Rendiconti di Matematica, Serie VII
Volume 27, Roma (2007), 105-122

Mathematical challenges of General Relativity

SERGIU KLAINERMAN

Abstract: *We give an overview of some of the main open problems in General Relativity as well as some new results concerning the bounded L^2 curvature conjecture.*

Together with Quantum Mechanics, General Relativity provides the conceptual framework of Modern Physics yet, unlike the former, General Relativity has received somewhat less attention from mathematicians. There is, however, no established physical theory, I contend, which has a more impressive mathematical pedigree or a more fertile mathematical ground. Indeed recall that Einstein discovered it in a purely theoretical attempt to find a theory which could reconcile Special Relativity with Newtonian Gravity. The reconciliation required, in a fundamental way, both the language of Riemannian Geometry and the reformulation, by Minkowski, of Special Relativity in the language of a Lorentz metric. Special Relativity itself was born in another grand theoretical effort to reconcile the Galilean invariance of Classical Mechanics with the Lorentzian invariance of the Maxwell equations. Both these physical theories have rich mathematical structures in their own right and have had, and continue to have, an extremely fruitful interaction with the rest of mathematics. It suffices to say, for example, that the theory of differential forms and Hodge theory were greatly influenced by Maxwell's theory of Electromagnetism while Calculus of Variations and Symplectic Geometry were born from a long and extremely fruitful attempt to unravel the mathematical structure of Classical Mechanics.

Key Words and Phrases: *Relativity – Cosmic censorship – Minkowski – Kerr – Schwarzschild – Causality*
A.M.S. Classification: 35J10

A lot more can be said about this impressive mathematical pedigree of General Relativity. Yet the goal of my talk is not to insist on its rich mathematical roots but rather to demonstrate that the theory provides an extremely rich ground for present day mathematical research. I would like to convince you that General Relativity has an impressive body of well formulated mathematical problems and conjectures that ought to fire our imagination. I will also try to describe some recent results in the field.

1 – The problem of evolution in GR

1.1 – Space-time

Recall that the main object of Einstein's general relativity is the space-time. This can be defined as a class of equivalence of differentiable, oriented four dimensional Lorentz manifolds $(\mathcal{M}, g)$. Two Lorentz manifolds $(\mathcal{M}, g)$, $(\mathcal{M}', g')$ are equivalent if there exists a diffeomorphism $\mathcal{M} \longrightarrow \mathcal{M}'$ such that $g' = \Phi_* g$. A space-time is simply a class of equivalence of such Lorentz manifolds. We recall that Lorentz metrics divide vectors X in a tangent space $T_p(\mathcal{M})$ into *timelike*, *null* and *space-like* according to whether $g(X, X)$ is, respectively, negative, zero or positive. A curve $\gamma(t)$ is said to be timelike, respectively null, if its tangent vector $\dot{\gamma}(t)$ is timelike or null. It is called *causal* if it is either time-like or null. Given a set $S \subset \mathcal{M}$ we denote by $\mathcal{I}^+(S)$ the set of all points in $\mathcal{M}$ which can be reached by timelike curves originating at S. It is called the chronological *future set* of S. The set $\mathcal{J}^+ S$, consisting of points which can be reached by causal curves from S, is called the *causal future* of S. One defines in the same manner the future and causal pasts $\mathcal{I}^-(S)$ and $\mathcal{J}^-(S)$. The future set of an event[(1)] p consists of all events in $\mathcal{M}$ which can be influenced by p. Its boundary $\mathcal{N}^+(p)$ consists of events in $\mathcal{M}$ which can be reached by null geodesic rays initiating at p. A hypersurface Σ is called*space-like*, respectively *null* if the direction normal to it is space-like, resp. null.

1.2 – Einstein equations

The space-time metric g has to satisfy the *Einstein Field Equations*,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}$$

with $R_{\alpha\beta}$ the Ricci curvature, R the scalar curvature of the metric and $T_{\alpha\beta}$ the energy-momentum tensor of some matter-field defined on $(\mathcal{M}, g)$. For simplicity we restrict ourselves to the particular case of vacuum i.e. $T \equiv 0$ in which case the equations take the form,

$$R_{\alpha\beta} = 0\,. \tag{1}$$

(1) Points in $\mathcal{M}$ are also called events of the space-time $\mathcal{M}$.

1.3 – Special solutions

The simplest example of an EV manifold is the flat Minkowski space $(\mathbb{R}^{1+3}, g_M)$ with metric,

$$g_M = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \,, \tag{2}$$

in standard coordinates $x^0 = t, x = (x^1, x^2, x^3) \in \mathbb{R}^3$. The Minkowski metric can be written in spherical coordinates $t, r = |x|$, $\omega \in \mathbb{S}^2$ in the form,

$$-dt^2 + dr^2 + r^2 d\sigma^2_{\mathbb{S}^2}$$

where $d\sigma^2$ represents the standard metric of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Another, very important, explicit solution of (EV) is given by the exterior Schwarzschild metric, of mass $m > 0$,

$$g_S = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} \,. \tag{3}$$

Though the metric seems singular at $r = 2m$ it turns out that one can glue together two regions $r > 2m$ and two regions $r < 2m$of the Schwarzschild metric to obtain a metric which is smooth along $\mathcal{E} = \{r = 2m\}$, see [15], called the Schwarzschild horizon.

