# On some quasilinear elliptic equations involving Hardy potential 

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Abstract: In this paper we consider nonlinear boundary value problems whose simplest model is the following:

$$
\begin{cases}-\Delta u+\nu|u|^{p-1} u=a \frac{u}{|x|^{2}}+f(x) & \text { in } \Omega  \tag{0.1}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $f(x)$ is a summable function in $\Omega$ (bounded open set in $\mathbb{R}^{N}, N>2$, containing the origin), $p>\frac{N}{N-2}$, and $\nu \in \mathbb{R}^{+}$.

## 1 - Introduction and main results

We are interested in existence and regularity of weak solutions for a class of quasilinear elliptic problems whose prototype is the following

$$
\begin{cases}-\Delta u+\nu|u|^{p-1} u=a \frac{u}{|x|^{2}}+f(x) & \text { in } \Omega  \tag{1.1}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $p>\frac{N}{N-2}, N>2, \Omega$ is a bounded open set in $\mathbb{R}^{N}$ containing the origin, $\nu$ and $a$ are positive constants.

If $a=0$, i.e., in absence of the Hardy potential, equations of the form (1.1) have been extensively studied in the sixties by F. Browner (see e.g. [17]) and by

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J. L. Lions (see [19]) for data $f \in H^{-1}$ using energy estimates and monotonicity methods. Later, the existence of solutions was also proved for $L^{1}$ data in a paper of Brezis and Strauss (see [16]) and the result was extended to more general problems involving nonlinear principal part, (as the $p$-laplacian), by Boccardo, Gallouet and Vazquez (see [10]). Many other results have been proved (see for example the well known paper of Benilan and Brezis [5] and the references therein). The interest for this kind of equations has various motivations; for example it comes from the study of the porous medium equation and in the same time it is related to the Thomas-Fermi equation (see again [5]).

Results for (1.1) involving the Hardy potential and in absence of the power term $\nu|u|^{p-1} u$ can be found, among others, in [12], [14], and [15]. Related results can be seen in [1], [2], [3] and in [4] in a more general framework. In particular [3] and [4] concern the following problem

$$
\begin{cases}-\Delta u \pm|\nabla u|^{p}=a \frac{u}{|x|^{2}}+f(x) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

and the aim of these papers is the study of the interaction between the power of the gradient of the solution $|\nabla u|^{p}, p>1$, and the zero order term involving the Hardy potential.

We recall that one of the most interesting phenomena that exhibit this problem if $\nu=0$ and $f$ is only an $L^{1}$ function is the absence of solutions as proved by Boccardo, Orsina and Peral (see [12]). Moreover if the datum $f$ is more summable, that is if $f \in L^{m}(\Omega)$ with $1<m<+\infty$, the existence of weak solutions in $L^{m^{* *}}(\Omega)$ is proved under the assumption that "a" is not too large, that is for $a<a_{0}$, with $a_{0}$ depending on $N$ and $m$; there are no bounded solutions even with datum $f$ bounded (see again [12]).

We prove here that the presence of the lower order term ensures the existence of a solution for every choice of $a$ and $f$, with $f$ only in $L^{1}(\Omega)$ or more in general in $M_{0}(\Omega)$ (i.e., bounded Radon measure continuous with respect to the standard harmonic capacity). Moreover we prove regularity results for such a solution (including the regularity $L^{m^{* *}}(\Omega)$ ) in terms of the summability of the datum $f$ without any restriction on the size of $a$.

What happens is that the term $|u|^{p-1} u$ allows the existence of a solution and in the same time has a regularizing effect on the solution itself, i.e. if $p$ is sufficiently large, that is if $p>\frac{N}{N-2 m}$, we obtain higher integrability properties for $u$ (that turns out to belong to $L^{p m}(\Omega)$, with $p m>m^{* *}$ ) and for its gradient with respect to the case $\nu=0$. This higher integrability will also occur for irregular data, as for example if $f \in L^{1+\varepsilon}(\Omega), \varepsilon>0$ and surprisingly, for $p$ suitably large, will assure a solution in $W_{0}^{1,2}(\Omega)$.

If $a \equiv 0$ and $\nu>0$ it is well known that a lower order term of the type $\nu|u|^{p-1} u$ has a regularizing effect on the solutions. As a matter of fact in [10]
it is proved an existence result in $W_{0}^{1, q}(\Omega)$, for every $1<q<\frac{2 p}{p+1}$ if $f$ is only a summable function (see also [18]).

This increased regularity, if there is a Hardy potential, is however only true up to the existence of unbounded solutions. Indeed if $a \neq 0$ and if $f \in$ $L^{m}$, with $m>\frac{N}{2}$ (which is the standard assumption which yields bounded solutions if $a=0$ ), then there exist solutions of (1.1) which are unbounded. This phenomenon holds true also replacing $|u|^{p-1} u$ with a continuous function $g(u)$ satisfying $g(u) u \geq 0$. Hence there are no lower order terms of the previous kind which can yield the boundedness of the solutions.

The existence of a solution if $f$ belongs to $L^{1}(\Omega)$ and $p \leq \frac{N}{N-2}$ is still an open problem, while bound $p>\frac{N}{N-2 m}$, that we assume to obtain the summablity $L^{m^{* *}}$, is optimal in the sense that, if it does not hold, we can prove the existence of less regular solutions (not belonging to $L^{m^{* *}}(\Omega)$ ). Indeed we can show how the regularity of the solution that we construct increases as $p$ varies in $\left(\frac{N}{N-2}, \frac{N}{N-2 m}\right)$.

We state now our results more in details. Let us consider the following problem

$$
\begin{cases}-\operatorname{div}(M(x, u) \nabla u)+\nu|u|^{p-1} u=a \frac{u}{|x|^{2}}+f(x)-\operatorname{div}(F) & \text { in } \Omega  \tag{1.2}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^{N}, N>2$, containing the origin and $a$ is a positive constant. We assume that $M(x, s)$ is a Carathéodory matrix (that is, measurable with respect to $x$ for every $s \in \mathbb{R}$, and continuous with respect to $s$ for almost every $x \in \Omega$ ) which satisfies, for some positive constants $\alpha$, $\beta$, a.e. in $x \in \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^{N}$,

$$
\begin{align*}
M(x, s) \xi \cdot \xi & \geq \alpha|\xi|^{2}  \tag{1.3}\\
|M(x, s)| & \leq \beta \tag{1.4}
\end{align*}
$$

We assume

$$
\begin{equation*}
p>\frac{N}{N-2}, \quad \text { and } \quad \nu>0 \tag{1.5}
\end{equation*}
$$

On the data we require that

$$
\begin{equation*}
f \in L^{1}(\Omega), \quad F(x) \in\left(L^{2}(\Omega)\right)^{N} \tag{1.6}
\end{equation*}
$$

Before enouncing our existence and regularity results we briefly introduce some notations and recall the definition of weak solution. If $m \in[1,+\infty]$ we denote with $m^{\prime}$, the value in $[1,+\infty]$ such that $\frac{1}{m}+\frac{1}{m^{\prime}}=1$, where we set " $\frac{1}{+\infty}=0$ ", and if $m<N$, we define $m^{*}$ as $\frac{1}{m^{*}}=\frac{1}{m}-\frac{1}{N}$.

