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A lower bound for the b-adic diaphony

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Abstract: *The b-adic diaphony is a quantitative measure for the irregularity of distribution of a point set in the s-dimensional unit cube. In this note we show that the b-adic diaphony (for prime b) of a point set consisting of N points in the s-dimensional unit cube is always at least of order* $(\log N)^{(s-1)/2}/N$. *This lower bound is best possible.*

1 – Introduction

As the (classical) diaphony (see [25] or [8, Definition 1.29] or [16, Exercise 5.27, p. 162]) the b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the s-dimensional unit cube. This notion was introduced by Hellekalek and Leeb [15] for $b = 2$ and later generalized by Grozdanov and Stoilova [11] for general integers $b \geq 2$. The main difference to the classical diaphony is that the trigonometric functions are replaced by b-adic Walsh functions. Before we give the exact definition of the b-adic diaphony we recall the definition of Walsh functions.

Let $b \geq 2$ be an integer. For a non-negative integer k with base b representation $k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_1 b + \kappa_0$, with $\kappa_i \in \{0, \ldots , b-1\}$ and $\kappa_{a-1} \neq 0$, we define the Walsh function ${}_b\,\mathrm{wal}_k : [0, 1) \to \mathbb{C}$ by

$$ {}_b\,\mathrm{wal}_k(x) := \mathrm{e}^{2\pi \mathrm{i}(x_1\kappa_0+\cdots+x_a\kappa_{a-1})/b}\,, $$

for $x \in [0, 1)$ with base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the x_i must be different from $b - 1$).

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For dimension $s \geq 2$, $x_1, \dots, x_s \in [0,1)$ and $k_1, \dots, k_s \in \mathbb{N}_0$ we define ${}_b\operatorname{wal}_{k_1,\dots,k_s} : [0,1)^s \to \mathbb{C}$ by

$$ {}_b\operatorname{wal}_{k_1,\dots,k_s}(x_1,\dots,x_s) := \prod_{j=1}^{s} {}_b\operatorname{wal}_{k_j}(x_j) \,. $$

For vectors $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \dots, x_s) \in [0,1)^s$ we write

$$ {}_b\operatorname{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := {}_b\operatorname{wal}_{k_1,\dots,k_s}(x_1,\dots,x_s) \,. $$

If it is clear which base we mean we simply write $\operatorname{wal}_{\boldsymbol{k}}(\boldsymbol{x})$. It is clear from the definitions that Walsh functions are piecewise constant. It can be shown that for any integer $s \geq 1$ the system $\{\operatorname{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L_2([0,1)^s)$, see for example [1], [17] or [20, Satz 1]. For more information on Walsh functions we refer to [1], [20], [24].

Now we give the definition of the b-adic diaphony (see [11] or [15]).

Definition 1. Let $b \geq 2$ be an integer. The b-adic diaphony of a point set $P_{N,s} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\} \subset [0,1)^s$ is defined as

$$ F_{b,N}(P_{N,s}) := \left(\frac{1}{(1+b)^s - 1} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ \boldsymbol{k} \neq \boldsymbol{0}}} r_b(\boldsymbol{k}) \left| \frac{1}{N} \sum_{h=0}^{N-1} {}_b\operatorname{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right|^2 \right)^{1/2} , $$

where for $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$, $r_b(\boldsymbol{k}) := \prod_{j=1}^{s} r_b(k_j)$ and for $k \in \mathbb{Z}$,

$$ r_b(k) := \begin{cases} 1 & \text{if } k = 0\,, \\ b^{-2a} & \text{if } b^a \leq k < b^{a+1} \text{ where } a \in \mathbb{N}_0\,. \end{cases} \tag{1} $$

Note that the b-adic diaphony is scaled such that $0 \leq F_{b,N}(P_{N,s}) \leq 1$ for all $N \in \mathbb{N}$, in particular we have $F_{b,1}(P_{1,s}) = 1$. If $b = 2$ we also speak of dyadic diaphony.

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence ω in the s-dimensional unit cube is uniformly distributed modulo one if and only if $\lim_{N\to\infty} F_{b,N}(\omega_N) = 0$, where ω_N is the point set consisting of the first N points of ω. This was shown in [15] for the case $b = 2$ and in [11] for the general case. Further it is shown in [5] that the b-adic diaphony is–up to a factor depending on b and s–the worst-case error for quasi-Monte Carlo integration of functions from a certain Hilbert space of functions.

More general notions of diaphony can be found in [10], [13], [14].

