# Casimir operators, abelian subspaces and $\mathfrak{u}$-cohomology 

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#### Abstract

We survey old and recent results by Kostant et al. on Casimir operators and abelian subspaces in $\mathbb{Z}_{2}$-graded algebras. Our approach stresses and exploits the connection with $\mathfrak{u}$-cohomology.


## 1 - Introduction

This note is an exposition of old and recent results of B. Kostant regarding the relationships between the exterior algebra of a simple Lie algebra $\mathfrak{g}$ and the action of the Casimir operator on it (see [8], [9], [10], [11]). A key role in this connection is played by the abelian subalgebras of $\mathfrak{g}$ and in particular by the abelian ideals of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. These objects have been intensively and thoroughly investigated after nice results of D. Peterson and subsequent work of several authors which link these ideals to discrete series, the theory of affine Weyl groups, combinatorics, and number theory.

The previous setting can be extended to a $\mathbb{Z}_{2}$-graded Lie algebra $\mathfrak{g}=\mathfrak{g}^{\overline{0}} \oplus$ $\mathfrak{g}^{\overline{1 / 2}}$, where the role of abelian subalgebras is played by the abelian subspaces of $\mathfrak{g}^{\overline{1 / 2}}$ (here $\mathfrak{g}^{\overline{0}}, \mathfrak{g}^{\overline{1 / 2}}$ denote the sets of fixed and antifixed points of the involution $\sigma$ on $\mathfrak{g}$ inducing the $\mathbb{Z}_{2}$-grading). In the following we will refer to this more general setting as the graded case. The framework we have described at the beginning

[^0]will be called the adjoint case (it is indeed a particular instance of the graded case: see Section 4).

The generalization of the results of Kostant to the graded case can be found in [13], [2], [4].

Recently (cf. [11]) Kostant pointed out a connection between his old results on abelian subalgebras and the generalization to the affine case due to GarlandLepowsky [3] of his classical results on $\mathfrak{u}$-cohomology [7]. In this paper, we exploit this connection to obtain a unified approach to Kostant's results and their graded generalizations. One of the advantages of our approach consists in avoiding any reference to the theory of Clifford algebras. Another useful device that we introduce in our treatment is the natural isomorphism

$$
\wedge^{p} \mathfrak{g}^{\overline{1 / 2}} \cong \wedge^{(p, p / 2)} \mathfrak{u}^{-}
$$

where $\mathfrak{u}^{-}=t^{-1} \mathfrak{g}^{\overline{0}}\left[t^{-1}\right] \oplus t^{-\frac{1}{2}} \mathfrak{g}^{\overline{1 / 2}}\left[t^{-\frac{1}{2}}\right]$ (see (3.3) for undefined notation). This explains the role played by affine Lie algebras and their cohomology.

Although the main results are individually known (we have tried to make precise attributions in Section 4), the new feature of our approach consists in exploting formula (3.1), which relates the action on $\wedge \mathfrak{u}^{-}$of the Casimir operator of $\mathfrak{k}$, the Laplacian, and the scaling element of the affine Lie algebra $\widehat{L}(\mathfrak{g}, \sigma)$. This formula is new in the graded case and it is known as Garland's formula in the adjoint case. The connection between Garland's formula and abelian ideals theory has been noticed by Kostant in [11].

Formula (3.1) is the cornerstone of the present work, for it allows us to give a clean, compact and unified treatment of the various contributions to the subject. The exposition is basically self-contained, with two exceptions: a "Laplacian calculation" which can be found in [12] and a technical lemma which is taken from [6]. The main results are Theorems 3.4, 3.5, 3.7.

## 2 - Setup

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $(\cdot, \cdot)$ be its Killing form. Let $\sigma$ be an involutive automorphism of $\mathfrak{g}$. If $j \in \mathbb{R}$ set $\bar{j}=j+\mathbb{Z} \in \mathbb{R} / \mathbb{Z}$. Let $\mathfrak{g}^{\bar{j}}$ be the $e^{2 \pi i j}$-eigenspace of $\sigma$, so that we can write $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}=\mathfrak{g}^{\overline{0}}$ and $\mathfrak{p}=\mathfrak{g}^{\overline{1 / 2}}$. Let $n$ be the rank of $\mathfrak{k}$ and $N$ its dimension. Fix a Borel subalgebra $\mathfrak{b}_{0}$ of $\mathfrak{k}$, with Cartan component $\mathfrak{h}_{0}$, and let $\Delta_{0}^{+}$be the set of positive roots of the root system $\Delta_{0}$ of $\mathfrak{k}$ corresponding to the previous choice.

Let $\widehat{L}(\mathfrak{g}, \sigma)$ be the twisted affine Kac-Moody Lie algebra corresponding to $\mathfrak{g}$ and $\sigma$ (cf. [5]). More precisely introduce a Cartan subalgebra $\widehat{\mathfrak{h}}$ by setting

$$
\mathfrak{h}^{\prime}=\mathfrak{h}_{0} \oplus \mathbb{C} c, \quad \widehat{\mathfrak{h}}=\mathfrak{h}_{0} \oplus \mathbb{C} c \oplus \mathbb{C} d
$$

and define

$$
\begin{aligned}
L(\mathfrak{g}, \sigma) & =\sum_{j \in \mathbb{Z}}\left(t^{j} \otimes \mathfrak{k}\right) \oplus \sum_{j \in \frac{1}{2}+\mathbb{Z}}\left(t^{j} \otimes \mathfrak{p}\right) \\
L^{\prime}(\mathfrak{g}, \sigma) & =L(\mathfrak{g}, \sigma)+\mathfrak{h}^{\prime} \\
\widehat{L}(\mathfrak{g}, \sigma) & =L(\mathfrak{g}, \sigma)+\widehat{\mathfrak{h}}
\end{aligned}
$$

