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Quasi-linear elliptic problems in L^1 with non homogeneous boundary conditions

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ABSTRACT: *We study quasi-linear elliptic problems with L^1 data and non homogeneous boundary conditions. Existence and uniqueness of entropy solutions are proved.*

1 – Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $1 < p < \infty$, and let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function such that (H_1) there exists $\lambda > 0$ such that $a(x,\xi)\cdot\xi \geq \lambda|\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, (H_2) there exists $c > 0$ and $g \in L^{p'}(\Omega)$ such that $|a(x,\xi)| \leq c(g(x) + |\xi|^{p-1})$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$, (H_3) $(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta) > 0$ for a.e. $x \in \Omega$ and for all $\xi,\eta \in \mathbb{R}^N,\ \xi \neq \eta$.

We are interested in the quasi-linear problem

$$(S) \qquad \begin{cases} -\text{div}\ a(.,Du) + u = \phi & \text{in } \Omega \\ a(.,Du)\cdot\eta + \beta(u) \ni \psi & \text{on } \partial\Omega\,, \end{cases}$$

where $\psi \in L^1(\partial\Omega), \phi \in L^1(\Omega)$ and β is a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \beta(0)$.

The main difficulties in the study of this problem are related to the non regularity of the data (see [4]) and to the condition on the boundary which is more general than the classical Dirichlet condition or the Neumann one.

KEY WORDS AND PHRASES: *Quasi-linear elliptic problem – Non homogneous boundary condition – Entropy solution – Accretive operator.*
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We solve problem (S) for $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$ when a is smooth or $D(\beta)$ is closed in the entropy sense introduced in [4] for problem (S) with homogeneous Dirichlet condition. The homogeneous case (that is $\psi \equiv 0$) was studied in [2] for particular graphs β. In the present paper, we overcome these restrictions on β using similar techniques than the ones employed in [2] and monotonicity arguments.

We also study the quasi-linear problem

$$\text{(P)} \qquad \begin{cases} -\text{div}\; a(., Du) = 0 & \text{in } \Omega \\ a(., Du)\cdot\eta + u = \psi & \text{on } \partial\Omega\,, \end{cases}$$

where $\psi \in L^1(\partial\Omega)$. We introduce a capacity operator which will be used to study parabolic problems with dynamical boundary conditions.

2 – Notations

As usual, λ_N denotes the Lebesgue measure in $\mathbb{R}^N$. For $1 \le p < +\infty$, $L^p(\Omega)$ and $W^{1,p}(\Omega)$ denote respectively the standard Lebesgue and Sobolev spaces, and $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. For $u \in W^{1,p}(\Omega)$, we denote by u or $\gamma(u)$ the trace of u on $\partial\Omega$ in the usual sense and by $W^{\frac{1}{p'},p}(\partial\Omega)$ the set $\gamma(W^{1,p}(\Omega))$.

In [4], the authors introduce the set

$$\mathcal{T}^{1,p}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1,p}(\Omega) \quad \forall k > 0\}\,,$$

where $T_k(s) = \sup(-k, \inf(s,k))$. They also prove that given $u \in \tau^{1,p}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$DT_k(u) = v\chi_{\{|v|<k\}} \quad \forall k > 0\,.$$

This function v will be denoted by Du for the function $u \in \mathcal{T}^{1,p}(\Omega)$. It is clear that if $u \in W^{1,p}(\Omega)$, then $v \in L^p(\Omega)$ and $v = Du$ in the usual sense. As in [2], $\mathcal{T}_{tr}^{1,p}(\Omega)$ denotes the set of functions u in $\mathcal{T}^{1,p}(\Omega)$ satisfying the following condition, there exists a sequence u_n in $W^{1,p}(\Omega)$ such that

(a) u_n converges to u a.e. in Ω,
(b) $DT_k(u_n)$ converges to $DT_k(u)$ in $L^1(\Omega)$ for all $k > 0$,
(c) there exists a measurable function v on $\partial\Omega$, such that $\gamma(u_n)$ converges a.e. in $\partial\Omega$ to v.

The function v is the trace of u in the generalized sense introduced in [2]. In the sequel we use the notations u or $\tau(u)$ to designate the trace of $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ on $\partial\Omega$. Let us recall that in the case $u \in W^{1,p}(\Omega)$, $\tau(u)$ coincides with $\gamma(u)$, the trace of u in the usual sense. Moreover $\gamma(T_k(u)) = T_k(\tau(u))$ for every $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ and $k > 0$, and if $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ and $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u - \phi \in \mathcal{T}_{tr}^{1,p}(\Omega)$ and $\tau(u - \phi) = \tau(u) - \gamma(\phi)$.

3 – Existence and uniqueness of solutions of problem (S)

We will prove existence and uniqueness of an entropy solution of problem (S) in the case $D(\beta)$ is closed or a is *smooth*, that is, for all $\phi \in L^\infty(\Omega)$, there exists $g \in L^1(\partial\Omega)$ such that the solution of the homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div} a(., Du) = \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

is a solution of the Neumann problem

$$\begin{cases} -\operatorname{div} a(., Du) = \phi & \text{in } \Omega \\ a(., Du) \cdot \eta = g & \text{on } \partial\Omega . \end{cases}$$

Functions a corresponding to linear operators with smooth coefficients and p-Laplacian type operators are smooth.

Definition 3.1. A measurable function u in Ω is an entropy solution of problem (S) if $u \in L^1(\Omega) \cap \mathcal{T}^{1,p}_{tr}(\Omega)$ and there exists $w \in L^1(\partial\Omega)$, $w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$, such that

$$\begin{aligned} &\int_\Omega a(., Du) \cdot DT_k(u-v) + \int_\Omega uT_k(u-v) + \int_{\partial\Omega} wT_k(u-v) \leq \\ &\leq \int_{\partial\Omega} \psi T_k(u-v) + \int_\Omega \phi T_k(u-v) \quad \forall k > 0 , \end{aligned} \tag{3.1}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $v(x) \in D(\beta)$ a.e. in $\partial\Omega$.

As we will see in the existence results, when a is smooth it is possible to remove the condition $v(x) \in D(\beta)$ a.e. in $\partial\Omega$ for the test functions in the above definition.

We prove the following result of existence and uniqueness of entropy solutions of problem (S).

Theorem 3.2. *Let $D(\beta)$ be closed or a smooth.*

(i) *For any $\phi \in L^1(\Omega)$, $\psi \in L^1(\partial\Omega)$, there exists a unique entropy solution of problem* (S)*.*

(ii) *If u_1 is the entropy solution of problem* (S) *corresponding to $\phi_1 \in L^1(\Omega)$ and $\psi_1 \in L^1(\partial\Omega)$ and u_2 is the entropy solution of problem* (S) *corresponding to $\phi_2 \in L^1(\Omega)$ and $\psi_2 \in L^1(\partial\Omega)$ then there exist $w_1 \in L^1(\partial\Omega)$, $w_1(x) \in \beta(u_1(x))$ a.e. in $\partial\Omega$, and $w_2 \in L^1(\partial\Omega)$, $w_2(x) \in \beta(u_2(x))$ a.e. in $\partial\Omega$, such that*

$$\begin{aligned} &\int_\Omega a(., Du_i) \cdot DT_k(u_i-v) + \int_\Omega u_iT_k(u_i-v) + \int_{\partial\Omega} w_iT_k(u_i-v) \leq \\ &\leq \int_{\partial\Omega} \psi_i T_k(u_i-v) + \int_\Omega \phi_i T_k(u_i-v) \quad \forall k > 0 , \end{aligned}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$*,* $v(x) \in D(\beta)$ *a.e. in* $\partial\Omega$*,* $i = 1, 2$*. Moreover*

$$\int_\Omega (u_1 - u_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \le \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+ .$$

To prove the above theorem we will proceed by approximation.