Stoilova [22] proved that the b-adic diaphony of a (t, m, s)-net in base b is bounded by

$$F_{b,N}(P) \leq c(b,s) b^t \frac{(m-t)^{\frac{s-1}{2}}}{b^m},$$

where $c(b, s) > 0$ only depends on b and s. For the definition of (t, m, s)-nets in base b we refer to Niederreiter [18], [19]. These are point sets consisting of $N = b^m$ points in the s-dimensional unit cube with outstanding distribution properties if the parameter $t \in \{0, \ldots, m\}$ is small. However, the optimal value $t = 0$ is not possible for all parameters $s \geq 1$ and $b \geq 2$. Niederreiter [18] proved that if a $(0, m, s)$-net in base b exists, then we have $s - 1 \leq b$. Faure [9] provided a construction of $(0, m, s)$-nets in prime base $p \geq s-1$ and Niederreiter [18] extended Faure's construction to prime power bases $p^r \geq s - 1$. Hence if $b \geq s - 1$ is a prime power we obtain for any $m \in \mathbb{N}$ the existence of $N = b^m$ points in $[0, 1)^s$ whose b-adic diaphony is bounded by

$$F_{b,N}(P) \leq c'(b,s) \frac{(\log N)^{\frac{s-1}{2}}}{N},$$

with $c'(b, s) > 0$. See also [6] where a similar bound on the dyadic diaphony of digital (t, m, s)-nets in base 2 (a subclass of (t, m, s)-nets) is shown.

The question for a general lower bound for the b-adic diaphony was pointed out in [22], see also [12]. In the following section we show that for prime b, the b-adic diaphony of an N-element point set in $[0, 1)^s$ is always at least of order $\frac{(\log N)^{\frac{s-1}{2}}}{N}$, which shows that the above given upper bounds are best possible.

2 – A general lower bound for the b-adic diaphony

In the following we prove a lower bound on the b-adic diaphony for prime b. This is done using Roth's lower bound on the $\mathcal{L}_2$ discrepancy, which is another measure for the distribution properties of a point set.

THEOREM 1. *Let b be a prime. For any dimension $s \geq 1$ there exists a constant $\overline{c}(s, b) > 0$, depending only on the dimension s and b, such that the b-adic diaphony of any point set $P_{N,s}$ consisting of N points in $[0, 1)^s$ satisfies*

$$F_{b,N}(P_{N,s}) \geq \overline{c}(s,b) \frac{(\log N)^{\frac{s-1}{2}}}{N}.$$

In the proof of our theorem below we use the generalized notion of weighted $\mathcal{L}_2$ discrepancy, which was introduced in [23]. In the following let D denote the index set $D = \{1, 2, \dots, s\}$ and let $\gamma = (\gamma_1, \gamma_2, \dots)$ be a sequence of non-negative real numbers. For $\mathfrak{u} \subseteq D$ let $|\mathfrak{u}|$ be the cardinality of $\mathfrak{u}$ and for a vector $\boldsymbol{x} \in [0,1)^s$ let $\boldsymbol{x}_\mathfrak{u}$ denote the vector from $[0,1)^{|\mathfrak{u}|}$ containing all components of $\boldsymbol{x}$ whose indices are in $\mathfrak{u}$. Further let $\gamma_\mathfrak{u} = \prod_{j\in\mathfrak{u}} \gamma_j$, $\mathrm{d}\boldsymbol{x}_\mathfrak{u} = \prod_{j\in\mathfrak{u}} \mathrm{d}x_j$, and let $(\boldsymbol{x}_\mathfrak{u}, 1)$ be the vector $\boldsymbol{x}$ from $[0,1)^s$ with all components whose indices are not in $\mathfrak{u}$ replaced by 1. Then the weighted $\mathcal{L}_2$ discrepancy of a point set $P_{N,s} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ is defined as

$$\mathcal{L}_{2,\gamma}(P_{N,s}) = \left(\sum_{\substack{\mathfrak{u}\subseteq D\\ \mathfrak{u}\neq\emptyset}} \gamma_\mathfrak{u} \int_{[0,1]^{|\mathfrak{u}|}} \Delta((\boldsymbol{x}_\mathfrak{u},1))^2 \,\mathrm{d}\boldsymbol{x}_\mathfrak{u} \right)^{1/2},$$

where

$$\Delta(t_1, \dots, t_s) = \frac{A_N([0,t_1) \times \dots \times [0,t_s))}{N} - t_1 \cdots t_s\,,$$

where $0 \le t_j \le 1$ and $A_N([0,t_1) \times \dots \times [0,t_s))$ denotes the number of indices n with $\boldsymbol{x}_n \in [0,t_1) \times \dots \times [0,t_s)$. We can see from the definition of the weighted $\mathcal{L}_2$ discrepancy that the weights $\gamma_\mathfrak{u} = \prod_{j\in\mathfrak{u}} \gamma_j$ modify the importance of different projections (see [7], [23] for more information on weights).

