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On some stability results of localized atomic decompositions

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In honor of Prof. Laura Gori for her 70th birthday

Abstract: *This paper is concerned with sufficient conditions ensuring that a perturbation of a frame is again a frame. We emphasize how stability of frames is fundamental for numerical applications and we discuss in particular the connection between stability conditions and localization principles for atomic decompositions in Banach spaces.*

1 – Introduction

Stable redundant nonorthogonal expansions in Hilbert spaces have been introduced by Duffin and Schaeffer [12] under the name of *frames*. Besides traditional and relevant applications of *wavelet and Gabor frames* [9], [10], [11], [15], [16], [22] in signal processing, image processing, data compression, pattern matching, sampling theory, communication and data transmission, recently the use of frames also in numerical analysis for the solution of operator equations and PDE is investigated [8], [25]. Therefore, not only the characterization by frames of functions in $L^2(\mathbb{R}^d)$ is relevant but also that of (smoothness) Banach function spaces is crucial to have a correct formulation of effective and stable numerical

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schemes. *Atomic decompositions* as an extension of frames in *coorbit spaces* have been introduced in [14], [21] and it will be an important concept in this paper. The problem when one is dealing with numerical applications is that one cannot use exact representations of functions, but only approximations can be available. Therefore any frame that one wants concretely to use in applications will be affected by a perturbation. At this point it is not ensured that the perturbed system is a frame anymore. For example, recent papers [6], [8] proposed efficient methods to compute canonical dual frames. Such systems are fundamental for the computation of the frame expansion coefficients by scalar products. On the one hand the proposed algorithms compute a nice and accurate approximation of each individual element of the canonical dual frame, producing a global system that can be interpreted as a perturbation of the exact canonical dual. On the other hand, no proof is given yet whether such new system is again a frame and how much the perturbation produced by the numerics can affect global expansions.

In this paper we want to discuss certain stability results of frames under perturbations, combining some of the well-known results due to Casazza, Christensen, and Heil [1], [3], [5], [15] with some new insights in the frame theory. In particular, the emphasis here will be on *localization of frames* introduced by Gröchenig in [23] and on its generalizations due to Gröchenig and the author in [20]. We will show that a suitable perturbation of an intrinsically localized frame can produce again an intrinsically localized atomic decomposition for a class of associated Banach spaces. Applications of such result will be presented in subsequent contributions.

The paper is organized as follows: Section 2 recalls the concept of a frame and some relevant instances of the localization of frame theory and Section 3 collects the perturbation and stability results.

2 – Frames and Schur localization

In this section we recall the concept of a frame, how it can be used to define certain associated Banach spaces, and how to obtain stable decompositions in these Banach spaces.

A subset $\mathcal{G} = \{g_n\}_{n\in\mathbb{Z}^d}$ of a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}^d} |\langle f, g_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}\,, \tag{1}$$

for some constants $0 < A \leq B < \infty$.

Equivalently, we could define a frame by the requirement that the frame operator $S = DC$ is boundedly invertible (positive and self-adjoint) on $\mathcal{H}$, where $C = C_{\mathcal{G}}$ defined by $Cf = (\langle f, g_n\rangle)_n$ is the corresponding bounded *analysis*

operator from $\mathcal{H}$ into $\ell^2(\mathbb{Z}^d)$, and $D = D_{\mathcal{G}} = C^*$, $D\mathbf{c} = \sum_n c_n g_n$, is the bounded *synthesis operator* from $\ell^2(\mathbb{Z}^d)$ into $\mathcal{H}$. The set $\tilde{\mathcal{G}} = \{\tilde{g}_n = S^{-1}g_n\}_{n\in\mathbb{Z}^d}$ is again a frame for $\mathcal{H}$ and it is called the *canonical dual frame* playing an important role in the reconstruction of $f \in \mathcal{H}$ from the frame coefficients, because we have

$$f = SS^{-1}f = \sum_n \langle f, S^{-1}g_n\rangle g_n = S^{-1}Sf = \sum_n \langle f, g_n\rangle S^{-1}g_n\,. \tag{2}$$

Since in general a frame is overcomplete, the coefficients in this expansion are in general not unique (unless $\mathcal{G}$ is a Riesz basis, we have $\ker(D) \neq \{0\}$) and there exist many possible dual frames $\{\tilde{g}_n\}_{n\in\mathbb{Z}^d}$ in $\mathcal{H}$ such that

$$f = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{g}_n\rangle g_n$$

with the norm equivalence $\|f\|_{\mathcal{H}} \asymp \|\langle f, \tilde{g}_n\rangle\|_2$. We refer the reader, for example, to the book [4] for a more complete literature, details, and examples.