Theorem 3.3. *Let* $D(\beta)$ *be closed and* $m, n \in \mathbb{N}$*,* $m \le n$*.*

(i) *For* $\phi \in L^\infty(\Omega)$ *and* $\psi \in L^\infty(\partial\Omega)$*, there exist* $u = u_{\phi,\psi,m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and* $w = w_{\phi,\psi,m,n} \in L^\infty(\partial\Omega)$*,* $w(x) \in \beta(u(x))$ *a.e. on* $\partial\Omega$*, such that*

$$\begin{aligned} &\int_\Omega a(., Du) \cdot D(u - v) + \int_\Omega u(u - v) + \int_{\partial\Omega} w(u - v) + \\ &+ \frac{1}{m} \int_{\partial\Omega} u^+(u - v) - \frac{1}{n} \int_{\partial\Omega} u^-(u - v) \le \\ &\le \int_{\partial\Omega} \psi(u - v) + \int_\Omega \phi(u - v) , \end{aligned} \tag{3.2}$$

for all $v \in W^{1,p}(\Omega)$*,* $v(x) \in D(\beta)$ *a.e. on* $\partial\Omega$*, and all* $k > 0$*. Moreover,*

$$\int_\Omega |u| + \int_{\partial\Omega} |w| \le \int_{\partial\Omega} |\psi| + \int_\Omega |\phi| . \tag{3.3}$$

(ii) *If* $m_1 \le m_2 \le n_2 \le n_1$*,* $\phi_1, \phi_2 \in L^\infty(\Omega)$*,* $\psi_1, \psi_2 \in L^\infty(\partial\Omega)$ *then*

$$\begin{aligned} &\int_\Omega (u_{\phi_1,\psi_1,m_1,n_1} - u_{\phi_2,\psi_2,m_2,n_2})^+ + \int_{\partial\Omega} (w_{\phi_1,\psi_1,m_1,n_1} - w_{\phi_2,\psi_2,m_2,n_2})^+ \le \\ &\le \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+ . \end{aligned}$$

Proof. Observe that $\frac{1}{m}s^+ - \frac{1}{n}s^- = \frac{1}{m}s + \left(\frac{1}{m} - \frac{1}{n}\right) s^- = \left(\frac{1}{m} - \frac{1}{n}\right) s^+ + \frac{1}{n}s$. For $r \in \mathbb{N}$, it is easy to see that the operator $B_r : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ defined by

$$\begin{aligned} \langle B_r u, v\rangle = &\int_\Omega a(x, D(u)) \cdot Dv + \int_\Omega T_r(u)v + \frac{1}{r} \int_\Omega |u|^{p-2}uv + \\ &+ \int_{\partial\Omega} T_r(\beta_r(u))v + \frac{1}{m} \int_{\partial\Omega} T_r(u^+)v - \frac{1}{n} \int_{\partial\Omega} T_r(u^-)v - \\ &- \int_{\partial\Omega} \psi v - \int_\Omega \phi v , \end{aligned} \tag{3.4}$$

where β_r is the Yosida approximation of β, is bounded, coercive, monotone and hemicontinuous. On the other hand, since $D(\beta)$ is closed,

$$W^{1,p}_{\beta}(\Omega) := \{u \in W^{1,p}(\Omega), u(x) \in D(\beta) \text{ } a.e. \text{ on } \partial\Omega\}$$

is a closed convex subset of $W^{1,p}(\Omega)$. Then, by a classical result of Browder ([9]), there exists $u_r = u_{\phi,\psi,m,n,r} \in W^{1,p}(\Omega)$, $u_r(x) \in D(\beta)$ a.e. on $\partial\Omega$, such that

$$\begin{aligned} &\int_\Omega a(x,Du_r)\cdot D(u_r - v) + \int_\Omega T_r(u_r)(u_r - v) + \frac{1}{r}\int_\Omega |u_r|^{p-2}u_r(u_r - v) + \\ &+ \int_{\partial\Omega} T_r(\beta_r(u_r))(u_r - v) + \frac{1}{m}\int_{\partial\Omega} T_r((u_r)^+)(u_r - v) - \\ &- \frac{1}{n}\int_{\partial\Omega} T_r((u_r)^-)(u_r - v) \le \int_{\partial\Omega}\psi(u_r - v) + \int_\Omega \phi(u_r - v)\,, \end{aligned} \tag{3.5}$$

for all $v \in W^{1,p}(\Omega)$, $v(x) \in D(\beta)$ a.e. in $\partial\Omega$.

Taking $v = u_r - T_k((u_r - mM)^+)$ in (3.5), where $M = \|\phi\|_\infty + \|\psi\|_\infty$, dropping nonnegative terms, dividing by k, and taking limits as k goes to 0, we get

$$\begin{aligned} &\frac{1}{m}\int_\Omega T_r(u_r)\mathrm{sgn}^+(u_r - mM) + \frac{1}{m}\int_{\partial\Omega} T_r(u_r)\mathrm{sgn}^+(u_r - mM) \le \\ &\le \int_{\partial\Omega}\psi\mathrm{sgn}^+(u_r - mM) + \int_\Omega \phi\mathrm{sgn}^+(u_r - mM)\,, \end{aligned}$$

consequently

$$\begin{aligned} &\int_\Omega (T_r(u_r) - mM)\mathrm{sgn}^+(u_r - mM) + \int_{\partial\Omega}(T_r(u_r) - mM)\mathrm{sgn}^+(u_r - mM) \le \\ &\le \int_{\partial\Omega}(m\psi - mM)\mathrm{sgn}^+(u_r - mM) + \int_\Omega (m\phi - mM)\mathrm{sgn}^+(u_r - mM) \le 0\,, \end{aligned}$$

therefore, for r large enough,

$$u_r(x) \le mM \quad \text{a.e in } \Omega\,.$$

Similarly, taking $v = u_r + T_k((u_r + nM)^-)$ in (3.5), we get

$$u_r(x) \ge -nM \quad \text{a.e in } \Omega\,.$$

Consequently, for r large enough, and taking into account that $m \le n$,

$$\|u_r\|_\infty \le nM\,. \tag{3.6}$$

Taking $v = 0$ as test function in (3.5) and using (H_1) and (3.6), it follows that

$$\int_\Omega |Du_r|^p \leq \frac{1}{\lambda} nM \left(\int_{\partial\Omega} |\psi| + \int_\Omega |\phi| \right) . \tag{3.7}$$

As a consequence of (3.6) and (3.7) we can suppose that there exists a subsequence, still denoted u_r, such that

$$\begin{aligned} &u_r \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } u \in W^{1,p}(\Omega)\,, \\ &u_r \text{ converges in } L^q(\Omega) \text{ and a.e. on } \Omega \text{ to } u, \text{ for any } q \geq 1\,, \\ &u_r \text{ converges in } L^p(\partial\Omega) \text{ and a.e. to } u\,. \end{aligned}$$

Next we show that $T_r(\beta_r(u_r))$ is weakly convergent in $L^1(\partial\Omega)$. Since $u_r(x) \in D(\beta)$,

$$|\beta_r(u_r)(x)| \leq \inf\{|r|, r \in \beta(u_r(x))\}\,.$$

If $D(\beta) = \mathbb{R}$,

$$\sup\{\beta(-nM)\} \leq \beta_r(u_r) \leq \inf\{\beta(mM)\}\,.$$

In the case $D(\beta)$ is a bounded interval $[a, b], a < b$,

$$\sup\{\beta(a)\} \leq \beta_r(u_r) \leq \inf\{\beta(b)\}\,.$$

If $D(\beta) = [a, +\infty), a \leq 0$,

$$\sup\{\beta(a)\} \leq \beta_r(u_r) \leq \inf\{\beta(M)\}\,.$$

The case $D(\beta) = (-\infty, a],\ a \geq 0$ can be treated similarly. Consequently, for r large enough, $T_r(\beta_r(u_r)) = \beta_r(u_r)$ is uniformly bounded and there exists a subsequence, denoted in the same way, $L^1(\partial\Omega)$-weakly convergent to some $w \in L^\infty(\partial\Omega)$. From here, since $u_r \to u$ in $L^1(\partial\Omega)$, applying [7, Lemma G], it follows that $w \in \beta(u)$ a.e. on $\partial\Omega$.

