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Eigenvalues of the Laplacian under the Ricci flow

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Abstract: *We derive, under a technical assumption, the first variation formula for the eigenvalues of the Laplacian on a closed manifold evolving by the Ricci flow and give some applications.*

1 – Introduction

Given a closed manifold M^n endowed with a Riemannian metric g_0, the Ricci flow determines a one parameter family of metrics $g(t)$ via the geometric evolution equation

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}\,(g) \tag{1}$$

with initial condition

$$g(0) = g_0\,.$$

Short time existence and uniqueness for solution to the Ricci flow was first shown in Hamilton's seminal paper [6], using the Nash-Moser theorem, and shortly after by DeTurck in [10] who substantially simplified the proof. Strictly related to the Ricci flow is the evolution equation

$$\frac{\partial g}{\partial t} = \frac{2}{n} r g - 2\mathrm{Ric}\,(g) \tag{2}$$

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where $r = \frac{\int_M R d\mu}{\int_M d\mu}$, which is called normalized Ricci flow since has the remarkable property to preserve the volume of the initial Riemannian manifold. As shown in [6], the evolution equations (1) and (2) differ only by a change of scale in space and a change of parametrization in time. Since its introduction, the Ricci flow has been a very effective tool for studying the topology of manifolds. In the most of first applications [6], [7], [8], [9], the normalized flow is shown to converge to a canonical metric, e.g. in [6] Hamilton proves that the normalized flow on any closed 3-manifold of positive Ricci curvature exists for all time and converges to a positive constant sectional curvature metric. In this sense, the Ricci flow may be regarded as a natural homotopy between a given metric of positive Ricci curvature and a canonical metric of constant sectional curvature in the same volume class. Analogously, whenever the Ricci flow converges, it can be considered as a natural homotopy between the initial metric and the limit metric that is necessarily Einstein

$$\mathrm{Ric}\,(g_\infty) = \frac{1}{n} \frac{\displaystyle\int_M R_\infty d\mu_\infty}{\displaystyle\int_M d\mu_\infty} g_\infty\,.$$

It is then natural to use the Ricci flow to study properties of geometrical meaningful objects, such as the eigenvalues of the Laplacian. In this paper we study, under a technical assumption, the behavior of the spectrum of the Laplacian when we deform a given initial metric by the Ricci flow.

Throughout the paper, let M^n be a closed manifold of dimension n, and $C^\infty(S_2^+TM)$ the space of smooth Riemannian metrics on M^n. For $g \in C^\infty(S_2^+TM)$, let $\Delta_g = Tr_g \nabla^2$ be the Laplacian operator on $C^\infty(M)$ and

$$\mathrm{Spec}\,(g) = \{0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \ldots \leq \lambda_k(g) \leq \ldots\}$$

the spectrum of Δ_g.

Given $g_0 \in C^\infty(S_2^+TM)$, let $g(t)$ be the smooth 1-parameter family of metrics who solves the Ricci flow with initial metric g_0. We can then regard each eigenvalue $\lambda(g(t))$ as a function of one real parameter, and try to study its evolution at least locally in time. This problem is clearly related to the general problem concerning the behavior of the spectrum under a suitable deformation $g(t)$ of g_0 in $C^\infty(S_2^+TM)$. In this direction a classical result is the following

Theorem 1.1. *For $g \in C^\infty(S_2^+TM)$ and $h \in C^\infty(S_2TM)$, let $g(t) = g + th$, $|t| < \epsilon$ for sufficiently small $\epsilon > 0$. Let λ be an eigenvalue of Δ_g with multiplicity l. Then there exist $f_i(t) \in C^\infty(M)$, $i = 1, \ldots, l$, such that*

(1) $\lambda_i(t)$ *and* $f_i(t)$ *depend real analytically on* t, $|t| < \epsilon$, *for each* $i = 1, \ldots, l$,
(2) $\Delta_{g(t)} f_i(t) + \lambda_i(t) f_i(t) = 0$, *for each* $i = 1, \ldots, l$ *and* t,
(3) $\lambda_i(0) = \lambda$, $i = 1, \ldots, l$, *and*
(4) $\{f_i(t)\}_{i=1}^{l}$ *is orthonormal with respect to* $\langle , \rangle_{g(t)}$ *for each* t.