The exterior Schwarzschild metrics are special examples of a two parameter family of explicit solutions, called exterior Kerr metrics. In Boyer-Lindquist coordinates these take the form,

$$\begin{aligned} g_K = &- \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi + \\ &+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2 d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned}$$

with $0 \leq a < m, \Sigma = r^2 + a^2 \cos^2\theta, \Delta = r^2 + a^2 - 2mr$. Moreover for the exterior Kerr metric we take $r > r_+ = m + (m^2 - a^2)^{1/2}$. As in the Schwarzschild case, the exterior Kerr metric can be smoothly extended across $\mathcal{E} = \{r = r_+\}$ which is called Kerr event horizon. The region $\ll \mathcal{M} \gg = \{r > r_+\}$ is called the *domain of outer communication* of the Kerr space-time. It can be shown that the future and past sets of any point in this set intersects any timelike curve, passing through points of arbitrary large values of r, in finite time as measured relative to proper time along the curve. This fact is violated by points in the region $r \leq r_+$, which defines the *black hole* region of the space-time. Thus physical signals which initiate at points in $r \leq r_+$ cannot be registered by far away observers. Moreover the black hole region is singular at $r = 0$. Fortunately this singular region cannot be in any way experienced by far away observer, since no physical signals can escape the black hole.

The exterior Kerr metrics are *stationary*, which means, roughly, that the coefficients of the metric are independent of the time variable t. One can reformulate this by saying that the vectorfield $T = \partial_t$ is Killing[(2)] and time-like at points with r large. One can also easily check that T is tangent to the horizon $\mathcal{E}$, which is itself a null hypersurface, i.e. the restriction of the metric to the tangent space to $\mathcal{E}$ is degenerate. In addition to being stationary the coefficients of the Kerr metric are independent of the circular variable ϕ. We say that Kerr is *axially symmetric*. The Schwarzschild metrics, corresponding to $a = 0$, are not just axially symmetric but spherically symmetric, which means that the metric is left invariant by the whole rotation group of the standard sphere $\mathbb{S}^2$. A well known theorem of Birkhoff, shows that they are the only such solutions of the Einstein equations. Another peculiarity of a Schwarzschild metric, not true in the case of Kerr, is that the stationary Killing vectorfield $T = \partial_t$ is orthogonal to the hypersurface $t = 0$. A stationary spacetime which has this property is called *static*. Moreover T is timelike for all $r > 2m$ and null along the Schwarzschild horizon $\mathcal{E} = \{r = 2m\}$.

This is not the case for Kerr solutions in which case $T = \partial_t$ is only time-like for $r > m+(m^2-a^2\cos^2\theta)^{1/2}$, null for $r = m+(m^2-a^2\cos^2\theta)^{1/2}$ and space-like in the region between r_+ and $r = m + (m^2 - a^2\cos^2\theta)^{1/2}$, called the *ergosphere*.

1.4 – Initial value problem

To solve the Einstein equations in vacuum we start with an initial data set $(\Sigma, g_{(0)}, k_{(0)})$ with Σ a three dimensional manifold, $g_{(0)}$ a Riemannian metric and $k_{(0)}$ a symmetric 2-tensor. To solve the the initial value problem, (I.V.P.), for the Einstein vacuum equations amounts to find a $3 + 1$ dimensional manifold $\mathcal{M}$ together with a Ricci flat (i.e. verifying (1)), Lorentz metric g on $\mathcal{M}$ and an embedding of $\Sigma_{(0)}$ to $\mathcal{M}$ whose first (i.e. induced metric) and second fundamental forms coincide with $g_{(0)}$ and $k_{(0)}$. One can easily see that $g_{(0)}, k_{(0)}$ cannot be arbitrary; to be compatible with (1) they have to satisfy a set of constraints, called *constraint equations*. Here are some more precise definitions.

Definition 1. An initial data set is a triple $(\Sigma_{0)}, g_{(0)}, k_{(0)})$ with Σ a three dimensional manifold, $g_{(0)}$ a Riemannian metric and $k_{(0)}$ a symmetric 2-tensor which satisfy a set of relations called *constraint equations*,

$$\begin{aligned} \operatorname{div} k_{(0)} - \nabla\ \operatorname{tr} k_{(0)} &= 0 \\ R_{(0)} - |k_{(0)}|^2 + (\operatorname{tr} k_{(0)})^2 &= 0\,. \end{aligned}$$

Here ∇ denotes the induced covariant derivative, div the usual divergence of a symmetric 2-tensor, defined with respect to ∇, and $R_{(0)}$ the scalar curvature of

[(2)] A vectorfield X is said to be Killing if its locally induced one parameter flow consists of isometries of g, i.e. the Lie derivative of the metric g with respect to X vanishes, $\mathcal{L}_X g = 0$.

the metric $g_{(0)}$. Moreover $|k_{(0)}|$ and $\operatorname{tr} k_{(0)}$ are the riemannian norm and trace of $k_{(0)}$ with respect to $g_{(0)}$.

We can view the hypersurface $t = 0$ in Minkowski space $(\mathbb{R}^{1+3}, m)$ as initial manifold $\Sigma_{(0)}$. In that the metric induced by m on $\Sigma_{(0)} = \mathbb{R}^3$ is $g_{(0)} = e$, where e is the standard euclidean metric, and the tensor $k_{(0)} = 0$ is the second fundamental form of the embedding $\Sigma_{(0)} \subset \mathbb{R}^{1+3}$. In this way we get the flat initial data set $(\mathbb{R}^3, e, 0)$. Similarly, in the case of the Schwarzschild metric (3), we can take $\Sigma_{(0)}$ to be the hypersurface $t = 0$, $g_{(0)}$ the metric $(1-\frac{2M}{r})^{-1}dr^2+r^2 d\sigma_{\mathbb{S}^2}$, induced on $\Sigma_{(0)}$, and $k_{(0)} = 0$. We obtain this way the standard initial data set of the Schwarzschild metric.