Definition 1.1. We say that $u \in W_{0}^{1,1}(\Omega) \cap L^{p}(\Omega)$ is a weak solution of (1.2) if for every $\varphi \in W_{0}^{1, \infty}(\Omega)$ we have

$$
\begin{align*}
& \int_{\Omega} M(x, u) \nabla u \nabla \varphi d x+\int_{\Omega} \nu|u|^{p-1} u \varphi d x= \\
& =\int_{\Omega} a \frac{u}{|x|^{2}} \varphi d x+\int_{\Omega} f(x) \varphi d x+\int_{\Omega} F \nabla \varphi \tag{1.7}
\end{align*}
$$

We note that, since $p>\frac{N}{N-2}, \frac{1}{|x|^{2}} \in L^{p^{\prime}}$ and so $\frac{u}{|x|^{2}}$ belongs to $L^{1}$.
We start from the existence result for irregular data.
Theorem 1.2. Assume that (1.3)-(1.6) hold true. Then there exists a weak solution $u \in W_{0}^{1, q}(\Omega)$ of (1.2) for every $1 \leq q<q_{1}$ where

$$
\begin{equation*}
q_{1}=\max \left\{\frac{N}{N-1}, \frac{2 p}{1+p}\right\}=\frac{2 p}{1+p} \tag{1.8}
\end{equation*}
$$

Remark 1.3. Notice that if $a \equiv 0$ in [8] it is proved the existence of a solution belonging to $W_{0}^{1, q}(\Omega)$ for every $1<q<1^{*}=\frac{N}{N-1}$ with the term $\nu|u|^{p-1} u$ replaced by a function $g(x, u)$ satisfying the weaker assumption $g(x, s) s \geq 0$. Moreover in [10] (see Theorem 5) it was improved the regularity of such a solution if $g(x, s)=\nu|u|^{p-1} u$, proving that it belongs also to $W_{0}^{1, q}(\Omega)$ for every $1<q<\frac{2 p}{p+1}$. Hence here we obtain the existence of a solution having the same regularity proved in the case $a \equiv 0$ but under the assumption that $a \not \equiv 0$.

Remark 1.4. By Sobolev imbedding theorem the solution constructed in Theorem 1.2 belongs also to $L^{s}(\Omega)$, for every $s<\left(\frac{2 p}{p+1}\right)^{*}$.

Remark 1.5. As every bounded Radon measure $\mu \in M_{0}$ (i.e. continuous respect to the standard harmonic capacity) can be decomposed as follows

$$
\mu=f_{0}-\operatorname{div}\left(F_{0}\right), \quad f_{0} \in L^{1}(\Omega), \quad F \in\left(L^{2}(\Omega)\right)^{N}
$$

(see [9]), then Theorem 1.2 ensures the existence of a weak solution of (1.2) with data a measure of $M_{0}$. Notice that if $a=0, f$ is the Dirac mass $\delta_{x_{0}}$ and $p>\frac{N}{N-2}$, there are no solutions of (1.1) (see [5]). For this reason we have not considered the case of singular measure data.

As just said, if $f$ has an higher integrability and $p$ grows in dependence of the summability exponent $m$ of $f$, we have more regular solutions. More in details we have the following result.

Theorem 1.6. Assume that (1.3)-(1.5) hold true, that $f$ belongs to $L^{m}(\Omega)$, where $1<m<\frac{N}{2}$ and that $F \equiv 0$. If $p>\frac{N}{N-2 m}$ then there exists a weak solution $u \in W_{0}^{1, q}(\Omega) \cap L^{p m}(\Omega)$ of (1.2) where

$$
\begin{equation*}
q=\min \left\{2, \frac{2 p m}{1+p}\right\} \tag{1.9}
\end{equation*}
$$

Remark 1.7. Notice that it results

$$
\begin{equation*}
p m \geq m^{* *} \Longleftrightarrow p \geq \frac{N}{N-2 m} \tag{1.10}
\end{equation*}
$$

Hence the summability obtained in Theorem 1.6 is higher than the summability that have the solutions of (1.2) when $a=\nu=0$.

Remark 1.8. We have that

$$
q=\min \left\{2, \frac{2 p m}{1+p}\right\}=2 \Longleftrightarrow p \geq \frac{1}{m-1} .
$$

Hence the regularity (1.9) obtained in Theorem 1.6 coincides with the regularity proved in [18] (see Theorem 3) if $a \equiv 0$ and $p>\frac{1}{m-1}$.

Remark 1.9. For the sake of simplicity we have stated Theorem 1.6 for $F \equiv 0$ but it also holds true for nonzero $F$ belonging to $\left(L^{r}(\Omega)\right)^{N}$ where

$$
r \geq \max \left\{2, \frac{2 p m}{1+p}\right\}
$$

(see Proposition 3.1 in Section 3 below). We can treat also the case when $r$ do not satisfy the previous condition: for further details see Proposition 3.2.

Remark 1.10. Notice that if $p<\frac{N}{N-2 m}$ then may exist solutions that do not belong to $L^{p m}(\Omega)$. As a matter of fact let us consider the model problem (1.1) with $\Omega=B(0,1)$, where $B(0,1)$ is the sphere centered in the origin with radius one, that is

$$
\begin{cases}-\Delta u+|u|^{p-1} u=a \frac{u}{|x|^{2}}+f(x) & \text { in } B(0,1)  \tag{1.11}\\ u=0 & \text { on } \partial B(0,1)\end{cases}
$$

Notice that if we look for radial non-negative solutions $w(r)$ the previous problem becomes

$$
-w^{\prime \prime}-(N-1) \frac{w^{\prime}}{r}+w^{p}=a \frac{w}{r^{2}}+f
$$

and thus, if we set $w(r)=A\left(\frac{1}{r^{\delta}}-1\right), w$ is a radial positive solution of (1.11) if we do the following choices

$$
\begin{aligned}
\delta & =\frac{2}{p-1}, \\
f & =a \frac{A}{r^{2}}+A^{p}\left[\left(\frac{1}{r^{\delta}}-1\right)^{p}-\frac{1}{r^{\delta p}}\right], \\
A^{p-1} & =\delta^{2}, \quad a=\delta(N-2) .
\end{aligned}
$$

Notice that the assumption

$$
p>\frac{N}{N-2}
$$

guarantees that $w$ belongs to $L^{p}(\Omega)$, and hence our equations is satisfied in the distributional sense. Moreover if $p>\frac{N}{N-2 m}$, then $u$ belongs to $L^{p m}(\Omega) \cap L^{m^{* *}}(\Omega)$, while if

$$
p<\frac{N}{N-2 m}
$$

then the previous radial solution does not belong to $L^{m^{* *}}(\Omega)$. By (1.10) $w$ does not belong to $L^{p m}(\Omega)$, while the datum $f$ belongs to $L^{m}(\Omega)$ for every $m<\frac{N}{2}$.