If $x \in \mathfrak{g}^{\bar{r}}$, we set $x_{r}=t^{r} \otimes x$ for any $r \in \bar{r}$. With this notation the bracket of $\widehat{L}(\mathfrak{g}, \sigma)$ is defined by

$$
\left[x_{r}+a c+b d, x_{s}^{\prime}+a^{\prime} c+b^{\prime} d\right]=\left[x, x^{\prime}\right]_{s+r}+s b x_{s}^{\prime}+r b^{\prime} x_{r}+\delta_{r,-s} r\left(x, x^{\prime}\right) c
$$

for $a, a^{\prime}, b, b^{\prime} \in \mathbb{C}$. Let $\widehat{\Delta}$ denote the set of roots of $\widehat{L}(\mathfrak{g}, \sigma)$ and

$$
\widehat{\Delta}^{+}=\Delta_{0}^{+} \cup\{\alpha \in \widehat{\Delta} \mid \alpha(d)>0\}
$$

Then $\widehat{\Delta}^{+}$is a set of positive roots for $\widehat{\Delta}$. We let $\widehat{\Pi}$ be the corresponding set of simple roots. It follows from [5, Exercise 8.3] that $\widehat{\Pi}$ is a finite linearly independent subset of $\widehat{\mathfrak{h}}^{*}$ with exactly $n+1$ elements. We set $\widehat{\Pi}=\left\{\alpha_{0}, \alpha_{1}, \ldots, \alpha_{n}\right\}$.

If $\lambda \in \widehat{\mathfrak{h}}^{*}$, we denote by $\bar{\lambda}$ the restriction of $\lambda$ to $\mathfrak{h}_{0}$. Define $\delta \in \widehat{\mathfrak{h}}^{*}$ by setting $\delta\left(\mathfrak{h}_{0}\right)=\delta(c)=0$ and $\delta(d)=1$. It is easy to check that $(\cdot, \cdot)$ is nondegenerate when restricted to $\mathfrak{h}_{0}$. Thus for $\mu \in \mathfrak{h}_{0}^{*}$ we can define $h_{\mu}$ to be the unique element of $\mathfrak{h}_{0}$ such that $\mu(h)=\left(h_{\mu}, h\right)$ for all $h \in \mathfrak{h}_{0}$. Then one can define a bilinear form on $\mathfrak{h}_{0}^{*}$ - still denoted by $(\cdot, \cdot)$ - by setting $(\mu, \eta)=\left(h_{\mu}, h_{\eta}\right)$.

Write $\alpha_{i}=s_{i} \delta+\bar{\alpha}_{i}$. By [5, Exercise 8.3] we have that $\bar{\alpha}_{i} \neq 0$. Set $h_{i}=\frac{2}{\left(\overline{\alpha_{i}}, \bar{\alpha}_{i}\right)} h_{\bar{\alpha}_{i}}$ and fix $e_{i}=t^{s_{i}} \otimes X_{i}, f_{i}=t^{-s_{i}} \otimes Y_{i}$ in the root spaces $\widehat{L}(\mathfrak{g}, \sigma)_{\alpha_{i}}, \widehat{L}(\mathfrak{g}, \sigma)_{-\alpha_{i}}$ respectively, in such a way that $\left(X_{i}, Y_{j}\right)=\delta_{i, j} \frac{2}{\left(\bar{\alpha}_{i}, \bar{\alpha}_{i}\right)}$. Then $\left[X_{i}, Y_{j}\right]=\delta_{i, j} h_{i}$. Set $\alpha_{i}^{\vee}=\frac{2 s_{i}}{\left(\bar{\alpha}_{i}, \bar{\alpha}_{i}\right)} c+h_{i}$ and $\widehat{\Pi}^{\vee}=\left\{\alpha_{0}^{\vee}, \ldots, \alpha_{n}^{\vee}\right\}$. It follows that $\left[e_{i}, f_{j}\right]=\delta_{i, j} \alpha_{i}^{\vee}$.

Denote by $\mathfrak{h}_{\mathbb{R}}$ the real span of $\alpha_{0}^{\vee}, \ldots, \alpha_{n}^{\vee}$ and let $\widehat{L}(\mathfrak{g}, \sigma)_{\mathbb{R}}$ be the real algebra generated by $\mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R} d$ together with the Chevalley generators $e_{i}, f_{i}, 1 \leq i \leq n$. Let conj be the conjugation of $\widehat{L}(\mathfrak{g}, \sigma)$ corresponding to the real form $\widehat{\widehat{L}}(\mathfrak{g}, \sigma)_{\mathbb{R}}$ and define the conjugate linear antiautomorphism $\sigma_{o}$ of $\widehat{L}(\mathfrak{g}, \sigma)$ by setting $\sigma_{o}(h)=$ $\operatorname{conj}(h)$ for $h \in \hat{\mathfrak{h}}, \sigma_{o}\left(e_{i}\right)=f_{i}$, and $\sigma_{o}\left(f_{i}\right)=e_{i}$. We extend the form $(\cdot, \cdot)$ to $\widehat{L}(\mathfrak{g}, \sigma)$ by setting

$$
\left(x_{r}, y_{s}\right)=\delta_{r,-s}(x, y),(L(\mathfrak{g}, \sigma), d)=\left(L^{\prime}(\mathfrak{g}, \sigma), c\right)=(d, d)=0,(c, d)=1
$$

It is easy to check that $(\cdot, \cdot)$ is a nondegenerate invariant form on $\widehat{L}(\mathfrak{g}, \sigma)$. In particular, it is nondegenerate when restricted to $\widehat{\mathfrak{h}}$. We let $\nu: \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^{*}$ be the isomorphism induced by $(\cdot, \cdot)$, i.e. $\nu(h)(k)=(h, k)$.

