{|P(w)| \ : \ w \in I_\pi\right\} = 8 \tag{4.10}$$

and observe that

$$H(w) = 1, \qquad w \in I_\pi. \tag{4.11}$$

LEMMA 4.4. *There exists a positive constant* c *such that for any* $w \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *we have that*

(a) $\left|\frac{1}{2} a_n \left(e^{-iw}\right)\right|^2 \le R(w)$.
(b) $|H(w)| \le c\,(1 + |w|)^{-3}$.
(c) $\left|\frac{1}{2} a_n \left(e^{-iw}\right) - 1\right| \le \min\left\{1, \frac{2}{\pi}|w|\right\}$.

PROOF. The first claim follows directly from the definition of a_n and R while (b) is a consequence of equation (4.10) used to estimate G in Theorem 4.2. For (c), we note for all $w \in \mathbb{R}$ that

$$\left|\frac{1}{2}a_n\left(e^{-iw}\right)-1\right| = \frac{|\sin\frac{w}{2}|^{2n+1}}{\left[\left(\cos^2\frac{w}{2}\right)^{2n+1}+\left(\sin^2\frac{w}{2}\right)^{2n+1}\right]^{1/2}} \leq 1,$$

while for $|w| \leq \pi/2$ we have that $\left(\tan\frac{|w|}{2}\right)^{2n+1} \leq \left(\frac{2}{\pi}|w|\right)^{2n+1}$. Therefore, we conclude for $|w| \leq \pi/2$ that

$$\left|\frac{1}{2}a_n\left(e^{-iw}\right)-1\right| = \frac{\left|\tan\frac{w}{2}\right|^{2n+1}}{\left[1+\left(\tan^2\frac{w}{2}\right)^{2n+1}\right]^{1/2}} \leq \left(\tan\frac{|w|}{2}\right)^{2n+1} \leq \frac{2}{\pi}|w|. \qquad \square$$

LEMMA 4.5.

(a) *The infinite product*

$$\prod_{\ell\in\mathbb{N}}\frac{1}{2}a_n\left(e^{iw/2^\ell}\right)$$

converges uniformly in $n \in \mathbb{N}$ and ω in any compact subset of $\mathbb{R}$.

(b) *We have that*

$$\lim_{n\to\infty}\widehat{A}_n(w) = \widehat{S}(w), \qquad \text{a.e. } w \in \mathbb{R}.$$

PROOF. (a) Fix a compact interval I. Choose a positive integer k so that $|w/2^k| \leq \pi/2$ for all $w \in I$. By Lemma 4.4 we obtain that

$$\sum_{\ell\in\mathbb{N}}\left|\frac{1}{2}a_n\left(e^{-iw/2^\ell}\right)-1\right| =$$
$$= \sum_{\ell\in\mathbb{Z}_k}\left|\frac{1}{2}a_n\left(e^{-iw/2^{\ell+1}}\right)-1\right| + \sum_{\ell\in\mathbb{Z}_\infty}\left|\frac{1}{2}a_n\left(e^{-iw/2^{\ell+k+1}}\right)-1\right| \leq k+\frac{2}{\pi}\frac{|w|}{2^k},$$

uniformly in n. This estimate proves the claim.

(b) Fix $w \in I_\pi$ and $\epsilon > 0$. There exists a positive integer k such that for all $n \in \mathbb{N}$ we have that

$$\left|\widehat{A}_n(w) - \prod_{\ell\in\mathbb{Z}_k}\frac{1}{2}a_n\left(e^{-iw/2^{\ell+1}}\right)\right| < \epsilon.$$

Therefore, we obtain that

$$\begin{aligned}&\left|\widehat{A}_n(w) - \widehat{S}(w)\right| \leq \\ &\leq \left|\widehat{A}_n(w) - \prod_{\ell \in \mathbb{Z}_k} \frac{1}{2} a_n\left(e^{-iw/2^{\ell+1}}\right)\right| + \left|\prod_{\ell \in \mathbb{Z}_k} \frac{1}{2} a_n\left(e^{-iw/2^{\ell+1}}\right) - 1\right| < \\ &< \epsilon + \left|\prod_{\ell \in \mathbb{Z}_k} \frac{1}{2} a_n\left(e^{-iw/2^{\ell+1}}\right) - 1\right|.\end{aligned}$$

