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On the motion of a convex body interacting with a perfect gas in the mean-field approximation

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Abstract: *We consider a convex body in $\mathbb{R}^3$, moving along the x-axis, immersed in an infinitely extended perfect gas in the mean-field approximation. We assume that the gas particles interact with the body by means of elastic collisions. Giving to the body an initial velocity V_0, we prove that, for $|V_0|$ small enough, $|V(t)| \approx C\, t^{-5}$ for large t, being C a positive constant depending on the medium and on the shape of the obstacle. The power law approach to the equilibrium $V = 0$, instead of the exponential one (typical in viscous friction problems), is due to the long memory effect of the recollisions. This paper completes the analysis made in previous papers (see [7] and [8]), in which for simplicity the body was assumed to be a disk.*

1 – Introduction

We study the behavior of the motion of a body immersed in a homogeneous fluid. We want to present a model which describes the macroscopic features of this physical system, from a microscopic point of view.

We assume the medium to be an infinitely extended perfect gas, taken in the mean-field approximation. Such approximation consists in taking the limit in which the mass of the particles constituting the free gas goes to zero, while the number of particles per unit volume diverges, in such a way that the mass density stays finite. Such a limit is well known for interacting particle systems in case of finite total mass ([1], [9], [13], [14]), and for one-dimensional particle systems with unbounded mass ([3]).

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We assume here that the body has a general convex shape, and its interaction with the gas particles happens by means of elastic collisions. We want to generalize the analysis made in previous papers (see [7] and [8]) in which the shape of the obstacle was assumed for simplicity to be a disk. In Ref. [7] it is present also an external constant force which acts on the disk, while in Ref. [8] it is considered the case in which the external force is absent (as in the present paper), and the case in which the external force is of elastic type.

The techniques here used are the same as in Refs. [7]-[8], with some non trivial difficulties due to the more general geometry of the body. In fact in case that the body is a disk orthogonal to the x-axis and moving along the same axis, the velocity component of a gas particle parallel to the disk is conserved after one or several collisions with the disk, while it is not so in case that the body has a convex shape, due to different orientations of the tangent plane to the body at different points of collision. In particular the convex shape determines a different coefficient in the upper bound (2.19) with respect to Ref. [8] (γ^7 instead of γ^5). We have not thought to a possible proof which includes also concave shapes, because of some further difficulties. In fact the convex shape of the body allows us to obtain inequality (3.10), which is widely used in the proof of the result of the present paper. Moreover a convex shape allows us to separate contributions between left and right recollisions, since a particle which hits the body on its left (or right) face the first time will always hit the same face, and this is not true in case of a concave shape.

The result of the present paper, as that of Refs. [7] and [8], is somehow surprising. If the initial velocity of the body is sufficiently close to its limiting velocity V_∞ (here $V_\infty = 0$), then for large t,

$$|V(t) - V_\infty| \approx \frac{C}{t^{d+2}}\,, \tag{1.1}$$

where the constant C depends on the medium and on the shape of the obstacle, and $d = 1, 2, 3$ is the dimension of the physical space.

This unexpected behavior, instead of the exponential approach to V_∞, considered in viscous friction problems, is due to the recollisions between the gas particles and the obstacle, which produce a long memory effect during the motion. Neglecting recollisions we obtain an exponential approach to the equilibrium. The probability that a gas particle delivers multiple collisions with the obstacle depends on the data of the physical system, considering that, to have multiple collisions, the velocity of a gas particle has to be close to the velocity of the body. Hence, to have an important effect of recollisions, the root mean square velocity of the gas particles (proportional to the square root of the temperature) has to be close to the velocity of the body.

We conjecture also that this memory effect is destroyed if the medium is an interacting particle system. In this case it is reasonable that the asymptotic

approach to the equilibrium is exponential, but we expect that the power law remains valid as a transient long-time behavior.

We address the reader to Ref. [11] for heuristic considerations related to the subject of the present paper, and to Refs. [2], [4], [5], [6], for the rigorous analysis of models similar to the one here considered. In particular the model of the present paper has been previously introduced in connection with the so called piston problem (see [10] and also [12], and references quoted therein).

The plan of the paper is the following. In Section 2 we introduce the model and we state the result, and we prove it in Section 3.

2 – Model and results

We consider a bounded convex solid, Ω, in $\mathbb{R}^3$. We take for simplicity Ω with unitary mass and constant density. Let us denote by R the diameter of Ω (i.e. the maximum distance between two points on its boundary $\partial\Omega$) and by $X(t)$ the position of its center of mass at time t. We constrain the center of mass to move along the x-axis, and we impose that the solid cannot undergo any kind of rotation. If the solid has a rotational symmetry around the x-axis, these constraints are superfluous, for the symmetry of the problem we are going to present. The outward unit normal to $\partial\Omega$ is denoted by $\hat{n}$. We require that $\hat{n}(\xi)$, with $\xi \in \partial\Omega$, is a continuous function for almost every $\xi \in \partial\Omega$. We denote by $\partial\Omega^+$ the right face of the solid, on which $\hat{n} \cdot \hat{x} \geq 0$ (being $\hat{x}$ the unit vector of the x-axis), and by $\partial\Omega^-$ the left face of the solid, on which $\hat{n} \cdot \hat{x} < 0$, and we assume that there exists a subset of $\partial\Omega^+$, and a subset of $\partial\Omega^-$, having positive measure, on which $\hat{n} \cdot \hat{x} \neq 0$.

The solid is immersed in a perfect gas in equilibrium at inverse temperature proportional to β and with constant density ρ. The gas is also assumed in the mean-field approximation.

We assign to the solid an initial small velocity, and we want to investigate how its velocity vanishes in time. Clearly the solid modifies the equilibrium of the gas, which starts to evolve according to the free transport equation: denoting by $f(x, v; t)$ the mass density in the phase space of each particle of the gas, it evolves according to:

$$(\partial_t + v \cdot \nabla_x) f(x, v; t) = 0, \quad \text{for} \quad x \notin \overline{\Omega}(t)\,. \tag{2.1}$$

Here $\Omega(t)$ denotes the domain of $\mathbb{R}^3$ occupied by the solid at time t.

Together with equation (2.1) we consider the boundary conditions, requiring the continuity of f along the trajectories with elastic reflection on $\partial\Omega(t)$. A necessary condition for which a gas particle with velocity v hits the solid at time t is that, at the collision point $P \in \partial\Omega(t)$, it results

$$v_n \leq V_n(t)\,, \tag{2.2}$$

denoting by $v_n = v \cdot \hat{n}$, and by $V_n(t) = V(t)\,\hat{n} \cdot \hat{x}$, where $V(t) = \dot{X}(t)$ and $\hat{n}$ is calculated at P. Imposing elastic reflection in P, we have for the outgoing velocity v':

$$v'_n = 2V_n(t) - v_n, \quad v'_{n_\perp} = v_{n_\perp}\,, \tag{2.3}$$

denoting by $v_{n_\perp} = v - v_n\hat{n}$. The collision law (2.3) takes into account the fact that the ratio between the mass of the gas particle and the mass of the solid is negligible, and for a derivation of it see Ref. [7]. We set

$$f_+(x, v'; t) = f_-(x, v; t), \quad \text{for} \quad x \in \partial\Omega(t) \tag{2.4}$$

where

$$f_\pm(x, v; t) = \lim_{\varepsilon \to 0^+} f(x \pm \varepsilon v, v; t \pm \varepsilon), \quad \text{for} \quad x \in \partial\Omega(t)\,. \tag{2.5}$$

Equation (2.4) describes the continuity of f along the collisions.