Let us see now that Du_r converges in measure to Du. We follow the technique used in [8] (see also [2]). Since Du_r converges to Du weakly in $L^p(\Omega)$, it is enough to show that Du_r is a Cauchy sequence in measure. Let t and $\epsilon > 0$. For some $A > 1$, we set

$$C(x, A, t) := \inf\{(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) : |\xi| \leq A,\ |\eta| \leq A,\ |\xi - \eta| \geq t\ \}\,.$$

Having in mind that the function $\xi \to a(x, \xi)$ is continuous (since ψ denotes a datum) for almost all $x \in \Omega$ and the set $\{(\xi,\eta)\ :\ |\xi| \leq A,\ |\eta| \leq A,\ |\xi - \eta| \geq t\ \}$ is compact, the infimum in the definition of $C(x, A, t)$ is a minimum. Hence, by (H_3), it follows that

$$C(x, A, t) > 0 \quad \text{for almost all } \ x \in \Omega\,. \tag{3.8}$$

Now, for $r, s \in \mathbb{N}$ and any $k > 0$, the following inclusion holds

$$\begin{aligned} &\{|Du_r - Du_s| > t\} \subset \\ &\subset \{|Du_r| \geq A\} \cup \{|Du_s| \geq A\} \cup \{|u_r - u_s| \geq k^2\} \cup \{C(x, A, t) \leq k\} \cup G\,, \end{aligned} \tag{3.9}$$

where

$G = \{|u_r - u_s| \leq k^2,\ C(x, A, t) \geq k,\ |Du_r| \leq A,\ |Du_s| \leq A,\ |Du_r - Du_s| > t\}\,.$

Since the sequence Du_r is bounded in $L^p(\Omega)$ we can choose A large enough in order to have

$$\lambda_N(\{|Du_r| \geq A\} \ \cup \ \{|Du_s| \geq A\}) \leq \frac{\epsilon}{4} \quad \text{for all} \quad r, s \in \mathbb{N}\,. \tag{3.10}$$

By (3.8), we can choose k small enough in order to have

$$\lambda_N(\{C(x, A, t) \leq k\}) \leq \frac{\epsilon}{4}\,. \tag{3.11}$$

On the other hand, if we use $u_r - T_k(u_r - u_s)$ and $u_s + T_k(u_r - u_s)$ as test functions in (3.5) for u_r and u_s respectively, we obtain

$$\begin{aligned} &\int_\Omega a(x, Du_r) \cdot DT_k(u_r - u_s) + \int_\Omega u_r T_k(u_r - u_s) + \frac{1}{r}\int_\Omega |u_r|^{p-2} u_r T_k(u_r - u_s) + \\ &+ \int_{\partial\Omega} \beta_r(u_r) T_k(u_r - u_s) + \frac{1}{m}\int_{\partial\Omega} u_r^+ T_k(u_r - u_s) - \\ &- \frac{1}{n}\int_{\partial\Omega} u_r^- T_k(u_r - u_s) \leq \int_{\partial\Omega} \psi T_k(u_r - u_s) + \int_\Omega \phi T_k(u_r - u_s)\,, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &- \int_\Omega a(x, Du_s) \cdot DT_k(u_r - u_s) - \int_\Omega u_s T_k(u_r - u_s) - \\ &- \frac{1}{s}\int_\Omega |u_s|^{p-2} u_s T_k(u_r - u_s) - \\ &- \int_{\partial\Omega} \beta_s(u_s) T_k(u_r - u_s) - \frac{1}{m}\int_{\partial\Omega} u_s^+ T_k(u_r - u_s) + \\ &+ \frac{1}{n}\int_{\partial\Omega} u_s^- T_k(u_r - u_s) \leq - \int_{\partial\Omega} \psi T_k(u_r - u_s) - \int_\Omega \phi T_k(u_r - u_s)\,. \end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13), we get

$$\begin{aligned} &\int_\Omega (a(x, Du_r) - a(x, Du_s)) \cdot DT_k(u_r - u_s) \leq \\ &\quad \leq - \int_\Omega \left(\frac{1}{r}|u_r|^{p-2} u_r - \frac{1}{s}|u_s|^{p-2} u_s\right) T_k(u_r - u_s) - \\ &\quad\quad - \int_{\partial\Omega} (\beta_r(u_r) - \beta_s(u_s))\, T_k(u_r - u_s)\,. \end{aligned}$$

Consequently, there exists a constant $\hat{M}$ independent of r and s such that

$$\int_\Omega (a(x, Du_r) - a(x, Du_s)) \cdot DT_k(u_r - u_s) \leq k\hat{M}\,.$$

Hence

$$\begin{aligned}
&\lambda_N(G) \leq \\
&\leq \lambda_N(\{|u_r - u_s| \leq k^2,\ (a(x, Du_r) - a(x, Du_s)) \cdot D(u_r - u_s) \geq k\}) \leq \\
&\leq \frac{1}{k} \int_{\{|u_r - u_s| < k^2\}} (a(x, Du_r) - a(x, Du_s)) \cdot D(u_r - u_s) = \\
&= \frac{1}{k} \int_\Omega (a(x, Du_r) - a(x, Du_s)) \cdot DT_{k^2}(u_r - u_s) \leq \frac{1}{k}\ k^2 \hat{M} \leq \frac{\epsilon}{4}
\end{aligned} \tag{3.14}$$

for k small enough.

Since A and k have been already chosen, if r_0 is large enough we have for $r, s \geq r_0$ the estimate $\lambda_N(\{|u_r - u_s| \geq k^2\}) \leq \frac{\epsilon}{4}$. From here, using (3.9), (3.10), (3.11) and (3.14), we can conclude that

$$\lambda_N(\{|Du_r - Du_s| \geq t\}) \leq \epsilon \quad \text{for} \ \ r, s \geq r_0\,.$$

From here, up to extraction of a subsequence, we also have $a(., Du_r)$ converges in measure and a.e. to $a(., Du)$. Now, by (H_2) and (3.7),

$$a(., Du_r) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ to } a(., Du)\,.$$

Finally, letting $r \to +\infty$ in (3.5), we prove (3.2).

In order to prove (ii), let us put $u_{1,r} = u_{\phi_1,\psi_1,m_1,n_1,r}$ and $u_{2,r} = u_{\phi_2,\psi_2,m_2,n_2,r}$. Taking $u_{1,r} - T_k((u_{1,r} - u_{2,r})^+)$, with r large enough, as test function in (3.5) for $u_{1,r}$, $m = m_1$ and $n = n_1$, we get

$$\begin{aligned}
&\int_\Omega a(., Du_{1,r}) \cdot DT_k((u_{1,r} - u_{2,r})^+) + \int_\Omega u_{1,r} T_k((u_{1,r} - u_{2,r})^+) + \\
&\quad + \frac{1}{r} \int_\Omega |u_{1,r}|^{p-2} u_{1,r} T_k((u_{1,r} - u_{2,r})^+) + \int_{\partial\Omega} \beta_r(u_{1,r}) T_k((u_{1,r} - u_{2,r})^+) + \\
&\quad + \frac{1}{m_1} \int_{\partial\Omega} u_{1,r}^+ T_k((u_{1,r} - u_{2,r})^+) - \frac{1}{n_1} \int_{\partial\Omega} u_{1,r}^- T_k((u_{1,r} - u_{2,r})^+) \leq \\
&\leq \int_{\partial\Omega} \psi_1 T_k((u_{1,r} - u_{2,r})^+) + \int_\Omega \phi_1 T_k((u_{1,r} - u_{2,r})^+)\,,
\end{aligned} \tag{3.15}$$

and taking $u_{2,r}+T_k(u_{1,r}-u_{2,r})^+$ as test function in (3.5) for $u_{2,r}$, $m=m_2$ and $n=n_2$, we get

$$
\begin{aligned}
&-\int_\Omega a(.,Du_{2,r})\cdot DT_k((u_{1,r}-u_{2,r})^+)-\int_\Omega u_{2,r}T_k((u_{1,r}-u_{2,r})^+)-\\
&-\frac{1}{r}\int_\Omega |u_{2,r}|^{p-2}u_{2,r}T_k((u_{1,r}-u_{2,r})^+)-\int_{\partial\Omega}\beta_r(u_{2,r})T_k((u_{1,r}-u_{2,r})^+)-\\
&-\frac{1}{m_2}\int_{\partial\Omega}u_{2,r}^+T_k((u_{1,r}-u_{2,r})^+)+\frac{1}{n_2}\int_{\partial\Omega}u_{1,r}^-T_k((u_{1,r}-u_{2,r})^+)\le\\
&\le-\int_{\partial\Omega}\psi_2T_k((u_{1,r}-u_{2,r})^+)-\int_\Omega\phi_2T_k((u_{1,r}-u_{2,r})^+)\,.
\end{aligned}
\tag{3.16}
$$

Adding these two inequalities, dropping some nonnegative terms, dividing by k, and letting $k\to 0$, we get

$$
\begin{aligned}
&\int_\Omega (u_{1,r}-u_{2,r})^+ + \int_{\partial\Omega}(\beta_r(u_{1,r})-\beta_r(u_{2,r}))^+\le\\
&\le\int_{\partial\Omega}(\psi_{1,r}-\psi_{2,r})^+ + \int_\Omega(\phi_{1,r}-\phi_{2,r})^+\,.
\end{aligned}
\tag{3.17}
$$

From here, taking into account the above convergences, (ii) can be obtained.