We now appeal to equation (4.9) and choose a positive integer m so that

$$\left|\prod_{\ell \in \mathbb{Z}_k} \frac{1}{2} a_n\left(e^{-iw/2^{\ell+1}}\right) - 1\right| < \epsilon, \qquad n \in m + \mathbb{Z}_\infty.$$

Therefore we conclude that

$$\lim_{n \to \infty} \widehat{A}_n(w) = \widehat{S}(w), \qquad \text{a.e. } w \in I_\pi.$$

We extend the convergence to $\mathbb{R}$ by comparing the refinement equation $\widehat{A}_n(2\cdot) = \frac{1}{2} a_n\left(e^{-i\cdot}\right) \widehat{A}_n$ for A_n to the one for S given in (1.5). □

We are now ready to complete the proof of Theorem 4.3.

Proof (Theorem 4.3). By part (a) of Lemma 4.4 we conclude for $w \in \mathbb{R}$ that $\left|\widehat{A}_n(w)\right|^2 \leq H(w)$. Since $H^{1/2}$ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the result follows from the dominated convergence theorem. □

5 – Stable filters

Recall that the rational function a defined by (2.15) always has a pole at $z = 0$. We want to determine a so that it is analytic in a neighborhood of $\overline{\Delta} \setminus \{0\}$. In this case, a gives rise to an infinite impulse response (IIR) digital filter which can be efficiently implemented by means of *delayed feedback* whenever it is *stable*, see [7] for additional details. Recall that the Butterworth autocorrelation filter in (2.16) is *not* stable. In this section we present a method for the construction of stable filters of the type (2.15). Our remarks in this section support the following conjecture.

Conjecture 5.1. If $k \in \mathbb{N}$ is an odd integer and $k \leq n$ then there exists a real symmetric polynomial q of degree $2n + 1 - k$ such that the polynomial d in (4.5) has a k–fold zero at zero and all its other zeros are outside $\overline{\Delta}$.

Before we address the details and numerical computations that support this conjecture, we explain how we approach it. First, we recall the celebrated *Hurwitz Criterion* for a polynomial to have zeros in the left half plane [5]. This is expressed in terms of determinants formed from the coefficients of the polynomial in its monomial basis. Recall that the linear fractional transformation $z = (w-1)(w+1)^{-1}$ maps the left half plane to the exterior of $\overline{\Delta}$. We associate with a coefficient vector $p = (p_j : j \in \mathbb{Z}_{n+1})$ two different polynomials. The first one which we call p_{be} uses the coefficient vector with the Bernstein basis relative to the interval $[-1,1]$,

$$p_{be}(z) = \sum_{j\in\mathbb{Z}_{n+1}} p_j\,(1-z)^j\,(1+z)^{n-j}, \qquad z \in \mathbb{C}, \tag{5.1}$$

and the other polynomial p_{mo} uses it with respect to the monomial basis,

$$p_{mo}(w) = \sum_{j\in\mathbb{Z}_{n+1}} p_j\,w^{n-j}, \qquad w \in \mathbb{C}. \tag{5.2}$$

These two polynomials are connected by means of the formula

$$p_{be}(z) = \left(\frac{2}{w+1}\right)^n p_{mo}(w), \qquad z = \frac{w-1}{w+1}.$$

Therefore, the Hurwitz criterion applied to p_{mo} will determine whether or not the zeros of p_{be} are outside $\overline{\Delta}$. The representation (5.1) is also convenient for the study of the polynomial

$$d(z) = (1+z)^k q(z) - (1-z)^k q(-z). \tag{5.3}$$

We wish to identify q so that $d(z) = z^k\,r(z)$ and the polynomial r has all its zeros outside of $\overline{\Delta}$. So, the procedure is to first express q in its Bernstein form and then find the Bernstein representation of r. This computation is burdensome because of the factor z^k. Let us not be discouraged by that and begin this task.