To conclude we show how the regularity of the solution constructed in Theorem 1.2 increases with respect to $p \in\left(\frac{N}{N-2}, \frac{N}{N-2 m}\right)$. For the sake of simplicity we enounce here our result for $F \equiv 0$; we refer to Propositions 3.5 and 3.6 in Section 3 below for the case $F \not \equiv 0$.

Theorem 1.11. Assume that (1.3)-(1.5) hold true and $F=0$. If it results

$$
\begin{equation*}
\frac{N}{N-2}<p \leq \frac{N}{N-2 m}, \quad f \in L^{m}(\Omega), \quad 1<m<\frac{N}{2} \tag{1.12}
\end{equation*}
$$

then there exists a solution $u$ of (1.2) belonging to $L^{s}(\Omega) \cap W_{0}^{1, q}(\Omega)$, for every choice of $s$ and $q$ satisfying

$$
\begin{aligned}
& 1 \leq s<s_{0} \equiv p+\left(p-\frac{N}{N-2}\right) \frac{N-2}{2} \\
& 1 \leq q<q_{1} \equiv \frac{2 p}{p+1}+\left(p-\frac{N}{N-2}\right) \frac{N-2}{p+1} \quad \text { if } \quad p \leq \frac{N+2}{N-2} \\
& q=2 \quad \text { if } \quad p>\frac{N+2}{N-2} .
\end{aligned}
$$

Remark 1.12. Notice that $q_{1}=2$ if $p=\frac{N+2}{N-2}$. Moreover, if $p=\frac{N}{N-2 m}$ it results $s_{0}=p m=m^{* *}$. Hence there is continuity for the results of Theorems 1.11 and 1.6. Indeed there is continuity also for the results of Theorems 1.11 and 1.2 because as $p$ tends to $p_{0} \equiv \frac{N}{N-2}$, we have that $q_{1}$ tends to $\frac{2 p_{0}}{p_{0}+1}=\frac{N}{N-1}$.

Observe also that $s_{0}$ and $q_{1}$ are increasing functions of $p$ and thus, as we expect, the regularity of $u$ increases as $p$ grows.

Remark 1.13. If $\frac{N}{2}<m<+\infty$, as said before, we can have unbounded solution as the following example shows.

By contradiction let us assume that there exists a bounded solution $u \in$ $H_{0}^{1}(\Omega)$ of the following problem

$$
\begin{cases}-\Delta u+|u|^{p-1} u=a \frac{u}{|x|^{2}}+\frac{1}{|x|^{\alpha}} & \text { in } B(0,1)  \tag{1.13}\\ u=0 & \text { on } \partial B(0,1)\end{cases}
$$

where $\alpha<2$ so that the datum $f=\frac{1}{|x|^{\alpha}}$ belongs to $L^{m}(B(0,1))$ for every $\frac{N}{2}<m<\frac{N}{\alpha}$. Notice that, being the datum $f$ non-negative, if we choose $0<a<H$, where $H=\frac{N-2}{2}$ is the Hardy constant, it follows that also the solution $u$ is non-negative as it can be easily proved taking $-u_{-}$as test function in (1.13). Being $u$ bounded, there exists a subset of $B(0,1)$ of the type $B(0, \epsilon)$ in which it results

$$
f-u^{p} \geq \frac{1}{2|x|^{\alpha}}
$$

This fact implies that

$$
-\Delta u \geq \frac{1}{2|x|^{\alpha}} \quad \text { in } \quad B(0, \epsilon)
$$

and hence

$$
u \geq \delta>0 \quad \text { in } \quad B\left(0, \frac{\epsilon}{2}\right)
$$

But from this last inequality it follows that

$$
\begin{cases}-\Delta u \geq \frac{a \delta}{|x|^{2}} & \text { in } B\left(0, \frac{\epsilon}{2}\right) \\ u \geq 0 & \text { in } B\left(0, \frac{\epsilon}{2}\right)\end{cases}
$$

which implies that $u \notin L^{\infty}\left(B\left(0, \frac{\epsilon}{2}\right)\right)$ which contradicts the boundedness of $u$.
For the sake of simplicity, we have studied just a model case but all the previous results hold true even if the principal part is nonlinear also with respect to the gradient. Moreover, another easy generalization that can be done is to replace the terms $\nu|u|^{p-1} u$ and $a \frac{u}{|x|^{2}}$ with, respectively, the Caratheodory
functions $g(x, u)$ and $b(x, u)$ verifying the following conditions a.e. in $x \in \Omega$, (bounded open subset of $\mathbb{R}^{N}$ ), and $\forall s \in \mathbb{R}$,

$$
\begin{gathered}
|b(x, s)| \leq a(x)|s|, \quad a(x) \in L^{\frac{N}{2}, \infty}(\Omega) \\
g(x, s) s \geq \nu|s|^{p+1}, \quad \forall s \in \mathbb{R}, \quad \text { with } \quad p>\frac{N}{N-2}, \quad \nu>0 .
\end{gathered}
$$

For further details see Remark 2.2, while for the basic properties on the Lorentz space $L^{\frac{N}{2}, \infty}(\Omega)$ see, for example, [7] and [11].

The plan of the paper is as follows: in Section 2 there are the proofs of the previous results while in Section 3 we generalize the regularity Theorems 1.6 and 1.11 to nonzero data $F$.

## 2 - Proof of Theorems

## 2.1 - Proof of Theorems 1.2

The proof follows the outline of that in [13]. Let us define for $n \in \mathbb{N}$, the following approximating problems

$$
\begin{cases}\left.-\operatorname{div}\left(M\left(x, u_{n}\right) \nabla u_{n}\right)\right)+\nu\left|u_{n}\right|^{p-1} u_{n}=b_{n}\left(x, u_{n}\right)+f_{n}-\operatorname{div}\left(F_{n}\right) & \text { in } \Omega  \tag{2.1}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where

$$
b_{n}(x, s) \equiv \frac{b(x, s)}{1+\frac{1}{n}|b(x, s)|}, \quad b(x, s) \equiv a \frac{s}{|x|^{2}}
$$

and

$$
f_{n}(x) \equiv \frac{f(x)}{1+\frac{1}{n}|f(x)|}, \quad F_{n}(x) \equiv \frac{F(x)}{1+\frac{1}{n}|F(x)|}
$$

It results

$$
\begin{equation*}
\left|b_{n}(x, s)\right| \leq n, \quad\left|b_{n}(x, s)\right| \leq a \frac{|s|}{|x|^{2}}, \quad\left|f_{n}(x)\right| \leq n, \quad\left|F_{n}(x)\right| \leq n \tag{2.2}
\end{equation*}
$$

and hence (see for example [13]), there exists $u_{n} \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$ weak solution of (2.1).

