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Geometric construction of generalized cubic splines

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ABSTRACT: *We study the Bézier-like, geometric properties of four dimensional spaces of the form span* $< u_k, \mathcal{P}_1, v_k >$*, where* u_k *and* v_k *are subject to general conditions, and describe the geometric construction of the corresponding spline spaces*

1 – Introduction

If we ask a mathematician the definition of a spline we probably have nowadays as answer – a piecewise function which has a certain degree of smoothness at the break points – which is much wider than the original idea contained the Shoenberg's pioneering papers (see e.g. [2] and [8]). Indeed, several kinds of function spaces have substituted polynomials to form the pieces of new classes of *splines*: exponential, trigonometric, rational, Tchebycheffian splines are only some of the most famous names (see, e.g., [5], [4], [6], [9]). As usual, these researches have been motivated both by theoretical and by practical reasons. Among the last ones, we want to concentrate on the solicitations induced by their applications in data interpolation and approximation and in computer aided design. The limit of polynomial splines in data approximation or interpolation relies essentially in the lack of reproduction of the shape of the data set. The consequences of this drawback can be serious and the first attempt to avoid them led to the so-called *tension splines* (see, e.g., [4], [5], [10] and references quoted therein) which can be forced to assume a piecewise linear shape. On the other hand, the necessity of an easy description and manipulation of real objects by means of curves and surfaces has led to other extensions of classical splines.

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In our opinion, the trend of the recent researches (especially those motivated by the two fields of applications described above) seems to indicate that any new class of *spline functions* should possess two main properties: to have a *geometric construction* and to have an *adaptive shape*.

One of the main advantages of polynomial splines (see, e.g., [3]) is that the *Bernstein-Bézier control points* (which give the representation of the polynomial pieces in term of the Bernstein basis) can be obtained via convex combinations of the *de Boor control points* (which give the representation of the spline in term of the B-spline basis). In addition, both the Bernstein and the B-spline basis are totally positive and sum up to one, so that the (difficult) task of controlling the shape of the polynomials pieces is basically reduced to the (much easier) control of the shape of the *de Boor control polygon* (the polygonal line connecting the de Boor control points).

At the same time, there are several applications in which we wish to model objects or phenomena with parts of completely different shape; therefore we would like to incorporate in our CAD or approximation tools different function spaces – trigonometric, exponential, variable degree polynomials etc. – still maintaining the simplicity of the original polynomial structure.

With regard to this, several results on splines with segments in particular function spaces have been established (see e.g. [4], [5], [6], [7]). However, an unified approach to the geometric construction and analysis of properties of such spline spaces seems of interest but not yet available.

Recently, as a first step in this direction, in [1] the main properties of n-degree Bernstein polynomials, which form a basis of the space $\mathcal{P}_n = \text{span} < (1-t)^n, \mathcal{P}_{n-2}, t^n >$, have been extended to suitable bases of spaces of the form $\text{span} < u(t), P_{n-2}, v(t) >$, where the functions u and v are subject to very general constraints.

The aim of the present paper is to give a further contribution in the same direction. Given a sequence of knots $\{x_0, \dots, x_n\}$, we study with more details the geometric properties of *cubic-like* spaces of the form $\text{span} < u_k, \mathcal{P}_1, v_k >$ and provide the geometric construction for the corresponding *adaptive shape*, *cubic-like* splines (u_k and v_k can vary in each subinterval).

The content is divided in four sections. In the next one we analyze a generalized Bézier-Bernstein representation for the elements of the space $\text{span} < u_k, \mathcal{P}_1, v_k >$ and in Section 3 we develop the geometric spline construction. Section 4 is then dedicated to final examples.

2 – Generalized Bézier representation

Let us consider the space

$$\mathcal{P}_{u,v} := \text{span} < 1, t, u(t); v(t) >, \quad t \in [0,1] \tag{1}$$

where we assume that $u, v \in C^2([0,1])$,

$$\dim(\mathcal{P}_{u,v}) = 4\,, \tag{2}$$

and, denoting by ψ any element of $\mathcal{P}_{u,v}$,

$$\text{if}\ \ \psi''(t_1) = \psi''(t_2) = 0,\ t_1, t_2 \in [0,1],\ t_1 \neq t_2,\ \text{then}\ \ \psi''(t) = 0, t \in [0,1]\,. \tag{3}$$

We notice that, considering (2), (3) is the same as saying that $\{u'', v''\}$ is a Tchebycheff system in $[0,1]$.