We write $k = 2m+1$ and let q be a symmetric polynomial of degree $N = 2n+1-k$, given as

$$q(z) = \sum_{j\in\mathbb{Z}_{n-m+1}} q_j\,(1-z)^{2j}\,(1+z)^{N-2j}, \qquad z \in \mathbb{C}. \tag{5.4}$$

Similarly, with $M = 2n+1$, we express d in the form

$$d(z) = \sum_{j\in\mathbb{Z}_{M+1}} d_j\,(1-z)^j\,(1+z)^{M-j}. \tag{5.5}$$

Lemma 5.2. *The polynomial d in* (5.5) *satisfies* (5.3) *for some q of the form* (5.4) *if and only if*

$$d_j = -d_{M-j}, \quad j \in \mathbb{Z}_{M+1}, \qquad d_{2j+1} = 0, \quad j \in \mathbb{Z}_m. \tag{5.6}$$

PROOF. Substituting (5.4) into (5.3) and comparing coefficients we obtain that d and q are related by the equations

$$d_{2j} = q_j, \qquad d_{2(j+m)+1} = -q_{n-m-j}, \qquad j \in \mathbb{Z}_{n-m+1}, \tag{5.7}$$

where we also require that the coefficients of d not appearing above are zero. From these formulas the claim follows. ∎

A consequence of Lemma 5.2 is that for $k > 1$ the polynomial d must have zeros inside Δ. Indeed, Lemma 5.2 implies that d has zero coefficients whenever $k > 1$ while the Hurwitz criterion yields that p_{mo} must have all nonzero coefficients of the same sign when it has its zeros in the left half plane. In addition, (5.6) tells us that every positive coefficient of d also implies the occurrence of a negative coefficient.

Motivated by this observation, we fix the order of zero of d at $z = 0$ and adjust the remaining factor to have all its zeros outside $\overline{\Delta}$. Our experience suggests that we choose the order of the zero at zero to be k. Hence, we write d as $d(z) = z^k\, r(z)$ for some polynomial r of degree $2n + 1 - k$ whose zeros will be outside the unit circle.

For the next result we introduce the backwards difference operator ∇ defined on a bi–infinite sequence $\lambda = (\lambda_j \,:\, j \in \mathbb{Z})$ as $(\nabla\lambda)_j = \lambda_j - \lambda_{j-1}$, $j \in \mathbb{Z}$.

THEOREM 5.3. *If $k \le n$ and r is a polynomial of degree N, written as*

$$r(z) = \sum_{j\in\mathbb{Z}_{N+1}} r_j\, (1-z)^j\, (1+z)^{N-j}\,, \tag{5.8}$$

then there exists a symmetric polynomials q of degree N such that $d(z) = z^k\, r(z)$ if and only if $r_j = r_{N-j}$, $j \in \mathbb{Z}_{N+1}$, and there exist coefficients α_j, $j \in \mathbb{Z}_m$, such that for $\ell \in \mathbb{Z}_m$ we have that

$$\begin{aligned} r_{2\ell} &= \sum_{j\in\mathbb{Z}_{\ell+1}} \alpha_j \binom{k}{2\ell-2j}, \\ r_{2\ell+1} &= \sum_{j\in\mathbb{Z}_{\ell+1}} \alpha_j \binom{k}{2\ell+1-2j}. \end{aligned} \tag{5.9}$$

PROOF. First, we set $r_j = 0$, $j \in \mathbb{Z} \setminus \mathbb{Z}_{N+1}$, and use the fact that $z = \frac{1}{2}\left[(1+z) - (1-z)\right]$ to conclude that

$$z\, r(z) = \frac{1}{2} \sum_{j\in\mathbb{Z}_{N+2}} (\nabla r)_j\, (1-z)^j\, (1+z)^{N-j+1}\,.$$