The concept of a frame can be extended to Banach spaces as follows: an *atomic decomposition* for a separable Banach space $\mathcal{B}$ is a sequence $\mathcal{G} = \{g_n\}_{n\in\mathbb{Z}^d}$ in $\mathcal{B}$ with an associated sequence space $\mathcal{B}_d$ such that the following properties hold.

(a) There exists a *coefficient operator* C defined by $Cf = (\langle f, \tilde{g}_n\rangle_{n\in\mathbb{Z}^d})$ bounded from $\mathcal{B}$ into $\mathcal{B}_d$, where $\tilde{\mathcal{G}} = \{\tilde{g}_n\}_{n\in\mathbb{Z}^d}$ is in $\mathcal{B}'$;

(b) norm equivalence: for all $f \in \mathcal{B}$

$$A_{\mathcal{B}}\|f\|_{\mathcal{B}} \leq \|\langle f, \tilde{g}_n\rangle_{n\in\mathbb{Z}^d}\|_{\mathcal{B}_d} \leq B_{\mathcal{B}}\|f\|_{\mathcal{B}}, \quad A_{\mathcal{B}}, B_{\mathcal{B}} > 0\,;$$

(c) the following series expansion converges unconditionally in $\mathcal{B}$

$$f = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{g}_n\rangle g_n, \quad \text{for all } f \in \mathcal{B}\,.$$

We want to illustrate in the following that there exists a natural choice of Banach spaces $\mathcal{B}$ such that suitable frames $\mathcal{G}$ for $\mathcal{H}$ extend to atomic decompositions for $\mathcal{B}$ with coefficient map $C = C_{\tilde{\mathcal{G}}}$ where $\tilde{\mathcal{G}}$ denotes the canonical dual frame. For "suitable" we mean that the frame should have additional *localization properties*.

The theory of localization of frames has been introduced and developed by Gröchenig *et al.* [7], [20], [23] in order to illustrate general principles for the frame characterization of Banach spaces. Recently it has been also recognized that localized frames are "good" not only for theoretical purposes but also for numerical applications in signal and image processing and in PDE [6], [8], [15], [16], [25].

In this paper we work only with a particular type of localization and we refer to [20] for a more general theory and results. In particular we shall work with the *Schur algebra* $\mathcal{A}_s^1$ [24] which is defined as the class of matrices $A = (a_{kl}), k, l \in \mathbb{Z}^d$, such that

$$\|A\|_{\mathcal{A}_s^1} := \max \left\{ \sup_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} |a_{kl}| v_s(k-l), \sup_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a_{kl}| v_s(k-l) \right\} < \infty\,,$$

where $v_s(x) = (1+|x|)^s$ for $s \geq 0$. For $s = 0$ we denote $\mathcal{A}_s^1$ with $\mathcal{A}^1$. The Schur algebra endowed with the norm $\|\cdot\|_{\mathcal{A}_s^1}$ is a Banach $*$-algebra, where the involution is the transpose-conjugate operator and the following properties hold

(A0) $\mathcal{A}_s^1 \subset \mathcal{B}(\ell_m^p(\mathbb{Z}^d))$, i.e., each $A \in \mathcal{A}_s^1$ defines a bounded operator on $\ell_m^p(\mathbb{Z}^d)$ for $1 \leq p \leq \infty$ and for any v_s-moderate weight m, i.e., $0 < m(x+y) \leq C v_s(x) m(y)$ for all $x, y \in \mathbb{R}^d$.

(A1) If $s > 0$ and $A \in \mathcal{A}_s^1$ is invertible on $\ell^2(\mathbb{Z}^d)$, then $A^{-1} \in \mathcal{A}_s^1$ as well. In the language of Banach algebras, $\mathcal{A}_s^1$ is called inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. It is not known whether such property holds even for $s = 0$;

(A2) $\mathcal{A}_s^1$ is solid: i.e., if $A \in \mathcal{A}_s^1$ and $|b_{kl}| \leq |a_{kl}|$ for all $k, l \in \mathbb{Z}^d$, then $B \in \mathcal{A}_s^1$ as well.