We prove now the following a priori estimate.
Lemma 2.1. Under the assumptions of Theorem 1.2, (i.e., if (1.3)-(1.6) hold true), there exists a positive constant $c_{0}$, independent on $n$, such that the
following estimates hold true

$$
\begin{align*}
\int_{\Omega}\left|\nabla u_{n}\right|^{q} & \leq C, \quad \forall 1 \leq q<\frac{2 p}{p+1},  \tag{2.3}\\
\int_{\Omega} \frac{\left|\nabla u_{n}\right|^{2}}{\left(1+\left|u_{n}\right|\right)^{\lambda}} & \leq C, \quad \forall 1<\lambda<p \frac{(2-q)}{q}, \\
\int_{\Omega}\left|u_{n}\right|^{p} & \leq C, \\
\int_{\Omega}\left(\frac{\left|u_{n}\right|}{|x|^{2}}\right)^{h} & \leq C, \quad \forall 1<h<\frac{p N}{N+2 p} . \tag{2.6}
\end{align*}
$$

Proof. As in [8] let us take as test function, $\varphi=\left[1-\frac{1}{\left(1+\left|u_{n}\right|\right)^{\lambda-1}}\right] \operatorname{sign}\left(u_{n}\right)$, where $\lambda>1$ will be chosen later. Observe that $|\varphi| \leq 1$. We obtain, using the structure assumptions and (2.2)

$$
\begin{align*}
& \alpha(\lambda-1) \int_{\Omega} \frac{\left|\nabla u_{n}\right|^{2}}{\left(1+\left|u_{n}\right|\right)^{\lambda}}+\nu \int_{\Omega}\left|u_{n}\right|^{p}\left[1-\left(1+\left|u_{n}\right|\right)^{1-\lambda}\right] \leq \\
& \leq a \int_{\Omega} \frac{\left|u_{n}\right|}{|x|^{2}}+\int_{\Omega}|f|+(\lambda-1) \int_{\Omega} \frac{\left|F_{n}\right|\left|\nabla u_{n}\right|}{\left(1+\left|u_{n}\right|\right)^{\lambda}} . \tag{2.7}
\end{align*}
$$

Notice that

$$
\begin{aligned}
& (\lambda-1) \int_{\Omega} \frac{\left|F_{n}\right|\left|\nabla u_{n}\right|}{\left(1+\left|u_{n}\right|\right)^{\lambda}}=(\lambda-1) \int_{\Omega} \frac{\left|F_{n}\right|}{\left(1+\left|u_{n}\right|\right)^{\lambda / 2}} \frac{\left|\nabla u_{n}\right|}{\left(1+\left|u_{n}\right|\right)^{\lambda / 2}} \leq \\
& \leq \frac{1}{\alpha}(\lambda-1) \int_{\Omega}|F|^{2}+\frac{\alpha}{4}(\lambda-1) \int_{\Omega} \frac{\left|\nabla u_{n}\right|^{2}}{\left(1+\left|u_{n}\right|\right)^{\lambda}} .
\end{aligned}
$$

Moreover, if $T$ is such that $1-(1+T)^{1-\lambda}=\frac{1}{2}$, we have

$$
\frac{1}{2} \int_{\left|u_{n}\right|>T}\left|u_{n}\right|^{p} \leq \int_{\left|u_{n}\right|>T}\left|u_{n}\right|^{p}\left[1-\left(1+\left|u_{n}\right|\right)^{1-\lambda}\right] \leq \int_{\Omega}\left|u_{n}\right|^{p}\left[1-\left(1+\left|u_{n}\right|\right)^{1-\lambda}\right]
$$

which implies

$$
\begin{aligned}
& \frac{1}{2} \int_{\Omega}\left|u_{n}\right|^{p} \leq \frac{1}{2} \int_{\left|u_{n}\right| \leq T}\left|u_{n}\right|^{p}+\frac{1}{2} \int_{\left|u_{n}\right|>T}\left|u_{n}\right|^{p} \leq \\
& \leq \frac{1}{2} T^{p}|\Omega|+\int_{\Omega}\left|u_{n}\right|^{p}\left[1-\left(1+\left|u_{n}\right|\right)^{1-\lambda}\right]
\end{aligned}
$$

Let $1 \leq q<2$ to be chosen later. It results

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u_{n}\right|^{q} \leq \frac{\alpha}{4}(\lambda-1) \int_{\Omega} \frac{\left|\nabla u_{n}\right|^{2}}{\left(1+\left|u_{n}\right|\right)^{\lambda}}+c_{1} \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\frac{\lambda q}{2-q}}, \tag{2.8}
\end{equation*}
$$

where $c_{1}=\left(1-\frac{q}{2}\right)\left[\frac{\alpha(\lambda-1)}{2 q}\right]^{-\frac{q}{2-q}}$. Choose $\lambda$ such that

$$
\begin{equation*}
\frac{\lambda q}{2-q}<p, \quad \text { that is } \quad 1<\lambda<\frac{p(2-q)}{q} \tag{2.9}
\end{equation*}
$$

Notice that

$$
\begin{equation*}
1<\frac{p(2-q)}{q} \Leftrightarrow q<\frac{2 p}{p+1} \tag{2.10}
\end{equation*}
$$

Hence we obtain

$$
c_{1} \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\frac{\lambda q}{2-q}} \leq \frac{\nu}{4} \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{p}+c_{2}
$$

where $c_{2}$ is a positive constant depending only on $c_{1}, p, q, \nu$ and $|\Omega|$. Putting together all the previous estimates we get

$$
\begin{aligned}
& \frac{\alpha}{2}(\lambda-1) \int_{\Omega} \frac{\left|\nabla u_{n}\right|^{2}}{\left(1+\left|u_{n}\right|\right)^{\lambda}}+\frac{\nu}{4} \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{p}+\int_{\Omega}\left|\nabla u_{n}\right|^{q} \leq \\
& \leq a \int_{\Omega} \frac{\left|u_{n}\right|}{|x|^{2}}+c_{3}
\end{aligned}
$$

where $c_{3}=c_{2}+\frac{\nu}{2} T^{p}|\Omega|+\frac{1}{\alpha}(\lambda-1) \int_{\Omega}|F|^{2}+\int_{\Omega}|f|$. Denoting with $\epsilon$ a positive constant to be chosen later, we obtain

$$
\begin{align*}
& a \int_{\Omega} \frac{\left|u_{n}\right|}{|x|^{2}} \leq \epsilon \int_{\Omega}\left|u_{n}\right|^{p}+a^{p^{\prime}} c(\epsilon) \int_{\Omega} \frac{1}{|x|^{2 p^{\prime}}} \leq  \tag{2.11}\\
& \leq \epsilon \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{p}+c_{4}
\end{align*}
$$

where $c(\epsilon)=\epsilon^{-\frac{1}{p-1}}$ and $c_{4}=a^{p^{\prime}} c(\epsilon) \int_{\Omega} \frac{1}{|x|^{2 p^{\prime}}}$ is a finite constant as the assumption $p>\frac{N}{N-2}$ is equivalent to require $2 p^{\prime}<N$. Thus, choosing $\epsilon=\frac{\nu}{8}$, from the previous estimates we obtain

$$
\begin{equation*}
\frac{\alpha}{2}(\lambda-1) \int_{\Omega} \frac{\left|\nabla u_{n}\right|^{2}}{\left(1+\left|u_{n}\right|\right)^{\lambda}}+\frac{\nu}{8} \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{p}+\int_{\Omega}\left|\nabla u_{n}\right|^{q} \leq c_{3}+c_{4} \tag{2.12}
\end{equation*}
$$

which implies estimates (2.3)-(2.5). Moreover, for every $1<h<p$ it results

$$
\int_{\Omega}\left(\frac{\left|u_{n}\right|}{|x|^{2}}\right)^{h} \leq \int_{\Omega}\left|u_{n}\right|^{p}+\int_{\Omega} \frac{1}{|x|^{2 h \frac{p}{p-h}}}
$$

where the last integral is finite if $\frac{2 p h}{p-h}<N$, that is if $h<h_{0} \equiv \frac{N p}{2 p+N}$. Notice that the requirement $\frac{N p}{2 p+N}>1$ is fullfilled as it is equivalent to require $p>\frac{N}{N-2}$ and obviously $h_{0}<p$. Hence also (2.6) holds true.