Repeating this process, it follows that

$$z^k\, r(z) = \frac{1}{2^k} \sum_{j\in\mathbb{Z}_{M+1}} \left(\nabla^k r\right)_j\, (1-z)^j\, (1+z)^{M-j} \tag{5.10}$$

and comparing coefficients with (5.5) we get that

$$d_j = 2^{-k} \left(\nabla^k r\right)_j, \qquad j \in \mathbb{Z}_{M+1}. \tag{5.11}$$

We first prove that the existence of q such that $d(z) = z^k\, r(z)$ is equivalent to the requirement that $r_j = r_{N-j}$, $j \in \mathbb{Z}_{N+1}$, and

$$r_{2\ell+1} = \sum_{j\in\mathbb{Z}_{2\ell+1}} (-1)^j \binom{k}{2\ell+1-j} r_j, \qquad \ell \in \mathbb{Z}_m. \tag{5.12}$$

We begin the proof by showing how to compute r_0 and r_N from q_0. This is accomplished by choosing $j = 0$ and $j = 2n+1$ in (5.11) and upon simplification we get $r_0 = r_N = 2^k\, q_0$. Similarly, we choose $j = 1$ and $j = 2n$ in (5.11) and using the fact that $d_1 = d_{2n} = 0$ from Lemma 5.2, we obtain $r_1 = r_{N-1} = k2^k\, q_0$. Thus, we have proved the cases $\ell = 0, 1$ of the following identity (5.13) that we claim to hold for all $\ell \in \mathbb{Z}_k$ and determines r_ℓ and $r_{N-\ell}$ uniquely and symmetrically in terms of q_j, $j \in \mathbb{Z}_{\ell/2+1}$:

$$r_\ell = r_{N-\ell} = \begin{cases} 2^k\, q_{\ell/2} - \displaystyle\sum_{j\in\mathbb{Z}_\ell} (-1)^j \binom{k}{\ell-j} r_j, & \ell \in 2\,\mathbb{Z}_\infty, \\ \displaystyle\sum_{j\in\mathbb{Z}_\ell} (-1)^j \binom{k}{\ell-j} r_j, & \ell \in 2\,\mathbb{Z}_\infty + 1. \end{cases} \tag{5.13}$$

In particular, the second part of (5.13) yields (5.12). We prove this by induction and advance the induction hypothesis by setting $j = \ell+1$ and $j = M-\ell-1$ in (5.11), isolating $r_{\ell+1}$ and $r_{N-\ell-1}$, respectively. An application of the induction hypothesis of symmetry on these expressions then shows that (5.13) also holds with ℓ replaced by $\ell+1$. Processing with the same argument we also obtain for $k \le \ell \le n$ that

$$r_\ell = r_{N-\ell} = \sum_{j\in\mathbb{Z}_k} (-1)^j \binom{k}{j} r_{\ell-k+j} + 2^k \begin{cases} q_m, & \ell \in 2\mathbb{Z}_\infty, \\ -q_{N-m}, & \ell \in 2\mathbb{Z}_\infty + 1. \end{cases} \tag{5.14}$$

Hence, the coefficients of q uniquely determine the coefficients of r. Conversely, the above computations also show that for any polynomial r of degree N whose coefficient vector r satisfies (5.12), there exists a symmetric polynomial q of degree N such that

$$z^k\, r(z) = (1+z)^k\, q(z) - (1-z)^k\, q(-z), \qquad z \in \mathbb{C},$$

which completes the proof.

It remains to relate (5.12) to (5.9). To that end we write (5.12) in matrix form as $M\widetilde{r} = 0$, where $\widetilde{r} = (r_j \;:\; j \in \mathbb{Z}_{2m})$ is the initial segment of r relevant for (5.12) and

$$M = \begin{bmatrix} \binom{k}{1} & -\binom{k}{0} & 0 & 0 & \dots & 0 & 0 \\ \binom{k}{3} & -\binom{k}{2} & \binom{k}{1} & -\binom{k}{0} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{k}{2m-1} & -\binom{k}{2m-2} & \binom{k}{2m-3} & \binom{k}{2m-4} & \dots & \binom{k}{1} & -\binom{k}{0} \end{bmatrix}$$