The above theorem is due to Berger [13], and it is one of the first attempt in studying the behavior of the spectrum under a perturbation of the metric. On the other hand Bando and Urakawa have extended the above theorem to a family of metrics which depends real analytically on time, see [14]. Unfortunately, the theorems of Berger and Bando-Urakawa does not apply directly to a smooth 1-parameter family of metrics generated by solving an IVP of the Ricci flow type. Since the Ricci flow equation is a parabolic equation for the metric see [6], its solutions are expected to not depend real analytically on the time variable. More precisely, the Ricci flow equation implies that the scalar curvature function of the evolving metric satisfies an heat-type scalar equation. Hence if the solution flow were analytic in time, we could analytically continue backwards in time, which implies that the associated scalar curvature equation admits solution for negative time.

In what follows we then assume that, under a Ricci flow deformation $g(t)$ of a given initial metric, a result of the Berger-Bando-Urakawa type holds, that is we assume the existence and C^1-differentibility of the elements $\lambda_i(t)$ and $f_i(t)$.

We remark that the problem of the spectrum variation under a deformation of the metric given by a parabolic equation seems not to be presented explicitely before. However in the celebrated paper [16], a similar geometric problem is cosidered from a different point of view. In this paper Perelman introduces a riemannian functional whose gradient flow is the Ricci flow modulo diffeomorphisms. It turns out that this functional is the lowest eigenvalue λ_1 of the operator $-4\Delta + R$, therefore proving that the quantity $\lambda_1(g(t))$ is nondecreasing along the flow. For the details and applications of this important observation we refer to the original paper.

2 – Variation formulas

Let $(M^n, g(t))$ be a solution of the Ricci flow on the maximal time interval $[0, T)$ and consider the associated Rayleigh-Ritz quotient

$$\lambda(t) = \frac{\displaystyle\int_M |\nabla f_t|^2 d\mu_t}{\displaystyle\int_M f_t^2 d\mu_t},$$

which defines the evolution of an eigenvalue of the Laplacian under the flow. We can then consider the change rate of any eigenvalue taking the time derivative of the Rayleigh-Ritz quotient which defines it. For simplicity we consider normalized eigenfunctions i.e.

$$\int_M f_t d\mu = 0, \quad \int_M f_t^2 d\mu = 1\,, \tag{3}$$

then

$$\begin{aligned}
\frac{d\lambda}{dt} &= \int_M \frac{d}{dt}|\nabla f|^2 d\mu + \int_M |\nabla f|^2 \frac{d}{dt} d\mu \\
&= \int_M \frac{d}{dt}(g^{ij})\nabla_i f \nabla_j f d\mu + 2\int_M \left\langle \nabla f^{'}, \nabla f \right\rangle d\mu - \int_M R|\nabla f|^2 d\mu \\
&= 2\int_M g^{ik}g^{jl}R_{kl}\nabla_i f \nabla_j f d\mu + 2\int_M \left\langle \nabla f^{'}, \nabla f \right\rangle d\mu - \int_M R|\nabla f|^2 d\mu \\
&= 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu + 2\int_M \left\langle \nabla f^{'}, \nabla f \right\rangle d\mu - \int_M R|\nabla f|^2 d\mu\,,
\end{aligned}$$

for the time derivative of the volume element see [6]. Now, using (3) we get the following two integrability conditions

$$\int_M f^{'} d\mu = \int_M f R d\mu \tag{4}$$

$$\int_M f^{'} f d\mu = \frac{1}{2}\int_M f^2 R d\mu\,. \tag{5}$$

Finally, by (5) we have

$$\begin{aligned}
2\int_M \left\langle \nabla f^{'}, \nabla f \right\rangle d\mu &= -\,2\int_M f^{'}\Delta f d\mu \\
&= 2\lambda \int_M f^{'} f d\mu = \lambda \int_M f^2 R d\mu\,.
\end{aligned}$$

We have thus proved the following proposition

PROPOSITION 2.1. *Let $(M^n, g(t))$ be a solution of the unnormalized Ricci on the smooth manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Ricci flow, then*

$$\frac{d\lambda}{dt} = \lambda \int_M f^2 R d\mu - \int_M R|\nabla f|^2 d\mu + 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu\,.$$

where f is the associated normalized evolving eigenfunction.