By the previous estimates it follows that there exists a subsequence of $u_{n}$, that we denote again $u_{n}$, and a function $u \in W_{0}^{1, q}(\Omega) \cap L^{p}(\Omega)$, where $q$ is as before, such that as $n \rightarrow+\infty$ it results

$$
\begin{array}{lll}
u_{n} \rightharpoonup u & \text { weakly in } & W_{0}^{1, q}(\Omega), \\
u_{n} \rightharpoonup u & \text { weakly in } & L^{p}(\Omega), \\
u_{n} \rightarrow u & \text { strongly in } & L^{q}(\Omega), \\
u_{n} \rightarrow u & \text { a.e. in } & \Omega, \\
\frac{\left|u_{n}\right|}{|x|^{2}} \rightharpoonup \frac{|u|}{|x|^{2}} & \text { weakly in } & L^{h}(\Omega) .
\end{array}
$$

Hence, passing to the limit in the approximating problem (2.1), we conclude the proof.

Remark 2.2. As it has been noticed in the introduction, the previous existence result holds true replacing the terms $\nu|u|^{p-1} u$ and $a \frac{u}{|x|^{2}}$ with, respectively, the Caratheodory functions $g(x, u)$ and $b(x, u)$ verifying the following conditions a.e. in $x \in \Omega$, (bounded open subset of $\mathbb{R}^{N}$ ), and $\forall s \in \mathbb{R}$,

$$
\begin{align*}
& |b(x, s)| \leq a(x)|s|, \quad a(x) \in L^{\frac{N}{2}, \infty}(\Omega)  \tag{2.13}\\
& g(x, s) s \geq \nu|s|^{p+1}, \quad \forall s \in \mathbb{R}, \quad \text { with } \quad p>\frac{N}{N-2}, \quad \nu>0 \tag{2.14}
\end{align*}
$$

As a matter of fact, the estimates of Lemma 2.1 remain true for the solutions $u_{n} \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$ of the following problem

$$
\begin{cases}\left.-\operatorname{div}\left(M\left(x, u_{n}\right) \nabla u_{n}\right)\right)+g\left(x, u_{n}\right)=b_{n}\left(x, u_{n}\right)+f_{n}-\operatorname{div}\left(F_{n}\right) & \text { in } \Omega  \tag{2.15}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $b_{n}, f_{n}$ and $F_{n}$ are defined as before. Notice that the existence of $u_{n} \in$ $H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$ solutions of (2.15) is guaranteed, for example, by the results in [13]. Moreover, to pass to the limit in the approximating problems we only need to prove that

$$
\begin{equation*}
\left\|g\left(x, u_{n}\right)-g(x, u)\right\|_{L^{1}(\Omega)} \rightarrow 0, \quad n \rightarrow+\infty \tag{2.16}
\end{equation*}
$$

But the proof of the previous convergence result is substantially equal to that of Lemma 2.3 of [13]. The only change regards the estimate of the Hardy potential $b_{n}\left(x, u_{n}\right)$ on the set $\left\{t+\varepsilon \leq\left|u_{n}\right|\right\}$ and this can be done as follows.

Being $a(x) \in L^{\frac{N}{2}, \infty}(\Omega)$ and $p>\frac{N}{N-2}$, there exists $r>1$ verifying

$$
\left\{\begin{array}{l}
\frac{1}{p}+\frac{1}{r}<1 \\
r<\frac{N}{2}
\end{array}\right.
$$

for which it results

$$
\begin{aligned}
& \int_{t+\varepsilon \leq\left|u_{n}\right|} a(x)\left|u_{n}\right| \leq\left\|u_{n}\right\|_{L^{p}(\Omega)}\|a(x)\|_{L^{r}(\Omega)}\left|\Omega \cap\left\{\mid u_{n} \geq t\right\}\right|^{1-\frac{1}{p}-\frac{1}{r}} \leq \\
& \leq C\left|\Omega \cap\left\{\left|u_{n}\right| \geq t\right\}\right|^{1-\frac{1}{p}-\frac{1}{r}}=\varepsilon(t)
\end{aligned}
$$

## 2.2 - Proof of Theorem 1.6

Let $u$ be the solution of problem (1.2) constructed as in the previous proof, that is, as the limit of the approximating solutions $u_{n} \in L^{\infty}(\Omega) \cap H_{0}^{1}(\Omega)$ of (2.1).

As by assumption $p>\frac{N}{N-2}$, there exists $\delta>0$ such that

$$
\begin{equation*}
p>\frac{2 \delta+N}{N-2} \tag{2.7}
\end{equation*}
$$

Let us take $\varphi=\left[\left(1+\left|u_{n}\right|\right)^{\delta}-1\right] \operatorname{sign}\left(u_{n}\right)$ as test function in (2.1), where $\delta$ is as before. We get, using the structure assumptions (1.3) and (1.5)

$$
\begin{align*}
& \delta \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+\nu \int_{\Omega}\left|u_{n}\right|^{p}\left[\left(1+\left|u_{n}\right|\right)^{\delta}-1\right] \leq \\
& \leq a \int_{\Omega} \frac{\left|u_{n}\right|}{|x|^{2}}\left[\left(1+\left|u_{n}\right|\right)^{\delta}-1\right]+\int_{\Omega}|f|\left(1+\left|u_{n}\right|\right)^{\delta} \tag{2.18}
\end{align*}
$$

We estimate now the terms in the previous equation. As it results

$$
\lim _{s \rightarrow+\infty} \frac{s^{p}\left[(1+s)^{\delta}-1\right]}{(1+s)^{\delta+p}}=1
$$

there exists a positive constant $k_{0}$, depending only on $\delta$ and $p$, such that

$$
s^{p}\left[(1+s)^{\delta}-1\right] \geq \frac{(1+s)^{\delta+p}}{2}, \quad \forall s \geq k_{0}
$$

Using the previous inequality, we deduce

$$
\nu \int_{\Omega}\left|u_{n}\right|^{p}\left[\left(1+\left|u_{n}\right|\right)^{\delta}-1\right] \geq \frac{\nu}{2} \int_{\left|u_{n}\right|>k_{0}}\left(1+\left|u_{n}\right|\right)^{\delta+p}
$$