Coupled to equation (2.1) we consider the evolution equation for the body:

$$\begin{aligned} &\dot{X}(t) = V(t), \quad \dot{V}(t) = -F(t)\,, \\ &X(0) = 0, \qquad V(0) = V_0\,, \end{aligned} \tag{2.6}$$

where the action of the gas on the body is described by the viscous friction term

$$\begin{aligned} F(t) = 2 &\int_{\partial\Omega^+} d\sigma \int_{v_n \le V_n(t)} dv\,(V_n(t) - v_n)^2\,\hat{n} \cdot \hat{x}\, f_-(x, v; t) \\ &+ 2 \int_{\partial\Omega^-} d\sigma \int_{v_n \le V_n(t)} dv\,(V_n(t) - v_n)^2\,\hat{n} \cdot \hat{x}\, f_-(x, v; t)\,, \end{aligned} \tag{2.7}$$

being $d\sigma$ the surface element on $\partial\Omega$. equation (2.7) takes into account the transfer of momentum from the gas particles to the body (from right and left collisions), and for a heuristic derivation of it see Refs. [7] and [8]. Notice that in (2.7) it appears the scalar product $\hat{n}\cdot\hat{x}$, since we have to consider the projection of the force along the x-axis. Moreover, the first integral in (2.7) is positive ($\hat{n} \cdot \hat{x} \ge 0$ on $\partial\Omega^+$) and the second one is negative ($\hat{n} \cdot \hat{x} < 0$ on $\partial\Omega^-$).

As initial state for the gas distribution we assume the thermal equilibrium, namely

$$\begin{aligned} &f_+(x, v; 0) = \rho \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}, \quad \text{for} \quad x \notin \overline{\Omega}(0)\,, \\ &f_+(x, v; 0) = 0, \qquad\qquad\qquad\qquad \text{for} \quad x \in \overline{\Omega}(0)\,, \end{aligned} \tag{2.8}$$

for $\beta > 0$.

Summarizing, we define a solution to our problem any pair (f, V) where $V = V(t)$ solves, for almost all $t \in \mathbb{R}^+$, equations (2.6)-(2.7) and f satisfies equation (2.9) below

$$\frac{d}{dt} f(x + vt, v; t) = 0 \quad \text{a.e. } (x, v)\,, \tag{2.9}$$

together with boundary conditions (2.4) and initial condition (2.8).

We first observe that equation (2.1) can be solved by means of characteristics. More precisely, knowing the evolution of the body $X(t)$, $V(t)$, we can trace back the time evolution of position and velocity of the gas particle $x(s, t; x, v)$, $v(s, t; x, v)$ at time $s \le t$, having position and velocity x, v at time t. Such backward evolution is the free motion up to the last instant $\tau < t$ at which the particle hits the body. On the surface of the body we impose the elastic collision (2.3). Then we continue backward in time, up to the previous collision, impose again the reflection condition and so on. At the end we obtain

$$\begin{aligned} F(t) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \le V_n(t)} dv\, (V_n(t) - v_n)^2\, \hat{n} \cdot \hat{x}\, e^{-\beta v^2(0,t;x,v)} \\ + k \int_{\partial\Omega^-} d\sigma \int_{v_n \le V_n(t)} dv\, (V_n(t) - v_n)^2\, \hat{n} \cdot \hat{x}\, e^{-\beta v^2(0,t;x,v)}\,, \end{aligned}$$

where $k = 2\rho(\beta/\pi)^{3/2}$. Note that to compute $F(t)$ we need to evaluate $v(0, t; x, v)$ and hence to know all the previous history $\{X(s), V(s),\ s < t\}$. On the other hand, if a light particle goes back without undergoing any collision, then

$$v(0, t; x, v) = v\,.$$

In this case we say, for obvious reasons, that the gas particle has no recollisions. In absence of recollisions the friction term is easily computed:

$$\begin{aligned} F_0(V) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \le V_n} dv\, (V_n - v_n)^2\, \hat{n} \cdot \hat{x}\, e^{-\beta v^2} \\ + k \int_{\partial\Omega^-} d\sigma \int_{v_n \le V_n} dv\, (V_n - v_n)^2\, \hat{n} \cdot \hat{x}\, e^{-\beta v^2}\,. \end{aligned} \tag{2.10}$$

We will show in Lemma 2.1 that F_0 is an increasing function, null in zero.

Let us show that, without recollisions, our problem can be trivially solved. Indeed replacing F by F_0 in equation (2.6) we have:

$$\begin{aligned} &\dot{X}(t) = V(t), \quad \dot{V}(t) = -F_0(V(t)) = -K(t)V(t)\,, \\ &X(0) = 0, \qquad V(0) = V_0\,, \end{aligned} \tag{2.11}$$

where

$$K(t) = \frac{F_0(V(t))}{V(t)}\,. \tag{2.12}$$

We take, without loss of generality, $V_0 > 0$ (the case $V_0 < 0$ is the symmetrical one). It results, for $V \in [0, V_0]$,

$$C_2 \le K(t) \le C_1 \tag{2.13}$$

where

$$0 < \min_{V\in[0,V_0]} F_0'(V) = C_2 < C_1 = \max_{V\in[0,V_0]} F_0'(V)\,.$$

The solution to equation (2.11) can be almost explicitly computed. It can be easily proved that V is decreasing in time and by a comparison argument

$$V_0\, e^{-C_1 t} \le V(t) \le V_0\, e^{-C_2 t}\,. \tag{2.14}$$

The Vlasov equation (2.1) is then solved by characteristics.

In the full problem, where we include recollisions, the long memory effect makes the problem much more difficult.

Let us rewrite the full friction term F as:

$$F(t) = F_0(V(t)) + r^+(t) + r^-(t) \tag{2.15}$$

where $r^+(t)$ and $r^-(t)$ are:

$$r^+(t) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \le V_n(t)} dv\, (v_n - V_n(t))^2\, \hat{n}\cdot\hat{x}\, (e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}) \tag{2.16}$$

and

$$r^-(t) = -k \int_{\partial\Omega^-} d\sigma \int_{v_n \le V_n(t)} dv\, (v_n - V_n(t))^2\, \hat{n}\cdot\hat{x}\, (e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)})\,. \tag{2.17}$$

The quantities ρ, β, R and $\gamma = V_0$ are the data of the problem.

We are now in the position to state the main result of the present paper.

THEOREM 2.1. *There exists $\gamma_0 = \gamma_0(\rho, \beta, R) > 0$ sufficiently small such that, for any initial velocity $V_0 = \gamma \in (0, \gamma_0)$ there exists at least one solution $(V(t), f(t))$ to problem* (2.1)-(2.9). *Moreover there exist two positive constants A_1, A_2 independent of γ, such that any solution $(V(t), f(t))$ satisfies the following properties*:

(i) *for any $t \ge 0$ it is*:

$$V(t) \ge \gamma\, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^{d+2}}\,, \tag{2.18}$$

(ii) *there exists a sufficiently large $\bar{t}$, depending on γ, such that for any $t \ge 0$:*

$$V(t) \le \gamma\, e^{-C_2 t} - \gamma^7 \frac{A_2}{t^{d+2}} \chi(\{t \ge \bar{t}\}) \tag{2.19}$$

where $\chi(\{\dots\})$ is the characteristics function of the set $\{\dots\}$.