As shown below, an analogous equation holds for the normalized Ricci flow

PROPOSITION 2.2. *Let $(M^n, g(t))$ be a solution of the normalized Ricci on the smooth manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the normalized Ricci flow, then*

$$\frac{d\lambda}{dt} = -\frac{2}{n}r\lambda + \lambda\int_M f^2 R d\mu - \int_M R|\nabla f|^2 d\mu + 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu\,.$$

where f is the associated normalized evolving eigenfunction.

PROOF. In the normalized case, the integrability conditions read as follows

$$\int_M f' d\mu = \int_M f R d\mu\,, \tag{6}$$

$$2\int_M f f' d\mu = \int_M f^2 R d\mu - r\,, \tag{7}$$

since

$$\frac{d}{dt}d\mu = \frac{1}{2}Tr_g\left(\frac{2}{n}rg - 2\mathrm{Ric}\right)d\mu = (r-R)d\mu\,.$$

We can then write

$$\begin{aligned}
\frac{d\lambda}{dt} =& \int_M \frac{d}{dt}|\nabla f|^2 d\mu + \int_M |\nabla f|^2 \frac{d}{dt}d\mu \\
=& \int_M \frac{d}{dt}(g^{ij})\nabla_i f \nabla_j f d\mu + 2\int_M \left\langle \nabla f', \nabla f\right\rangle d\mu \\
&+ \int_M (r-R)|\nabla f|^2 d\mu \\
=& -\int_M g^{ik}g^{jl}\left(\frac{2r}{n}g_{kl} - 2R_{kl}\right)\nabla_i f \nabla_j f d\mu + 2\int_M \left\langle \nabla f', \nabla f\right\rangle d\mu \\
&+ \int_M (r-R)|\nabla f|^2 d\mu \\
=& 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu - \frac{2r}{n}\int_M |\nabla f|^2 d\mu + 2\lambda\int_M f f' d\mu \\
&+ \int_M (r-R)|\nabla f|^2 d\mu \\
=& 2\int_M Ric(\nabla f, \nabla f) d\mu - \frac{2r}{n}\lambda + \lambda\left(\int_M f^2 R d\mu - r\right) + r\lambda \\
&- \int_M R\,|\nabla f|^2\, d\mu \\
=& -\frac{2}{n}r\lambda + \lambda\int_M f^2 R d\mu - \int_M R|\nabla f|^2 d\mu + 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu\,. \qquad \square
\end{aligned}$$

It is now interesting to write down Proposition 2.1 and Proposition 2.2 in some remarkable particular case.

COROLLARY 2.3. *Let $(M^2, g(t))$ be a solution of the unnormalized Ricci flow on a closed surface, then*

$$\frac{d\lambda}{dt} = \lambda \int_M f^2 R d\mu \,.$$

PROOF. In dimension $n = 2$ we have

$$\mathrm{Ric} = \frac{1}{2} R g \,,$$

then

$$\begin{aligned} \frac{d\lambda}{dt} &= \lambda \int_M f^2 R d\mu - \int_M R|\nabla f|^2 d\mu + 2 \int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \\ &= \lambda \int_M f^2 R d\mu - \int_M R|\nabla f|^2 d\mu + \int_M R|\nabla f|^2 d\mu \\ &= \lambda \int_M f^2 R d\mu \,. \end{aligned}$$

□

COROLLARY 2.4. *Let $(M^2, g(t))$ be a solution of the normalized Ricci flow on a closed surface with normalized initial metric, then*

$$\frac{d\lambda}{dt} = \lambda \int_M f^2 R d\mu - r\lambda = \lambda \int_M R(f^2 - 1) d\mu \,.$$

PROOF. Obvious. □

REMARK 2.5. Because of the Gauss-Bonnet theorem the above variation formula can be written as

$$\frac{d\lambda}{dt} = \lambda \int_M f^2 R d\mu - \lambda 4\pi \chi(M) \,,$$

where $\chi(M)$ is the Euler characteristic of the surface.