Recalling that $f$ belongs to $L^{m}(\Omega), m>1$, we obtain

$$
\int_{\Omega}|f|\left(1+\left|u_{n}\right|\right)^{\delta} \leq c_{0}\left(\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\delta m^{\prime}}\right)^{\frac{1}{m^{\prime}}}
$$

where $c_{0}=\|f\|_{L^{m}(\Omega)}$. Let $\gamma$ be a real number satisfying

$$
\left\{\begin{array}{l}
2<\gamma<N  \tag{2.19}\\
(\delta+1)\left(\frac{\gamma}{2}\right)^{\prime} \leq p+\delta
\end{array}\right.
$$

Notice that (2.19) is equivalent to require that

$$
\begin{equation*}
\frac{2(p+\delta)}{p-1} \leq \gamma<N \tag{2.20}
\end{equation*}
$$

and hence there exists $\gamma$ satisfying (2.20) as

$$
\frac{2(p+\delta)}{p-1}<N \quad \Leftrightarrow \quad p>\frac{N+2 \delta}{N-2}
$$

that is (2.17). Thanks to the previous choices we have

$$
\begin{aligned}
& a \int_{\Omega} \frac{\left|u_{n}\right|}{|x|^{2}}\left[\left(1+\left|u_{n}\right|\right)^{\delta}-1\right] \leq a \int_{\Omega} \frac{\left(\left|u_{n}\right|+1\right)^{\delta+1}}{|x|^{2}} \leq \\
& \leq \frac{\nu}{4} \int_{\Omega}\left(\left|u_{n}\right|+1\right)^{(\delta+1)\left(\frac{\gamma}{2}\right)^{\prime}}+c_{1} \int_{\Omega}\left(\frac{a}{|x|^{2}}\right)^{\frac{\gamma}{2}} \leq \frac{\nu}{4} \int_{\Omega}\left(\left|u_{n}\right|+1\right)^{p+\delta}+c_{2}
\end{aligned}
$$

where $c_{1}$ and $c_{2}$ are positive constants depending on $\nu$ and independent on $n$. Putting together all the previous estimates we obtain

$$
\begin{align*}
& \delta \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+\frac{\nu}{4} \int_{\Omega}\left(\left|u_{n}\right|+1\right)^{p+\delta} \leq \\
& \leq c_{0}\left(\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\delta m^{\prime}}\right)^{\frac{1}{m^{\prime}}}+c_{2} \tag{2.21}
\end{align*}
$$

Choose now

$$
\begin{equation*}
\delta=p(m-1) \quad \Leftrightarrow \quad \delta m^{\prime}=p+\delta \tag{2.22}
\end{equation*}
$$

Notice that the previous choice is possible as in such a case (2.17) becomes

$$
p>\frac{N}{N-2 m}
$$

that is our assumption on $p$. From (2.21), applying Hoelder inequality, we get

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{p(m-1)-1}+\int_{\Omega}\left(\left|u_{n}\right|+1\right)^{p m} \leq c_{3} \tag{2.23}
\end{equation*}
$$

where $c_{3}$ is independent on $n$. Thus it follows that

$$
\begin{equation*}
\int_{\Omega}|u|^{p m} \leq c_{3} \tag{2.24}
\end{equation*}
$$

If $p(m-1)-1 \geq 0$ the result follows immediately as in this case $2=\min \left\{2, \frac{2 p m}{1+p}\right\}$ and estimate (2.23) implies that $u_{n}$ is equibounded not only in $L^{p m}(\Omega)$, but also in the energy space $H_{0}^{1}(\Omega)$.

If otherwise $p(m-1)-1<0$ we reason as follows. Let $q \in(1,2)$ to be determined. Recalling the definition (2.22) of $\delta$, it results

$$
\begin{align*}
& \int_{\Omega}\left|\nabla u_{n}\right|^{q}=\int_{\Omega} \frac{|\nabla u|^{q}}{\left(1+\left|u_{n}\right|\right)^{\frac{q(1-\delta)}{2}}}\left(1+\left|u_{n}\right|\right)^{\frac{q(1-\delta)}{2}} \leq \\
& \leq \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\frac{q(1-\delta)}{2-q}}=  \tag{2.25}\\
& =\int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{p(m-1)-1}+\int_{\Omega}\left(\left|u_{n}\right|+1\right)^{\frac{q[1-p(m-1)]}{2-q}}
\end{align*}
$$

and hence by (2.23) we get the result choosing $\frac{q[1-p(m-1)]}{2-q}=p m$, that is $q=$ $\frac{2 p m}{1+p}$.

## 2.3 - Proof of Theorem 1.11

Proceeding exactly as in the proof of Theorem 1.6 we conclude that (2.21) holds true for every choice of $\delta$ satisfying (2.17), that is for every $\delta$ such that

$$
\begin{equation*}
0<\delta<\left(p-\frac{N}{N-2}\right) \frac{N-2}{2} \tag{2.26}
\end{equation*}
$$

Notice that the assumption $p \leq \frac{N}{N-2 m}$ is equivalent to require that

$$
\begin{equation*}
\left(p-\frac{N}{N-2}\right) \frac{N-2}{2} \leq p(m-1) \tag{2.27}
\end{equation*}
$$

Moreover it results

$$
\begin{equation*}
\delta m^{\prime} \leq p+\delta \quad \Leftrightarrow \quad \delta \leq p(m-1) \tag{2.28}
\end{equation*}
$$

Hence by (2.27) and (2.28) it follows that every $\delta$ satisfying (2.26) also satisfies

$$
\delta m^{\prime} \leq p+\delta
$$

Using this last inequality in (2.21) we get that $u_{n}$ is equibounded in $L^{p+\delta}(\Omega)$ for every $\delta$ as in (2.26), that is $u_{n}$ is equibounded in $L^{s}(\Omega)$ for every $s<s_{0}$, where

$$
\begin{equation*}
s_{0}=p+\left(p-\frac{N}{N-2}\right) \frac{N-2}{2} . \tag{2.29}
\end{equation*}
$$

Notice that if it is possible to choose $\delta \geq 1$, that is if $p>\frac{N+2}{N-2}$, from (2.21) we also deduce that $\left|\nabla u_{n}\right|$ is equibounded in $L^{2}(\Omega)$ and thus

$$
\int_{\Omega}|\nabla u|^{2} \leq c_{6}
$$

If otherwise $\delta<1$ we estimate the summability of the gradient proceeding as in (2.25). Hence let $q \in(1,2)$ to be determined; it results

$$
\int_{\Omega}\left|\nabla u_{n}\right|^{q} \leq \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\frac{q(1-\delta)}{2-q}}
$$

from which, choosing $\frac{q(1-\delta)}{2-q}=\delta+p$, that is $q=\frac{2(\delta+p)}{1+p}$, we get that

$$
\int_{\Omega}|\nabla u|^{q} \leq c_{7}
$$

Notice that by (2.26) we can choose every value of $q$ satisfying

$$
q<q_{1}=\frac{2 p}{p+1}+\frac{N-2}{1+p}\left(p-\frac{N}{N-2}\right) .
$$

## 3 - The case $F \not \equiv 0$

We study here what happens in Theorems 1.6 and 1.11 if we do not assume $F \equiv 0$.