Thanks to (3), in the following we assume, without loss of generality, that

$$u(1) = u'(1) = u''(1) = 0, \quad u(0) = 1\,, \tag{4}$$

$$v(0) = v'(0) = v''(0) = 0, \quad v(1) = 1\,. \tag{5}$$

Lemma 1. *Let u, v be the unique elements in $\mathcal{P}_{u,v}$ satisfying* (4) *and* (5) *respectively. Then*

$$u'(t) < 0, u''(t) > 0 t \in [0,1), v'(t) > 0, v''(t) > 0; t \in (0,1]\,. \tag{6}$$

Moreover,

$$u'(0) < -1, \quad v'(1) > 1, \quad 1 - \frac{1}{v'(1)} > -\frac{1}{u'(0)}\,. \tag{7}$$

Proof. From

$$-1 = \int_0^1 u'(s)ds \tag{8}$$

we have that $u'(t) < 0$ for some $t \in (0,1)$. If there exists $t_1 \in [0,1)$ such that $u'(t_1) \geq 0$ then u' has at least one zero in $(0,1)$. Since $u'(1) = 0$, we have that u'' has at least one zero in $(0,1)$. That contradicts (3) since $u''(0) = 0$ and $u'' \not\equiv 0$. Thus $u'(t) < 0, t \in [0,1)$. Similarly we obtain the remaining inequalities of (6).

Now, from (6), u' is strictly increasing in $(0,1)$ so, from (8) we obtain the first inequality of (7). Similarly we conclude for the second one.

Finally, let us consider the function

$$g(t) := t - \frac{u(t)}{u'(0)} - \frac{v(t)}{v'(1)} \in \mathcal{P}_{u,v}\,.$$

From (4), (5), (6), and the first two inequalities of (7) we have

$$g(0) = -\frac{1}{u'(0)}, g(1) = 1 - \frac{1}{v'(1)}, g'(0) = 0, g'(1) = 0, g''(0) > 0, g''(1) < 0\,.$$

If $g(1) \le g(0)$ the previous inequalities imply that g'' has at least three distinct zeros in $(0,1)$ and that contradicts (3) since $g'' \not\equiv 0$. Thus the last inequality in (7) holds.

Following [1], we have that the space $\mathcal{P}_{u,v}$ has a *generalized Bernstein basis* $\{B_0, B_1, B_2, B_3\}$. The elements of the basis form a partition of unity, are strictly positive in $(0,1)$ and characterized by the following relations

$$\begin{aligned} B_0(0) = 1, &\qquad B_0(1) = B_0'(1) = B_0''(1) = 0\,,\\ B_1(0) = 0, B_1'(0) = -u'(0), &\qquad B_1(1) = B_1'(1) = 0\,,\\ B_2(0) = B_2'(0) = 0, &\qquad B_2(1) = 0, B_2'(1) = -v'(1)\,,\\ B_3(0) = B_3'(0) = B_3''(0) = 0, &\qquad B_3(1) = 1\,, \end{aligned}$$

so that

$$B_0 = u, \qquad B_3 = v\,.$$

With some computation we obtain the following

THEOREM 1. *Let B_0, B_1, B_2, B_3 the elements of the generalized Bernstein basis of $\mathcal{P}_{u,v}$ constructed according to* [1]*. Thus*

$$t \equiv \sum_{i=0}^{3} \tau_i B_i(t)$$

where

$$\tau_0 := 0, \quad \tau_1 := -\frac{1}{u'(0)}, \quad \tau_2 := 1 - \frac{1}{v'(1)}, \quad \tau_3 := 1\,.$$

Let

$$\psi(t) = \sum_{i=0}^{3} b_i B_i(t)$$

be any element of $\mathcal{P}_{u,v}$. We will refer to the coefficients b_j, $j = 0, 1, 2, 3$ as the *generalized Bézier coefficients* of ψ and to the polygonal line having vertices

$$\begin{pmatrix}\tau_j\\ b_j\end{pmatrix}, \quad j=0,1,2,3$$

as the *generalized Bézier control polygon* of ψ.