Definition 2. An initial data set is said to be flat, or trivial, if it corresponds to a space-like hypersurface in Minkowski space with its induced metric and second fundamental form. An initial data set is said to be asymptotically flat (AF) if there exists a system of coordinates (x^1, x^2, x^3), defined outside a sufficiently large compact set[(3)] $K \subset \Sigma$, relative to which the metric $g_{(0)}$ approaches the euclidian metric and $k_{(0)}$ approaches zero[(4)] as $r = \sqrt{(x^1)^2 + (x^2)^2 + (x_3)^2} \to \infty$.

Definition 3. A Cauchy development of $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ is a Ricci flat space-time manifold $(\mathcal{M}, g)$ and an embedding $i : \Sigma \longrightarrow \mathcal{M}$ such that $i_*(g_{(0)})$, $i_*(k_{(0)})$ are the first and second fundamental forms of $i(\Sigma_{(0)})$ in $\mathcal{M}$. A development is required to be also globally hyperbolic. This means that $i(\Sigma_{(0)})$ is a Cauchy hypersurface, i.e. each causal curve in $\mathcal{M}$ intersects $i(\Sigma_{(0)})$ at precisely one point.

A future development of $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ consists of a globally hyperbolic manifold $(\mathcal{M}, g)$ with boundary, verifying the Einstein equations, and an embedding i as before which identifies Σ to the boundary of $\mathcal{M}$.

Throughout the remaining of this paper we shall only consider globally hyperbolic space-times with a Cauchy hypersurface $\Sigma_{(0)}$ which is asymptotically flat and has compact interior.

The most primitive question asked about the initial value problem, solved in a satisfactory way, for very large classes of evolution equations, is that of local existence and uniqueness of solutions. For the Einstein equations this type of result was first established by Y. Choquet-Bruhat [7] with the help of wave coordinates which allowed her to cast the Einstein equations in the form of a system of nonlinear wave equations to which one can apply the standard

(3) such that $\Sigma \setminus K$ is diffeomorphic to the complement of a ball in $\mathbb{R}^3$.

(4) Because of the constraint equations the asymptotic behavior cannot be arbitrarily prescribed. A precise definition of asymptotic flatness has to involve the ADM mass of (Σ, g). Taking the mass into account we write $g_{(0)} = (1 + \frac{2M}{r})\delta + o(r^{-1})$. According to the positive mass theorem $M \geq 0$ and $M = 0$ implies that the initial data set is flat.

theory of symmetric hyperbolic systems. The optimal, classical[(5)], result, due to Hughes-Kato-Marsden(1976) [16], states the following

THEOREM (Local existence). *Let $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ be an initial data set for the Einstein vacuum equations. Assume that $\Sigma_{(0)}$ can be covered by a locally finite system of coordinate charts U_α related to each other by C^1 diffeomorphisms, such that $(g_{(0)}, k_{(0)}) \in H^s_{\text{loc}}(U_\alpha) \times H^{s-1}_{\text{loc}}(U_\alpha)$ with $s > \frac{5}{2}$. Then there exists a unique (up to an isometry[(6)]) globally hyperbolic, development $(\mathcal{M}, g)$ for which $\Sigma_{(0)}$ is a Cauchy hypersurface.*

In the case of nonlinear systems of differential equations the local existence and uniqueness result leads, through a straightforward extension argument, to a global result. The formulation of the same type of result for the Einstein equations is a little more subtle; it was done by Y. Choque-Bruhat and R. P. Geroch in [8].

THEOREM (Bruhat-Geroch). *For each smooth initial data set there exists a unique maximal future Cauchy development.*

Thus any construction, obtained by an evolutionary approach from a specific initial data set, must be necessarily contained in its maximal development. This may be said to solve the problem of global[(7)] existence and uniqueness in General Relativity; all further questions may be said to concern the qualitative properties of the maximal Cauchy developments. The central issue becomes that of existence and character of singularities. First we can define a regular maximal development as one which is complete in the sense that all future time-like and null geodesics are complete. Roughly speaking this means that any freely moving observer in M can be extended indefinitely, as measured relative to its proper time. It turns out that any initial data set, which is sufficiently close to the flat one, admits a regular maximal Cauchy development, see [12].

THEOREM (Global Stability of Minkowski). *Any asymptotically flat initial data set which is sufficiently close to the trivial one has a complete maximal future development. Moreover the curvature of the development is globally small and tends to zero at infinity, along any direction.*

---

(5) Based on only energy estimates and classical Sobolev inequalities.

(6) The uniqueness up to an isometry requires additional regularity, $s > \frac{5}{2} + 1$, on the data. One has uniqueness, however, without additional regularity for the reduced Einstein equations system in wave coordinates

(7) This is of course misleading, for equations defined in a fixed background global is a solution which exists for all time. In general relativity, however, we have no such background as the spacetime itself is the unknown. The connection with the classical meaning of a global solution requires a special discussion concerning the proper time of timelike geodesics.