$$\in \mathbb{R}^{m\times 2m}$$

has rank m. Next, we note that $MM' = 0$ for the matrix

$$M' = \begin{bmatrix} \binom{k}{0} & 0 & \dots & 0 \\ \binom{k}{1} & 0 & \dots & 0 \\ \binom{k}{2} & \binom{k}{0} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k}{2m-3} & \binom{k}{2m-5} & \dots & 0 \\ \binom{k}{2m-2} & \binom{k}{2m-4} & \dots & \binom{k}{1} \\ \binom{k}{2m-1} & \binom{k}{2m-3} & \dots & \binom{k}{0} \end{bmatrix} \in \mathbb{R}^{2m\times m},$$

which follows by direct computations taking into account that

$$\begin{aligned}
&\sum_{t=2j}^{2\ell+1}(-1)^t\binom{k}{2\ell+1-t}\binom{k}{t-2j} = \\
&= \frac{1}{2}\left[\sum_{t=2j}^{2\ell+1}(-1)^t\binom{k}{2\ell+1-t}\binom{k}{t-2j} + \right. \\
&\quad \left. + \sum_{t=2j}^{2\ell+1}(-1)^{2\ell+1+2j+t}\binom{k}{t-2j}\binom{k}{2\ell+1-t}\right] = \\
&= \frac{1}{2}\sum_{t=2j}^{2\ell+1}\binom{k}{2\ell+1-t}\binom{k}{t-2j}\left((-1)^t+(-1)^{t+1}\right) = 0.
\end{aligned}$$

Thus, the linearly independent columns of M' span the nullspace of M and therefore $\widetilde{r} = M'\alpha$, for some $\alpha = (\alpha_j \ : \ j \in \mathbb{Z}_m)$. Written explicitely, this is (5.9). □

Now we can attack the construction of stable filters, that is, the construction of a Hurwitz polynomial r that satisfies the conditions of Theorem 5.3. We will describe a construction for the case $k = n$.

By Theorem 5.3, the coefficients of r must satisfy the symmetry relations

$$r_j = r_{k+1-j}, \qquad j \in \mathbb{Z}_{m+1} \setminus \mathbb{Z}_3.$$

Substituting (5.9) into these relations we obtain $m-1$ homogeneous equations in the m unknowns α_j, $j \in \mathbb{Z}_m$, from (5.9), i.e., a system of the form

$$M\alpha = 0, \qquad M \in \mathbb{R}^{m-1\times m}, \quad \alpha = (\alpha_j \ : \ j \in \mathbb{Z}_m).$$

Symbolic computations which we performed for all odd values up to $k = 111$ confirm that the rank of M is $m-1$ and therefore setting $\alpha_0 = 1$, $\alpha_1 = t$, $t \in \mathbb{R}$, uniquely determines the remaining values of α. In our experiments we found two values for t useful, namely

$$t_0 = \frac{k+1}{2k-2}, \qquad t_1 = \frac{k+1}{2k}.$$

In both cases, we obtained that all the zeros of the resulting polynomial r were outside the unit circle for all the values of k that we tested, which we verified by checking the Hurwitz criterion symbolically. Indeed, the computational results strongly support Conjecture 5.1 as it turns out the minimal value of all Hurwitz determinants of a given order k was *strictly increasing* with respect to k in all the cases we considered. The use of t_1 resulted in raising the order of the zero of a by two, thus leading to even smoother functions. We close this section by showing some graphs of the associated refinable functions and the placement of the nonzero poles of a for various parameters of k.

Another interesting properties of this construction which is suggested by the numerical computations is that there exists a limit distribution of the poles as $k \to \infty$ which is depicted in fig. 2.