Since $(\cdot, \cdot)$ is real on $\mathfrak{h}_{\mathbb{R}}$, we have that $\left(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}\right) \subset \mathbb{R}$. Following [12, Definition 2.3.9], we can therefore define the Hermitian form $\{\cdot, \cdot\}$ on $\widehat{L}(\mathfrak{g}, \sigma)$ by setting

$$
\begin{equation*}
\{x, y\}=\left(x, \sigma_{o}(y)\right) \tag{2.1}
\end{equation*}
$$

This form is contravariant, i.e. $\{[a, x], y\}=-\left\{x,\left[\sigma_{o}(a), y\right]\right\}$.
We set

$$
\begin{aligned}
\mathfrak{m} & =\mathfrak{k}+\widehat{\mathfrak{h}} \\
\mathfrak{u} & =\sum_{\alpha(d)>0} \widehat{L}(\mathfrak{g}, \sigma)_{\alpha} \\
\mathfrak{q} & =\mathfrak{m} \oplus \mathfrak{u}
\end{aligned}
$$

We also set $\mathfrak{u}^{-}=\sum_{\alpha(d)<0} \widehat{L}(\mathfrak{g}, \sigma)_{\alpha}, \mathfrak{q}^{-}=\mathfrak{m} \oplus \mathfrak{u}^{-}$; note that $\sigma_{o}(\mathfrak{u})=\mathfrak{u}^{-}$. Since $(\mathfrak{u}, \mathfrak{q})=0$ and the form $(\cdot, \cdot)$ is nondegenerate on $\widehat{L}(\mathfrak{g}, \sigma)$, it follows that $\{\cdot, \cdot\}$ defines a nondegenerate Hermitian form on $\mathfrak{u}^{-}$. By Theorem 2.3.13 of [12], this form is positive definite. Extend $\{\cdot, \cdot\}$ to $\wedge \mathfrak{u}^{-}$in the usual way: elements in $\wedge^{r} \mathfrak{u}^{-}$are orthogonal to elements of $\wedge^{s} \mathfrak{u}^{-}$if $r \neq s$ whereas

$$
\left(X_{1} \wedge \cdots \wedge X_{r}, Y_{1} \wedge \cdots \wedge Y_{r}\right)=\operatorname{det}\left(\left\{X_{i}, Y_{j}\right\}\right)
$$

Similarly, we can extend $(\cdot, \cdot)$ to define a symmetric bilinear form on $\wedge \widehat{L}(\mathfrak{g}, \sigma)$. If we extend $\sigma_{o}$ to $\wedge^{k} \widehat{L}(\mathfrak{g}, \sigma)$ by setting $\sigma_{o}\left(x^{1} \wedge \cdots \wedge x^{k}\right)=\sigma_{o}\left(x^{1}\right) \wedge \cdots \wedge \sigma_{o}\left(x^{k}\right)$, then obviously (2.1) still holds with $x, y \in \wedge \mathfrak{u}^{-}$.

Set $\partial_{p}: \wedge^{p} \mathfrak{u}^{-} \rightarrow \wedge^{p-1} \mathfrak{u}^{-}$to be the usual Chevalley-Eilenberg boundary operator defined by

$$
\partial_{p}\left(X_{1} \wedge \ldots \wedge X_{p}\right)=\sum_{i<j}(-1)^{i+j}\left[X_{i}, X_{j}\right] \wedge X_{1} \ldots \widehat{X}_{i} \ldots \widehat{X}_{j} \cdots \wedge X_{p}
$$

if $p>1$ and $\partial_{1}=\partial_{0}=0$ and let $H_{p}\left(\mathfrak{u}^{-}, \mathbb{C}\right)$ be its homology. Let $L_{p}: \wedge^{p} \mathfrak{u}^{-} \rightarrow$ $\wedge^{p} \mathfrak{u}^{-}$be the corresponding Laplacian:

$$
L_{p}=\partial_{p+1} \partial_{p+1}^{*}+\partial_{p}^{*} \partial_{p}
$$

where $\partial_{p}^{*}$ denotes the adjoint of $\partial_{p}$ with respect to $\{\cdot, \cdot\}$.
We shall use the following two basic properties of $L_{p}$ (see e.g. [7, Section 2])

$$
\begin{equation*}
\text { Ker } L_{p} \cong H_{p}\left(\mathfrak{u}^{-}\right) \tag{2.2}
\end{equation*}
$$

$$
\begin{equation*}
\left(\operatorname{Ker} L_{p}\right)^{\perp}=\operatorname{Im} \partial_{p}^{*}+\operatorname{Im} \partial_{p+1} \tag{2.3}
\end{equation*}
$$