At the opposite end of this result, when the initial data set is very far form flat, we have the following singularity theorem of Penrose, see [32].

THEOREM (Penrose). *If the manifold support of an initial data set is non-compact and contains a closed trapped surface the corresponding maximal development is incomplete.*

The notion of a trapped surface $S \subset \Sigma$, can be rigorously defined in terms of a local condition on S. The flat initial data sets have, of course, no such surfaces. On the other hand, for the Schwarzschild initial data set, any surface $r = r_0$, with $r_0 < 2m$ is trapped. One can also check that the Schwarzschild metric has a genuine singularity at $r = 0$, where the curvature tensor becomes infinite. This is a lot stronger than just saying that space-time is incomplete. In fact all Kerr solutions, with the exception of the flat Minkowski space itself, have trapped surfaces and curvature singularities.

1.5 – Main Conjectures

The unavoidable presence of singularities, for sufficiently large initial data sets, has led Penrose to formulate two conjectures which go under the name of the *weak and strong cosmic censorhip conjectures.* The first asserts that for all, but possibly an exceptional set of initial conditions, no singularities may be observed from infinity. In other words, the singularities in General Relativity are hidden by regions of space-time, called black-holes, in which all future causal geodesics remain necessarily trapped. To get a feeling for this consider the difference between the Minkowski and the black hole region $\{r \leq 2m\}$ of a Schwarzschild space-time. In Minkowski space light originating at any point $p = (t_0, x_0)$ propagates, towards future, along the null rays of the null cone $t - t_0 = |x - x_0|$. Any free observer in $\mathbb{R}^{1+3}$, following a straight time-like line, will necessarily meet the this light cone in finite time, thus experiencing the event p. On the other hand, any point p in the trapped region $r < 2m$ of the Schwarzschild space, is such that all all null rays initiating at p remain trapped in the region $r < 2m$. In particular events connected to the singularity at $r = 0$ cannot influence events in the domain of outer communication. The region $r > 2m$, the domain of outer communication, is entirely free of singularities. The so called Weak Cosmic Censorship conjecture is an optimistic extension of this fact to the future developments of general, asymptotically flat initial data. The desired conclusion of the conjecture is that any such development, with the possible exception of a non-generic set of initial conditions, has the property that any *sufficiently distant observer* will never encounter singularities or any other effects propagating from singularities. To make this more precise one needs define what a sufficiently distant observer means. This is typically done by introducing the notion of future null infinity $\mathcal{I}^+$ which, roughly speaking, provides end points for the null geodesics which propagate to asymptotically large distances. The future null infinity is formally constructed by conformally embedding the physical spacetime $\mathcal{M}$ under consideration to a larger space-time $\overline{\mathcal{M}}$ with a null boundary $\mathcal{S}^+$.

Definition. The future null infinity $\mathcal{S}^+$ is said to be complete if any future null geodesics along it can be indefinitely extended relative an afine parameter.

Given this enlarged space-time, with complete $\mathcal{S}^+$, one defines the black hole region to be

$$\mathcal{B} = \mathcal{M} - \mathcal{I}^-(\mathcal{S}^+) \tag{4}$$

with the chronological past I^- defined relative to the enlarged, non-physical space-time $\overline{\mathcal{M}}$. The event horizon $\mathcal{E}$ of the black hole is defined to be the boundary of $\mathcal{B}$ in $\mathcal{M}$. The requirement that space-time $\mathcal{M}$ has a complete future null infinity can be informally reformulated, by saying that the complement of the black hole region should be free of singularities. Indeed singularities outside the black hole region will affect the completeness of $\mathcal{S}^+$. The black hole region, however, can only be defined a-posteriori after the completeness of $\mathcal{S}^+$ has been established.

Here is now a more precise formulation of the Weak Cosmic Censorship (WCC) conjecture.

Conjecture 1 (WCC Conjecture). Generic asymptotically flat initial data have maximal future developments possessing a complete future null infinity.

The WCC conjecture was formulated in order to guarantee the unique predictability of observations visible from infinity. It does not preclude, however, the possibility that singularities may be visible by local observers inside the black hole region. Since predictability is a fundamental requirement of all classical physics it seems reasonable to want it valid throughout spacetime. Predictability is known to fail, however, within the black hole of a Kerr solution[(8)] in which case the maximum domain of development of any complete spacelike hypersurface has a future boundary, called a Cauchy horizon, where the Kerr solution is perfectly smooth and yet beyond which there are many possible smooth extensions. This failure of predictability is due to a global pathology of the geometry of characteristics and not to a loss of local regularity. It is to avoid this pathology and ensure uniqueness that we want the maximum domain of development of generic data to be in-extendible. This motivation has led Penrose to introduce the following conjecture, called Strong Cosmic Censorship (SCC). Since Kerr itself, however, violates this requirement we can only hope that the conjecture holds for generic data.

[(8)] Or the Reissner-Nordstrom solution of the Einstein -Maxwell equations.

Conjecture 2 (SCC Conjecture). Generic asymptotically flat or compact initial data sets have maximal future developments which are locally inextendible.

The formulation above leads open the sense in which the maximal future developments are *inextendible*. The precise notion of extendibility, which is to be avoided by SCC, is a subtle issue which, I believe, can only be settled together with a complete solution of the conjecture. There have been various proposals among which I will only mention two, see [10] for a more thorough discussion.