For notational convenience let us denote

$$\xi := \tau_1, \quad \eta := 1-\tau_2, \quad \nu := 1-\xi-\eta\,.$$

From Theorem 1 we have $0<\xi;\nu;\eta<1$. Moreover, with some additional computations we obtain

Corollary 1. *Let $\psi(t)=\sum_{i=0}^{3} b_i B_i(t)$ be any element of $\mathcal{P}_{u,v}$ expressed with respect to the generalized Bernstein basis of the space. Then,*

$$\begin{gathered}\psi(0)=b_0 \quad \psi(1)=b_3\,,\\ \psi'(0)=(b_1-b_0)\frac{1}{\xi}, \quad \psi'(1)=(b_3-b_2)\frac{1}{\eta}\,,\\ \psi''(0)=u''(0)[(b_2-b_1)\frac{\xi}{\nu}-(b_1-b_0)], \quad \psi''(1)=v''(1)\left[(b_3-b_2)-(b_2-b_1)\frac{\eta}{\nu}\right].\end{gathered}$$

In the remaining part of this section we give a *de Casteljau like* algorithm. It is well known that any cubic polynomial p can be evaluated – starting from its Bézier coefficients b_0,b_1,b_2,b_3 – using the de Casteljau algorithm, which, in matrix form, can be expressed as

$$p(t)=(1-t\, t)\begin{pmatrix}1-t & t & 0\\ 0 & 1-t & t\end{pmatrix}\begin{pmatrix}1-t & t & 0 & 0\\ 0 & 1-t & t & 0\\ 0 & 0 & 1-t & t\end{pmatrix}\begin{pmatrix}b_0\\ b_1\\ b_2\\ b_3\end{pmatrix}.$$

Similarly, for the elements of the general space $\mathcal{P}_{u,v}$, it is simple to check the following result.

Theorem 2. *Let $\psi\in\mathcal{P}_{u,v}$, with $\psi=\sum_{i=0}^{3} b_i B_i$. Then, for any $t\in[0,1]$,*

$$\psi(t)=(1-t\, t)\begin{pmatrix}1-t & t & 0\\ 0 & 1-t & t\end{pmatrix}\begin{pmatrix}m_{1,1} & m_{1,2} & m_{1,3} & 0\\ 0 & m_{2,2} & m_{2,3} & 0\\ 0 & m_{3,2} & m_{3,3} & m_{3,4}\end{pmatrix}\begin{pmatrix}b_0\\ b_1\\ b_2\\ b_3\end{pmatrix} \tag{9}$$

where

$$
\begin{aligned}
m_{1,1} &= \frac{u(t)}{(1-t)^2}, \quad m_{3,4} = \frac{v(t)}{t^2}\\
m_{1,2} &= \frac{1}{u'(0)v'(1)-u'(0)+v'(1)}\left(u'(0)(1-v'(1))\left(\frac{u(t)}{(1-t)^2}-1\right)\right)-t\\
m_{1,3} &= -\frac{1}{u'(0)v'(1)-u'(0)+v'(1)}\left(v'(1)\left(\frac{u(t)}{(1-t)^2}-1\right)\right)+t\\
m_{2,2} &= (1-t)-\frac{1}{2}\frac{u'(0)+v'(1)}{u'(0)v'(1)-u'(0)+v'(1)}\\
m_{2,3} &= t+\frac{1}{2}\frac{u'(0)+v'(1)}{u'(0)v'(1)-u'(0)+v'(1)}\\
m_{3,2} &= \frac{1}{u'(0)v'(1)-u'(0)+v'(1)}\left(u'(0)\left(\frac{v(t)}{t^2}-1\right)\right)+(1-t)\\
m_{3,3} &= -\frac{1}{u'(0)v'(1)-u'(0)+v'(1)}\left(v'(1)(1+u'(0))\left(\frac{v(t)}{t^2}-1\right)\right)-(1-t)\,.
\end{aligned}
$$