Note that (2.19) establishes the power law approach to the equilibrium state.

We underline that in our model the body slows down its velocity in an unexpected way, in spite of what intuition suggests. The velocity is initially positive and it decreases to zero in a finite time, then $V(t)$ becomes negative and, from negative values, it tends asymptotically to zero (as it can be seen by (2.19)). The fact that $V(t)$ changes sign is due to the memory of the recollisions, whose effect is contained in the terms $r^+(t)$ and $r^-(t)$.

We discuss now the announced properties of F_0.

LEMMA 2.1. *F_0 is an increasing function, null in zero.*

PROOF. With the previous definitions of v_n and $v_{n_\perp}$, defining

$$G = \int dv_{n_\perp}\, e^{-\beta v_{n_\perp}^2}\,, \tag{2.20}$$

we have, by (2.10):

$$\begin{aligned} F_0(V) = k\,G \int_{\partial\Omega^+} d\sigma\, \hat{n}\cdot\hat{x} \int_{-\infty}^{V_n} dv_n\,(V_n - v_n)^2 e^{-\beta v_n^2} \\ + k\,G \int_{\partial\Omega^-} d\sigma\, \hat{n}\cdot\hat{x} \int_{-\infty}^{V_n} dv_n\,(V_n - v_n)^2 e^{-\beta v_n^2}\,, \end{aligned} \tag{2.21}$$

$$\begin{aligned} F_0'(V) = 2k\,G \int_{\partial\Omega^+} d\sigma\,(\hat{n}\cdot\hat{x})^2 \int_{-\infty}^{V_n} dv_n\,(V_n - v_n) e^{-\beta v_n^2} \\ + 2k\,G \int_{\partial\Omega^-} d\sigma\,(\hat{n}\cdot\hat{x})^2 \int_{-\infty}^{V_n} dv_n\,(V_n - v_n) e^{-\beta v_n^2}\,, \end{aligned} \tag{2.22}$$

from which it follows $F_0(0) = 0$ (since $\int_{\partial\Omega^+} \hat{n}\cdot\hat{x}\, d\sigma = -\int_{\partial\Omega^-} \hat{n}\cdot\hat{x}\, d\sigma$), and $F_0'(V) > 0\ \forall V \in \mathbb{R}$. ∎

3 – Proof of Theorem 2.1

In the sequel we will denote by C any positive constant, possibly depending on β, ρ, R, but not on γ, which is our small parameter.

For any $\gamma \in (0, \gamma_0)$ with γ_0 sufficiently small, we introduce an a.e. differentiable function with bounded derivative, $t \to W(t)$, such that $W(0) = V_0$, $\lim_{t\to\infty} W(t) = 0$, and satisfying the following properties:

(i) W is decreasing in any time interval in which $W(t) > 0$.
(ii) There exist two positive constants A_1, A_2 such that, for any $t \geq 0$, it is:

$$W(t) \geq \gamma\, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^5} \equiv f_1(t) \tag{3.1}$$

and

$$W(t) \leq \gamma\, e^{-C_2 t} - \gamma^7 \frac{A_2}{t^5} \chi(\{t \geq \bar{t}\}) \equiv f_2(t)\,. \tag{3.2}$$

The two constants A_1 and A_2, independent of each other and also of γ and γ_0, will be fixed later on. The time $\bar{t}$ will be fixed in (3.16).

The idea of the proof of Theorem 2.1 is the following. We assign the velocity W of the body with the properties just stated. We solve the free Vlasov equation outside a body moving with velocity W and compute the terms r_W^+ and r_W^- defined below. Then we solve equation (2.6) for the body with assigned r_W^+ and r_W^-, finding a new velocity V_W. The solution of our problem is the fixed point of the map $W \to V_W$, thus we have to prove for V_W the same properties established above for W.

Let $X(t) = \int_0^t W(\tau) d\tau$ be the position of the body at time t. Consider the modified problem:

$$\dot{V}_W(t) = -K(t)\, V_W(t) - r_W^+(t) - r_W^-(t)\,, \tag{3.3}$$

where $K(t)$ is the function introduced in (2.12) with $W(t)$ in place of $V(t)$.

We define

$$r_W^+(t) = k \int_{\partial\Omega^+} d\sigma \int_{v_n \leq W_n(t)} dv\, (v_n - W_n(t))^2\, \hat{n}\cdot\hat{x}\, (e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}) \tag{3.4}$$

and

$$r_W^-(t) = -k \int_{\partial\Omega^-} d\sigma \int_{v_n \leq W_n(t)} dv (v_n - W_n(t))^2\, \hat{n}\cdot\hat{x} (e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)})\,, \tag{3.5}$$

where $W_n(t) = W(t)\, \hat{n}\cdot\hat{x}$. We notice that as long as W is decreasing $r_W^+(t) = 0$, so it appears only for negative velocities and moreover, by the collision law (2.3), it is negative. The analysis of the sign of $r_W^-(t)$ is more involved, and it will be done later. We will show that, in any way, the sum of r_W^+ and r_W^- is positive.

The velocities of the light particles $v(s,t;x,v)$, $s < t$, are computed according to the evolution of the body moving with velocity W and to the law of elastic reflection (2.3).

We may ask whether equation (3.3) is well posed. The following proposition, proved in Ref. [7], shows that this dynamical system is well defined for almost

all initial data and almost all $t \in \mathbb{R}^+$. More precisely we can neglect in the sequel all the initial configurations giving rise to infinitely many or tangential collisions, namely those for which there exists a time $s < t$ such that $x \in \partial\Omega(t)$, $x(s,t;x,v) \in \partial\Omega(s)$ and $v(s,t;x,v) \cdot \hat{n}_s = W(s)\,\hat{n}_s \cdot \hat{x}$, being $\hat{n}_s$ the normal calculated in $x(s,t;x,v)$.

PROPOSITION 3.1. *Consider the dynamics of the body with given velocity $W = W(t)$ and the fluid trajectories $x(s,t;x,v)$, $v(s,t;x,v)$ computed according to the evolution of the body and the law of the elastic reflection* (2.3). *Assume W differentiable for almost all t and such that*

$$\operatorname{ess\,sup}_{t\in\mathbb{R}^+}(|W(t)| + |\dot{W}(t)|) = L < +\infty\,. \tag{3.6}$$

Then the set of all $t \in \mathbb{R}^+$, $x \in \partial\Omega(t)$, $v \in \mathbb{R}^3$ for which $x(s,t;x,v)$, $v(s,t;x,v)$, $0 \le s < t$, delivers infinitely many backward collisions, or has a tangential collision, has vanishing Lebesgue measure.

We want to show now that V_W behaves like W. To this aim we have to estimate $r_W^{\pm}$.

For $0 \le s < t$, we set

$$\langle W\rangle_{s,t} = \frac{1}{t-s}\int_s^t W(\tau)\,d\tau \tag{3.7}$$

and

$$\langle W\rangle_{0,t} = \langle W\rangle_t\,. \tag{3.8}$$

We indicate by $\langle W_n\rangle_{s,t}$ and $\langle W_n\rangle_t$ the analogous expressions obtained by (3.7) and (3.8) replacing W by W_n.