Finally, observe that when $\phi_2=0$ and $\psi_2=0$, taking $v=0$ in (3.5) for $\phi=\phi_2$ and $\psi=\psi_2$, we get $u_{2,r}=0$. Therefore, from (3.17) we get (3.3).

THEOREM 3.4. *Let a be smooth and $m,n\in\mathbb{N}$, $m\le n$.*

(i) *For $\phi\in L^\infty(\Omega)$ and $\psi\in L^\infty(\partial\Omega)$, there exist $u=u_{\phi,\psi,m,n}\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $w=w_{\phi,\psi,m,n}\in L^1(\partial\Omega)$, $w(x)\in\beta(u(x))$ a.e. on $\partial\Omega$, such that*

$$
\begin{aligned}
&\int_\Omega a(.,Du)\cdot D(u-v)+\int_\Omega u(u-v)+\int_{\partial\Omega}w(u-v)+\\
&\quad+\frac{1}{m}\int_{\partial\Omega}u^+(u-v)-\frac{1}{n}\int_{\partial\Omega}u^-(u-v)\le\\
&\le\int_{\partial\Omega}\psi(u-v)+\int_\Omega\phi(u-v)\,,
\end{aligned}
$$

for all $v\in W^{1,p}(\Omega)$ and all $k>0$. Moreover,

$$\int_\Omega|u|+\int_{\partial\Omega}|w|\le\int_{\partial\Omega}|\psi|+\int_\Omega|\phi|\,.$$

(ii) *If $m_1\le m_2\le n_2\le n_1$, $\phi_1,\phi_2\in L^\infty(\Omega)$, $\psi_1,\psi_2\in L^\infty(\partial\Omega)$ then*

$$
\begin{aligned}
&\int_\Omega(u_{\phi_1,\psi_1,m_1,n_1}-u_{\phi_2,\psi_2,m_2,n_2})^+ + \int_{\partial\Omega}(w_{\phi_1,\psi_1,m_1,n_1}-w_{\phi_2,\psi_2,m_2,n_2})^+\le\\
&\le\int_{\partial\Omega}(\psi_1-\psi_2)^+ + \int_\Omega(\phi_1-\phi_2)^+\,.
\end{aligned}
$$

Proof. Applying Theorem 3.3 to β_r, the Yosida approximation of β, there exists $u_r = u_{\phi,\psi,m,n,r} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that

$$\begin{aligned}(3.18)\qquad &\int_\Omega a(.,Du_r)\cdot D(u_r-v) + \int_\Omega u_r(u_r-v) + \int_{\partial\Omega} \beta_r(u_r)(u_r-v) + \\ &+ \frac{1}{m}\int_{\partial\Omega} u_r^+(u_r-v) - \frac{1}{n}\int_{\partial\Omega} u_r^-(u_r-v) \leq \\ &\leq \int_{\partial\Omega}\psi(u_r-v) + \int_\Omega \phi(u_r-v)\,,\end{aligned}$$

for all $v \in W^{1,p}(\Omega)$. Moreover, u_r is uniformly bounded by $n\,(\|\phi\|_\infty + \|\psi|_\infty)$.

Let $\hat{u}$ be the solution of the Dirichlet problem

$$\begin{cases} -\text{div } a(x,D\hat{u}) + \hat{u} = \phi & \text{in } \Omega \\ \hat{u} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Since a is smooth, there exists $\hat{\psi} \in L^1(\partial\Omega)$ such that

$$(3.19)\qquad \int_\Omega a(.,D\hat{u})\cdot D(\hat{u}-v) + \int_\Omega \hat{u}(\hat{u}-v) = \int_{\partial\Omega}\hat{\psi}(\hat{u}-v) + \int_\Omega \phi(\hat{u}-v)\,,$$

for any $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Taking $v = u_r - \rho(\beta_r(u_r - \hat{u}))$ as test function in (3.18), where $\rho \in C^\infty(\mathbb{R})$, $0 \leq \rho' \leq 1$, $\text{supp}(\rho')$ is compact and $0 \notin \text{supp}(\rho)$ ($\text{supp}(\rho)$ being the support of ρ), and $\hat{u} + \rho(\beta_r(u_r - \hat{u}))$ as test function in (3.19), and adding both inequalities we get, after dropping nonnegative terms, that

$$\int_{\partial\Omega}\beta_r(u_r)\rho(\beta_r(u_r)) \leq \int_{\partial\Omega}(\psi - \hat{\psi})\rho(\beta_r(u_r))\,,$$

which implies, see [6], that

$$\lim_{r\to+\infty}\beta_r(u_r) = w \quad \text{weakly in } L^1(\partial\Omega)\,.$$

Now, arguing as in the proof of Theorem 3.3, we obtain (i).

To prove (ii), by Theorem 3.3 applied to β_r, we have that, denoting $u_{i,r} = u_{\phi_i,\psi_i,m_i,n_i,r}$, $i = 1,2$,

$$\begin{aligned}(3.20)\qquad &\int_\Omega (u_{1,r}-u_{2,r})^+ + \int_{\partial\Omega}(\beta_r(u_{1,r}) - \beta_r(u_{2,r}))^+ \leq \\ &\leq \int_{\partial\Omega}(\psi_1-\psi_2)^+ + \int_\Omega(\phi_1-\phi_2)^+\,.\end{aligned}$$

Taking limits in (3.20) as r goes to $+\infty$ we can get (ii).

Proof of Theorem 3.2. *Existence.* Let us approximate ϕ in $L^1(\Omega)$ by $\phi_{m,n} = \sup\{\inf\{m,\phi\},-n\}$, which is bounded, non decreasing in m and non increasing in n, and ψ in $L^1(\partial\Omega)$ by $\psi_{m,n} = \sup\{\inf\{m,\psi\},-n\}$. Then, if $m \le n$, by Theorem 3.3 and Theorem 3.4, there exist $u_{m,n} \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $w_{m,n} \in L^1(\partial\Omega)$, $w_{m,n}(x) \in \beta(u_{m,n}(x))$ a.e. on $\partial\Omega$, such that

$$\begin{aligned}&\int_\Omega a(.,Du_{m,n})\cdot D(u_{m,n}-v)+\int_\Omega u_{m,n}(u_{m,n}-v)+\int_{\partial\Omega} w_{m,n}(u_{m,n}-v)+\\ &+\frac{1}{m}\int_{\partial\Omega} u^+_{m,n}(u_{m,n}-v)-\frac{1}{n}\int_{\partial\Omega} u^-_{m,n}(u_{m,n}-v)\le\\ &\le\int_{\partial\Omega}\psi_{m,n}(u_{m,n}-v)+\int_\Omega\phi_{m,n}(u_{m,n}-v)\,,\end{aligned}\tag{3.21}$$

for any $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, $v(x) \in D(\beta)$ a.e. on $\partial\Omega$. Moreover

$$\int_\Omega |u_{m,n}|+\int_{\partial\Omega}|w_{m,n}|\le\int_{\partial\Omega}|\psi_{m,n}|+\int_\Omega|\phi_{m,n}|\le\int_{\partial\Omega}|\psi|+\int_\Omega|\phi|\,.\tag{3.22}$$

Fixed $m \in \mathbb{N}$, by Theorem 3.3 (ii) and Theorem 3.4 (ii), $\{u_{m,n}\}_{n=m}^\infty$ and $\{w_{m,n}\}_{n=m}^\infty$ are monotone non increasing. Then, by (3.22) and the Monotone convergence theorem, there exists $\hat{u}_m \in L^1(\Omega)$, $\hat{w}_m \in L^1(\partial\Omega)$ and a subsequence $n(m)$, such that

$$\|u_{m,n(m)}-\hat{u}_m\|_1\le\frac{1}{m}$$

and

$$\|w_{m,n(m)}-\hat{w}_m\|_1\le\frac{1}{m}\,.$$

Thanks to Theorem 3.3 (ii) and Theorem 3.4 (ii), $\hat{u}_m$ and $\hat{w}_m$ are non decreasing in m. Now, by (3.22), we have that $\int_\Omega|\hat{u}_m|$ and $\int_{\partial\Omega}|\hat{w}_m|$ are bounded. Using again the Monotone convergence theorem, there exist $u \in L^1(\Omega)$ and $w \in L^1(\partial\Omega)$ such that

$$\hat{u}_m \text{ converges a.e. and in } L^1(\Omega) \text{ to } u$$

and

$$\hat{w}_m \text{ converges a.e. and in } L^1(\partial\Omega) \text{ to } w\,.$$