Since $\mathfrak{u}^{-}$is stable under $a d(\mathfrak{m})$ we have an action of $\mathfrak{m}$ on $\mathfrak{u}^{-}$. Restricting this action to $\mathfrak{k}$ we get an action of $\mathfrak{k}$ on $\mathfrak{u}^{-}$. Notice also that, since $c$ is a central
element, the action of $c$ on $\mathfrak{u}^{-}$is trivial. Recall that the Casimir operator $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ is the element of the universal enveloping algebra of $\mathfrak{k}$ defined by setting

$$
\Omega_{\mathfrak{k}}=\sum_{i=1}^{N} b_{i} b_{i}^{\prime},
$$

where $\left\{b_{1}, \ldots, b_{N}\right\},\left\{b_{1}^{\prime}, \ldots, b_{N}^{\prime}\right\}$ are dual bases of $\mathfrak{k}$ with respect to $(\cdot, \cdot)$. Set $\left\{u_{1}, \ldots, u_{n}\right\}$ and $\left\{u^{1}, \ldots, u^{n}\right\}$ to be bases of $\mathfrak{h}$ dual to each other with respect to $(\cdot, \cdot)$. It is well known that $\Omega_{\mathfrak{k}}$ can be rewritten as

$$
\Omega_{\mathfrak{k}}=\sum_{i=1}^{n} u_{i} u^{i}+2 \nu^{-1}\left(\rho_{0}\right)+\sum_{\alpha \in \Delta_{0}^{+}} x_{-\alpha} x_{\alpha}
$$

where $\rho_{0}=\frac{1}{2} \sum_{\alpha \in \Delta_{0}^{+}} \alpha$ and $x_{\alpha}$ is a root vector in $\mathfrak{k}$.
Define $\Lambda_{0} \in \widehat{\mathfrak{h}}^{*}$ by setting $\Lambda_{0}\left(\mathfrak{h}_{0}\right)=\Lambda_{0}(d)=0$ and $\Lambda_{0}(c)=1$. Set

$$
\begin{equation*}
\rho=\frac{1}{2} \Lambda_{0}+\rho_{0} . \tag{2.4}
\end{equation*}
$$

Notice that, since $\sigma^{2}=I d$ and $(\cdot, \cdot)$ is the Killing form, then $\rho$ coincides with the element $\widehat{\rho}_{\sigma}$ defined in [6, (4.27)]. In particular, by [6, Lemma 5.3]), we have that $\rho\left(\alpha_{i}^{\vee}\right)=1$ for $i=0, \ldots, n$ and $\rho(d)=0$.

## 3 - The results

First we make explicit the relationship between the Casimir element and the Laplacian.

Proposition 3.1. For $x \in \wedge \mathfrak{u}^{-}$we have

$$
\begin{equation*}
L_{p}(x)=-\frac{1}{2}\left(d+\Omega_{\mathfrak{k}}\right)(x) . \tag{3.1}
\end{equation*}
$$

Proof. Note that $\left\{u_{1}, \ldots, u_{n}, c, d\right\}$ and $\left\{u^{1}, \ldots, u^{n}, d, c\right\}$ are bases of $\widehat{\mathfrak{h}}$ dual to each other with respect to $(\cdot, \cdot)$. Then, following [12], we set

$$
\Omega=\sum_{i=1}^{n} u_{i} u^{i}+2 c d+2 \nu^{-1}(\rho)+\sum_{\alpha \in \Delta_{0}^{+}} x_{-\alpha} x_{\alpha} .
$$

By (2.4), we have that

$$
\Omega=\Omega_{\mathfrak{k}}+d+2 d c
$$

The Laplacian calculation done at p. 105 of [12] applied to $\wedge \mathfrak{u}^{-} \simeq \mathbb{C} \otimes \wedge \mathfrak{u}^{-}$gives that, if $x \in \wedge \mathfrak{u}^{-}$, then $L_{p}(x)=-\frac{1}{2} \Omega(x)$. Hence, by observing that $c$ acts trivially on $\wedge \mathfrak{u}^{-}$, the result follows.

We need to recall a key tool in what follows, namely Garland-Lepowsky generalization of Kostant's theorem on the cohomology of the nilpotent radical. We need some more notation. If $\lambda \in \widehat{\mathfrak{h}}^{*}$ is such that $\bar{\lambda}$ is dominant integral for $\Delta_{0}^{+}$, denote by $V(\lambda)$ be the irreducible $\mathfrak{m}$-module of highest weight $\lambda$. Denote by $\widehat{W}$ the Weyl group of $\widehat{L}(\mathfrak{g}, \sigma)$. If $w \in \widehat{W}$ set

$$
N(w)=\left\{\beta \in \widehat{\Delta}^{+} \mid w^{-1}(\beta) \in-\widehat{\Delta}^{+}\right\} .
$$

Set

$$
\widehat{W}^{\prime}=\left\{w \in \widehat{W} \mid w^{-1}\left(\Delta_{0}^{+}\right) \subset \widehat{\Delta}^{+}\right\} .
$$

The following is a special case of Theorem 3.2.7 from [12], which is an extended version of Garland-Lepowsky result [3].

Theorem 3.2.

$$
H_{p}\left(\mathfrak{u}^{-}\right)=\bigoplus_{\substack{w \in \widehat{W}^{\prime} \\ \ell(w)=p}} V(w(\rho)-\rho) .
$$

Moreover a representative of the highest weight vector of $V(w(\rho)-\rho)$ is given by

$$
\begin{equation*}
X_{-\beta_{1}} \wedge \cdots \wedge X_{-\beta_{p}} \tag{3.2}
\end{equation*}
$$

where $N(w)=\left\{\beta_{1}, \ldots, \beta_{p}\right\}$ and the $X_{-\beta_{i}}$ are root vectors in $\widehat{L}(\mathfrak{g}, \sigma)$.
We now define

$$
\begin{equation*}
\wedge^{(r, s)} \mathfrak{u}^{-}=\operatorname{Span}\left\{x_{i_{1}}^{1} \wedge x_{i_{2}}^{2} \wedge \cdots \wedge x_{i_{r}}^{r} \mid-\sum_{i=1}^{r} i_{j}=s\right\} . \tag{3.3}
\end{equation*}
$$