We want to establish preliminarily some necessary conditions for which a gas particle can have recollisions with the body. Let $s < t$ be the first backward recollision time, and let us denote by $P \in \partial\Omega$ the collision point at time t, and by $Q \in \partial\Omega$ the collision point at time s. The condition to have two subsequent collisions is the following:

$$v_n(t-s) = \overrightarrow{QP} \cdot \hat{n} + \langle W_n\rangle_{s,t}(t-s) \ge \langle W_n\rangle_{s,t}(t-s) \tag{3.9}$$

where $\hat{n}$ is calculated at P and $\overrightarrow{QP}$ is the vector joining the points Q and P at time s. In fact $v_n(t-s)$ is the space along the $\hat{n}$ direction covered by the gas particle in the time interval $[s,t]$, $\langle W_n\rangle_{s,t}(t-s)$ is the space along the $\hat{n}$ direction covered by the body in the time interval $[s,t]$, and $\overrightarrow{QP} \cdot \hat{n}$ is the distance along the $\hat{n}$ direction between the two points Q and P at time s, which, by the convex shape of the body, is always non-negative.

Hence a first necessary condition to have a recollision is

$$v_n \geq \langle W_n \rangle_{s,t} \,. \tag{3.10}$$

Another necessary condition is the following:

$$|v_{n_\perp}| \, (t-s) \leq 2R + |\langle W \rangle_{s,t}| \, (t-s) \,. \tag{3.11}$$

In fact the r.h.s. of (3.11) collects the maximum displacement that a particle can undergo along the x-direction, and along any direction orthogonal to the x-axis, to have a recollision with the body. Then from (3.11) we obtain

$$|v_{n_\perp}| \leq \frac{C}{t-s} \,. \tag{3.12}$$

Summarizing, we have condition (3.10) on v_n, and condition (3.12) on $v_{n_\perp}$.

Let us estimate now $r_W^+(t)$, proving the following lemma:

Lemma 3.1. *For any* $t \geq 0$

$$0 \leq -r_W^+(t) \leq C \frac{\gamma^9 A_1^3}{(1+t)^5} \, \chi(\{t > t_0\}) \tag{3.13}$$

where

$$t_0 = K_0 \log \frac{1}{\gamma} \tag{3.14}$$

and K_0 *is a constant satisfying* $1/C_1 \leq K_0 < 2/C_1$.

Proof. First of all let us notice that $r_W^+(t) = 0$ as far as W is decreasing (i.e. as far as $W(t) > 0$), so let us give an upper and lower bound for the first time t^* for which $W(t^*) = 0$. For $t = t_0$ it results

$$f_1(t_0) = \gamma^{1+K_0 C_1} - \gamma^3 \frac{A_1}{\left(1 + K_0 \log \frac{1}{\gamma}\right)^5} > 0 \,, \tag{3.15}$$

the last inequality being satisfied by taking γ sufficiently small. By the properties (i) and (ii) of the function W this implies that, for $0 \leq t \leq t_0$, $W(t) \geq W(t_0) \geq f_1(t_0) > 0$. Let us set

$$\bar{t} = \bar{K} \log \frac{1}{\gamma} \tag{3.16}$$

where $\bar{K} > 6/C_2$. Then it results

$$f_2(\bar{t}) = \gamma^{1+\bar{K}C_2} - \gamma^7 \frac{A_2}{\left(\bar{K}\log\frac{1}{\gamma}\right)^5} < 0 \tag{3.17}$$

for γ sufficiently small. Then $W(t) < 0$ for $t \geq \bar{t}$, and so the first time t^* for which $W(t^*) = 0$ satisfies $t^* \in (t_0, \bar{t})$; moreover $W(t) \leq 0$ for $t^* \leq t \leq \bar{t}$.

This explains the characteristic function in (3.13). It is also evident from the law of elastic reflection (2.3) that $r_W^+(t) \leq 0$, since it appears for negative velocities. Let us establish then an upper bound for $|r_W^+(t)|$. Recalling the necessary condition on v_n (3.10) to have a recollision, we have by (3.1):

$$\begin{aligned} \langle W\rangle_{s,t} &\geq \frac{1}{t-s}\int_s^t \Big(\gamma e^{-C_1\tau} - \gamma^3 \frac{A_1}{(1+\tau)^5}\Big) d\tau \\ &\geq \frac{1}{t-s}\int_s^t \Big(-\gamma^3 \frac{A_1}{(1+\tau)^5}\Big) d\tau \geq -CA_1 \frac{\gamma^3}{1+t} \end{aligned} \tag{3.18}$$

for $s < t/2$. By (3.4), (3.10), (3.12), and (3.18), putting

$$\chi_{v_n} = \chi\left(\left\{-CA_1\frac{\gamma^3}{1+t}\hat{n}\cdot\hat{x} \leq v_n \leq W_n(t)\right\}\right), \tag{3.19}$$

we have that a first contribution to the estimate of $|r_W^+(t)|$, for $s < t/2$ and $t > t^*$, is:

$$C\int_{\partial\Omega^+} d\sigma \int dv_n\,(v_n - W_n(t))^2\,\chi_{v_n} \int_{|v_{n_\perp}|<\frac{C}{t}} dv_{n_\perp}\,. \tag{3.20}$$

For $s < t/2$ we have

$$\int_{|v_{n_\perp}|<\frac{C}{t}} dv_{n_\perp} \leq \frac{C}{(1+t)^2}\,, \tag{3.21}$$

so that,

$$\begin{aligned} (3.20) &\leq \frac{C}{(1+t)^2}\int_{\partial\Omega^+} d\sigma \int dv_n\,(v_n - W_n(t))^2\,\chi_{v_n} \\ &\leq \frac{C}{(1+t)^2}\Big(W(t) + \frac{CA_1\gamma^3}{1+t}\Big)^3 \\ &\leq C\frac{A_1^3\gamma^9}{(1+t)^5} \end{aligned} \tag{3.22}$$

since $W(t) < 0$. Hence (3.22) constitutes the first contribution, for $s < t/2$, to the estimate of $|r_W^+(t)|$.

If $s \geq t/2$ and $t > t^*$ we have by (3.1):

$$\begin{aligned} \langle W \rangle_{s,t} &\geq \frac{1}{t-s} \int_s^t \Big(-\gamma^3 \frac{A_1}{(1+\tau)^5} \Big) d\tau \\ &\geq -C \frac{A_1 \gamma^3}{(1+t)^5} \,, \end{aligned} \tag{3.23}$$

hence, by (3.4) and (3.10), the second contribution to the estimate of $|r_W^+(t)|$ is:

$$\begin{aligned} & C \int_{\partial\Omega^+} d\sigma \int dv_n \, (v_n - W_n(t))^2 \, \chi \left(\left\{ -C \frac{A_1 \gamma^3}{(1+t)^5} \hat{n} \cdot \hat{x} \leq v_n \leq W_n(t) \right\} \right) \\ & \leq C \left(\frac{A_1 \gamma^3}{(1+t)^5} \right)^3 , \end{aligned} \tag{3.24}$$

where we have used the fact that $e^{-\beta v^2(0,t;x,v)}$ is integrably decreasing with respect to v. This can be seen for example by using the conservation of the total momentum of the system gas+solid, whose component along the x-axis is

$$\rho \left(\frac{\beta}{\pi} \right)^{3/2} \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dv \, v_x \, e^{-\beta v^2(0,t;x,v)} + V(t) = V_0 \,, \tag{3.25}$$

being V_0 the value that it takes at $t = 0$. Hence, by the boundedness of $V(t)$, the integral of $e^{-\beta v^2(0,t;x,v)}$ with respect to v is bounded by a constant.