Consequently,

$$u_m := u_{m,n(m)} \text{ converges a.e. and in } L^1(\Omega) \text{ to } u$$

and

$$w_m := w_{m,n(m)} \text{ converges a.e. and in } L^1(\partial\Omega) \text{ to } w\,.\tag{3.23}$$

Taking $v = u_m - T_k(u_m)$ in (3.21) with $n = n(m)$,

$$\lambda \int_\Omega |DT_k(u_m)|^p \leq k\left(\|\phi\|_1 + \|\psi\|_1\right), \forall k \in \mathbb{N}\,. \tag{3.24}$$

From (3.24), we deduce that $T_k(u_m)$ is bounded in $W^{1,p}(\Omega)$. Then, we can suppose that

$$T_k(u_m) \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } T_k(u)\,,$$
$$T_k(u_m) \text{ converges in } L^p(\Omega) \text{ and a.e. on } \Omega \text{ to } T_k(u)$$

and

$$T_k(u_m) \text{ converges in } L^p(\partial\Omega) \text{ and a.e. on } \partial\Omega \text{ to } T_k(u)\,.$$

Taking $G = \{|u_m - u_n| \leq k^2,\ |u_m| \leq A,\ |u_n| \leq A,\ C(x, A, t) \geq k,\ |Du_m| \leq A,\ |Du_n| \leq A,\ |Du_m - Du_n| > t\}$, and arguing as in Theorem 3.3, it is not difficult to see that Du_m is a Cauchy sequence in measure. Similarly we can prove that $DT_k(u_m)$ converges in measure to $DT_k(u)$. Then, up to extraction of a subsequence, Du_m converges to Du a.e. in Ω. From here,

$$a(., DT_k(u_m)) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ and a.e. in } \Omega \text{ to } a(., DT_k(u)). \tag{3.25}$$

Let us see now that $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$. Obviously, $u_m \to u$ a.e. in Ω. On the other hand, since $DT_k(u_m)$ is bounded in $L^p(\Omega)$ and $DT_k(u_m) \to DT_k(u)$ in measure, it follows from [4, Lemma 6.1] that $DT_k(u_m) \to DT_k(u)$ in $L^1(\Omega)$. Finally, let us see that $\gamma(u_m)$ converges a.e. in $\partial\Omega$. For every $k > 0$, let

$$A_k := \{x \in \partial\Omega : |T_k(u)(x)| < k\} \text{ and } C := \partial\Omega \sim \cup_{k>0} A_k\,.$$

Then, by (3.22), (3.24) and the Trace theorem, there exists positive constants $M_1,\ M_2$ such that

$$\begin{aligned} &\lambda_{N-1}(\{x \in \partial\Omega : |T_k(u)(x)| = k\}) \leq \frac{1}{k^p}\int_{\partial\Omega} |T_k(u)|^p \leq \\ &\leq \frac{M_1}{k^p}\left(\int_\Omega |T_k(u)||T_k(u)|^{p-1} + \int_\Omega |DT_k(u)|^p\right) \leq \frac{M_2}{k^p}(k^{p-1} + k)\,. \end{aligned} \tag{3.26}$$

Hence, $\lambda_{N-1}(C) = 0$. Thus, if we define in $\partial\Omega$ the function v by

$$v(x) = T_k(u)(x) \quad \text{ if } x \in A_k\,,$$

it is easy to see that

$$u_n \to v =: \tau(u) \quad \text{ a.e. in } \partial\Omega\,. \tag{3.27}$$

Therefore, $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and moreover, by (3.26), $u \in M^{p_0}(\partial\Omega)$, $p_0 = \inf\{p-1, 1\}$, where $M^{p_0}(\partial\Omega)$ is the Marcinkiewicz space of exponent p_0 (see, for instance, [5]).

Since $w_m(x) \in \beta(u_m(x))$ a.e. on $\partial\Omega$, from (3.23), (3.27) and from the maximal monotonicity of β, we deduce that $w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$.

Finally let us pass to the limit in (3.21) to prove that u is an entropy solution of (S). For this step, we introduce the class $\mathcal{F}$ of functions $S \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$S(0) = 0,\ 0 \le S' \le 1,\ S'(s) = 0 \text{ for } s \text{ large enough}\,,$$
$$S(-s) = -S(s),\ \text{and } S''(s) \le 0 \text{ for } s \ge 0\,.$$

Let $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v(x) \in D(\beta)$ a.e. if $D(\beta)$, and $S \in \mathcal{F}$. Taking $u_m - S(u_m - v)$ as test function in (3.21) we get

$$\begin{aligned}
&\int_\Omega a(x, Du_m) \cdot DS(u_m - v) + \int_\Omega u_m S(u_m - v) + \int_{\partial\Omega} w_m S(u_m - v) + \\
&+ \frac{1}{m} \int_{\partial\Omega} u_m^+ S(u_m - v) - \frac{1}{n(m)} \int_{\partial\Omega} u_{n(m)}^- S(u_m - v) \le \\
&\le \int_{\partial\Omega} \psi_m S(u_m - v) + \int_\Omega \phi_m S(u_m - v)\,.
\end{aligned} \tag{3.28}$$

We can write the first term of (3.28) as

$$\int_\Omega a(x, Du_m) \cdot Du_m S'(u_m - v) - \int_\Omega a(x, Du_m) \cdot Dv S'(u_m - v)\,. \tag{3.29}$$

Since $u_m \to u$ and $Du_m \to Du$ a.e., Fatou's lemma yields

$$\int_\Omega a(x, Du) \cdot Du S'(u - v) \le \liminf_{m\to\infty} \int_\Omega a(x, Du_m) \cdot Du_m S'(u_m - v)\,.$$

The second term of (3.29) is estimated as follows. Let $r := \|v\|_\infty + \|S\|_\infty$. By (3.25)

$$a(x, DT_r u_m) \to a(x, DT_r u) \quad \text{weakly in } L^{p'}(\Omega)\,. \tag{3.30}$$

On the other hand,

$$|Dv S'(u_m - v)| \le |Dv| \in L^p(\Omega)\,.$$

Then, by the Dominated Convergence theorem, we have

$$Dv S'(u_m - v) \to Dv S'(u - v) \quad \text{in } L^p(\Omega)^N\,. \tag{3.31}$$

Hence, by (3.30) and (3.31), it follows that

$$\lim_{m\to\infty}\int_\Omega a(x,Du_m)\cdot DvS'(u_m-v)=\int_\Omega a(x,Du)\cdot DvS'(u-v)\,.$$

Therefore, applying again the Dominated Convergence theorem in the other terms of (3.28), we obtain

$$\begin{aligned}\int_\Omega a(x,Du)\cdot DS(u-v)+\int_\Omega uS(u-v)+\int_{\partial\Omega} wS(u-v)\le\\ \le\int_{\partial\Omega}\psi S(u-v)+\int_\Omega \phi S(u-v)\,.\end{aligned}$$

From here, to conclude, we only need to apply the technique used in the proof of [4, Lemma 3.2].

Uniqueness. Let u be an entropy solution of problem (S), taking $T_h(u)$ as test function in (3.1), $h>0$, we have

$$\begin{aligned}\int_\Omega a(x,Du)\cdot DT_k(u-T_h(u))+\int_\Omega uT_k(u-T_h(u))+\int_{\partial\Omega} wT_k(u-T_h(u))\le\\ \le\int_{\partial\Omega}\psi T_k(u-T_h(u))+\int_\Omega \phi T_k(u-T_h(u))\,.\end{aligned}$$

Now, using (H_1) and the positivity of the second and third terms, it follows that

$$\lambda\int_{\{h<|u|<h+k\}}|Du|^p\le k\int_{\partial\Omega\cap\{|u|\ge h\}}|\psi|+k\int_{\Omega\cap\{|u|\ge h\}}|\phi|\,. \tag{3.32}$$