Note that, because of (4), (5) and (7), the elements $m_{i,j}$ are well defined, continuous functions of $t \in [0,1]$. It is easy to check that

$$\sum_{j=1}^{4} m_{i,j} = 1, \qquad i = 1,2,3\,; \tag{10}$$

but the elements of the matrix are, in general not positive. Now, as it happens in many interesting examples, assume that the functions u and v smoothly depend on a parameter α, possibly scaled so that $\alpha \in [0,\infty)$, that is

$$u = u(t) = u(t;\alpha), \qquad v = v(t) = v(t;\alpha)$$

and, in addition to properties (4), (5), (6), assume that they satisfy the following conditions:

$$-\frac{\partial u(0,\alpha)}{\partial t} = \frac{\partial v(1,\alpha)}{\partial t} = \omega > 0\,; \tag{11}$$

$$\lim_{\alpha\to\infty} u(t,\alpha) = 0, 0 < a \le t \le 1; \quad \lim_{\alpha\to\infty} v(t,\alpha) = 0, 0 \le t \le b < 1\,; \tag{12}$$

$$u(t,0) = (1-t)^3, \qquad v(t,0) = t^3\,; \tag{13}$$

$$\frac{\partial u(t,\alpha)}{\partial \alpha} \le 0, \qquad \frac{\partial v(t,\alpha)}{\partial \alpha} \le 0\,. \tag{14}$$

Then the elements of the matrix used in algorithm (9) are

$$m_{1,1} = \frac{u(t,\alpha)}{(1-t)^2}, \qquad m_{3,4} = \frac{v(t,\alpha)}{t^2} \tag{15}$$

$$m_{1,2} = \frac{\omega-1}{\omega-2}\left(1 - \frac{u(t,\alpha)}{(1-t)^2}\right) - t \tag{16}$$

$$m_{1,3} = \frac{1}{\omega-2}\left(\frac{u(t,\alpha)}{(1-t)^2} - 1\right) + t \tag{17}$$

$$m_{2,2} = (1-t), \qquad m_{2,3} = t \tag{18}$$

$$m_{3,2} = \frac{1}{\omega-2}\left(\frac{v(t,\alpha)}{t^2} - 1\right) + (1-t) \tag{19}$$

$$m_{3,3} = \frac{\omega-1}{\omega-2}\left(1 - \frac{v(t,\alpha)}{t^2}\right) - (1-t) \tag{20}$$

and we have the following result.

THEOREM 3. *Let the functions* $u = u(t) = u(t,\alpha)$ *and* $v = v(t) = v(t,\alpha)$ *satisfy properties* (11)-(14) *and let the elements* $m_{i,j}$ *be defined by* (15)-(20). *Then* $m_{i,j} \geq 0$.

PROOF. Using (11)-(14) we have $\omega \geq 3$ and

$$\begin{aligned} m_{1,2} = m_{1,2}(t,\alpha) &= \frac{\omega-1}{\omega-2}\left(1 - \frac{u(t,\alpha)}{(1-t)^2}\right) - t \geq \\ &\geq \frac{\omega-1}{\omega-2}\left(1 - \frac{u(t;0)}{(1-t)^2}\right) - t = \\ &= \frac{\omega-1}{\omega-2}(1-(1-t)) - t = \frac{1}{\omega-2}t \geq 0\,. \end{aligned} \tag{21}$$

Let us now consider $m_{1,3} = m_{1,3}(t;\alpha)$. From (4), (5), (6) and (12) we have

$$-\frac{\partial^2 u(0;\alpha)}{\partial t \partial \alpha} = \frac{\partial^2 v(1;\alpha)}{\partial t \partial \alpha} = \frac{d\omega}{d\alpha} \geq 0$$

and $m_{1,3}(t;0) = 0$. Thus, $m_{1,3}(t;\alpha) \geq 0$ since from (14)

$$\frac{\partial m_{1,3}(t;\alpha)}{\partial \alpha} = -\frac{d\omega}{d\alpha}\frac{1}{(\omega-2)^2}\frac{1}{(1-t)^2}\frac{\partial u(t;\alpha)}{\partial \alpha} \geq 0\,.$$

Similar arguments hold, respectively, for $m_{3,2}$ and $m_{3,3}$.