Collecting estimates (3.22) and (3.24) we finally achieve the proof of Lemma 3.1. ∎

For $r_W^-(t)$ we have an upper bound expressed by the following lemma:

Lemma 3.2. *For any* $t \geq 0$

$$r_W^-(t) \leq C \frac{(\gamma + A_1 \gamma^3)^3}{(1+t)^5} \,. \tag{3.26}$$

Proof. We work here on $\partial\Omega^-$, and it is convenient to introduce the inward normal $\hat{u} = -\hat{n}$. Hence the collision condition (2.2) becomes $v_u \geq W_u(t)$, recalling that on $\partial\Omega^-$ $\hat{u} \cdot \hat{x} > 0$. By (3.10) (with $\hat{u}$ in place of $\hat{n}$) and (3.2) we obtain:

$$\begin{aligned} v_u \leq \langle W_u \rangle_{s,t} &\leq \frac{\hat{u} \cdot \hat{x}}{t-s} \int_s^t \Big[\gamma \, e^{-C_2 \tau} - \gamma^7 \frac{A_2}{\tau^5} \chi(\{\tau \geq \bar{t}\}) \Big] d\tau \\ &\leq \hat{u} \cdot \hat{x} \, \frac{\gamma}{t-s} \frac{e^{-C_2 s} - e^{-C_2 t}}{C_2} \,, \end{aligned} \tag{3.27}$$

then we have

$$v_u \le C\frac{\gamma}{1+t}\,\hat{u}\cdot\hat{x} \qquad \text{if} \quad s<\frac{t}{2} \tag{3.28}$$

$$v_u \le C\,\gamma\, e^{-C_2 t/2}\,\hat{u}\cdot\hat{x} \qquad \text{if} \quad s\ge\frac{t}{2}\,. \tag{3.29}$$

Hence a first contribution to the estimate of $r_W^-(t)$ is, in case of condition (3.28), recalling (3.1) and (3.21) for what concerns the integration over $v_{n_\perp}$:

$$\begin{aligned}
&\frac{C}{(1+t)^2}\int_{\partial\Omega^-} d\sigma \int dv_u (v_u - W_u(t))^2\,\chi\Big(\Big\{W_u(t)\le v_u\le \frac{C\gamma\,\hat{u}\cdot\hat{x}}{1+t}\Big\}\Big)\\
&\le \frac{C}{(1+t)^2}\Big(\frac{C\gamma}{1+t}-W(t)\Big)^3 \le \frac{C}{(1+t)^2}\Big(\frac{C\gamma}{1+t}-\gamma e^{-C_1 t}+\gamma^3\frac{A_1}{(1+t)^5}\Big)^3\\
&\le C\frac{(\gamma+A_1\gamma^3)^3}{(1+t)^5}\,.
\end{aligned} \tag{3.30}$$

If $s\ge t/2$, using (3.29), we have the second contribution to the estimate of $r_W^-(t)$:

$$\begin{aligned}
&C\int_{\partial\Omega^-} d\sigma \int dv_u (v_u - W_u(t))^2\,\chi(\{W_u(t)\le v_u\le C\gamma e^{-C_2 t/2}\,\hat{u}\cdot\hat{x}\})\\
&\le C\big(C\gamma e^{-C_2 t/2}-W(t)\big)^3 \le C\Big(\frac{A_1\gamma^3+\gamma}{(1+t)^5}\Big)^3\,,
\end{aligned} \tag{3.31}$$

therefore, collecting (3.30) and (3.31), we obtain the thesis. ∎

It follows, by the collision law (2.3), that $r_W^-(t)\ge 0$ for any $t\le t^*$ (as long as $W(t)\ge 0$). Actually the positivity of $r_W^-(t)$ for any $t\ge 0$ is not obvious, since for negative velocities of the body, $r_W^-(t)$ could change sign. We can prove that this is not the case. Moreover we can show that the sum $(r_W^+(t)+r_W^-(t))$ is not negative for any $t\ge 0$, which is a key ingredient in the proof of Theorem 2.1.

Lemma 3.3. *Suppose γ sufficiently small. Then, for $t\ge t_0$ we have*:

$$r_W^-(t)\ge C\frac{\gamma^7}{t^5}\,. \tag{3.32}$$

PROOF. The "bad" contributions to $r_W^-(t)$, i.e. those which tend to transform $r_W^-(t)$ into a negative quantity, come uniquely from those particles for which at least one recollision with the disk happens at a time instant $s < t$ at which W is negative. Let $s < t$ be the largest collision time for a particle having velocity v at time t. For what said, we require $W(s) < 0$, and denoting by $\tilde{r}_W^-(t)$ such "bad" contributions, let us give an upper bound for $|\tilde{r}_W^-(t)|$. Considering the inward normal $\hat{u} = -\hat{n}$, condition (3.10) becomes

$$v_u \le \langle W_u \rangle_{s,t} < 0\,, \tag{3.33}$$

which is negative, since $W(s) < 0$, and so W is negative in the whole interval $[s, t]$. Therefore, from the definition (3.5) of $r_W^-(t)$ and using (3.1):

$$\begin{aligned} |\tilde{r}_W^-(t)| &\le C \int_{\partial\Omega^-} d\sigma \int_{W_u(t)}^{0} (v_u - W_u(t))^2 dv_u \le C(-W(t))^3 \\ &\le C\Big(-\gamma e^{-C_1 t} + \gamma^3 \frac{A_1}{(1+t)^5} \Big)^3 \\ &\le C \frac{\gamma^9 A_1^3}{(1+t)^{15}}\,. \end{aligned} \tag{3.34}$$

Up to now we could obviously write

$$r_W^-(t) \ge -|\tilde{r}_W^-(t)|\,. \tag{3.35}$$

To improve this lower bound, let us denote by $\hat{r}_W^-(t)$ a term which contains some "good" contributions to $r_W^-(t)$, namely those coming from recollisions in the past that happen for some $s < t$ such that $W(s) > 0$. Hence we have

$$r_W^-(t) \ge \hat{r}_W^-(t) - |\tilde{r}_W^-(t)| \tag{3.36}$$

and the difficulty now shifts to get a lower bound for $\hat{r}_W^-(t)$. To this end we restrict the analysis to a subset of the "good" contributions, as we are going to see.

Let us fix $s_0 < t_0$. Then, for $s \le s_0$ and for γ small enough, by (3.1) it is

$$W(s) \ge W(s_0) \ge C\gamma\,. \tag{3.37}$$

Consider a light particle which hits the body on $\partial\Omega^-$ at time t with velocity v, and let s be the time of its previous collision with the body.

Once fixed the body, with the characteristics established at the beginning of Section 2, we pick a point $\xi_0 \in \partial\Omega^-$, such that $\hat{u}(\xi)$ is continuous at $\xi = \xi_0$ and it results $\lambda_0 \equiv \hat{u}(\xi_0) \cdot \hat{x} > 0$. Let us restrict the velocities of the particles hitting the body at time t to the following set:

$$\Gamma = \left\{ (\hat{u}, v) : \langle W_u \rangle_{s_0,t} \le v_u \le \langle W_u \rangle_t, |v_{n_\perp}| \le C\frac{\gamma}{t}, \frac{3\lambda_0}{4} < \hat{u}\cdot\hat{x} < \frac{5\lambda_0}{4} \right\}. \tag{3.38}$$

The set Γ is non-empty, and it generates, for each $\hat{u} \in \Gamma$, a subfamily of characteristics whose first backward recollision time $s < t$ satisfies $s \leq s_0$.