As said in Remark 1.9, the regularity result of Theorem 1.6 may remain true also for nonzero $F$. More in details we have the following.

Proposition 3.1. Assume that (1.3)-(1.5) hold true, that $f$ belongs to $L^{m}(\Omega)$, where $1<m<\frac{N}{2}$. Let $F \in\left(L^{r}(\Omega)\right)^{N}$ where

$$
\begin{equation*}
r \geq \max \left\{2, \frac{2 p m}{1+p}\right\} \equiv r_{0} \tag{3.1}
\end{equation*}
$$

If $p>\frac{N}{N-2 m}$ then there exists a weak solution $u \in W_{0}^{1, q}(\Omega) \cap L^{p m}(\Omega)$ of (1.2) where

$$
\begin{equation*}
q=\min \left\{2, \frac{2 p m}{1+p}\right\} \tag{3.2}
\end{equation*}
$$

Proof. Proceeding exactly as in the proof of Theorem 1.6, it appears in the right-hand side of (2.18) the following new term

$$
\begin{equation*}
\delta \int_{\Omega}|F|\left|\nabla u_{n}\right|\left(1+\left|u_{n}\right|\right)^{\delta-1} \tag{3.3}
\end{equation*}
$$

To estimate the previous integral we need to distinguish two cases: the case $\delta>1$ and the case $0<\delta \leq 1$, where we recall that $\delta=p(m-1)$ (see (2.22)). If $\delta>1$, that is if $r_{0}>2$, assumption (3.1) implies

$$
\begin{equation*}
r \geq 2\left(\frac{\delta+p}{\delta-1}\right)^{\prime} \tag{3.4}
\end{equation*}
$$

and hence we can estimate the new term (3.3) as follows

$$
\begin{align*}
& \delta \int_{\Omega}|F|\left|\nabla u_{n}\right|\left(1+\left|u_{n}\right|\right)^{\delta-1} \leq \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+ \\
& \quad+\frac{\delta}{2} \int_{\Omega}|F|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1} \leq \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+  \tag{3.5}\\
& \quad+\frac{\nu}{8} \int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\delta+p}+c_{4} \int_{\Omega}|F|^{2\left(\frac{\delta+p}{\delta-1}\right)^{\prime}},
\end{align*}
$$

where $c_{4}$ is a positive constant depending on $\nu$ and $\delta$ that is independent on $n$.
If otherwise $0<\delta \leq 1$, then it results $r_{0}=2$ and we can estimate (3.3) as follows

$$
\begin{align*}
& \delta \int_{\Omega}|F|\left|\nabla u_{n}\right|\left(1+\left|u_{n}\right|\right)^{\delta-1}=\delta \int_{\Omega} \frac{|F|}{\left(1+\left|u_{n}\right|\right)^{\frac{1-\delta}{2}}} \frac{\left|\nabla u_{n}\right|}{\left(1+\left|u_{n}\right|\right)^{\frac{1-\delta}{2}}} \leq \\
& \leq \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+c(\delta) \int_{\Omega} \frac{|F|^{2}}{\left(1+\left|u_{n}\right|\right)^{1-\delta}} \leq  \tag{3.6}\\
& \leq \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+c(\delta) \int_{\Omega}|F|^{2}
\end{align*}
$$

Then, proceeding exactly as in the proof of Theorem 1.6, the result follows.
We can also deal the case in which $r$ does not satisfy (3.1). More in details we have the following.

Proposition 3.2. Assume (1.3)-(1.5), $f \in L^{m}(\Omega), 1<m<\frac{N}{2}$, and $F \in\left(L^{r}(\Omega)\right)^{N}$ where

$$
\begin{equation*}
2<r \leq \frac{2 p m}{1+p}, \quad r<N \tag{3.7}
\end{equation*}
$$

If

$$
\begin{equation*}
p>\frac{N+r}{N-r}, \quad \text { and } \quad p \geq \frac{1}{m-1} \tag{3.8}
\end{equation*}
$$

then there exists a solution $u$ of (1.2) belonging to $L^{\frac{r(p+1)}{2}}(\Omega) \cap H_{0}^{1}(\Omega)$.

Remark 3.3. Assumption $p \geq \frac{1}{m-1}$ is equivalent to require that $2 \leq \frac{2 p m}{1+p}$ and thus assumption (3.7) can be fullfilled. Moreover, as in Theorem 1.6, (see Remark 1.8), the assumption $p \geq \frac{1}{m-1}$ guarantees that $u$ belongs to $\left.H_{0}^{1}(\Omega)\right)$. Notice also that if we only know that $F \in\left(L^{2}(\Omega)\right)^{N}$ then Proposition 3.2 assures the existence of a solution belonging to $L^{p+1}(\Omega) \cap H_{0}^{1}(\Omega)$ if $p>\max \left\{\frac{N+2}{N-2}, \frac{1}{m-1}\right\}$.

Remark 3.4. As we expect, since in Proposition 3.2 we are assuming less regularity on $F$ we obtain a less regular solution. As a matter of fact assumption (3.7) implies that $\frac{r(p+1)}{2} \leq p m$. Moreover the summability exponent of $u$ satisfies $\frac{r(p+1)}{2}>p$.

Observe also that it results $p_{0} \equiv \max \left\{\frac{N+r}{N-r}, \frac{1}{m-1}\right\}<\frac{N}{N-2 m}$ if and only if $m>m_{0} \equiv \max \left\{\frac{2 N}{N+2}, \frac{r N}{N+r}\right\}$. Hence, if $m>m_{0}$ the assumption on $p$ in Proposition 3.2 is weaker than the assumption on $p$ done in Proposition 3.1, and so, as just noticed, we obtain less summability regularity on $u$.

Proof of Proposition 3.2. We proceed exactly as in the proof of Theorem 1.6. As just noticed before, it appears the new term (3.3) containing $F$ that we estimate as follows using the assumption $r>2$

$$
\begin{align*}
& \delta \int_{\Omega}|F|\left|\nabla u_{n}\right|\left(1+\left|u_{n}\right|\right)^{\delta-1} \leq \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+ \\
& \quad+\frac{\delta}{2} \int_{\Omega}|F|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1} \leq \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+  \tag{3.9}\\
& \quad+\frac{\delta}{2}\||F|\|_{L^{r}(\Omega)}^{2}\left(\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\frac{(\delta-1) r}{r-2}}\right)^{1-\frac{2}{r}}
\end{align*}
$$