We refer to [24] where a characterization of a large class of algebras with properties (A0-2) is presented. We also observe here that a major part of the results presented in this paper can be generalized considering those algebras instead of $\mathcal{A}_s^1$.

Given two subsets $\mathcal{G} = \{g_n\}_{n\in\mathbb{Z}^d}$ and $\mathcal{F} = \{f_x\}_{x\in\mathbb{Z}^d}$ in the Hilbert space $\mathcal{H}$, the (cross-) Gramian matrix $A = A(\mathcal{G}, \mathcal{F})$ of $\mathcal{G}$ with respect to $\mathcal{F}$ is the $\mathbb{Z}^d \times \mathbb{Z}^d$-matrix with entries

$$a_{nx} = \langle f_x, g_n \rangle\,.$$

A frame $\mathcal{G}$ for $\mathcal{H}$ is called $\mathcal{A}_s^1$*-localized* with respect to another frame $\mathcal{F}$ if $A(\mathcal{G}, \mathcal{F}) \in \mathcal{A}_s^1$. In this case we write $\mathcal{G} \sim_{\mathcal{A}_s^1} \mathcal{F}$. If $\mathcal{G} \sim_{\mathcal{A}_s^1} \mathcal{G}$, then $\mathcal{G}$ is called $\mathcal{A}_s^1$*-self-localized* or *intrinsically* $\mathcal{A}_s^1$*-localized*. By exploiting property (A1) one can prove [20] the following

THEOREM 2.1. *For $s > 0$, any $\mathcal{A}_s^1$-self-localized frame $\mathcal{G}$ has $\mathcal{A}_s^1$-self-localized canonical dual.*

Let $(\mathcal{G}, \tilde{\mathcal{G}})$ be a pair of dual $\mathcal{A}_s^1$-self-localized frames for $\mathcal{H}$. Assume that m is a v_s-moderate weight and $\ell_m^p(\mathbb{Z}^d) \subseteq \ell^2(\mathbb{Z}^d)$. Then the Banach space $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$ is defined by

$$\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}}) := \{f \in \mathcal{H} : \quad f = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{g}_n \rangle g_n, \quad (\langle f, \tilde{g}_n \rangle)_{n\in\mathbb{Z}^d} \in \ell_m^p(\mathbb{Z}^d)\} \tag{3}$$

with the norm $\|f\|_{\mathcal{H}^p_m} := \|(\langle f, \tilde{g}_n\rangle)_{n\in\mathbb{Z}^d}\|_{\ell^p_m}$. Since $\ell^p_m(\mathbb{Z}^d) \subseteq \ell^2(\mathbb{Z}^d)$, $\mathcal{H}^p_m$ is a dense subspace of $\mathcal{H}$. If $\ell^p_m(\mathbb{Z}^d) \not\subseteq \ell^2(\mathbb{Z}^d)$ and $p < \infty$ then we define $\mathcal{H}^p_m$ to be the completion of the subspace $\mathcal{H}_0$ of all finite linear combinations in $\mathcal{G}$ with respect to the norm $\|f\|_{\mathcal{H}^p_m} = \|(\langle f, \tilde{g}_n\rangle)_{n\in\mathbb{Z}^d}\|_{\ell^p_m}$. If $p = \infty$ then we define $\mathcal{H}^\infty_m$ as the completion of $\mathcal{H}_0$ in the $\sigma(\mathcal{H}, \mathcal{H}_0)$-topology.

Remark. The definition of $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ does not depend on the particular $\mathcal{A}^1_s$-self localized dual chosen, and any other $\mathcal{A}^1_s$-self-localized frame $\mathcal{F}$ which is also localized to $\mathcal{G}$ generates in fact the same spaces. In particular one has that $\mathcal{H}^2(\mathcal{G}, \tilde{\mathcal{G}}) = \mathcal{H}$.

Then the following theorem [20] holds.

Theorem 2.2. *Assume that $\mathcal{G}$ is an $\mathcal{A}^1_s$-self-localized frame for $\mathcal{H}$, for $s > 0$. Then $\mathcal{G}$ and its canonical dual frame $\tilde{\mathcal{G}}$ are atomic decompositions for $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ with $C_{\tilde{\mathcal{G}}}$ and $C_{\mathcal{G}}$ as corresponding and respective coefficient operators.*

Throughout the rest of the paper we assume that $\mathcal{G}$ is a $\mathcal{A}^1_s$-self-localized frame for $\mathcal{H}$, for $s > 0$.