To prove that Γ is non-empty, we have to show that $\langle W \rangle_t - \langle W \rangle_{s_0,t} > 0$ for $t > s_0$. Indeed we have

$$\begin{aligned} \langle W \rangle_t - \langle W \rangle_{s_0,t} &= \frac{1}{t} \int_0^t W(\tau)\, d\tau - \frac{1}{t - s_0} \int_{s_0}^t W(\tau)\, d\tau \\ &= \left(\frac{1}{t} - \frac{1}{t - s_0} \right) \int_0^t W(\tau)\, d\tau + \frac{1}{t - s_0} \int_0^{s_0} W(\tau)\, d\tau \\ &= \frac{s_0}{t - s_0} \left[\frac{1}{s_0} \int_0^{s_0} W(\tau)\, d\tau - \frac{1}{t} \int_0^t W(\tau)\, d\tau \right] \\ &= \frac{s_0}{t - s_0} \left(\langle W \rangle_{s_0} - \langle W \rangle_t \right), \end{aligned} \tag{3.39}$$

moreover

$$\frac{d}{dt} \langle W \rangle_t = -\frac{1}{t^2} \int_0^t W(\tau)\, d\tau + \frac{1}{t} W(t) = \frac{1}{t} (W(t) - \langle W \rangle_t) < 0 \tag{3.40}$$

as long as W is decreasing, that is for $t < t^*$ (defined just below (3.17)), therefore the r.h.s. of (3.39) is positive for $t \in (s_0, t^*)$. For $t \geq t^*$, taking γ sufficiently small and t^* consequently large, by (3.2) and (3.37) we have

$$\begin{aligned} \langle W \rangle_{s_0} - \langle W \rangle_t &\geq C\gamma + \frac{1}{t} \int_0^t \left(-\gamma e^{-C_2 \tau} \right) d\tau \\ &\geq C\gamma - \gamma \frac{1 - e^{-C_2 t}}{C_2 t} \geq C\gamma, \end{aligned} \tag{3.41}$$

so that the r.h.s. of (3.39) is positive also for $t \geq t^*$.

Let us show now that $s \leq s_0$. We have by (3.10), replacing $\hat{n}$ by $\hat{u} = -\hat{n}$,

$$v_u \leq \langle W_u \rangle_{s,t}, \tag{3.42}$$

and we are going to prove that, for $s > s_0$, $\langle W_u \rangle_{s,t} < \langle W_u \rangle_{s_0,t}$, obtaining thus a contradiction with the fact that v_u belongs to Γ, while for $s \leq s_0$ it is $\langle W_u \rangle_{s_0,t} \leq \langle W_u \rangle_{s,t} \leq \langle W_u \rangle_t$. Since $\hat{u} \cdot \hat{x} > 0$ on $\partial\Omega^-$, we can consider W in place of $W_u = \hat{u} \cdot \hat{x}\, W$.

Computing

$$\frac{d}{ds} \langle W \rangle_{s,t} = \frac{1}{t - s} [\langle W \rangle_{s,t} - W(s)] \tag{3.43}$$

we have that the r.h.s. of (3.43) is obviously negative for $s < t \le t^*$, since W is decreasing. For $s < t^*$ (for which $W(s) > W(t^*) = 0$) and $t > t^*$, the r.h.s. of (3.43) is also negative, in fact

$$\begin{aligned}\langle W\rangle_{s,t} - W(s) &= \frac{1}{t-s}\left(\int_s^{t^*} W(\tau)\,d\tau + \int_{t^*}^{t} W(\tau)\,d\tau\right) - W(s)\\ &= \left[\frac{1}{t^*-s}\int_s^{t^*} W(\tau)\,d\tau - W(s)\right]\\ &\quad + \left[\left(\frac{1}{t-s} - \frac{1}{t^*-s}\right)\int_s^{t^*} W(\tau)\,d\tau\right]\\ &\quad + \left[\frac{1}{t-s}\int_{t^*}^{t} W(\tau)\,d\tau\right] < 0\end{aligned} \tag{3.44}$$

since all the terms in square brackets in the r.h.s. of (3.44) are negative. Up to now we have proved that $\langle W\rangle_{s,t}$ is a decreasing function with respect to s for $s < t^*$ (and obviously $t > s$). Finally, for $s \ge t^*$, it results always $\langle W\rangle_{s,t} < \langle W\rangle_{s_0,t}$, in fact by (3.1), for $t > t^*$,

$$\begin{aligned}\langle W\rangle_{s_0,t} &= \frac{1}{t-s_0}\int_{s_0}^{t} W(\tau)\,d\tau \ge \frac{1}{t-s_0}\int_{s_0}^{\infty} W(\tau)\,d\tau\\ &\ge \frac{C}{t-s_0}(\gamma - A_1\gamma^3)\end{aligned} \tag{3.45}$$

which shows that $\langle W\rangle_{s_0,t} > 0$ for γ suitably small; moreover $\langle W\rangle_{s,t} < 0$, since $W(\tau)$ is negative for $\tau > t^*$. This concludes the proof that $s \le s_0$.

The remaining conditions on $|v_{n_\perp}|$ (to be compared with (3.12)) and $\hat{u}\cdot\hat{x}$ appearing in (3.38) are due to the following reason. We will need to give a lower bound of the term $[e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}]$ appearing in the definition of r^-, and the previous conditions allow us to do this in a simple way, as we are going to see. We have that, by (3.1) and (3.2), for $v \in \Gamma$,

$$|v| \le C\frac{\gamma}{t}\,, \tag{3.46}$$

therefore, taking γ small enough, we obtain that the collision at time s happens in a subset of $\partial\Omega^-$ such that

$$\frac{\lambda_0}{2} < \hat{u}_s\cdot\hat{x} < \frac{3\lambda_0}{2}\,, \tag{3.47}$$

where $\hat{u}_s$ denotes the inward normal at the collision point at time s. Let us put, following the usual notation, $v_{u_s} = v\cdot\hat{u}_s$, $W_{u_s}(s) = \hat{u}_s\cdot\hat{x}\,W(s)$. Notice that,

being v the outgoing velocity in the collision point at time s, by (2.2) and (2.3) it satisfies $v_{u_s} \leq W_{u_s}(s)$. By (3.46) and (3.47) we obtain a better condition, that is

$$v_{u_s} \leq \frac{1}{2} W_{u_s}(s)\,, \tag{3.48}$$

in fact

$$v_{u_s} \leq |v| \leq C\frac{\gamma}{t}\,, \tag{3.49}$$

and, for t large, the r.h.s. of (3.49) is certainly smaller than $1/2\, W_{u_s}(s)$, since, by (3.37) and (3.47),

$$\frac{1}{2}\,\hat{u}_s \cdot \hat{x}\, W(s) \geq \frac{\lambda_0}{4} W(s_0) \geq C\gamma\,. \tag{3.50}$$