Note that the map $x_{-\frac{1}{2}}^{1} \wedge \ldots \wedge x_{-\frac{1}{2}}^{r} \mapsto x^{1} \wedge \ldots \wedge x^{r}$ affords a canonical identification

$$
\begin{equation*}
\wedge^{(r, r / 2)} \mathfrak{u}^{-} \xrightarrow{\cong} \wedge^{r} \mathfrak{p} \tag{3.4}
\end{equation*}
$$

that intertwines the adjoint action of $\mathfrak{k}$.
Remark 3.1. Recall that there is a standard linear isomorphism $\tau: \operatorname{so}(\mathfrak{p}) \rightarrow$ $\wedge^{2} \mathfrak{p}$ given by $\tau(\varphi)=-\frac{1}{4} \sum_{i} \varphi\left(p_{i}\right) \wedge p^{i}$, where $\left\{p_{i}\right\},\left\{p^{i}\right\}$ are dual basis of $\mathfrak{p}$ with respect to $(\cdot, \cdot)_{\mid \mathfrak{p}}$. The adjoint action $a d_{\mathfrak{p}}$ of $\mathfrak{k}$ on $\mathfrak{p}$ defines an embedding $\theta: \mathfrak{k} \rightarrow s o(\mathfrak{p})$. Observe that $\operatorname{Im} \tau \circ \theta$ corresponds, under the identification (3.4), to $\partial_{2}^{*}\left(\wedge^{(1,1)} \mathfrak{u}^{-}\right)$. Infact, for $x \in \mathfrak{k}$, a formal calculation affords

$$
\partial_{2}^{*}\left(x_{-1}\right)=-\frac{1}{2} \sum_{t=1}^{\operatorname{dim} \mathfrak{p}}\left[x, p_{t}\right]_{-\frac{1}{2}} \wedge p_{-\frac{1}{2}}^{t}
$$

Lemma 3.3. Given linearly independent elements $x^{1}, \ldots, x^{p}$ of $\mathfrak{p}$, set $v=$ $x_{-\frac{1}{2}}^{1} \wedge \ldots \wedge x_{-\frac{1}{2}}^{p}$. Then $\partial_{p}(v)=0$ if and only if $\left[x^{i}, x^{j}\right]=0$ for all $i, j$.

Proof. This follows readily from the definition of $\partial_{p}$ :

$$
\partial_{p}(v)=\sum(-1)^{i+j}\left[x^{i}, x^{j}\right]_{-1} \wedge x_{-\frac{1}{2}}^{1} \ldots \widehat{x_{-\frac{1}{2}}^{i}} \ldots \wedge \widehat{x_{-\frac{1}{2}}^{\hat{j}}} \ldots \wedge x_{-\frac{1}{2}}^{p} .
$$

For a $p$-dimensional subspace $\mathfrak{a}=\bigoplus_{i=1}^{p} \mathbb{C} v^{i}$ of $\mathfrak{p}$ define

$$
\begin{aligned}
v_{\mathfrak{a}} & =v^{1} \wedge \ldots \wedge v^{p} \in \wedge^{p} \mathfrak{p} \\
\widehat{v}_{\mathfrak{a}} & =v_{-\frac{1}{2}}^{1} \wedge \ldots \wedge v_{-\frac{1}{2}}^{p} \in \wedge^{(p, p / 2)} \mathfrak{u}^{-}
\end{aligned}
$$

THEOREM 3.4. The maximal eigenvalue for the action of $\Omega_{\mathfrak{k}}$ on $\wedge^{p} \mathfrak{p}$ is at most $p / 2$. Equality holds if and only if there exists a commutative subspace $\mathfrak{a}$ of $\mathfrak{p}$ of dimension $p$. In such a case, $v_{\mathfrak{a}}$ is an eigevector for $\Omega_{\mathfrak{k}}$ relative to the eigenvalue $p / 2$.

Proof. To prove the first statement, remark that $L_{p}$ is self-adjoint and positive semidefinite on $\wedge \mathfrak{u}^{-}$with respect to $\{$,$\} . Since, by Proposition 3.1,$ $\Omega_{\mathfrak{e}}=-d-2 L_{p}$, the claim follows.

Suppose that $\mathfrak{a}$ is an abelian subspace of $\mathfrak{p}$ of dimension $p$. Then, by Lemma 3.3, $\partial_{p}\left(\widehat{v}_{\mathfrak{a}}\right)=0$. Since $\widehat{v}_{\mathfrak{a}} \in \wedge^{(p, p / 2)} \mathfrak{u}^{-}$, we have that $\partial_{p+1}^{*}\left(\widehat{v}_{\mathfrak{a}}\right)=0$, hence $L_{p}\left(\widehat{v}_{\mathfrak{a}}\right)=0$. Therefore, by (3.1), we have $\Omega_{\mathfrak{k}}\left(v_{\mathfrak{a}}\right)=p / 2 v_{\mathfrak{a}}$. Conversely, if $\Omega_{\mathfrak{k}}$ has eigenvalue $p / 2$ on $\wedge^{p} \mathfrak{p}$, then Ker $L_{p} \cap \wedge^{(p, p / 2)} \mathfrak{u}^{-} \neq 0$. Using (2.2) and Theorem 3.2, we know that Ker $L_{p}$ decomposes with multiplicity one. Since $\wedge^{(p, p / 2)} \mathfrak{u}^{-}$is $\mathfrak{m}$-stable, we deduce that one of the highest weight vectors (3.2), say $x_{-\frac{1}{2}}^{1} \wedge \cdots \wedge x_{-\frac{1}{2}}^{p}$, must belong to Ker $L_{p} \cap \wedge^{(p, p / 2)} \mathfrak{u}^{-}$. Since $\partial_{p}^{*} \partial_{p}=0$ implies that $\partial_{p}=0$, Lemma 3.3 gives that $\operatorname{Span}\left(x^{1}, \ldots, x^{p}\right)$ is the required abelian subspace.