Let now u_1 and u_2 be entropy solutions of problem (S), following the lines of [4], we shall see that $u_1=u_2$. Let $w_1,w_2\in L^1(\partial\Omega)$ with $w_1(x)\in\beta(u_1(x))$ and $w_2(x)\in\beta(u_2(x))$ a.e. on $\partial\Omega$ such that for every $h>0$,

$$\begin{aligned}&\int_\Omega a(x,Du_1)\cdot DT_k(u_1-T_h(u_2))+\int_\Omega u_1T_k(u_1-T_h(u_2))+\\ &\quad+\int_{\partial\Omega} w_1T_k(u_1-T_h(u_2))\le\int_{\partial\Omega}\psi T_k(u_1-T_h(u_2))+\int_\Omega \phi T_k(u_1-T_h(u_2))\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega a(x,Du_2)\cdot DT_k(u_2-T_h(u_1))+\int_\Omega u_2T_k(u_2-T_h(u_1))+\\ &\quad+\int_{\partial\Omega} w_2T_k(u_2-T_h(u_1))\le\int_{\partial\Omega}\psi T_k(u_2-T_h(u_1))+\int_\Omega \phi T_k(u_2-T_h(u_1))\,.\end{aligned}$$

Adding both inequalities and taking limits when h goes to ∞, on account of the monotonicity of β, we get

$$-\int_\Omega (u_1-u_2)T_k(u_1-u_2) \geq \liminf_{h\to\infty} I_{h,k}\,,$$

where

$$I_{h,k} := \int_\Omega a(x,Du_1)\cdot DT_k(u_1-T_h(u_2)) + \int_\Omega a(x,Du_2)\cdot DT_k(u_2-T_h(u_1))\,.$$

Then, in order to prove that $u_1=u_2$, it is enough to prove that

$$\liminf_{h\to\infty} I_{h,k} \geq 0 \quad \text{for any } k\,. \tag{3.33}$$

To prove this, we split

$$I_{h,k} = I^1_{h,k} + I^2_{h,k} + I^3_{h,k} + I^4_{h,k}\,,$$

where

$$I^1_{h,k} := \int_{\{|u_1|<h,\ |u_2|<h\}} (a(x,Du_1)-a(x,Du_2))\cdot DT_k(u_1-u_2) \geq 0\,,$$

$$\begin{aligned} I^2_{h,k} := &\int_{\{|u_1|<h,\ |u_2|\geq h\}} a(x,Du_1)\cdot DT_k(u_1-h\,\mathrm{sgn}(u_2))+ \\ &+\int_{\{|u_1|<h,\ |u_2|\geq h\}} a(x,Du_2)\cdot DT_k(u_2-u_1) \geq \\ &\geq \int_{\{|u_1|<h,\ |u_2|\geq h\}} a(x,Du_2)\cdot DT_k(u_2-u_1)\,, \end{aligned}$$

$$\begin{aligned} I^3_{h,k} := &\int_{\{|u_1|\geq h,\ |u_2|<h\}} a(x,Du_1)\cdot DT_k(u_1-u_2)+ \\ &+\int_{\{|u_1|\geq h,\ |u_2|<h\}} a(x,Du_2)\cdot DT_k(u_2-h\,\mathrm{sgn}(u_1)) \geq \\ &\geq \int_{\{|u_1|\geq h,\ |u_2|<h\}} a(x,Du_1)\cdot DT_k(u_1-u_2)\,, \end{aligned}$$

$$\begin{aligned} I^4_{h,k} := &\int_{\{|u_1|\geq h,\ |u_2|\geq h\}} a(x,Du_1)\cdot DT_k(u_1-h\,\mathrm{sgn}(u_2))+ \\ &+\int_{\{|u_1|\geq h,\ |u_2|\geq h\}} a(x,Du_2)\cdot DT_k(u_2-h\,\mathrm{sgn}(u_1)) \geq 0\,. \end{aligned}$$

Combining the above estimates we get

$$I_{h,k} \geq L^1_{h,k} + L^2_{h,k}\,,$$

where

$$L^1_{h,k} := \int_{\{|u_1|<h,\ |u_2|\geq h\}} a(x, Du_2) \cdot DT_k(u_2 - u_1)$$

and

$$L^2_{h,k} := \int_{\{|u_1|\geq h,\ |u_2|<h\}} a(x, Du_1) \cdot DT_k(u_1 - u_2)\,.$$

Now, if we put

$$C(h,k) := \{h < |u_1| < k + h\} \cap \{h - k < |u_2| < h\}\,,$$

we have

$$\begin{aligned} |L^2_{h,k}| &\leq \int_{\{|u_1-u_2|<k,\ |u_1|\geq h,\ |u_2|<h\}} |a(x, Du_1) \cdot (Du_1 - Du_2)| \leq \\ &\leq \int_{C(h,k)} |a(x, Du_1) \cdot Du_1| + \int_{C(h,k)} |a(x, Du_1) \cdot Du_2|\,. \end{aligned}$$

Then, by Hölder's inequality, we get

$$\begin{aligned} |L^2_{h,k}| \leq \Big(\int_{C(h,k)} |a(x, Du_1)|^{p'}\Big)^{1/p'} \Big(\Big(\int_{C(h,k)} |Du_1|^p\Big)^{1/p} + \\ + \Big(\int_{C(h,k)} |Du_2|^p\Big)^{1/p}\Big)\,. \end{aligned}$$

Now, by (H_2),

$$\begin{aligned} \Big(\int_{C(h,k)} |a(x, Du_1)|^{p'}\Big)^{1/p'} &\leq \Big(\int_{C(h,k)} c^{p'}\Big(g(x) + |Du_1|^{p-1}\Big)^{p'}\Big)^{1/p'} \leq \\ &\leq c2^{\frac{1}{p}}\Big(\|g\|^{p'}_{p'} + \int_{\{h<|u_1|<k+h\}} |Du_1|^p\Big)^{1/p'}\,. \end{aligned}$$

On the other hand, applying (3.32), we obtain

$$\int_{\{h<|u_1|<k+h\}} |Du_1|^p \leq \frac{k}{\lambda}\Big(\int_{\{|u_1|\geq h\}} |\psi| + \int_{\{|u_1|\geq h\}} |\phi|\Big)$$

and

$$\int_{\{h-k<|u_2|<h\}} |Du_2|^p \leq \frac{k}{\lambda}\left(\int_{\{|u_2|\geq h-k\}} |\psi| + \int_{\{|u_2|\geq h-k\}} |\phi|\right).$$

Then, since $u_1, u_2, \phi, \psi \in L^1(\Omega)$ and $u_1, u_2 \in M^{p_0}(\partial\Omega)$, we have that

$$\lim_{h\to\infty} L^2_{h,k} = 0\,.$$

Similarly, $\lim_{h\to\infty} L^1_{h,k} = 0$. Therefore (3.33) holds.

Finally, let u_1 be the entropy solution of problem (S) corresponding to $\phi_1 \in L^1(\Omega)$ and $\psi_1 \in L^1(\partial\Omega)$ and let u_2 be the entropy solution of problem (S) corresponding to $\phi_2 \in L^1(\Omega)$ and $\psi_2 \in L^1(\partial\Omega)$. As a consequence of uniqueness we can construct u_1 and u_2 following the proof of (i), then, taking into account Theorem 3.3 (ii) and Theorem 3.4 (ii), we prove (ii).

Definition 3.5. Let us suppose that $D(\beta)$ is closed or a is smooth. For $\psi \in L^1(\partial\Omega)$, let us define the operator $\mathcal{A}$ in $L^1(\Omega) \times L^1(\Omega)$ by $(u, \phi) \in \mathcal{A}$ if $u \in L^1(\Omega) \cap \mathcal{T}^{1,p}_{tr}(\Omega)$, $\phi \in L^1(\Omega)$ and there exists $w \in L^1(\partial\Omega)$, $w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$, such that

$$\int_\Omega a(.,Du)\cdot DT_k(u-v) + \int_{\partial\Omega} wT_k(u-v) \leq$$
$$\leq \int_{\partial\Omega} \psi T_k(u-v) + \int_\Omega \phi T_k(u-v)$$

for all $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v(x) \in D(\beta)$ a.e. in $\partial\Omega$, and all $k > 0$.

By Theorem 3.2 we have that $\mathcal{A}$ is an m-accretive operator. Moreover, it is not difficult to see that $\overline{D(\mathcal{A})} = L^1(\Omega)$. Then by the Nonlinear Semigroup Theory it is possible to solve in the mild sense the evolution problem in $L^1(\Omega)$

$$\begin{cases} u_t + \mathcal{A}u \ni 0 & \text{in } \Omega\times]0,+\infty[\,, \\ u(0) = u_0 \in L^1(\Omega)\,. \end{cases}$$

The mild solution of the above problem in the case $\psi = 0$ is characterized in [3] in the entropy sense for particular graphs β.