Denoting by $v(s^-, t; x, v)$ the velocity of the gas particle just before hitting the body at time s, for $s \leq s_0$ by (3.37), which shows that the velocity of the body is positive, and by the collision law (2.3), we have

$$\begin{aligned} e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)} &\geq e^{-\beta v^2} - e^{-\beta v^2(s^-,t;x,v)} \\ &= e^{-\beta v^2}\left[1 - e^{\beta(v^2 - v^2(s^-,t;x,v))}\right]. \end{aligned} \tag{3.51}$$

We give now lower and upper bounds for the quantity $[v^2 - v^2(s^-, t; x, v)]$, for particles belonging to Γ. We easily obtain, by the collision law (2.3), (3.48) and (3.47),

$$\begin{aligned} v^2(s^-, t; x, v) - v^2 &= (-v_{u_s} + 2W_{u_s}(s))^2 - v_{u_s}^2 \\ &\geq 2\,(\hat{u}_s \cdot \hat{x})^2\, W^2(s) \geq \frac{\lambda_0^2}{2} W^2(s)\,, \end{aligned} \tag{3.52}$$

moreover, using $v_{u_s} \geq -|v|$ and (3.49),

$$\begin{aligned} v^2(s^-, t; x, v) - v^2 &= (-v_{u_s} + 2W_{u_s}(s))^2 - v_{u_s}^2 \\ &= 4W_{u_s}(s)(W_{u_s}(s) - v_{u_s}) \\ &\leq 4W_{u_s}(s)\left(W_{u_s}(s) + C\frac{\gamma}{t}\right)\,. \end{aligned} \tag{3.53}$$

Recalling that, in proving Lemma 3.3, we consider t large (namely $t \geq t_0$), we can write, by (3.52) and (3.53),

$$-8V_0^2 \leq v^2 - v^2(s^-, t; x, v) < 0\,, \tag{3.54}$$

and in this interval it results, by (3.52) and (3.37),

$$1 - e^{\beta(v^2 - v^2(s^-,t;x,v))} \geq -C\beta\left(v^2 - v^2(s^-,t;x,v)\right) \geq CW^2(s) \geq C\gamma^2\,. \tag{3.55}$$

This bound will be used later on.

We consider now the restriction of $r_W^-(t)$ to the set Γ, setting

$$I(t) = \int_{\partial\Omega^-} d\sigma \int dv\,(v_u - W_u(t))^2\,\hat{u}\cdot\hat{x}\,(e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)})\,\chi(\Gamma) \tag{3.56}$$

where $dv = dv_u\,dv_{n_\perp}$.

By definition of $\hat{r}_W^-(t)$ and (2.3) it results that $\hat{r}_W^-(t)$ is non-negative. Moreover it is $W_u(t) < \langle W_u\rangle_{s_0,t}$ (we prove it just below), then it is obviously

$$\hat{r}_W^-(t) \geq CI(t)\,. \tag{3.57}$$

Let us show that, for $t > s_0$,

$$W(t) - \langle W\rangle_{s_0,t} < 0\,. \tag{3.58}$$

We have, by (3.1) and (3.2), for t large enough,

$$\begin{aligned} W(t) - \langle W\rangle_{s_0,t} &= W(t) - \frac{1}{t-s_0}\int_{s_0}^t ds\,W(s) \\ &\leq \gamma e^{-C_2 t} + \frac{1}{t-s_0}\int_{s_0}^t ds\,\left[-\gamma e^{-C_1 s} + \gamma^3\frac{A_1}{(1+s)^5}\right] \\ &\leq \gamma\left[e^{-C_2 t} - \frac{e^{-C_1 s_0} - e^{-C_1 t}}{C_1(t-s_0)} + \frac{1}{t}CA_1\gamma^2\right]\,. \end{aligned} \tag{3.59}$$

For γ small enough, so that t_0 is sufficiently large and $A_1\gamma^2$ is small, we obtain that the r.h.s. of (3.59) is negative for $t > t_0$. For $s_0 < t \leq t_0$ $W(t)$ is decreasing, so that (3.58) holds for any $t > s_0$.

Let us go back to the investigation of $I(t)$. It results

$$\int dv_{n_\perp}\,\chi\left(\left\{|v_{n_\perp}| \leq C\frac{\gamma}{t}\right\}\right) e^{-\beta v_{n_\perp}^2} \geq C\frac{\gamma^2}{t^2}\,, \tag{3.60}$$

so that we have, by (3.51) and (3.55):

$$\begin{aligned} I(t) &\geq C\gamma^2\int_{\partial\Omega^-} d\sigma \int dv\,(v_u - W_u(t))^2\,\hat{u}\cdot\hat{x}\,e^{-\beta v^2}\,\chi(\Gamma) \\ &\geq \frac{C\gamma^4}{t^2}\int_{\partial\Omega^-} d\sigma \int_{\langle W_u\rangle_{s_0,t}}^{\langle W_u\rangle_t} dv_u\,(v_u - W_u(t))^2\,e^{-\beta v_u^2}\,\chi(\Gamma) \\ &\geq \frac{C\gamma^4}{t^2}\int_{\partial\Omega^-} d\sigma\,\left[(\langle W_u\rangle_t - W_u(t))^3 - (\langle W_u\rangle_{s_0,t} - W_u(t))^3\right]\,\chi(\Gamma) \end{aligned}$$

By (3.39), (3.40), and (3.41), taking the integral over the set Γ, we obtain:

$$I(t) \geq \frac{C\gamma^4}{t^2}\Big[(\langle W\rangle_t - \langle W\rangle_{s_0,t})\,(\langle W\rangle_t - W(t))^2\Big]\,. \tag{3.61}$$

We now estimate both differences appearing in (3.61), showing that they are both $O(\frac{1}{t})$. By (3.39) and (3.41) we obtain that

$$\langle W\rangle_t - \langle W\rangle_{s_0,t} \geq C\frac{\gamma}{t}\,. \tag{3.62}$$

For the other term in (3.61), proceeding as in (3.59) we have:

$$\langle W\rangle_t - W(t) \geq -\gamma\Big[e^{-C_2 t} - \frac{1-e^{-C_1 t}}{C_1 t} + \frac{1}{t}CA_1\gamma^2\Big]\,, \tag{3.63}$$

therefore, for γ sufficiently small (so that $A_1\gamma^2$ is small enough) and t large,

$$\langle W\rangle_t - W(t) \geq C\frac{\gamma}{t}\,. \tag{3.64}$$

Inserting estimates (3.62) and (3.64) in (3.61), by (3.57) we conclude that, for γ sufficiently small, t_0 consequently large, and $t \geq t_0$,

$$\hat{r}^-_W(t) \geq C\frac{\gamma^7}{t^5}\,. \tag{3.65}$$

Recalling (3.36), for $t \geq t_0$ and γ sufficiently small, by (3.65) and (3.34),

$$r^-_W(t) \geq \hat{r}^-_W(t) - |\tilde{r}^-_W(t)| \geq C\frac{\gamma^7}{t^5} - C\frac{\gamma^9 A_1^3}{(1+t)^{15}}\,, \tag{3.66}$$

so that, for $\gamma^2 A_1^3$ small enough,

$$r^-_W(t) \geq C\frac{\gamma^7}{t^5}\,. \tag{3.67}$$

□

We remark that, from (3.13) and (3.32) it follows immediatly, for γ small and any t,

$$r^+_W(t) + r^-_W(t) \geq 0\,. \tag{3.68}$$

We remark also that, due to the convex shape of the obstacle, more complex compared with the simplified shape of a disk considered in Ref. [8], we obtain the bound (2.19) with γ^7, instead of γ^5.