Remark 1. Theorem 3 and equality (10) guarantee that the intermediate control points given at the k-th step (matrix multiplication) of the scheme (9) are obtained as a convex combination of the intermediate control points of step $k-1$.

Remark 2. Assume that $u(t) = v(1-t)$ (which implies (11)); then (12), (13) and (14) describe the usual tension properties that are required for the construction of cubic-like tension splines.

Remark 3. For the special case $u(t) = (1-t)^{\alpha}$, $v(t) = t^{\alpha}$ we have the results stated in Subsection 6.1 of [1].

3 – Generalized cubic B-splines

Let us consider the (extended) sequence of knots

$$x_{-3} < \ldots < x_0 < \ldots < x_n < \ldots < x_{n+3}\,,$$

and the corresponding grid-spacings

$$h_k := x_{k+1} - x_k, \qquad k = -3, \ldots, n+2\,.$$

For any $k = -3, \ldots, n+2$ let us consider a pair of functions

$$u_k(t), v_k(t),\ t \in [0,1]$$

and the corresponding spaces $\mathcal{P}_{u_k,v_k}$ (see (1)) so that hypotheses (2) and (3) are satisfied. Let us denote by $\mathbf{U}(\mathbf{V})$ the collection of functions $u_k(v_k)$, $k = -3, \ldots, n+2$ respectively.

Finally, let us consider the space of functions of class C^2 in $[x_{-3}, x_{n+3}]$ belonging "piecewise" to $\mathcal{P}_{u_k,v_k}$, that is

$$\begin{aligned}\mathcal{S}_{\mathbf{U},\mathbf{V}} := \{s \in C^2[x_{-3}, x_{n+3}] \text{ s.t. } & s_{j[x_k,x_{k+1}]}(x) := s_k((x-x_k)/h_k),\\ & s_k \in \mathcal{P}_{u_k,v_k}, k = -3, \ldots, n+2\}\,.\end{aligned}$$

For each segment $[x_k, x_{k+1}]$ we have

$$s(x) = s_k(t) = \sum_{i=0}^{3} b_{i,k} B_{i,k}(t)$$

where $t = (x - x_k)/h_k$ and $B_{j,k}, j = 0,1,2,3$ denote the elements of the generalized Bernstein basis in $\mathcal{P}_{u_k,v_k}$. Obviously, s is continuous in x_k if and only if

$$b_{3,k-1} = b_{0,k}\,.$$

Setting

$$\xi_k := -\frac{1}{u_k'(0)}, \quad \eta_k := \frac{1}{v_k'(1)}, \quad \nu_k := 1 - \xi_k - \eta_k\,,$$

from Corollary 1 we have that $s'(x_k^-) = s'(x_k^+)$ if and only if

$$(b_{3,k-1} - b_{2,k-1})\frac{1}{h_{k-1}\eta_{k-1}} = (b_{1,k} - b_{0,k})\frac{1}{h_k\xi_k}\,,$$

while $s''(x_k^-) = s''(x_k^+)$ if and only if

$$\frac{v_{k-1}''(1)}{h_{k-1}^2}\left[(b_{3,k-1} - b_{2,k-1}) - (b_{2,k-1} - b_{1,k-1})\frac{\eta_{k-1}}{\nu_{k-1}}\right] = \\ = \frac{u_k''(0)}{h_k^2}\left[(b_{2,k} - b_{1,k})\frac{\xi_k}{\nu_k} - (b_{1,k} - b_{0,k})\right].$$

Finally, we set

$$\gamma_k^+ := \frac{h_{k-1}}{h_k}\frac{\xi_k}{\nu_k}u_k''(0)\gamma_k, \quad \gamma_k^- := \frac{h_k}{h_{k-1}}\frac{\eta_{k-1}}{\nu_{k-1}}v_{k-1}''(1)\gamma_k\,,$$
$$\gamma_k := \frac{h_{k-1}\eta_{k-1} + h_k\xi_k}{v_{k-1}''(1)h_k\eta_{k-1} + u_k''(0)h_{k-1}\xi_k}\,,$$

and define the sequence

$$\sigma_k := x_k + \xi_k h_k - \gamma_k^+ h_k\nu_k = x_k - \eta_{k-1}h_{k-1} + \gamma_k^- h_{k-1}\nu_{k-1}\,.$$