4 – Existence and uniqueness of solutions of problem (P)

Let us now study problem

$$\text{(P)} \qquad \begin{cases} -\text{div}\, a(.,Du) = 0 & \text{in } \Omega \\ a(.,Du)\cdot\eta + u = \psi & \text{on } \partial\Omega\,, \end{cases}$$

for any a satisfying (H_1), (H_2) and (H_3) and any $\psi \in L^1(\partial\Omega)$.

Using classical variational methods ([9], [10]), for every data $\psi \in L^\infty(\partial\Omega)$ this problem can be solved in $W^{1,p}(\Omega)$. In fact, let us define the following capacity operator

$$\mathcal{C} : W^{\frac{1}{p'},p}(\partial\Omega) \to W^{\frac{-1}{p'},p'}(\partial\Omega)$$

by

$$< \mathcal{C}f, g >= \int_\Omega a(., Du) \cdot Dv$$

where $u \in W^{1,p}(\Omega)$ is the solution of the Dirichlet problem

$$\text{(D)} \qquad \begin{cases} -\operatorname{div} a(., Du) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega\,, \end{cases}$$

and $v \in W^{1,p}(\Omega)$ is such that $\gamma(v) = g$. Function u is called the A-harmonic lifting of f, where A is the operator associated to the formal differential expression $-\text{div}a(x, Du)$. It is easy to see that the operator $\mathcal{C}$ is bounded from $W^{\frac{1}{p'},p}(\partial\Omega)$ to its dual $W^{\frac{-1}{p'},p'}(\partial\Omega)$, hemicontinuous and strictly monotone. Therefore,

$$\text{(4.34)} \qquad \mathcal{C}f + f = \psi \quad \text{has a unique solution } f \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)\,.$$

In the general case where $\psi \in L^1(\partial\Omega)$, the variational methods are not available. For this reason we introduce a new concept of solution, named entropy solution, and we will give an existence and uniqueness result of solutions in this sense.

Definition 4.1. A measurable function $u : \Omega \to \mathbb{R}$ is an entropy solution of (P) if $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$, $\tau(u) \in L^1(\partial\Omega)$ and

$$\int_\Omega a(., Du) \cdot DT_k(u - v) + \int_{\partial\Omega} uT_k(u - v) \le \int_{\partial\Omega} \psi T_k(u - v)$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ and all $k > 0$.

Theorem 4.2. *For any $\psi \in L^1(\partial\Omega)$, there exists a unique entropy solution of problem* (P).

Moreover, if u_1 is an entropy solution of problem (P) *corresponding to $\psi_1 \in L^1(\partial\Omega)$ and u_2 is an entropy solution of problem* (P) *corresponding to $\psi_2 \in L^1(\partial\Omega)$ then*

$$\int_{\partial\Omega} |u_1 - u_2| \le \int_{\partial\Omega} |\psi_1 - \psi_2|\,.$$

Proof. Let $n \in \mathbb{N}$, using Theorem 3.2 with $\beta(r) = r$ for all $r \in \mathbb{R}$ and $\phi = 0$, we have that, given $\psi \in L^1(\partial\Omega)$, there exists $u_n \in L^1(\Omega) \cap \mathcal{T}^{1,p}_{tr}(\Omega)$, $\tau(u_n) \in L^1(\partial\Omega)$, such that

$$\begin{aligned} &\int_\Omega a(., Du_n)\cdot DT_k(u_n - v) + \frac{1}{n}\int_\Omega u_n T_k(u_n - v) + \int_{\partial\Omega} u_n T_k(u_n - v) \leq \\ &\leq \int_{\partial\Omega} \psi T_k(u_n - v) \end{aligned} \tag{4.35}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ and all $k > 0$.

Taking $v = 0$ as test function in (4.35), and using (H_1), it is easy to see that

$$\frac{1}{k}\int_\Omega |DT_k(u_n)|^p \leq \frac{M}{\lambda} \quad \forall n \in \mathbb{N} \text{ and } \forall k > 0\,, \tag{4.36}$$

$$\int_{\partial\Omega} |u_n| \leq M \quad \forall n \in \mathbb{N} \tag{4.37}$$

and

$$\int_\Omega \frac{1}{n}|u_n| \leq M \quad \forall n \in \mathbb{N}\,, \tag{4.38}$$

where $M = ||\psi||_{L^1(\partial\Omega)}$. Then, by (4.36), we can suppose that

$$T_k(u_n) \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } \sigma_k \in W^{1,p}(\Omega)\,,$$

$$T_k(u_n) \text{ converges in } L^p(\Omega) \text{ and a.e. to } \sigma_k$$

and

$$T_k(u_n) \text{ converges in } L^p(\partial\Omega) \text{ and a.e. to } \sigma_k\,.$$

Since there exists $C_1 > 0$ such that, for all $n \in \mathbb{N}$ and for all $k > 0$,

$$\left(\int_\Omega |T_k(u_n)|^{p^*}\right)^{1/p^*} \leq C_1\left(\int_{\partial\Omega}|T_k(u_n)| + \left(\int_\Omega |DT_k(u_n)|^p\right)^{1/p}\right),$$

where $p^* = \frac{Np}{N-p}$, we deduce, thanks to (4.36) and (4.37), that there exists $C_2 > 0$ such that

$$\|T_k(u_n)\|_{L^{p^*}(\Omega)} \leq C_1\left(M + \left(\frac{Mk}{\lambda}\right)^{\frac{1}{p}}\right) \leq C_2 k^{\frac{1}{p}} \quad \forall k \geq 1\,.$$

Now,

$$\begin{aligned} &\lambda_N\{x \in \Omega : |\sigma_k(x)| = k\} \leq \int_\Omega \frac{|\sigma_k|^{p^*}}{k^{p^*}} \leq \\ &\leq \liminf_n \int_\Omega \frac{|T_k(u_n)|^{p^*}}{k^{p^*}} \leq C_2^{p^*}\frac{1}{k^{N(p-1)/(N-p)}} \quad \text{for all } k \geq 1\,. \end{aligned}$$

Hence, there exists $C_3 > 0$ such that

$$\lambda_N\{x \in \Omega : |\sigma_k(x)| = k\} \leq C_3 \frac{1}{k^{N(p-1)/(N-p)}} \quad \text{for all } k > 0\,.$$

Let u be defined on Ω by $u(x) = \sigma_k(x)$ on $\{x \in \Omega : |\sigma_k(x)| < k\}$. Then

$$u_n \text{ converges to } u \text{ a.e. in } \Omega\,,$$

and we can suppose that

$$T_k(u_n) \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } T_k(u) \in W^{1,p}(\Omega)\,,$$
$$T_k(u_n) \text{ converges in } L^p(\Omega) \text{ and a.e. to } T_k(u)\,,$$

and

$$T_k(u_n) \text{ converges in } L^p(\partial\Omega) \text{ and a.e. to } T_k(u)\,.$$

Consequently, $u \in \mathcal{T}^{1,p}(\Omega)$.

On the other hand, thanks to (4.37)

$$\lambda_{N-1}\{x \in \partial\Omega : |T_k(u)(x)| = k\} \leq \frac{1}{k}\int_{\partial\Omega} |T_k(u)| \leq$$
$$\leq \frac{1}{k} \liminf_n \int_{\partial\Omega} |T_k(u_n)| \leq \frac{M}{k}\,.$$

Therefore, if we define $v(x) = T_k(u)(x)$ on $\{x \in \partial\Omega : |T_k(u)(x)| < k\}$,

$$u_n \to v \quad \text{a.e. in } \partial\Omega\,.$$

Consequently, $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and, by (4.37), $u \in L^1(\partial\Omega)$.