1. The maximal future development is inextendible as a $C^{1,1}$ Lorentzian manifold. This means, in particular, that some components of the curvature tensor must become infinite(9)
2. The maximal future development is inextendible as a continuous Lorentzian manifold.

Though general, asymptotically flat, solutions of the Einstein vacuum equations are exceedingly complicated we expect that their asymptotic behavior is quite simple and is dictated in fact by the two parameter family of explicit Kerr solutions, corresponding to axially symmetric, rotating black holes. Here is a rough version of the conjecture.

Conjecture 3 (Final State Conjecture). Generic asymptotically flat initial data sets have maximal future developments which can be described, asymptotically, as a finite number of black holes, Kerr solutions, moving away from each other.

The simple motivation behind this conjecture is that one expects, due to gravitational radiation, that general, dynamic, solutions of the Einstein field equations settle down, asymptotically, into a stationary regime. A spacetime is said to be stationary if it admits a Killing vectorfield which is timelike in the asymptotic region, i.e. at space-like infinity.

Kerr solutions are obvious examples of stationary solutions but are they unique? Can there be, in other words, other stationary solutions of the Einstein vacuum equations? It has been shown, under very general conditions, that if the Killing vectorfield is also static, i.e. hypersurface orthogonal, than the spacetime must be Schwarzschild, see discussion and references in [5]. A less satisfactory uniqueness result holds true for stationary, real analytic space-times, see discussion and references in [5]. The condition of real analyticity is however very unnatural in General Relativity and ought to be removed.

(9)More precisely, along any future, inextendible, timelike geodesic of finite length the some components of the Riemann curvature tensor, expressed relative to a parallel transported orthonormal frame along the geodesic, become infinite as the value of the arc-length approaches its limiting value.

Conjecture 4 (Uniqueness of Kerr). Remove the analyticity assumption in the Hawking-Carter proof of uniqueness of the Kerr space-time among stationary solutions.

Another important open problem in general relativity, whose solution would have to be understood long before the full Final State conjecture is settled, is that of the nonlinear stability of the exterior Kerr metric.

Conjecture 5 (Global stability of Kerr). Any small perturbation of the initial data set of a Kerr space-time has a global future development with a complete future null infinity which, within its domain of outer communication[(10)], behaves asymptotically like a (another) Kerr solution.

A first, essential step, in the proof of stability of the Kerr solution would be to establish appropriate decay estimates for solutions to linear field equations in a fixed Kerr background. At this point we don't even have a satisfactory proof of the boundedness of solutions to the scalar wave equation,

$$\square_g \phi = 0 \tag{5}$$

in the exterior of the Kerr background metric g. The situation is somewhat better in the case of the Schwarzschild metric, see [13] for a recent result and relevant references.

2 – Bounded L^2 curvature conjecture.

An important general conjecture which has received some attention in recent years is the bounded L^2 curvature conjecture. As a helpful analogy consider the case of the Cheeger-Gromov non-collapse theory in Riemannian geometry. The theory shows how L^∞ bounds on the curvature tensor, lower bounds on the diameter and upper bounds for the volume of a compact manifold are sufficient to control the geometry of the manifold. Yet Einstein field equations are of hyperbolic character and as such L^∞ bounds for the curvature tensor are unnatural. Indeed, based on the Bianchi identities one can show that the curvature tensor satisfy a Maxwell type equation and thus we can only expect to establish L^2 energy type estimates along space-like hypersurfaces and null boundaries of future or past domains. It is thus natural to ask whether the boundedness of these quantities, plus reasonable initial conditions, suffices to control the geometry of a space-time.

The problem mentioned above is intimately tied to the issue of optimal well-posedness which can be formulated and addressed for the full Einstein -vacuum equations in the absence of any symmetry. The optimal local existence result

[(10)] That means, roughly, outside the black hole region.

result, see [16], requires an initial data set $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ such that, in well defined local coordinates, $g_{(0)} \in H^s_{(\mathrm{loc})}(\Sigma)$, $k_{(0)} \in H^{s-1}_{(\mathrm{loc})}(\Sigma_{(0)})$ with $s > \frac{5}{2}$. By only scaling considerations we expect to make sense of the initial value problem for $s \geq s_c = \frac{3}{2}$. A result of well posedness for the Einstein equations for the critical regularity, $s = s_c$, is not only completely out of reach but may very well be wrong[(11)]. A far more realistic goal at the present time is the following:

Conjecture (L^2-Bounded Curvature Conjecture BCC). The Einstein Vacuum equations are strongly, locally, well posed for initial data sets $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ with locally finite L^2 curvature and locally finite L^2 norm of the covariant derivatives of $k_{(0)}$.

It is important to emphasize here that the conjecture can be interpreted as a continuation argument for the Einstein equations; that is the spacetime constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and covariant derivatives of its second fundamental form remain L^2-bounded. The following, loosely formulated, result may be viewed as a possible corollary of the bounded L^2 curvature conjecture:

Corollary. *Consider a future Cauchy development, for the Einstein-vacuum equations, of smooth, regular, initial data set. If the curvature flux along any backward null cone initiating in the past of a point p is uniformly bounded the solution can be smoothly continued past p.*

Clearly, such a result would be an important step in understanding formation and structure of singularities for the $3 + 1$ Einstein equations. The description of a local continuation criterion stated above in terms of a curvature flux is natural from both geometric and physical point of view. On the other hand, an even more ambitious goal is to find geometrically meaningful dimensionless quantities whose boundedness ensures a unique local extension of the corresponding spacetime.