Let us now consider the behavior of the spectrum when we evolve an initial metric that is homogeneous. As shown in [6], the Ricci flow preserves the isometries of the initial Riemannian manifold. We conclude that the evolving metric remains homogeneous during the flow. This important observation implies the following

Corollary 2.6. *Let* $(M^n, g(t))$ *be a solution of the unnormalized Ricci on the smooth homogeneous manifold* (M^n, g_0)*. If* $\lambda(t)$ *denotes the evolution of an eigenvalue under the Ricci flow, then*

$$\frac{d\lambda}{dt} = 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \,.$$

Proof. Since the evolving metric remains homogeneous, the thesis follows from Proposition 2.1 and the fact that an homogeneous manifold has constant scalar curvature. ∎

Corollary 2.7. *Let* $(M^n, g(t))$ *be a solution of the normalized Ricci on the smooth homogeneous manifold* (M^n, g_0)*. If* $\lambda(t)$ *denotes the evolution of an eigenvalue under the Ricci flow, then*

$$\frac{d\lambda}{dt} = -\frac{2}{n}R\lambda + 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \,.$$

Proof. In the homogeneous case the average scalar curvature can be written as

$$r = \frac{\int_M R d\mu}{\int_M d\mu} = R \,.$$ ∎

3 – Applications

In this section we show how the variational formulas can be effectively applied to derive some interesting properties of the evolving spectrum. First we concentrate on 3-manifolds, next we discuss the Riemannian surface and in particular the 2-sphere.

Let (M^3, g_0) be a closed three manifold with positive Ricci curvature. It is well known that the Ricci flow on such manifold exists on a limited maximal time interval $[0, T)$, see [6]; we shall then show that the eigenvalues of the Laplacian diverges as $t \to T$. The result is independent of the technical assumption explained in the introduction.

Proposition 3.1. *Let* $(M^3, g(t))$ *be a solution of the Ricci flow on a closed* 3*-manifold whose Ricci curvature is positive initially, then*

$$\lim_{t \to T} \lambda(t) = \infty \,.$$

Proof. On a closed manifold M^n, for any smooth functions f holds the celebrated Reilly formula

$$\int_M |\nabla\nabla f|^2 \, d\mu + \int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu = \int_M (\Delta f)^2 d\mu \, .$$

Since

$$|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2 \, ,$$

we have the inequality

$$\frac{n-1}{n} \int_M (\Delta f)^2 d\mu \geq \int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \, . \tag{8}$$

For any solution of the Ricci flow on a closed three manifold with positive Ricci curvature there exists $\epsilon > 0$ such that the condition

$$\mathrm{Ric}\, \geq \epsilon R g$$

is preserved along the flow, see [6]. Therefore

$$\frac{2}{3}\lambda^2(t) \geq \int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \geq \epsilon \int_M R \, |\nabla f|^2 \, d\mu \geq \epsilon R_{\min}(t)\lambda(t) \, ,$$

and then

$$\lambda(t) \geq \frac{3}{2}\epsilon R_{\min}(t) \, .$$

The thesis follows since

$$\lim_{t \to T} R_{\min}(t) = \infty \, ,$$

see again [6]. ∎

Proposition 3.2. *Let $(M^3, g(t))$ be a solution to the Ricci flow on a closed manifold whose Ricci curvature is positive initially. Then there exists $\overline{t} \in [0, T)$ depending on g_0 such that for each $t \in [\overline{t}, T)$ the eigenvalues of the Laplacian are increasing.*

Proof. Let $0 < \epsilon \leq \frac{1}{3}$ be a constant such that

$$\mathrm{Ric} \geq \epsilon R g$$

is preserved under the flow. By Proposition 2.1 we can then write

$$\begin{aligned}\frac{d\lambda}{dt} &\geq \lambda \int_M f^2 R d\mu - \int_M R|\nabla f|^2 d\mu + 2\epsilon \int_M R\, |\nabla f|^2\, d\mu \\ &\geq \lambda \left\{R_{\min}(t) + (2\epsilon - 1) R_{\max}(t)\right\}\,.\end{aligned}$$