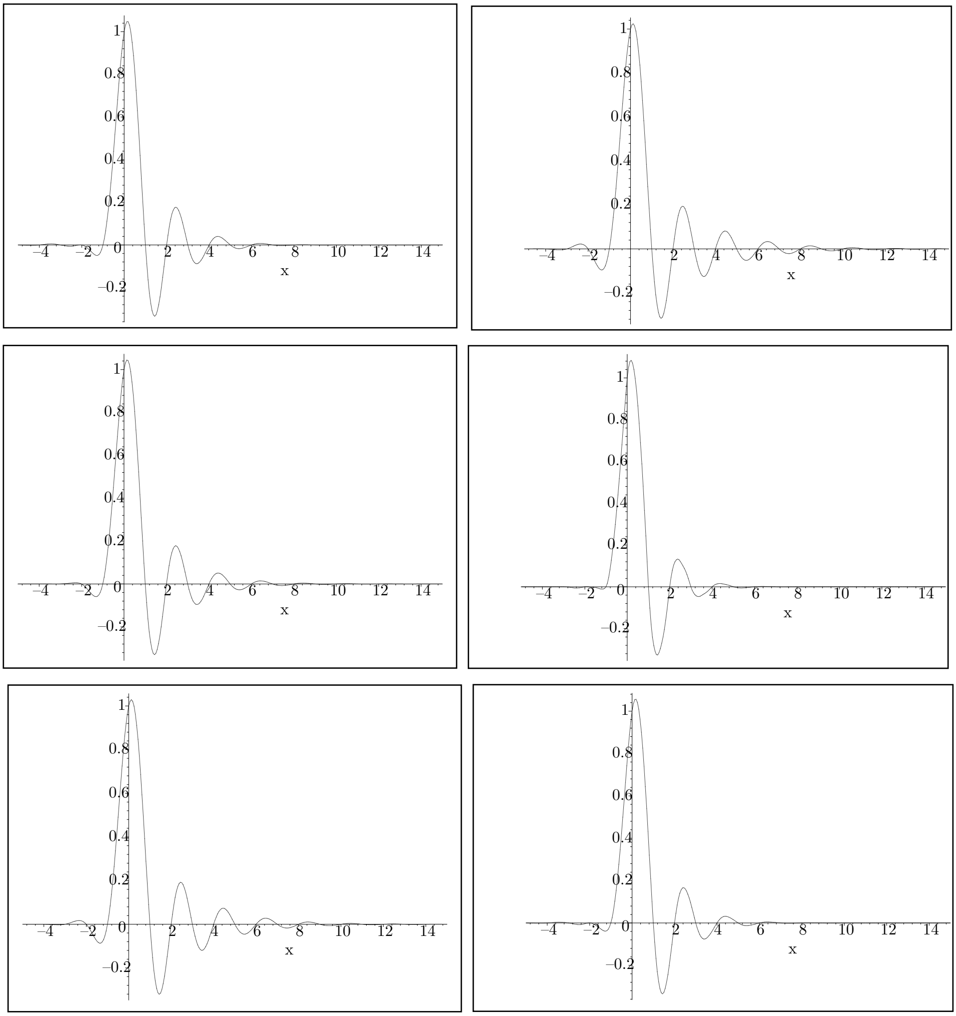

Fig. 1: The refinable functions for $k = 5, 9, 11$ and the choices $\alpha = (k+1)/(2k-2)$ (left) and $\alpha = (k+1)/2k$ (right).

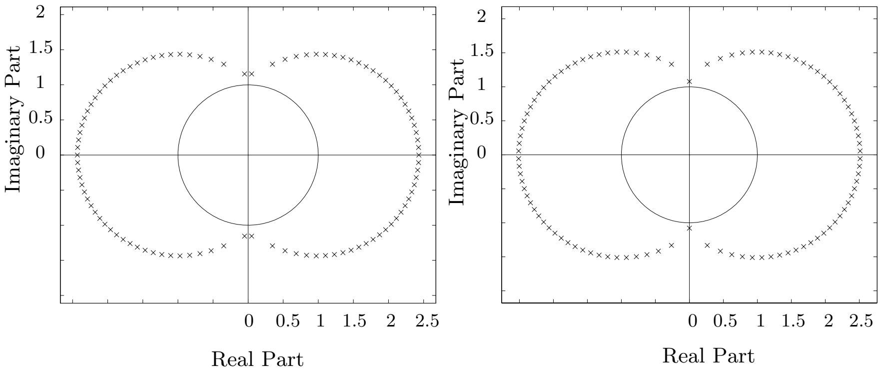

Fig. 2: Distribution of the poles for $k = 101$ and $\alpha = (k+1)/(2k-2)$ (left) and $\alpha = (k+1)/2k$ (right).