Problem. Find a dimensionless local extension criteria for solutions of the $3 + 1$ Einstein vacuum equations.

2.1 – Strategy for BCC

The conjecture, which was first proposed in [20], means that one can significantly improve the classical local existence mentioned above from $s > 5/2$ to $s = 2$ which corresponds to initial data sets with bounded L^2 curvature, result

---

(11) The causal structure seems to break-down for $s < 2$. Thus BCC, with $s = 2$, may be sharp.

which would be particularly satisfying in view of the naturalness of the norms involved. The conjecture was motivated by the progress made earlier on semilinear type equations such as Wave Maps and Yang Mills equations.

At that time the conjecture was made it seemed however completely out of reach. It was clear that in order to improve the exponent $s > 5/2$ one had to abandon the naive use of Sobolev inequalities of the classical argument and rely instead on Strichartz and vector-fields bilinear type estimates. The problem was that one needed to extend these estimates to wave operators on very rough background metrics. The first results below 5/2 are due to Bahouri-Chemin [2], Tataru [36] and Klainerman-Rodnianski [24] and relied, indeed, on proving Strichartz type estimates on such backgrounds. To do this they had to rely on adequate notions of approximate fundamental solutions or vector-fields in the case of [24]) for the corresponding wave operators based on an adapted version of the classical geometric optics construction. This construction depends heavily on the regularity properties of null hypersurfaces associated to these backgrounds and applies to general type of quasilinear wave equations including the reduced Einstein vacuum equations in wave coordinates. In [25] we were able to reach, for the particular case of Einstein vacuum equations, any exponent $s > 2$. A similar result was also obtained in [35] for the general class of quasilinear wave equations mentioned above.

The case $s = 2$ is far more difficult. First of all such a result cannot hold for general quasilinear wave equations. As the experience with semilinear wave equations demonstrates, see discussion in [??], to prove such a result we need the following ingredients:

1. Provide a system of coordinates relative to which (EV) verifies an appropriate version of the null condition.
2. Prove appropriate bilinear estimates for solutions to homogeneous wave equations, of the type $\square_{\mathbf{g}}\phi = 0$, on a fixed Einstein Vacuum background (endowed with the coordinate system indicated in 1.) with bounded L^2 curvature. In the flat case such estimates were proved in [21] and used to prove global existence for the Yang Mills equations in the energy norm, see [22].

To prove bilinear estimates we need a good notion of approximate solutions; this requires the third ingredient, typical to quasilinear equations,

3. Make sense of null hypersurfaces, on Einstein vacuum backgrounds with only L^2-bounds on their curvature tensor, and provide estimates on their geometry.

3 – A Break-down criterion

I report on recent work in collaboration with Igor Rodnianski concerning a *geometric* criterion for breakdown of solutions $(\mathcal{M}, g)$ of the vacuum Einstein equations. The main result discussed here is stated and proved in [31]; the proof depends however on the results and methods of [26], [27], [28], [29] which establish a lower bound for the radius of injectivity of null hypersurfaces with finite curvature flux as well as [30] in which we construct a Kirchoff-Sobolev type parametrix for solutions to covariant wave equations.

Assume that a part of space-time $\mathcal{M}_I \subset \mathcal{M}$ is foliated by the level hypersurfaces Σ_t of a time function t, monotonically increasing towards future in the interval $I \subset \mathbb{R}$, with lapse n and second fundamental form k defined by,

$$k(X,Y) = g(D_X T, Y)\,, \qquad n = (-g(Dt,Dt))^{-1/2} \tag{6}$$

where T is the future unit normal to Σ_t, D is the space-time covariant derivative associated with g, and X, Y are tangent to Σ_t. Let Σ_0 be a fixed leaf of the t foliation, corresponding to $t = t_0 \in I$, which we consider the initial slice. We assume that the space-time region $\mathcal{M}_I$ is globally hyperbolic, i.e. every causal curve from a point $p \in \mathcal{M}_I$ intersects Σ_0 at precisely one point. Assume also that the initial slice verifies the assumption.

A1. There exists a finite covering of Σ_0 by a finite number of charts U such that for any fixed chart, the induced metric g verifies

$$\Delta_0^{-1}|\xi|^2 \le g_{ij}(x)\xi_i\xi_j \le \Delta_0|\xi|^2, \qquad \forall x \in U \tag{7}$$

with Δ_0 a fixed positive number.

Though our work in [31] covers only the second of the following two situations below, it applies in principle to both.

1. The surfaces Σ_t are asymptotically flat and maximal.

$$\operatorname{tr} k = 0\,.$$

2. The surfaces Σ_t are compact, of Yamabe type -1, and of constant, negative mean curvature. They form what is called a (CMC) foliation.

$$\operatorname{tr} k = t, \qquad t < 0\,.$$

We may assume in what follows that the region $\mathcal{M}_I$ corresponds to the time interval $I = [t_0, t_*)$ with $t_* < 0$. Without loss of generality we shall identify the entire space-time $\mathcal{M}$ with $\mathcal{M}_I$. We can also assume that the initial hypersurface Σ_0 corresponds to $t_0 = -1$.