3 – Perturbation and localization of frames

In the last years several results on *stability under perturbation* of frames and Riesz bases have been investigated, for example, by Casazza, Christensen, and Heil [1], [3], [5], [15], to name some of the most prominent authors.

One of the first motivations and classical applications of the perturbation of frame theory is the study of "non-uniform" *coherent* frames generated by a strongly continuous (square-integrable) and irreducible representation π of some locally compact group by $\mathcal{U}(L^2(\mathbb{R}^d))$, the unitary operators on $L^2(\mathbb{R}^d)$, (see for example [14], [21]), i.e., $\mathcal{F} := \{\pi(x)g\}_{x\in H}$, where $H \subset G$. In most of the classical cases, namely Gabor and wavelet frames, there is usually a well structured canonical choice of the index subset H, maybe a discrete subgroup. The question is whether, given a frame $\mathcal{F} := \{\pi(x)g\}_{x\in H}$, a *perturbation* of H, namely $\tilde{H}$, might preserve the frame property, i.e., whether $\mathcal{E} := \{\pi(\tilde{x})g\}_{\tilde{x}\in\tilde{H}}$ is again a frame for $L^2(\mathbb{R}^d)$.

Here we want to emphasize that the perturbation of frame theory is indeed very relevant and important for numerical purposes. In fact, it is never possible to compute exactly a frame (of functions), in particular its canonical dual frame, and the numerical methods applied in this context perform approximations up to some prescribed tolerance, and numerical rounding errors are anyway present. Then it is clear that if such perturbations destroyed the frame property, then the use of such expansions for numerical applications would be potentially incorrect. Moreover, in many applications, for example in PDE numerical solution [8], [25],

it is relevant that the frames used can be stable in a wider sense, i.e., any small perturbation should preserve not only the Hilbert space frame properties, but even the atomic decomposition one, especially in the characterization of those Banach spaces where it is expected that the solution is sitting.

One of the first results on perturbation of frames has been proposed by Christensen in [3] and it reads as follows:

Theorem 3.1. *Let $\mathcal{F}$ be a frame for $\mathcal{H}$ with bounds A, B and let $\mathcal{E}$ be a subset in $\mathcal{H}$. If*

$$R := \sum_{n\in\mathbb{Z}^d} \|e_n - f_n\|_{\mathcal{H}}^2 < A\,, \tag{4}$$

then $\mathcal{E}$ is again a frame with bounds $A(1-\sqrt{\frac{R}{A}})^2, B(1-\sqrt{\frac{R}{B}})^2$.

Unfortunately condition (4) means that the approximation tolerated for each individual element of the frame is not uniform but should decrease quickly for n going to ∞. For frames "uniformly distributed" in the space and well structured, one would prefer not to *treat* differently one element of the frame with respect to others, but would accept a more "democratic" (uniform) approximation.

A stronger result has been shown by Christensen in [2] which overcomes this difficulty. A very general version of this result can be found in [1].

Theorem 3.2. *Let $\mathcal{F}$ be a frame for $\mathcal{H}$ with bounds A, B. Let $\mathcal{E}$ be a subset in $\mathcal{H}$, and assume that there exists a constant $\mu \geq 0$ such that $\frac{\mu}{\sqrt{A}} < 1$ and*

$$\left\| \sum_{n\in F} c_n(e_n - f_n) \right\| \leq \mu \left(\sum_{n\in F} |c_n|^2 \right)^{1/2}, \tag{5}$$

for any finite sequence $(c_n)_{n\in F}$ of scalars and $F \subset \mathbb{Z}^d$, $\#F < \infty$. Then $\mathcal{E}$ is a frame with bounds $A(1-\frac{\mu}{\sqrt{A}})^2, B(1+\frac{\mu}{\sqrt{B}})^2$.

Criterion (4) implies (5), and, in particular, the latter does not imply that the approximation should improve for n going to ∞. For general frames, to verify the condition (5) might be anyway difficult. We refer the reader to the works [13], [17], [18], [19] where perturbation techniques similar to (5) have been exploited in order to prove the existence of a large class of intermediate (wave-packet) frames between the more classical and well-known Gabor and wavelet frames.

We want to illustrate here sufficient conditions, usually easier to check for localized frames, which ensure the application of Theorem 3.2 as a useful tool in several concrete cases.