As proved in [6], for each $\eta > 0$, we can find $T_\eta \in [0, T)$ such that for $t \in [T_\eta, T)$

$$R \geq (1 - \eta) R_{\max}$$

hence the proposition follows easily. ∎

Proposition 3.3. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed homogeneous 3-manifold whose Ricci curvature is nonnegative initially, then the eigenvalues of the Laplacian are increasing.*

Proof. Since in dimension three the nonnegativity of the Ricci tensor is preserved under the Ricci flow, the thesis follows easily from Corollary 2.6. ∎

Proposition 3.4. *Let $(M^n, g(t))$ be a solution of the Ricci flow on a closed homogeneous n-manifold whose curvature operator is nonnegative initially, then the eigenvalues of the Laplacian are increasing.*

Proof. As shown in [7], in any dimension the nonnegativity of the curvature operator is preserved along the Ricci flow. Since the nonnegativity of the curvature operator implies the nonnegativity of the Ricci tensor the thesis follows from Corollary 2.6. ∎

Proposition 3.5. *Let $(M^n, g(t))$ be a solution of the Ricci flow on a closed homogeneous manifold, then*

$$\frac{d\lambda}{dt} \leq 2\frac{(n-1)}{n}\lambda^2\,.$$

Proof. By (8)

$$\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \leq \frac{n-1}{n}\int_M (\Delta f)^2\,,$$

we have thus

$$\begin{aligned}\frac{d\lambda}{dt} &= 2\int_M \mathrm{Ric}\,(\nabla f, \nabla f) d\mu \leq 2\frac{(n-1)}{n}\int_M (\Delta f)^2 \\ &= 2\frac{(n-1)}{n}\lambda^2\,.\end{aligned}$$

∎

We conclude studying the spectrum of Riemannian surface evolving under the flow. Our result is the following

PROPOSITION 3.6. *Let (M^2, g_0) be a closed surface with nonnegative scalar curvature, then the eigenvalues of the Laplacian are increasing under the Ricci flow.*

PROOF. Under the unnormalized Ricci flow on a surface, we have

$$\frac{\partial}{\partial t} R = \Delta R + R^2 \,,$$

see [4]. By the scalar maximum principle, the nonnegativity of the scalar curvature is preserved along the flow. We conclude using Corollary 2.3. ∎

Let us now consider the normalized flow on a surface and the associated variation of the eigenvalues of the Laplacian. Let $(\mathbb{S}^2, g)$ be a closed Riemannian manifold diffeomorphic to the two dimensional sphere. In [5], Chow proved that the normalized Ricci flow on $(\mathbb{S}^2, g)$ exists for all time and converges to a smooth metric g_S of constant curvature, that is the standard metric. On the other hand, a classical result of Hersch [12] states that g_S maximizes λ_1 among all the Riemannian metrics of the same volume. In particular, if g_S is the standard metric on $\mathbb{S}^2$ normalized to volume one, we have

$$\lambda_1(g_S) = 8\pi \,,$$

see [3]. We then conjecture that the smallest positive eigenvalue of the Laplacian $\lambda_1(t)$ is monotonically increasing along the Ricci flow $g(t)$ on $\mathbb{S}^2$.

CONJECTURE 3.7. Let $(\mathbb{S}^2, g_0)$ be a topological sphere endowed with a smooth metric normalized to volume one and let $g(t)$ be the unique solution of the normalized Ricci flow

$$\frac{\partial g}{\partial t} = (r - R)g$$
$$g(0) = g_0 \,.$$

Then $\lambda_1(g(t))$ is increasing for all $t \in [0, \infty)$ and converges to $\lambda_1(g_S) = 8\pi$.

Actually the convergence of $\lambda_1(g(t))$ to $\lambda_1(g_S) = 8\pi$ as $t \to \infty$ follows easily from the fact that $g(t) \to g_S$ in the C^∞-topology, see [5]. In fact, it is simple to prove that the spectrum of the Laplacian depends continuously on the metric even with respect to the C^0-topology, see again [3].