Given a t-foliation of $\mathcal{M}$ and $p \in \mathcal{M}$ we can define a point-wise norm $|\Pi(p)|$ of any space-time tensor Π via the decomposition,

$$X = -X^0 T + \underline{X}, \quad X \in T\mathcal{M}, \quad \underline{X} \in T(\Sigma_t)\,.$$

We denote by $\|\Pi(t)\|_{L^p}$ the L^p norm of Π on Σ_t. More precisely,

$$\|\Pi(t)\|_{L^p} = \int_{\Sigma_t} |\Pi|^p dv_g$$

with dv_g the volume element of the metric g of Σ_t. The main result I want to report is the following,

THEOREM 3.1 (Klainerman-Rodnianski). *Let $(\mathcal{M}, g)$ be a globally hyperbolic development of Σ_0 foliated by the level hypersurfaces of a time function $t < 0$, verifying conditions* (1) *or* (2) *above, such that Σ_0 corresponds to the level surface $t = t_0$. Assume that Σ_0 verifies* A1. *Then the first time $T_* < 0$ of a breakdown is characterized by the condition*

$$\limsup_{t \to T_*^-} \big(\|k(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} \big) = \infty \,. \tag{8}$$

More precisely the space-time together with the foliation Σ_t can be extended beyond any value $t_* < 0$ for which[12],

$$\sup_{t \in [t_0, t_*)} \|k(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} = \Delta_0 < \infty \,. \tag{9}$$

Condition (9) can be reformulated in terms of the deformation tensor of the future unit normal T, $\pi = {}^{(\mathbf{T})}\pi = \mathcal{L}_T g$. By a simple calculation, expressed relative to an orthonormal frame $e_0 = T, e_1, e_2, e_3$, we find,

$$\pi_{00} = 0 \,, \quad \pi_{0i} = n^{-1} \nabla_i n \,, \quad \pi_{ij} = -2k_{ij} \,. \tag{10}$$

Thus condition (9) can be interpreted as the requirement that $\mathbf{T}$ is an approximate Killing vectorfield in the following sense,

A2. There exists a constant Δ_0 such that,

$$\sup_{t \in [t_0, t_*)} \|\pi(t)\|_{L^\infty} \le \Delta_0 \,. \tag{11}$$

In addition to the constant Δ_0 in A1, A2 the constant R_0, which bounds the L^2 norm of the spacetime curvature tensor R on Σ_0, plays an essential role,

$$\|R(t_0)\|_{L^2(\Sigma_0)} \le R_0 \,. \tag{12}$$

[12] For simplicity we use below the same constant Δ_0 as in (7).

To prove the theorem we have to show that if assumptions A1 and A2 are satisfied then the space-time $\mathcal{M}_I$, $I = [t_0, t_*)$, $t_* < 0$ can be extended beyond t_*. We want to emphasize that Theorem 3.1 is a large data result; indeed one need not make any smallness assumptions on the constants Δ_0 and R_0.

Our theorem is connected and partially motivated by the following three earlier breakdown criteria results:

1. The first is a result of M. Andersson [1], who showed that a breakdown can be tied to the condition that

$$\limsup_{t \to t_*^-} \|R(t)\|_{L^\infty} = \infty \,. \tag{13}$$

It is clear that condition (8) is formally weaker than (13) as it requires one degree less of differentiability. Moreover a condition on the boundedness of the L^∞ norm of R covers all the dynamical degree of freedom of the equations. Indeed, once we know that $\|R(t)\|_{L^\infty}$ is finite, one can find bounds for n, ∇n and k purely by elliptic estimates.

2. Our result can be also compared to the well known Beale-Kato-Majda criterion for breakdown of solutions of the incompressible Euler equation

$$\partial_t v + (v \cdot \nabla) v = -\nabla p \,, \qquad \operatorname{div} v = 0 \,,$$

with smooth initial data at $t = t_0$. A routine application of the energy estimates shows that a solution v blows up if and only if

$$\int_{t_0}^{t_*} \|\nabla v(t)\|_{L^\infty} dt = \infty \,. \tag{14}$$

The Beale-Kato-Majda criterion improves the blow up criterion by replacing it with the following condition on the vorticity $\omega = \operatorname{curl} v$:

$$\int_{t_0}^{t_*} \|\omega(t)\|_{L^\infty} dt = \infty \,. \tag{15}$$

Similarly, in the case of the Einstein equations energy estimates, expressed relative to a special system of coordinates (such as wave coordinates), show that breakdown does not occur unless

$$\int_{t_0}^{t_*} \|\partial g(t)\|_{L^\infty} dt = \infty \,.$$

This condition however is not geometric as it depends on the choice of a full coordinate system. Observe that both the spatial derivatives of the lapse ∇n and the components of the second fundamental form, $k_{ij} = -\frac{1}{2} n^{-1}\, \partial_t g_{ij}$, can be interpreted as components of ∂g. Note however that after prescribing k and ∇n we are still left with many more degrees of freedom in determining ∂g.

3. Finally, the result whose proof is closest in spirit to ours and which has played the main motivating role in developing our approach, is the proof of global regularity of solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$ by Eardley and Moncrief, see [14].

REFERENCES

[1] L. Andersson: *The global existence problem in General Relativity*, preprint.

[2] H. Bahouri – J. Y. Chemin: *Equations d'ondes quasilineaires et effect dispersif*, Amer. J. Math., **121** (1999), 1337-1377.

[3] H. Bahouri – J. Y. Chemin: *Equations d'ondes quasilineaires et estimations de Strichartz*, Int. Math. Res. Not., **21** (1999), 1141-1177.