PROPOSITION 3.3. *Let $\mathcal{F}$ be a frame for $\mathcal{H}$ with bounds A, B and let $\mathcal{E}$ be a subset in $\mathcal{H}$ satisfying*

$$\|A(\mathcal{E}-\mathcal{F},\tilde{\mathcal{G}})\|_{\mathcal{A}^1} = \max\left\{\sup_{n\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d}|\langle e_n - f_n, \tilde{g}_x\rangle|, \sup_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|\langle e_n - f_n, \tilde{g}_x\rangle|\right\} \le \varepsilon\,, \tag{6}$$

with $0 < \varepsilon < (\sqrt{A^{-1}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}})^{-1}$. Then $\mathcal{E}$ is a frame with bounds $A(1-\sqrt{A^{-1}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}}\varepsilon)^2$, $B(1+\sqrt{B^{-1}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}}\varepsilon)^2$.

PROOF. It is sufficient to show that $C^*_{\mathcal{E}-\mathcal{F}}$ is bounded from ℓ^2 to $\mathcal{H}$ with bound small enough.

$$\begin{aligned}\|C^*_{\mathcal{E}-\mathcal{F}}(\mathbf{c})\|^2 &= \left\|\sum_{n\in\mathbb{Z}^d} c_n(e_n - f_n)\right\|^2 = \\ &= \sum_{n,m\in\mathbb{Z}^d} c_n\overline{c_m}\langle e_n - f_n, e_m - f_m\rangle = \\ &= \sum_{n,m\in\mathbb{Z}^d} c_n\overline{c_m}\sum_{x,y\in\mathbb{Z}^d}\langle e_n - f_n, \tilde{g}_x\rangle\overline{\langle e_m - f_m, \tilde{g}_y\rangle}\langle g_x, g_y\rangle\,.\end{aligned}$$

Since $\mathcal{G}$ is assumed $\mathcal{A}^1_s$-intrinsically localized one has $A(\mathcal{G},\mathcal{G}) \in \mathcal{A}^1_s$. Moreover, since (6) holds one also has that $A(\mathcal{E}-\mathcal{F},\mathcal{G}) \in \mathcal{A}^1$ with norm $\|A(\mathcal{E}-\mathcal{F},\mathcal{G})\|_{\mathcal{A}^1} \le \varepsilon$. By the property (A0) of the Schur algebra one immediately has

$$\|C^*_{\mathcal{E}-\mathcal{F}}(\mathbf{c})\|^2 \le \|\mathbf{c}\|^2_{\ell^2}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}\varepsilon^2\,.$$

Therefore, since $\sqrt{A^{-1}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}}\varepsilon < 1$, one concludes by an application of Theorem 3.2. ∎

Observe that if $\mathcal{F} \sim_{\mathcal{A}^1} \mathcal{G}$ then the assumptions of Proposition 3.3 in particular imply that $\mathcal{E} \sim_{\mathcal{A}^1} \mathcal{G}$. In fact, the localization principle has been thought and invented as a measure of similarity or equivalence of frames. Thus, it is not surprising that the localization condition (6) can be used as a natural measure of the perturbation for a frame.

We conclude with a stability result of $\mathcal{E}$ as an atomic decomposition for $\mathcal{H}^p_m$ for all $p \in [1,\infty]$.

THEOREM 3.4. *Let $\mathcal{F}$ be a frame for $\mathcal{H}$.*

(i) *Assume $\mathcal{F} \sim_{\mathcal{A}^1_s} \{\mathcal{F}, \mathcal{G}\}$. This in particular implies that $\mathcal{F}$ is an atomic decomposition for $\mathcal{H}^p_m$ for any v_t-moderate weight m, for $0 \le t \le s$. Let us denote the corresponding atomic decomposition bounds $A_{p,m}, B_{p,m} > 0$ for all $p \in [1,\infty]$ and for any v_t-moderate weight m, and assume $B := \sup_{p\in[1,\infty],\ m} B_{p,m} < \infty$. Let $\mathcal{E}$ be a system in $\mathcal{H}$ satisfying*

$$\|A(\mathcal{E}-\mathcal{F},\tilde{\mathcal{G}})\|_{\mathcal{A}^1_t} = \max\left\{ \sup_{n\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} |\langle e_n - f_n, \tilde{g}_x\rangle| v_t(n-x), \right. \tag{7}$$
$$\left. \sup_{x\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} |\langle e_n - f_n, \tilde{g}_x\rangle| v_t(n-x) \right\} \le \varepsilon\,,$$