Now we prove that the function $V_W(t)$ satisfying equation (3.3) enjoys, for γ suitably small, the same properties as the function W, with the same constants A_1, A_2. After this the proof of Theorem 2.1 will follow easily.

From now on we proceed as in Ref. [8], and we repeat here the argument for the sake of completeness.

PROPOSITION 3.2. *Suppose γ sufficiently small. Then*:

(i) *$t \to V_W(t)$ is an a.e. differentiable function with bounded derivative, decreasing in any time interval in which $V_W(t) > 0$.*

(ii) *For any $t \geq 0$:*

$$V_W(t) > \gamma\, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^5}\,. \tag{3.69}$$

(iii) *For any $t > 0$:*

$$V_W(t) < \gamma\, e^{-C_2 t} - \gamma^7 \frac{A_2}{t^5} \chi(\{t \geq \bar{t}\})\,. \tag{3.70}$$

PROOF.

(i) From equation (3.3) and the Duhamel formula we have:

$$V_W(t) = \gamma e^{-\int_0^t K(\tau)d\tau} - \int_0^t ds\, e^{-\int_s^t K(\tau)d\tau} (r_W^+(s) + r_W^-(s))\,, \tag{3.71}$$

and since $r_W^+(t)$ and $r_W^-(t)$ are bounded, by (3.71) and (3.3) V_W is a.e. differentiable with bounded derivative. The fact that $V_W(t)$ is decreasing in any time interval in which $V_W(t) > 0$ is obvious by equation (3.3) and (3.68).

(ii) By (3.71), (3.13), and (3.26) it follows:

$$V_W(t) \geq \gamma e^{-C_1 t} - C(\gamma + A_1\gamma^3)^3 \int_0^t ds\, e^{-C_2(t-s)} \frac{1}{(1+s)^5}\,. \tag{3.72}$$

Let us evaluate the integral:

$$\begin{aligned}\int_0^t ds\, \frac{e^{C_2 s}}{(1+s)^5} &= \int_0^{\frac{t}{2}} (\cdot)\, ds + \int_{\frac{t}{2}}^t (\cdot)\, ds \\ &\leq \frac{e^{C_2 \frac{t}{2}} - 1}{C_2} + \frac{2^5}{(2+t)^5} \frac{e^{C_2 t} - e^{C_2 \frac{t}{2}}}{C_2}\,.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^t ds\,\frac{e^{-C_2(t-s)}}{(1+s)^5} &\le \frac{e^{-C_2\frac{t}{2}}-e^{-C_2t}}{C_2}+\frac{2^5}{(2+t)^5}\frac{1-e^{-C_2\frac{t}{2}}}{C_2}\\ &\le \frac{1}{C_2}\left(e^{-C_2\frac{t}{2}}+\frac{2^5}{(2+t)^5}\right)\le\frac{C}{(1+t)^5}\,.\end{aligned}\tag{3.73}$$

To conclude, there exists a constants $\bar{C}$ such that:

$$V_W(t)\ge\gamma e^{-C_1t}-\bar{C}(\gamma+A_1\gamma^3)^3\frac{1}{(1+t)^5}\,,\tag{3.74}$$

hence, to achieve the thesis, it is sufficient that

$$\bar{C}(\gamma+A_1\gamma^3)^3<A_1\gamma^3\,,\tag{3.75}$$

which is satisfied, for instance, by choosing $A_1=2\bar{C}$ (this fixes A_1) and γ consequently small (also to satisfy the previous constraints on A_1).

(iii) First, by (3.68) and (3.71), we have that, for any $t\ge 0$,

$$V_W(t)\le\gamma e^{-C_2t}\,.\tag{3.76}$$

By (3.71), (3.13), and (3.32), for γ suitably small and $t\ge\bar{t}>2t_0$ (where $\bar{t}$ is defined in (3.16) and t_0 in (3.14)), it follows:

$$\begin{aligned}V_W(t)&\le\gamma e^{-C_2t}+\int_0^t ds\,e^{-\int_s^t K(\tau)d\tau}\left(C\frac{\gamma^9}{(1+s)^5}-C\frac{\gamma^7}{s^5}\right)\chi(\{s>t_0\})\\ &\le\gamma e^{-C_2t}-C\gamma^7\int_0^t ds\,e^{-\int_s^t K(\tau)d\tau}\,\frac{1}{s^5}\,\chi(\{s>t_0\})\,.\end{aligned}\tag{3.77}$$

We have that

$$\begin{aligned}\int_0^t ds\,e^{-\int_s^t K(\tau)d\tau}\,\frac{1}{s^5}\,\chi(\{s>t_0\}) &\ge\int_{t_0}^t ds\,e^{-C_1(t-s)}\,\frac{1}{s^5}\\ &\ge\frac{1-e^{-C_1(t-t_0)}}{C_1t^5}\\ &\ge\frac{1-e^{-C_1t_0}}{C_1t^5}\ge\frac{1}{2C_1t^5}\,,\end{aligned}\tag{3.78}$$

since $t\ge\bar{t}>2t_0$.

Then, by (3.77) and (3.78),

$$V_W(t) \leq \gamma e^{-C_2 t} - C\frac{\gamma^7}{t^5}\,. \tag{3.79}$$

Last inequality enables us to choose A_2, in such a way that (3.70) is satisfied. This can be done in a consistent manner, since the constant C appearing in (3.79) does not depend of A_2. Actually it depends of A_1, nevertheless A_2 can be chosen independently of A_1 for γ sufficiently small. □

Using Proposition 3.2 we easily prove Theorem 2.1. We can construct a sequence $\{V_n\}_{n=1}^{\infty}$ defined by

$$V_n = V_{V_{n-1}}, \quad n \geq 2 \tag{3.80}$$

setting $V_1 = W$, being W any function with the properties established at the beginning of this section. By Proposition 3.2 such properties hold for the whole sequence (for suitable values of A_1, A_2, $\bar{t}$ independent of n). By compactness (the sequence is equibounded and equicontinuous), we can extract a subsequence $V_{n'}$ converging to a limit point $V = V(t)$. Moreover, for any $n \geq 1$ we can solve the free Vlasov equation with reflecting boundary conditions on the body moving according to the velocity $V_{n'}(t)$, by means of the characteristics which are a.e. defined. The convergence of $V_{n'}$ implies the convergence of almost all characteristics to a family of characteristics satisfying the reflecting boundary conditions on the body moving with velocity $V(t)$ (for a full explanation of this fact see [7]). This yields a solution to the Vlasov equation (2.1) producing the friction term (2.7). Therefore we have obtained a solution to the problem (2.1)-(2.9).

Moreover, any solution to this problem satisfies bounds (2.18) and (2.19). Consider in fact any solution (V, f) of the problem. By continuity of V there exists a time interval in which inequalities (2.18)-(2.19) hold strictly. Let T be the first time for which our strict inequalities are violated. The same arguments used in Proposition 3.2 (replacing W by V) show that (2.18)-(2.19) hold strictly in the interval $(0, T]$, since in this interval V enjoys the same properties as W. Then T must be infinite. This concludes the proof of Theorem 2.1.

We remark that the proof of the existence of a solution given above is not complete, and it has to be improved by the use of the Schauder fixed point theorem.

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