Note that, since $x_k + \xi_k h_k < x_{k+1} - \eta_k h_k$ and both $\gamma_k^+ h_k\nu_k$ and $\gamma_{k+1}^- h_k\nu_k$ are positive, for all k, $\sigma_k < \sigma_{k+1}$. We can associate to any element s of the space $\mathcal{S}_{\mathbf{U},\mathbf{V}}$ a polygonal line (the *generalized de Boor control polygon*) with vertices

$$\mathbf{P}_k := \begin{pmatrix}\sigma_k \\ P_k\end{pmatrix}, \tag{22}$$

where P_k are real numbers, so that the Bézier coefficients of s can be deduced from (22) according to the following rules (see also fig. 1)

$$b_{2,k-1} = P_k\frac{1+\gamma_{k-1}^+}{1+\gamma_{k-1}^+ + \gamma_k^-} + P_{k-1}\frac{\gamma_k^-}{1+\gamma_{k-1}^+ + \gamma_k^-}\,,$$
$$b_{1,k} = P_k\frac{1+\gamma_{k+1}^-}{1+\gamma_{k+1}^- + \gamma_k^+} + P_{k+1}\frac{\gamma_k^+}{1+\gamma_{k+1}^- + \gamma_k^+}\,,$$
$$b_{3,k-1} = b_{0,k} = b_{2,k-1}\frac{h_k\xi_k}{\xi_k h_k + \eta_{k-1}h_{k-1}} + b_{1,k}\frac{h_{k-1}\eta_{k-1}}{\xi_k h_k + \eta_{k-1}h_{k-1}}\,.$$

The family of functions $\{\mathcal{B}^{(i)}, i = -1, \dots, n+1\}$ determined by the polygons with vertices

$$\mathbf{P}_k := \begin{pmatrix} \sigma_k \\ 0 \end{pmatrix},\ k = -3, \dots, n+3,\ k \neq i,\ \mathbf{P}_i := \begin{pmatrix} \sigma_k \\ 1 \end{pmatrix},$$

has the usual properties of the family of C^2 cubic B-splines so they will be referred to as *generalized cubic B-splines.* In particular we have

$$\begin{aligned} \mathcal{B}^{(i)}(x) &\geq 0,\ x \in [x_{-3}, x_{n+3}]\,, \\ \mathcal{B}^{(i)}(x) &= 0,\ x \notin (x_{i-2}, x_{i+2})\,, \\ \sum_{i=-1}^{n+1} \mathcal{B}^{(i)}(x) &\equiv 1,\ x \in [x_0, x_n]\,. \end{aligned}$$

4 – Examples

In this section we present two examples to illustrate the geometric aspects of the construction of generalized cubic B-splines and the graphical behavior of the obtained functions. In the first example (see fig. 1) we have considered the sequence of knots $[x_0\ x_1\ x_2\ x_3\ x_4] := [0\ 2\ 3\ 3.5\ 5]$ and we have juxtaposed in the various subintervals classical cubic polynomials and exponential functions. More precisely, we have considered the following spaces $\mathcal{P}_{u_k,v_k}$

$$\begin{aligned} \mathcal{P}_{u_0,v_0} = \mathcal{P}_{u_3,v_3} &:= \text{span} < 1, t, (1-t)^3, t^3 > \\ \mathcal{P}_{u_1,v_1} &:= \text{span} < 1, t, \exp(2t), \exp(-2t) > \\ \mathcal{P}_{u_2,v_2} &:= \text{span} < 1, t, \exp(4t), \exp(-4t) > \ . \end{aligned}$$

Fig. 1 refers to the corresponding function $\mathcal{B}^{(2)}$. On the left we have its generalized de Boor control polygon and the geometric construction of the generalized Bézier control polygon where, for notational convenience we put

$$\alpha_k := \frac{h_k \xi_k}{\xi_k h_k + \eta_{k-1} h_{k-1}}\,.$$

The right part of the figure shows the graph of $\mathcal{B}^{(2)}$ with its control polygons.