We now relate the vectors $v_{\mathfrak{a}}$ to distinguished elements of $\widehat{W}$. Set $\Delta_{\mathfrak{p}}$ to be the set of $\mathfrak{h}_{0}$-weights of $\mathfrak{p}$ and suppose that $\mathfrak{i}$ is a $\mathfrak{h}_{0}$-stable subspace of $\mathfrak{p}$. Set

$$
\begin{aligned}
& \Phi_{\mathfrak{i}}=\left\{\alpha \in \Delta_{\mathfrak{p}} \mid \mathfrak{p}_{\alpha} \subset \mathfrak{i}\right\} \\
& \widehat{\Phi}_{\mathfrak{i}}=\left\{\left.\frac{1}{2} \delta-\alpha \right\rvert\, \alpha \in \Phi_{\mathfrak{i}}\right\}
\end{aligned}
$$

THEOREM 3.5. The following statements are equivalent

1) $\mathfrak{i}$ is an abelian $\mathfrak{b}_{0}$-stable subspace of $\mathfrak{p}$.
2) There is an element $w_{i} \in \widehat{W}$ such that $N\left(w_{i}\right)=\widehat{\Phi}_{\mathfrak{i}}$.
3) $\mathfrak{i}$ is a $\mathfrak{b}_{0}$-stable subspace of $\mathfrak{p}$ and $\Omega_{\mathfrak{k}} v_{\mathfrak{i}}=\frac{1}{2}(\operatorname{dim} \mathfrak{i}) v_{\mathfrak{i}}$.

Proof. 1) $\Longrightarrow 2)$. Set $p=\operatorname{dim} \mathfrak{i}$. Then, since $\mathfrak{i}$ is abelian, $\partial_{p}\left(\widehat{v}_{\mathfrak{i}}\right)=0$. Notice that $\widehat{v}_{\mathfrak{i}} \in \wedge^{(p, p / 2)} \mathfrak{u}^{-}$, so $\partial_{p}^{*}\left(\widehat{v}_{\mathfrak{i}}\right)=0$. It follows that $L_{p}\left(\widehat{v}_{\mathfrak{i}}\right)=0$. Since $\mathfrak{i}$ is $\mathfrak{b}_{0}$-stable, $\widehat{v}_{\mathfrak{i}}$ is a maximal vector for $\mathfrak{m}$ in $\wedge \mathfrak{u}^{-}$. By Theorem 3.2, there is an element $w_{\mathrm{i}} \in \widehat{W}$ such that

$$
\wedge_{\alpha \in N\left(w_{\mathfrak{i}}\right)} X_{-\alpha}=\widehat{v}_{\mathfrak{i}}
$$

and this implies that $N\left(w_{\mathfrak{i}}\right)=\widehat{\Phi}_{\mathfrak{i}}$.
$2) \Longrightarrow 3)$. By Theorem 3.2 we have that $\widehat{v}_{i}$ is a maximal vector for the action of $\mathfrak{m}$ on $\wedge \mathfrak{u}^{-}$, hence $\mathfrak{i}$ is a $\mathfrak{b}_{0}$-stable subspace of $\mathfrak{p}$. Moreover $L_{p}\left(\widehat{v}_{\mathfrak{i}}\right)=0$ therefore

$$
\Omega_{\mathfrak{k}}\left(\widehat{v}_{\mathfrak{i}}\right)=-\left(2 L_{p}+d\right)\left(\widehat{v}_{\mathfrak{i}}\right)=\frac{1}{2}(\operatorname{dim} \mathfrak{i}) \widehat{v}_{\mathfrak{i}}
$$

and this implies that $\Omega_{\mathfrak{k}} v_{\mathfrak{i}}=\frac{1}{2}(\operatorname{dim} \mathfrak{i}) v_{\mathfrak{i}}$.
$3) \Longrightarrow 1)$. This follows from Theorem 3.4.
Let $\widehat{A}_{p}$ denote the linear span of the vectors $\widehat{v}_{\mathfrak{a}}$ when $\mathfrak{a}$ ranges over the set of commutative subalgebras of $\mathfrak{p}$ of dimension $p$. Let also $\widehat{M}_{p}$ denote the eigenspace corresponding to the eigenvalue $p / 2$ for the action of $\Omega_{\mathfrak{k}}$ on $\wedge^{(p, p / 2)} \mathfrak{u}^{-}$.

Denote by $\mathfrak{a}_{1}, \ldots, \mathfrak{a}_{r}$ the abelian $\mathfrak{b}_{0}$-stable subspaces of $\mathfrak{p}$ of dimension $p$ and set $\mu_{i}=\sum_{\alpha \in \widehat{\Phi}_{\mathfrak{a}_{i}}} \alpha=-\frac{1}{2} \operatorname{dim}\left(\mathfrak{a}_{i}\right) \delta+\sum_{\alpha \in \Phi_{\mathfrak{a}_{i}}} \alpha$.

Set $\widehat{J}$ to be the ideal (for exterior multiplication) in $\wedge \mathfrak{u}^{-}$generated by $\partial_{2}^{*}\left(\mathfrak{u}^{-}\right)$ and set $\widehat{J}_{p}=\widehat{J} \cap \wedge^{(p, p / 2)} \mathfrak{u}^{-}$.