[4] R. Bartnik: *Existence of maximal hypersurfaces in asymptotically flat space-times*, Comm. Math. Phys., **94** n. 2 (1984), 155-175.

[5] R. Beig – P. Chrusciel: *Stationary black Holes*, arXiv:gr-qc/0502041 v1, 10 Feb. 2005.

[6] G. D. Birkhoff: *Relativity and Modern Physics*, Harvard University Press, 1923.

[7] Y. Choquet Bruhat: *Theoreme d'Existence pour certains systemes d'equations aux derivees partielles nonlineaires*, Acta Math., **88** (1952), 141-225.

[8] Y. Ch. Bruhat – R. P. Geroch: *Global aspects of the Cauchy problem in General Relativity*, Comm. Math. Phys., **14**, 329-335.

[9] M. Dafermos: *Princeton PhD thesis*, 2001.

[10] D. Christodoulou: *On the global initial value problem and the issue of singularities*, Class. Quant. Gr., (1999), A23-A35.

[11] D. Christodoulou: *The instability of naked singularities in the gravitational collapse of a scalar field*, Ann. of Math., **149** (2) n. 1 (1999), 183-217.

[12] D. Christodoulou – S. Klainerman: *The global nonlinear stability of the Minkowski space*, Princeton University Press, 1993.

[13] M. Dafermos – I. Rodnianski: *The red shift effect and radiation decay on black hole spacetimes*, arXiv:gr-qc/0512119 v1 21 Dec. 2005.

[14] D. M. Eardley – V. Moncrief: *The global existence of Yang-Mills fields in* 4 *dimensional Minkowski space I, II*, Comm. Math. Phys., **83** n. 2 (1982), 171-191, 193-212.

[15] S. Hawking – G. F. R. Ellis: *The large scale structure of spacetime*, Cambridge University Press, Cambridge, 1973.

[16] Hughes – T. Kato – J. Marsden: *Well posed quasilinear second order hyperbolic systems ...*, Arch. Rat. Mech. Anal., **63** n. 3 (1976), 273-294.

[17] W. Israel: *Event horizons in static vacuum spacetimes*, Phys. Rev., **164** (1967), 1776-1779.

[18] S. Klainerman: *The null condition and global existence to nonlinear wave equations*, Lect. Appl. Math., **23** (1986), 293-326.

[19] S. Klainerman: *A commuting vectorfield approach to Strichartz type inequalities and applications to quasilinear wave equations*, to appear in IMRN.

[20] S. Klainerman: *PDE as a unified subject*, Special Volume, GAFA 2000, 279-315.

[21] S. Klainerman – M. Machedon: *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math., **46** (1993), 1221-1268.

[22] S. Klainerman – M. Machedon: *Finite energy solutions for the Yang-Mills equations in* RR^{3+1}, Ann. Math., **142** (1995), 39-119.

[23] S. Klainerman – F. Nicolo: *On local and global aspects of the Cauchy problem in General Relativity*, Class. Quant. Grav., **16** (1999), R73-R157. There will be also a new book with a similar title.

[24] S. Klainerman – I. Rodnianski: *Improved local well posedness for quasilinear wave equations in dimension three*, Duke Math. J., **117** n. 1 (2003), 1-124.

[25] S. Klainerman – I. Rodnianski: *Improved local well posedness for the Einstein equations in wave coordinates*, Annals of Mathematics, **161** (2005), 1143-1193.

[26] S. Klainerman – I. Rodnianski: *Causal geometry of Einstein-Vacuum spacetimes with finite curvature flux*, Inventiones Math., **159** n. 3 (2005), 437-529.

[27] S. Klainerman – I. Rodnianski: *A geometric approach to Littlewood-Paley theory*, to appear in GAFA (Geom. and Funct. Anal.).

[28] S. Klainerman – I. Rodnianski: *Sharp trace theorems for null hypersurfaces on Einstein metrics with finite curvature flux*, to appear in GAFA.

[29] S. Klainerman – I. Rodnianski: *Lower bounds for the radius of injectivity of null hypersurfaces*, preprint.

[30] S. Klainerman – I. Rodnianski: *A Kirchoff-Sobolev parametrix for the wave equations in a curved space-time*, preprint.

[31] S. Klainerman – I. Rodnianski: *A breakdown criterion in General Relativity*, preprint.

[32] R. Penrose: *Singularities and time asymetry in General Relativity-an Einstein centenary survey*, S. Hawking, W. Israel (eds.), Cambridge Univ. Press, Cambridge, 1979.

[33] Poisson – Israel – Werner: *Internal structure of black holes*, Phys. Rev. Letters D (3), **41** n. 6 (1990), 1796-1809.

[34] A. D. Rendall: *Constant mean curvature foliations in cosmological space-times*, Journées Relativistes 96, Part II, Ascona, 1996.

[35] H. Smith – D. Tataru: *Sharp local well posedness results for the nonlinear wave equation*, To appear in Annals of Math.

[36] D. Tataru: *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, to appear in Amer. J. Math.

[37] R. Wald: *General Relativity*, University of Chicag Press, 1984.

Lavoro pervenuto alla redazione il 22 gennaio 2007
ed accettato per la pubblicazione il 31 gennaio 2007.
Bozze licenziate il 14 maggio 2007

Indirizzo dell'Autore:

S. Klainerman – Department of Mathematics – Princeton University – Princeton NJ 08544
E-mail: seri@math.princeton.edu