REFERENCES

[1] A. Z. Averbuch – A. B. Pevnyi – V. A. Zheludev: *Butterworth wavelet transforms derived from discrete interpolatory splines: recursive interpolation*, Signal Processing, **81** (2001), 2363-2382.

[2] A. S. Cavaretta – W. Dahmen – C. A. Micchelli: *Stationary subdivision*, Memoirs of the AMS, **93** (453) (1991), AMS.

[3] A. Cohen – I. Daubechies: *A new technique to estimate the regularity of refinable functions*, Rev. Mat. Iberioamericana, **12** (1996), 527-591.

[4] I. Daubechies: *Ten Lectures on Wavelets*, SIAM, 1992.

[5] F. R. Gantmacher: *Matrix Theory*, Chelsea Publishing Co, 1959.

[6] T. N. T. Goodman – C. A. Micchelli: *Orthonormal cardinal functions*, In "Wavelets : Theory, Algorithms and Applications", Chui, C. K., Montefusco, L. and Puccio, L., (eds.), Academic Press, 1994, pp. 53-88.

[7] R. W. Hamming: *Digital Filters*, Prentice–Hall, 1989.

[8] C. Herley – M. Vetterli: *Wavelets and recursive filter banks*, IEEE Transactions on Signal Processing, **41** (1993), 2536-2556.

[9] R. M. Lewis: *Cardinal interpolating multiresolution*, J. Approx. Theory, **76** (1994), 177-202.

[10] R. J. Marko II: *Introduction to Shannon Sampling and Interpolation Theory*, Springer Verlag, 1991.

[11] C. A. Micchelli – H. Prautzsch: *Uniform refinement of curves*, Linear Algebra and its Applications, **114/115** (1989), 841-870.

[12] C. A. Micchelli – T. Sauer: *Regularity of multiwavelets*, Advances Comput. Math., **7** (1997), 455-545.

[13] C. E. Shannon: *A mathematical theory of communication*, Bell System Tech. J., **27** (1948), 379-423.

[14] E. T. Whittaker: *On the functions which are represented by the expansion of the interpolation theory*, Proc. Roy. Soc. Edinburgh, sec. A, **35** (1915), 181-194.

[15] V. A. Zheludev: *Interpolatory subdivision schemes with infinite masks generated from splines*, Adv. Comput. Math., to appear.

Lavoro pervenuto alla redazione il 09 novembre 2004
ed accettato per la pubblicazione il 05 gennaio 2006.
Bozze licenziate il 26 settembre 2006

Indirizzo degli Autori:

Mariantonia Cotronei – DIMET, Università di Reggio Calabria – Via Graziella, I–89060 Reggio Calabria (Italia)
E-mail: mariantonia.cotronei@unirc.it

Maria Laura Lo Cascio – Dipartimento Me.Mo.Mat – Università di Roma "La Sapienza" – Via Scarpa 16, I–00161 Roma (Italia)
E-mail: locascio@dmmm.uniroma1.it

Hong Oh Kim* – Division of Applied Mathematics – KAIST – 373-1 Kusong-Dong Yusong-Gu Taejon 305-701 (Korea)
E-mail: hkim@amath.kaist.ac.kr

Charles A. Micchelli** – Department of Mathematics and Statistics – State University of New York – The University at Albany – Albany, New York (USA)
E-mail: cam@math.albany.edu

Tomas Sauer – Lehrstuhl für Numerische Mathematik – Justus-Liebig-Universität Gießen – Heinrich–Buff–Ring 44 – D–35392 Gießen (Germany)
E-mail: Tomas.Sauer@math.uni-giessen.de

*Supported by grant KRF–2002-070-C00004. **Supported by the US National Science Foundation under grant CCR–0312113.