Hence, instead of (2.21) now we get

$$
\begin{align*}
& \frac{\delta}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2}\left(1+\left|u_{n}\right|\right)^{\delta-1}+\frac{\nu}{4} \int_{\Omega}\left(\left|u_{n}\right|+1\right)^{p+\delta} \leq \\
& \leq c_{0}\left(\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\delta m^{\prime}}\right)^{\frac{1}{m^{\prime}}}+c_{5}\left(\int_{\Omega}\left(1+\left|u_{n}\right|\right)^{\frac{(\delta-1) r}{r-2}}\right)^{1-\frac{2}{r}}+c_{2} \tag{3.10}
\end{align*}
$$

where $c_{5}=\frac{\delta}{2}\||F|\|_{L^{r}(\Omega)}^{2}$. We recall that the previous estimate holds true for every choice of $\delta$ satisfying

$$
\begin{equation*}
p>\frac{N+2 \delta}{N-2} \tag{3.11}
\end{equation*}
$$

Let us choose

$$
\frac{(\delta-1) r}{r-2}=\delta+p \quad \Leftrightarrow \quad \delta=\frac{p(r-2)+r}{2}
$$

The previous choice of $\delta$ is an admissible choice as in this case (3.11) becomes

$$
p>\frac{N+r}{N-r}
$$

that is true by assumption (3.8). Notice that it results $\delta>1$ and that with this choice of $\delta$ assumption (3.7) is equivalent to require that $\delta m^{\prime} \leq \delta+p$. Hence by (3.10) it follows that

$$
\int_{\Omega}\left|\nabla u_{n}\right|^{2}+\int_{\Omega}\left(\left|u_{n}\right|+1\right)^{p+\delta} \leq c_{6}
$$

from which the assert follows.
The following result is the generalization of Theorem 1.11 to nonzero $F$.
Proposition 3.5. Assume that (1.3)-(1.5) hold true and $F \in\left(L^{r}(\Omega)\right)^{N}$ where

$$
\begin{array}{ll}
2 \leq r, & \text { if } p \leq \frac{N+2}{N-2} \\
\text { (3.12) } 2<r<N, & \text { if } \frac{N+2}{N-2}<p \leq \frac{N+r}{N-r}, \\
\frac{N(p-1)}{p+1}=\max \left\{2, \frac{N(p-1)}{p+1}\right\} \leq r<N, & \text { if } p>\frac{N+r}{N-r} .
\end{array}
$$

If it results

$$
\begin{equation*}
\frac{N}{N-2}<p \leq \frac{N}{N-2 m}, \quad f \in L^{m}(\Omega), \quad 1<m<\frac{N}{2} \tag{3.13}
\end{equation*}
$$

then there exists a solution $u$ of (1.2) belonging to $L^{s}(\Omega) \cap W_{0}^{1, q}(\Omega)$, for every choice of $s$ and $q$ satisfying

$$
\begin{aligned}
& 1 \leq s<s_{0} \equiv p+\left(p-\frac{N}{N-2}\right) \frac{N-2}{2} \\
& 1 \leq q<q_{1} \equiv \frac{2 p}{p+1}+\left(p-\frac{N}{N-2}\right) \frac{N-2}{p+1} \quad \text { if } \quad p \leq \frac{N+2}{N-2} \\
& q=2 \quad \text { if } \quad p>\frac{N+2}{N-2}
\end{aligned}
$$

Proof. Proceeding as in the proof of Theorem 1.11, it appears the new following term

$$
\begin{equation*}
\delta \int_{\Omega}|F|\left|\nabla u_{n}\right|\left(1+\left|u_{n}\right|\right)^{\delta-1} \tag{3.14}
\end{equation*}
$$

We recall that $\delta$ can be any number satisfying (2.26). Thus it results $\delta<1$ if

$$
\begin{equation*}
\left(p-\frac{N}{N-2}\right) \frac{N-2}{2}<1, \quad \Leftrightarrow \quad p<\frac{N+2}{N-2} . \tag{3.15}
\end{equation*}
$$

In this case the assumption on $r$ becomes $r \geq 2$ and allows us to estimate (3.14) exactly as in (3.6).

If otherwise $p>\frac{N+2}{N-2}$ by (3.15) it follows that we can choose $\delta>1$. We distinguish two cases.

If $\frac{N+2}{N-2}<p \leq \frac{N+r}{N-r}$ then we suppose $r>2$ and we can estimate (3.14) as in (3.9). Hence (3.10) holds true. Notice that

$$
\begin{equation*}
\frac{(\delta-1) r}{r-2} \leq \delta+p \quad \Leftrightarrow \quad \delta \leq \frac{p(r-2)+r}{2} \tag{3.16}
\end{equation*}
$$

Moreover it results

$$
\left(p-\frac{N}{N-2}\right) \frac{N-2}{2} \leq \frac{p(r-2)+r}{2} \Leftrightarrow p \leq \frac{N+r}{N-r} .
$$

Hence in this case (3.16) is satisfied and thus the result follows.
Finally if $\frac{N+r}{N-r}<p$, the assumption on $r$ becomes $r \geq \frac{N(p-1)}{p+1}$ and implies that (3.4) holds true. Thus we can estimate (3.14) exactly as in (3.5).

The only case of nonzero $F$ not considered in Proposition 3.5 is treated in the following result.

Proposition 3.6. Assume that (1.3)-(1.5) hold true, $f \in L^{m}(\Omega), \frac{2 N}{N+2}<$ $m<\frac{N}{2}$ and $F \in\left(L^{r}(\Omega)\right)^{N}$ where

$$
\begin{align*}
\frac{N+r}{N-r} & <p<\frac{N}{N-2 m}  \tag{3.17}\\
2 & <r<\min \left\{\frac{N(p-1)}{p+1}, N, m^{*}\right\} . \tag{3.18}
\end{align*}
$$

Then there exists a solution $u$ of (1.2) belonging to $L^{\frac{r(p+1)}{2}}(\Omega) \cap H_{0}^{1}(\Omega)$.

Remark 3.7. Notice that assumption $r<m^{*}$ is equivalent to require that $\frac{N+r}{N-r}<\frac{N}{N-2 m}$ and hence it is necessary to guarantee that condition (3.17) can be fullfilled. Moreover the assumption $m>\frac{2 N}{N+2}$ is equivalent to require $2<m^{*}$ and thus it is necessary to obtain a nonempty set in (3.18).

Remark 3.8. Notice that if $p>\frac{N+r}{N-r}$ and $r=\frac{N(p-1)}{p+1}$, by Proposition 3.5 it follows that there exists a solution $u$ belonging to $H_{0}^{1}(\Omega) \cap L^{s}(\Omega)$, for every $s<s_{0}=\frac{N(p-1)}{2}$. Moreover if $r \rightarrow \frac{N(p-1)}{p+1}$ then $\frac{r(p+1)}{2} \rightarrow \frac{N(p-1)}{2}$. Hence there is "continuity of the regularity results" of Propositions 3.5 and 3.6.

Proof of Proposition 3.6. Notice that condition $p<\frac{N}{N-2 m}$ implies that

$$
\frac{N(p-1)}{p+1}<\frac{2 p m}{1+p}
$$

Since the assumption $p \geq \frac{1}{m-1}$ in Proposition 3.2 is done only to guarantee that assumption (3.7) can be fulfilled, and being in our case

$$
2<\min \left\{\frac{N(p-1)}{p+1}, N, m^{*}\right\}
$$

because, as just noticed, we are assuming $p>\frac{N+r}{N-r}$ and $m>\frac{2 N}{N+2}$, we can repeat the proof of Proposition 3.2 and conclude that the result follows.

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