Taking $G = \{|u_m - u_n| \leq k^2,\ |u_m| \leq A,\ |u_n| \leq A,\ C(x, A, t) \geq k,\ |Du_m| \leq A,\ |Du_n| \leq A,\ |Du_m - Du_n| > t\}$, and arguing as in Theorem 3.3, it is not difficult to see that Du_m is a Cauchy sequence in measure. Similarly, $DT_k(u_m)$ converges in measure to $DT_k(u)$. Then, up to extraction of a subsequence, Du_m converges to Du a.e. in Ω. From here,

$$a(., DT_k(u_m)) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ and a.e. in } \Omega \text{ to } a(., DT_k(u))\,.$$

Let us see finally that

$$u_n \text{ converges to } u \text{ in } L^1(\partial\Omega)\,, \tag{4.39}$$

$$\frac{1}{n} u_n \text{ converges to } 0 \text{ in } L^1(\Omega)\,. \tag{4.40}$$

In fact, taking $v = T_h(u_n)$ as test function in (4.35), dividing by k and letting $k \to 0$, we get

$$\text{(4.41)} \quad \frac{1}{n}\int_{\{x\in\Omega:|u_n(x)|\geq h\}} |u_n| + \int_{\{x\in\partial\Omega:|u_n(x)|\geq h\}} |u_n| \leq \int_{\{x\in\partial\Omega:|u_n(x)|\geq h\}} |\psi| \,.$$

Now, by (4.37), $\lambda_{N-1}\{x \in \partial\Omega : |u_n(x)| \geq h\} \to 0$ as $h \to +\infty$. Then, by (4.41), it is easy to see that the sequence $\{\frac{1}{n}u_n\}$ is equiintegrable in $L^1(\Omega)$ and that the sequence $\{u_n\}$ is equiintegrable in $L^1(\partial\Omega)$. Since $\frac{1}{n}u_n \to 0$ a.e. in Ω and $u_n \to u$ a.e. in $\partial\Omega$, applying Vitali's convergence theorem we get (4.39) and (4.40).

We can then pass to the limit in (4.35) (as in the proof of Theorem 3.2) to conclude that u is an entropy solution of (P).

Let us prove now the uniqueness. Let u_1 be an entropy solution of problem (P) corresponding to $\psi_1 \in L^1(\partial\Omega)$ and u_2 be an entropy solution of problem (P) corresponding to $\psi_2 \in L^1(\partial\Omega)$. Working as in the proof of the uniqueness of Theorem 3.2, we get

$$\begin{aligned}
&\int_{\partial\Omega}(\psi_1-\psi_2)T_k(u_1-u_2) - \int_{\partial\Omega}(u_1-u_2)T_k(u_1-u_2) \geq \\
&\geq \liminf_{h\to+\infty}\left(\int_\Omega a(x,Du_1)\cdot DT_k(u_1-T_h(u_2))+\right.\\
&\quad\left.+\int_\Omega a(x,Du_2)\cdot DT_k(u_2-T_h(u_1))\right) \geq \\
\text{(4.42)}\quad &\geq \liminf_{h\to+\infty}\left(\int_{\{|u_1|<h,\ |u_2|<h\}}(a(x,Du_1)-a(x,Du_2))\cdot DT_k(u_1-u_2)+\right.\\
&\quad+\int_{\{|u_1|<h,\ |u_2|\geq h\}} a(x,Du_2)\cdot DT_k(u_2-u_1)+\\
&\quad\left.+\int_{\{|u_1|\geq h,\ |u_2|<h\}} a(x,Du_1)\cdot DT_k(u_1-u_2)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{h\to+\infty}\left(\int_{\{|u_1|<h,\ |u_2|\geq h\}} a(x,Du_2)\cdot DT_k(u_2-u_1)+\right.\\
&\quad\left.+\int_{\{|u_1|\geq h,\ |u_2|<h\}} a(x,Du_1)\cdot DT_k(u_1-u_2)\right)=0\,.
\end{aligned}$$

Since $\int_{\{|u_1|<h,\ |u_2|<h\}}(a(x,Du_1)-a(x,Du_2))\cdot DT_k(u_1-u_2)\geq 0$, dividing by k and letting $k\to 0$, we get that

$$\int_{\partial\Omega}|u_1-u_2| \leq \int_{\partial\Omega}|\psi_1-\psi_2|\,.$$

In order to prove that $u_1 = u_2$ in Ω if $\psi_1 = \psi_2$, it is enough to observe that the inequalities (4.42) become equalities. Consequently

$$\liminf_{h\to+\infty} \int_{\{|u_1|<h,\ |u_2|<h\}} (a(x, Du_1) - a(x, Du_2)) \cdot DT_k(u_1 - u_2) = 0\,.$$

From here, since $\int_{\{|u_1|<h,\ |u_2|<h\}}(a(x, Du_1)-a(x, Du_2))\cdot DT_k(u_1-u_2)$ is positive and non decreasing in h, it follows that $DT_h(u_1) = DT_h(u_2)$ a.e. in Ω for all h, but since $u_1 = u_2$ a.e. in $\partial\Omega$, we get $u_1 = u_2$ a.e. in Ω.

Definition 4.3. We define the following operator $\mathcal{B}$ in $L^1(\partial\Omega) \times L^1(\partial\Omega)$ by $(f, \psi) \in \mathcal{B}$ if $f, \psi \in L^1(\partial\Omega)$ and there exists $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ with $\tau(u) = f$ such that

$$\int_\Omega a(., Du) \cdot DT_k(u - v) \le \int_{\partial\Omega} \psi T_k(u - v)\,,$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ and all $k > 0$.

By Theorem 4.2, $\mathcal{B}$ is an m-accretive operator in $L^1(\partial\Omega)$. Now, on the one hand, operator $\mathcal{C}$ considered as an operator on $L^1(\partial\Omega) \times L^1(\partial\Omega)$, denoted again $\mathcal{C}$, is completely accretive (see [6]). In fact, let $\rho \in C^\infty(\mathbb{R})$, $0 \le \rho' \le 1$, $\mathrm{supp}(\rho')$ compact and $0 \notin \mathrm{supp}(\rho)$. If (f_1, ψ_1), $(f_2, \psi_2) \in \mathcal{C}$, then,

$$\begin{aligned}\int_{\partial\Omega} (\psi_1 - \psi_2)\rho(f_1 - f_2) &= \int_\Omega (a(., Du_1) - a(., Du_2)) \cdot D\rho(u_1 - u_2) = \\ &= \int_\Omega (a(., Du_1) - a(., Du_2)) \cdot D(u_1 - u_2)p'(u_1 - u_2) \ge \\ &\ge 0\,,\end{aligned}$$

where u_i is the A-harmonic lifting of f_i, $i = 1, 2$. Consequently, by (4.34), $\overline{\mathcal{C}}^{L^1(\partial\Omega)\times L^1(\partial\Omega)}$ is m-accretive in $L^1(\partial\Omega)$.

On the other hand, if $(f, \psi) \in \mathcal{C}$ then

$$< \psi, T_k(\hat{u} - v) >= \int_\Omega a(., D\hat{u}) \cdot DT_k(\hat{u} - v)\,,$$

for any $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, where $\hat{u} \in W^{1,p}(\Omega)$ is the solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(., D\hat{u}) = 0 & \text{in } \Omega \\ \hat{u} = f & \text{on } \partial\Omega\,. \end{cases}$$

Therefore

$$(f, \psi) \in \mathcal{B}\,,$$

and consequently, since $\mathcal{B}$ is m-accretive,

$$\overline{\mathcal{C}}^{L^1(\partial\Omega)\times L^1(\partial\Omega)} = \mathcal{B}\,.$$

Remark 4.4. In [1], the operator $\mathcal{B}$ is also characterized as follows, $(f,\psi) \in \mathcal{B}$ if $f,\psi \in L^1(\partial\Omega)$, $T_k(f) \in W^{\frac{1}{p'},p}(\partial\Omega)$ for all $k > 0$ and

$$< \mathcal{C}(g + T_k(f-g)), T_k(f-g) > \leq \int_{\partial\Omega} \psi T_k(f-g)\,,$$

for all $g \in L^\infty(\partial\Omega) \cap W^{\frac{1}{p'},p}(\partial\Omega)$ and for all $k > 0$.

Remark 4.5. It is not difficult to see that $D(\mathcal{B})$ is dense in $L^1(\partial\Omega)$. Then, by the Nonlinear Semigroup Theory, it is possible to solve in the mild sense the evolution problem in $L^1(\partial\Omega)$

$$\begin{cases} u_t + \mathcal{B}u = 0 & \text{in } \partial\Omega\times]0,+\infty[\,, \\ u(0) = u_0 \in L^1(\partial\Omega)\,, \end{cases}$$

which rewrites, from the point of view of Nonlinear Semigroup Theory, the following problem

$$\begin{cases} -\text{div}\ a(x,Du) = 0 & \text{in } \Omega\times]0,+\infty[\,, \\ u'(t) + a(x,Du)\cdot\eta = 0 & \text{on } \partial\Omega\times]0,+\infty[\,, \\ u(0) = u_0 \in L^1(\partial\Omega)\,. \end{cases}$$

In a forthcoming paper the mild solutions of the above problem will be characterized in the entropy sense.

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