for $0 < \varepsilon < B^{-1}$. Then $\mathcal{E} \sim_{\mathcal{A}^1_t} \{\mathcal{E}, \mathcal{F}, \mathcal{G}\}$ is an atomic decomposition for $\mathcal{H}^p_m$ with bounds $A_{p,m}(1+(\varepsilon B_{p,m}))^{-1}$, $B_{p,m}(1-(\varepsilon B_{p,m}))^{-1}$ for all $p \in [1,\infty]$ and for any v_t-moderate weight m. In particular there exists $\tilde{\mathcal{E}} \subset (\mathcal{H}^p_m)' = \mathcal{H}^{p'}_{1/m}$, $1/p + 1/p' = 1$, such that for all $f \in \mathcal{H}^p_m$

$$f = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{e}_n\rangle e_n\,,$$

with unconditional convergence in $\mathcal{H}^p_m$ and

$$\left\| f - \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{f}_n\rangle e_n \right\|_{\mathcal{H}^p_m} \le \varepsilon B \|f\|_{\mathcal{H}^p_m}\,, \tag{8}$$

for all $f \in \mathcal{H}^p_m$, $p \in [1,\infty]$, and v_t-moderate weight m.

(ii) *If (7) holds with $t = 0$, for $0 < \varepsilon < (\sqrt{A^{-1}_{2,1}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}})^{-1}$, and $\mathcal{E} \sim_{\mathcal{A}^1_s} \mathcal{G}$ then $\mathcal{E} \sim_{\mathcal{A}^1_s} \mathcal{E}$ and $\mathcal{E}$ is an atomic decomposition for $\mathcal{H}^p_m$ for all $p \in [1,\infty]$ and for any v_s-moderate weight m.*

PROOF.

(i) By similar arguments as in the proof of Proposition 3.3 one can show

that

$$\left\|\sum_{n\in\mathbb{Z}^d} c_n(e_n - f_n)\right\|_{\mathcal{H}^p_m} = \left\|\left\langle \sum_{n\in\mathbb{Z}^d} c_n(e_n - f_n), \tilde{g}_x \right\rangle\right\|_{\ell^p_m} =$$

$$= \left\|\sum_{n\in\mathbb{Z}^d} c_n\langle (e_n - f_n), \tilde{g}_x\rangle\right\|_{\ell^p_m} \leq$$

$$\leq \|A(\mathcal{E} - \mathcal{F}, \tilde{\mathcal{G}})\|_{\mathcal{A}^1_t}\|c\|_{\ell^p_m} \leq \varepsilon \left(\sum_{n\in\mathbb{Z}^d} |c_n|^p m(n)^p\right)^{1/p}.$$

By an application of [5, Theorem 2.3] one has that $\mathcal{E}$ is an atomic decomposition for $\mathcal{H}^p_m$ with bounds $A_{p,m}(1+(\varepsilon B_{p,m}))^{-1}$, $B_{p,m}(1-(\varepsilon B_{p,m}))^{-1}$ for all $p \in [1,\infty]$ and for all v_t-moderate weight m. Since $\mathcal{E} \sim_{\mathcal{A}^1_t} \mathcal{G}$ and $\tilde{\mathcal{G}} \sim_{\mathcal{A}^1_t} \mathcal{F}$ then by [20, Lemma 2.2] it is $\mathcal{E} \sim_{\mathcal{A}^1_t} \mathcal{F}$. Note that $|A(\mathcal{E},\mathcal{E})| \leq |A(\mathcal{E},\tilde{\mathcal{G}})||A(\mathcal{G},\mathcal{E})|$. All the matrices on the right-hand side are in $\mathcal{A}^1_t$ and then by (A2) one has that $\mathcal{E} \sim_{\mathcal{A}^1_t} \mathcal{E}$.

(ii) Assume now that $0 < \varepsilon < (\sqrt{A_2^{-1}\|A(\mathcal{G},\mathcal{G})\|_{\mathcal{A}^1}})^{-1}$. By an application of Proposition 3.3 one has that $\mathcal{E}$ is a frame for $\mathcal{H}$. If $\mathcal{E} \sim_{\mathcal{A}^1_s} \mathcal{G}$ then by [20, Lemma 2.2] it is also $\mathcal{E} \sim_{\mathcal{A}^1_s} \{\tilde{\mathcal{G}}, \mathcal{E}\}$ and by Theorem 2.2 the frame $\mathcal{E}$ is an atomic decomposition for $\mathcal{H}^p_m(\mathcal{E},\tilde{\mathcal{E}}) = \mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}}) = \mathcal{H}^p_m$. ∎