In [3] the authors considered point sets which are randomized in the following sense: for $b \ge 2$ let $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ and $\sigma = \frac{\sigma_1}{b} + \frac{\sigma_2}{b^2} + \cdots$ be the base b representation of x and σ. Then the digitally shifted point $y = x \oplus_b \sigma$ is given by $y = \frac{y_1}{b} + \frac{y_2}{b^2} + \cdots$, where $y_i = x_i + \sigma_i \in \mathbb{Z}_b$. For vectors $\boldsymbol{x}$ and $\boldsymbol{\sigma}$ we define the digitally shifted point $\boldsymbol{x} \oplus_b \boldsymbol{\sigma}$ component wise. Obviously, the shift depends on the base b. Now for $P_{N,s} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\} \subseteq [0,1)^s$ and $\boldsymbol{\sigma} \in [0,1)^s$ we define the point set $P_{N,s,\boldsymbol{\sigma}} = \{\boldsymbol{x}_0 \oplus_b \boldsymbol{\sigma}, \dots, \boldsymbol{x}_{N-1} \oplus_b \boldsymbol{\sigma}\}$.

Proof. In [3] it was shown that if one chooses $\boldsymbol{\sigma}$ uniformly from $[0,1)^s$, then the expected value of the weighted $\mathcal{L}_2$ discrepancy of a point set $P_{N,s,\boldsymbol{\sigma}}$ is given by

$$\mathbb{E}(\mathcal{L}^2_{2,\boldsymbol{\gamma}}(P_{N,s,\boldsymbol{\sigma}})) = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ \boldsymbol{k}\neq\boldsymbol{0}}} \rho_b(\boldsymbol{\gamma},\boldsymbol{k}) \left| \frac{1}{N} \sum_{h=0}^{N-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right|^2,$$

where $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_s) \in \mathbb{N}_0^s$, $\rho_b(\boldsymbol{\gamma},\boldsymbol{k}) = \prod_{j=1}^s \rho_b(\gamma_j, k_j)$, and

$$\rho_b(\gamma,k) = \begin{cases} 1 + \dfrac{\gamma}{3} & \text{if } k = 0\,, \\[2ex] \dfrac{\gamma}{2b^{2(a+1)}} \left(\dfrac{1}{\sin^2\left(\frac{\kappa_a \pi}{b}\right)} - \dfrac{1}{3} \right) & \text{if } b^a \le k < b^{a+1} \text{ and} \\[2ex] & \kappa_a = \left\lfloor \frac{k}{b^a} \right\rfloor, \text{ where } a \in \mathbb{N}_0\,. \end{cases}$$

If we take $\gamma_j = 3b^2$, for $j = 1, \dots, s$ we have, $\rho_b(\gamma_j, 0) = (1+b^2) = (1+b^2)r_b(0)$ and for $k \geq 1$ we have $\rho_b(\gamma_j, k) = \frac{3}{2} r_b(k) \left(\frac{1}{\sin^2(\frac{\kappa_a \pi}{b})} - \frac{1}{3} \right)$. Let us denote $d_b := \max_{1 \leq \kappa \leq b-1} \left(\frac{1}{\sin^2(\frac{\kappa \pi}{b})} - \frac{1}{3} \right)$ and $c_b := \max\{1+b^2, \frac{3}{2} d_b\}$.

For the above choice of the weights we have

$$\rho_b((3b^2), \boldsymbol{k}) = \prod_{i=1}^{s} \rho_b(3b^2, k_i) \leq c_b^s \prod_{i=1}^{s} r_b(k_i) = c_b^s r_b(\boldsymbol{k}) \,.$$

Hence from the definition of b-adic diaphony we obtain the inequality

$$\mathbb{E}(\mathcal{L}^2_{2,(3b^2)}(P_{N,s,\boldsymbol{\sigma}})) \leq c_b^s((1+b)^s - 1) F^2_{b,N}(P_{N,s}) \,. \tag{2}$$

Roth [21] proved that for any dimension $s \geq 1$ there exists a constant $\widehat{c}(s) > 0$ such that for any point set consisting of N points in the s-dimensional unit cube $[0,1)^s$ the classical $\mathcal{L}_2$ discrepancy of a point set satisfies

$$\mathcal{L}_2^2(P_{N,s}) \geq \widehat{c}(s) \frac{(\log N)^{s-1}}{N^2} \,.$$

Here we just note that the weights only change the constant $\widehat{c}(s)$, but do not change the convergence rate of the bound (see [2], [4], [23] for more information). Hence, for any point set $P_{N,s}$ consisting of N points in the s-dimensional unit cube there is a constant $\widetilde{c}(s,b)$, depending only on the dimension s, such that

$$\mathcal{L}^2_{2,(3b^2)}(P_{N,s}) \geq \widetilde{c}(s,b) \frac{(\log N)^{s-1}}{N^2} \,.$$

From (2) it follows that there is a constant $\overline{c}(s,b)$, depending only on the dimension and the prime number b, such that

$$F^2_{b,N}(P_{N,s}) \geq \overline{c}^2(s,b) \frac{(\log N)^{s-1}}{N^2} \,,$$

which completes the proof. □

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