Unfortunately, there are some difficulties in proving the above conjecture. It is in fact well known that the standard metric on $\mathbb{S}^2$ does not maximize λ_2, see [11]. Hence it is not possible to prove in general that the variation formula of Corollary 2.4 is positive on $\mathbb{S}^2$. This suggests that the "shape" of the evolving first eigenfunction of the Laplacian plays a fundamental role in proving such kind of conjecture.

4 – Examples

In this section we determine the behavior of the evolving spectrum on self-similar solutions to the Ricci flow, which are called *Ricci soliton*. Let $(M^n, g(t))$ be a solution to the Ricci flow with initial condition $g(0) = g_0$. The solution $g(t)$ is called Ricci soliton if there exist a smooth function $\sigma(t)$ and a 1-parameter family of diffeomorphisms $\{\psi_t\}$ of M^n such that

$$g(t) = \sigma(t)\psi_t^*(g_0) \tag{9}$$

with $\sigma(0) = 1$ and $\psi_0 = id_{M^n}$. Now, taking the time derivative in (9) and evaluating the result for $t = 0$, we get that the metric g_0 satisfies the identity

$$-2\mathrm{Ric}\,(g_0) = 2\epsilon g_0 + L_X g_0\,, \tag{10}$$

where $\epsilon = \frac{\sigma'(0)}{2}$ and X is the vector field on M^n generated by $\{\psi_t\}$ for $t = 0$. Conversely, given a metric g_0 which satisfies (10) there exist $\sigma(t)$ and $\{\psi_t\}$ such that $g(t) = \sigma(t)\psi_t^*(g_0)$ is a solution to the Ricci flow with initial condition $g(0) = g_0$. In particular we can choose $\sigma(t) = 1+2\epsilon t$ and $\{\psi_t\}$ as the 1-parameter family of diffeomorphisms generated by the vector fields

$$Y_t = \frac{1}{\sigma(t)} X\,,$$

see [4]. In summary, each self-similar solution to the Ricci flow can be written in the canonical form

$$g(t) = (1 + 2\epsilon t)\psi_t^*(g_0)\,.$$

We then say that the soliton is expanding, shrinking, or steady, if $\epsilon > 0$, $\epsilon < 0$, or $\epsilon = 0$ respectively.

Now, let (M, g) and (N, h) be two closed Riemannian manifolds and

$$\varphi : (M, g) \longrightarrow (N, h)$$

an isometry, we then have the following remarkable identity

$$\Delta_g \circ \varphi^* = \varphi^* \circ \Delta_h$$

see [3]. Given a diffeomorphism $\psi : M^n \longrightarrow M^n$ we have that

$$\psi : (M^n, \psi^* g) \longrightarrow (M^n, g)$$

is an isometry, we conclude that $(M^n, \psi^* g)$ and (M^n, g) have the same spectrum

$$\mathrm{Spec}\,(g) = \mathrm{Spec}\,(\psi^* g)$$

with eigenfunctions $\{f_k\}$ and $\{\psi^* f_k\}$ respectively. We have thus that if $g(t)$ is a Ricci soliton on (M^n, g_0) then

$$\operatorname{Spec}(g(t)) = \frac{1}{\sigma(t)} \operatorname{Spec}(g_0), \tag{11}$$

that is Spec $(g(t))$ is "proportional" to the initial spectrum Spec (g_0) and shrinks, is stationary, or expands depending on wheter ϵ is positive, zero, or negative. By (11), we can also compute explicitely the change rate of each eigenvalue of the Laplacian on a soliton

$$\frac{d\lambda}{dt} = -\frac{\sigma'(t)}{\sigma(t)^2} = -\frac{2\epsilon}{(1+2\epsilon t)^2}\,.$$

It could be now interesting to combine the above formula with Proposition 2.1 and try to study some geometrical aspect of the soliton. This will be studied elsewhere.

5 – Remark

The variation formula of Proposition 2.1 also appears in an equivalent form in [15], see Section 7 Theorem 7.3, where it is derived as a consequence of a general formula for the variation of the L^2-norm of a time dependent function on a manifold evolving under the Ricci flow.

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