Remark. For $t = 0$ Schur type localization it is not known whether there exists dual frames $\tilde{\mathcal{E}}$ with such localization to define the corresponding spaces $\mathcal{H}^p(\mathcal{E},\tilde{\mathcal{E}})$. Therefore, Theorem 3.4 (i) gives, by a perturbation argument, an alternative criterion to Theorem 2.2 in order to show that a $\mathcal{A}^1$-intrinsically localized frame $\mathcal{E}$ in fact can extend to an atomic decomposition for a class of non-trivial Banach spaces $\mathcal{H}^p$.

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REFERENCES

[1] P. G. Casazza – O. Christensen: *Perturbation of operators and applications of frame theory*, J. Fourier Anal. Appl., **3** no. 5 (1998), 543-557.

[2] O. Christensen: *A Paley-Wiener theorem for frames*, Proc. Am. Math. Soc., **123** no. 7 (1995), 2199-2201.

[3] O. Christensen: *Frame perturbation*, Proc. Amer. Math. Soc., **123** (1995), 1217-1220.

[4] O. Christensen: *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2003.

[5] O. Christensen – C. Heil: *Perturbation of Banach frames and atomic decompositions*, Math. Nachr., **185** (1997), 33-47.

[6] O. Christensen – T. Strohmer: *The finite section method and problems in frame theory*, J. Approx. Theory, **133** no. 2 (2005), 221-237.

[7] E. Cordero – K. Gröchenig: *Localization of frames, II*, Appl. Comput. Harmon. Anal., **17** no. 1 (2004), 29-47.

[8] S. Dahlke – M. Fornasier – T. Raasch: *Adaptive frame methods for elliptic operator equations*, Adv. Comput. Math., (2005), to appear.

[9] I. Daubechies: *Wavelets, time–frequency localization and signal analysis*, IEEE Trans. Inf. Th., **36** (1990), 961-1005.

[10] I. Daubechies: *Ten Lectures on Wavelets*, SIAM, 1992.

[11] I. Daubechies – A. Grossmann – Y. Meyer: *Painless nonorthogonal expansions.*, J. Math. Phys., **27** (1986), 1271-1283.

[12] R. J. Duffin – A. C. Schaeffer: *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341-366.

[13] H. G. Feichtinger – M. Fornasier: *Flexible Gabor–wavelet atomic decompositions for L^2–Sobolev spaces*, Ann. Mat. Pura Appl., (2003), to appear.

[14] H. G. Feichtinger – K. Gröchenig: *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Funct. Anal., **86** no. 2 (1989), 307-340.

[15] H. G. Feichtinger – T. Strohmer: *Gabor Analysis and Algorithms*, Birkhäuser (eds.), 1998.

[16] H. G. Feichtinger – T. Strohmer: *Advances in Gabor Analysis*, Birkhäuser (eds.), 2003.

[17] M. Fornasier: *Constructive Methods for Numerical Applications in Signal Processing and Homogenization Problems*, Ph.D. thesis, University of Padova and University of Vienna, 2002.

[18] M. Fornasier: *Banach frames for α–modulation spaces*, preprint 10/2004, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Roma "La Sapienza", 2004.

[19] M. Fornasier: *Quasi–orthogonal decompositions of structured frames*, J. Math. Anal. Appl., **289** no. 1 (2004), 180-199.

[20] M. Fornasier – K. Gröchenig: *Intrinsic localization of frames*, Constr. Approx., **22** (2005), 395-415.

[21] K. Gröchenig: *Describing functions: atomic decompositions versus frames*, Monatsh. Math., **112** no. 1 (1991), 1-42.

[22] K. Gröchenig: *Foundations of Time–Frequency Analysis*, Birkhäuser, 2000.

[23] K. Gröchenig: *Localization of frames, Banach frames, and the invertibility of the frame operator*, J. Fourier Anal. Appl., **10** no. 2 (2004), 105-132.

[24] K. Gröchenig – M. Leinert: *Symmetry and inverse-closedness of matrix algebras and symbolic calculus for infinite matrices*, Trans. Amer. Math. Soc., (2003), to appear.

[25] R. Stevenson: *Adaptive solution of operator equations using wavelet frames*, SIAM J. Numer. Anal., **41** no. 3 (2003), 1074-1100.

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