In the second example (see fig. 2) we have considered again the same sequence of knots but a different four dimensional space of functions, verifying (3), for each subinterval. More precisely,

$$\begin{aligned} \mathcal{P}_{u_0,v_0} &:= \text{span} < 1, t, (\cos(3t), \sin(3t) > \\ \mathcal{P}_{u_1,v_1} &:= \text{span} < 1, t, (1-t)^3, t^3 > \\ \mathcal{P}_{u_2,v_2} &:= \text{span} < 1, t, \exp(6t), \exp(-6t) > \\ \mathcal{P}_{u_3,v_3} &:= \text{span} < 1, t, (1-t)^{10}, t^3 > \ . \end{aligned}$$

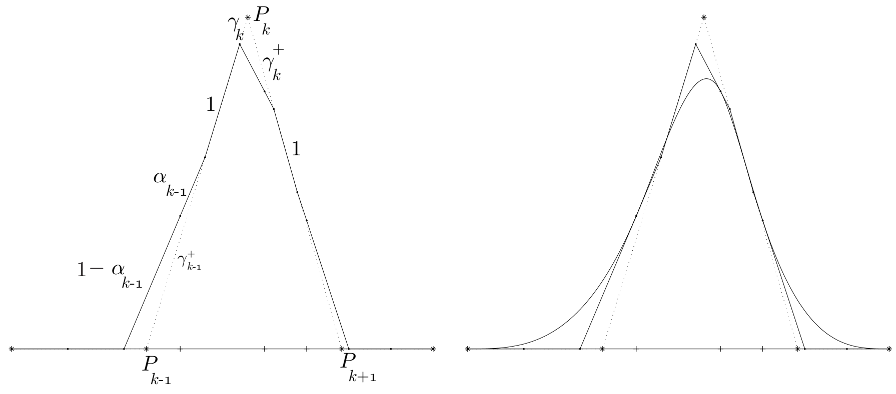

Fig. 1: Example 1: geometric construction of a generalized cubic B-spline. Left: generalized de Boor (dotted line) and Bézier control polygon. Right: control polygons and the corresponding B-spline.

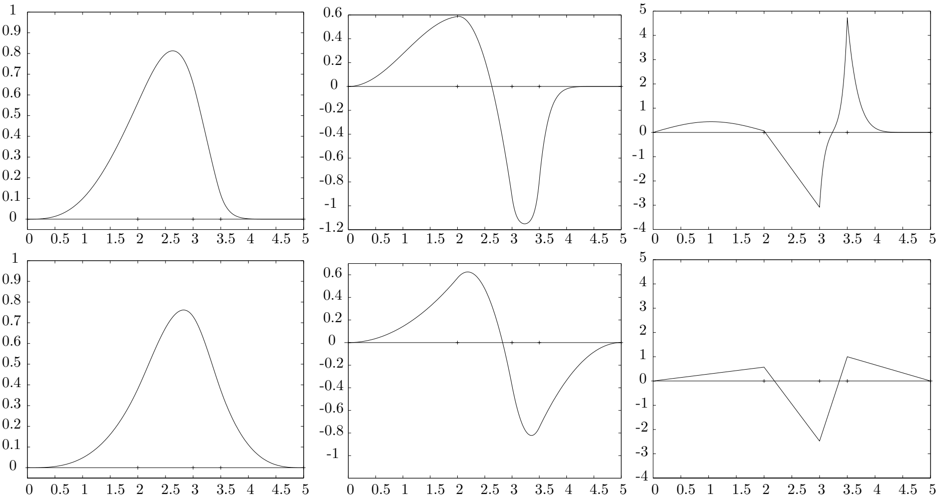

Fig. 2: Example 2: generalized (top) and classical (bottom) cubic B-spline and its first and second derivative.

The first line of fig. 2 shows the corresponding generalized cubic B-spline $\mathcal{B}^{(2)}$ and its first and second derivatives. For the sake of comparison the second line of the figure shows the corresponding graphs for the classical cubic spline on the same sequence of knots.

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