Proposition 3.6.

1) $\widehat{A}_{p}=\widehat{M}_{p}=\bigoplus_{i=1}^{r} V\left(\mu_{i}\right)=\operatorname{Ker}\left(L_{p}\right)$.
2) 

$$
\wedge^{(p, p / 2)} \mathfrak{u}^{-}=\widehat{A}_{p} \oplus \widehat{J}_{p}
$$

is the orthogonal decomposition with respect to $\{\cdot, \cdot\}$. In particular, letting $\mathcal{A}$ be the subalgebra of $\bigoplus_{p \geq 0} \wedge^{(p, p / 2)} \mathfrak{u}^{-}$generated by 1 and $\partial_{2}^{*}\left(\mathfrak{u}^{-}\right)$, then

$$
\bigoplus_{p \geq 0} \wedge \wedge^{(p, p / 2)} \mathfrak{u}^{-}=\mathcal{A} \wedge \sum_{p \geq 0} \widehat{A}_{p}
$$

Proof. 1). By Theorem 3.4, the linear generators of $\widehat{A}_{p}$ are eigenvectors for $\Omega_{\mathfrak{k}}$ of eigenvalue $p / 2$, hence $\widehat{A}_{p} \subseteq \widehat{M}_{p}$. Clearly, by (3.1), $\widehat{M}_{p} \subseteq \operatorname{Ker} L_{p}$. For any element $w \in \widehat{W}$, the following relation holds (see e.g. [12, Corollary 1.3.22]):

$$
w(\rho)-\rho=-\sum_{\alpha \in N(w)} \alpha
$$

Combining this observation with Theorem 3.2 and Theorem 3.5, we have that Ker $L_{p}=\bigoplus_{i=1}^{r} V\left(\mu_{i}\right)$. Finally, by Theorem 3.5, $V\left(\mu_{i}\right)$ is linearly generated by elements in $\widehat{A}_{p}$, hence $\bigoplus_{i=1}^{r} V\left(\mu_{i}\right) \subseteq \widehat{A}_{p}$.
2). We have

$$
\widehat{A}_{p}^{\perp}=\left(\operatorname{Ker} L_{p}\right)^{\perp}=\partial_{p}^{*}\left(\wedge^{(p-1, p / 2)} \mathfrak{u}^{-}\right)
$$

The first equality is clear from part 1), whereas the second follows combining (2.3) with the fact that $\wedge^{(p+1, p / 2)} \mathfrak{u}^{-}=0$. It remains to prove that $\partial_{p}^{*}\left(\wedge^{(p-1, p / 2)} \mathfrak{u}^{-}\right)=\widehat{J_{p}}$. Observe that, if $v \in \wedge^{(p-1, p / 2)} \mathfrak{u}^{-}$, then necessarily $v$ is a sum of decomposable elements of type $x_{-1}^{1} \wedge x_{-\frac{1}{2}}^{2} \wedge \cdots \wedge x_{-\frac{1}{2}}^{p-1}$. Assume that $v=x_{-1}^{1} \wedge x_{-\frac{1}{2}}^{2} \wedge \cdots \wedge x_{-\frac{1}{2}}^{p-1}$. Since $\partial^{*}$ is a skew-derivation and $\partial_{p-1}^{*}\left(x_{-\frac{1}{2}}^{2} \wedge \cdots \wedge x_{-\frac{1}{2}}^{p-1}\right) \in \wedge^{\left(p-1, \frac{p-2}{2}\right)} \mathfrak{u}^{-}=0$, we have

$$
\partial_{p}^{*}(v)=\partial_{2}^{*}\left(x_{-1}^{1}\right) \wedge x_{-\frac{1}{2}}^{2} \wedge \cdots \wedge x_{-\frac{1}{2}}^{p-1}
$$

so that $\partial_{p}^{*}(v) \in \widehat{J}_{p}$. Conversely, if $w \in \widehat{J}_{p}$, then $w$ is a sum of terms of type $\partial_{2}^{*}(x) \wedge y$ with $x \in \wedge^{(1, s)} \mathfrak{u}^{-}, y \in \wedge^{(p-2, r)} \mathfrak{u}^{-}$. Since $s+r=p / 2, r \geq \frac{p-2}{2}, s \geq 1$, we have necessarily $s=1, r=\frac{p-2}{2}$. Therefore $\partial_{p-1}^{*}(y)=0$, hence $w=\partial_{p}^{*}(x \wedge y) \in$ $\partial_{p}^{*}\left(\wedge^{(p-1, p / 2)} \mathfrak{u}^{-}\right)$.

Finally, if $x \in \bigoplus_{p \geq 0} \wedge^{(p, p / 2)} \mathfrak{u}^{-}$, then $x=a_{1}+\partial_{2}^{*}\left(j_{1}\right) \wedge b_{1}$ with $a_{1} \in \widehat{A}_{p}, j_{1} \in$ $\mathfrak{u}^{-}, b_{1} \in \wedge^{\left(p-2, \frac{p-2}{2}\right)} \mathfrak{u}^{-}$. In turn, we can write $b_{1}=a_{2}+\partial_{2}^{*}\left(j_{2}\right) \wedge b_{2}$ as above, and so on. The last claim now follows.

Using the map (3.4), the previous proposition can be restated as a result on the algebra $\wedge \mathfrak{p}$. We set $A_{p}$ to be the linear span of the vectors $v_{\mathfrak{a}}$ when $\mathfrak{a}$ ranges over the set of commutative subalgebras of $\mathfrak{p}$ of dimension $p, M_{p}$ to denote the eigenspace corresponding to the eigenvalue $p / 2$ for the action of $\Omega_{\mathfrak{k}}$ on $\wedge^{p} \mathfrak{p}$. Denote by $L(\xi)$ the irreducible $\mathfrak{k}$-module of highest weight $\xi$.

Set $J$ to be the ideal (for exterior multiplication) in $\wedge \mathfrak{p}$ generated by $(\tau \circ \theta)(\mathfrak{k})$ and set $J_{p}=J \cap \wedge^{p} \mathfrak{p}$.

Theorem 3.7 .

1) $A_{p}=M_{p}=\bigoplus_{i=1} L\left(\sum_{\alpha \in \Phi_{\mathfrak{a}_{i}}} \alpha\right)$.
2) We have

$$
\begin{equation*}
\wedge^{p} \mathfrak{p}=A_{p} \oplus J_{p} \tag{3.5}
\end{equation*}
$$

This is the orthogonal decomposition with respect to the form on $\wedge \mathfrak{p}$ defined by extending, by determinants, the Killing form of $\mathfrak{g}$. Moreover, letting A be the subalgebra of $\wedge \mathfrak{p}$ generated by 1 and $(\tau \circ \theta)(\mathfrak{k})$, then

$$
\wedge \mathfrak{p}=A \wedge \sum_{p \geq 0} A_{p}
$$

Proof. The only statement which does not follows directly from (3.4) and Remark 3.1 is that the decomposition (3.5) is orthogonal with respect to the form induced by the Killing form. Fix $x \in J_{p}$ and $v_{\mathfrak{a}}=x^{1} \wedge \cdots \wedge x^{p}$ with $\left[x^{i}, x^{j}\right]=0$. We want to show that $\left(x, v_{\mathfrak{a}}\right)=0$. Let $\hat{x}$ be the element of $\widehat{J}_{p}$ corresponding to $x$ under (3.4). Then

$$
\begin{equation*}
\left(x, x^{1} \wedge \cdots \wedge x^{p}\right)=\left(\hat{x}, x_{\frac{1}{2}}^{1} \wedge \cdots \wedge x_{\frac{1}{2}}^{p}\right)=\left\{\hat{x}, \sigma_{o}\left(x_{\frac{1}{2}}^{1} \wedge \cdots \wedge x_{\frac{1}{2}}^{p}\right)\right\} \tag{3.6}
\end{equation*}
$$

Set $\tilde{x}_{-\frac{1}{2}}^{i}=\sigma_{o}\left(x_{\frac{1}{2}}^{i}\right)$. We now observe that $\left[\tilde{x}^{i}, \tilde{x}^{j}\right]=0$. Indeed

$$
\begin{aligned}
& {\left[\tilde{x}^{i}, \tilde{x}^{j}\right]_{-1}=} \\
& =\left[\tilde{x}_{-\frac{1}{2}}^{i}, \tilde{x}_{-\frac{1}{2}}^{j}\right]=\left[\sigma_{o}\left(x_{\frac{1}{2}}^{i}\right), \sigma_{o}\left(x_{\frac{1}{2}}^{j}\right)\right]=-\sigma_{o}\left(\left[x_{\frac{1}{2}}^{i}, x_{\frac{1}{2}}^{j}\right]\right)=-\sigma_{o}\left(\left[x^{i}, x^{j}\right]_{1}\right),
\end{aligned}
$$

and the last term is zero since $\mathfrak{a}$ is abelian. Since (3.6) gets rewritten as

$$
\left(x, v_{\mathfrak{a}}\right)=\left\{\hat{x}, \tilde{x}_{-\frac{1}{2}}^{1} \wedge \cdots \wedge \tilde{x}_{-\frac{1}{2}}^{p}\right\}
$$

and $\hat{x} \in \widehat{J}_{p}$, Proposition 3.6 implies that $\left(x, v_{\mathfrak{a}}\right)=0$.

## 4-Remarks

1. If $\mathfrak{s}$ is a simple Lie algebra, consider the semisimple algebra $\mathfrak{g}=\mathfrak{s} \oplus \mathfrak{s}$, endowed with the switch automorphism $\sigma(x, y)=(y, x)$. Then we have $\mathfrak{k} \cong \mathfrak{p} \cong \mathfrak{s}$ and we recover Kostant's classical results on abelian ideals of Borel subalgebras. Theorems 3.4 appears in [8] as Theorem 5. The statements in Theorem 3.7 appear in $[8$, Theorem 8] and [9, Theorems A, B]. In all cases proofs are different from the ones we have presented.

Subsequently Kostant realized the connection of abelian ideals with Lie algebra homology (see [11]): our treatment is inspired by this approach.
2. In the graded case Theorem 3.4 appears in [13, Theorem 0.3], whereas Theorem 3.7 appears in [4, Theorems 1.1, 1.2]. Both authors do not exploit the connection with Lie algebra homology.
3. The so-called Peterson's abelian ideals theorem states that the number of abelian ideals of a Borel subalgebra of $\mathfrak{g}$ in $2^{\text {rank } \mathfrak{g}}$. This result shed a new light on the results from [8], as Kostant pointed out in [10]. The latter paper contains an outline of a proof of Peterson's result and a proof of equivalence 1) $\Leftrightarrow 2$ ) of Theorem 3.5 for abelian ideals (see [10, Section 2]).

A proof of Peterson's theorem using the geometry of alcoves is given in [1]. Combining this geometric approach with Garland-Lepowsky theorem, a uniform enumeration of abelian $\mathfrak{b}_{0}$-stable subspaces in $\mathfrak{p}$ has been obtained in [2]. The proof of Theorem 3.5 is taken from [2, Theorem 3.2].

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