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The dependence structure of log-fractional stable noise with analogy to fractional Gaussian noise

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Abstract: *We examine the process log-fractional stable motion (log-FSM), which is an α-stable process with $\alpha \in (1,2)$. Its tail probabilities decay like $x^{-\alpha}$ as $x \to \infty$, and hence it has a finite mean, but its variance is infinite. As a result, its dependence structure cannot be described by using correlations. Its increments, log-fractional noise (log-FSN), are stationary and so the dependence between any two points in time can be determined by a function of only the distance (lag) between them. Since log-FSN is a moving average and hence "mixing," the dependence between the two time points decreases to zero as the lag tends to infinity. Using measures such as the codifference and the covariation, which can replace the covariance when the variance is infinite, we show that the decay is so slow that log-FSN (or, conventionally, log-FSM) displays long-range dependence. This is compared to the asymptotic dependence structure of fractional Gaussian noise (FGN), a befitting circumstance since log-FSN and FGN share a number of features.*

1 – Introduction

The classical Central Limit Theorem deals with the convergence of normalized sums of independent and identically distributed random variables, and states that if these random variables have finite variance then the limit is Gaussian. The cases of infinite variance and triangular arrays are more involved. The limits are then infinitely divisible. Bruno de Finetti was one of the first to consider infinitely divisible distributions (see [3] as well as [2] and [10]). Since then the

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subject has developed in many directions. One of them concerns dependence. The dependence of infinitely divisible random variables is most conveniently described when they have an α-stable distribution, because linear combinations of α-stable random variables remain α-stable. Sequences and random processes with α-stable distributions then can be readily defined and their dependence structure investigated.

An α-stable process with index of stability $0 < \alpha < 2$ has tail probabilities that decrease to zero hypergeometrically, that is, like the power function $x^{-\alpha}$, as $x \to \infty$. These are the proverbial "heavy" tails since the rate of decrease can be very slow. Moments of the process that have order $p < \alpha$ are necessarily finite, but they are infinite if $p \geq \alpha$. An important case is when $\alpha \in (1, 2)$, so that the mean is finite but the variance is infinite. This contrasts markedly with the more familiar Gaussian process (by convention, the case $\alpha = 2$), which has exponentially "light" tails of order $c_1 x^{-1} e^{-c_2 x^2} (c_i > 0)$, and hence has all moments finite. Unlike the Gaussian distribution, which is symmetric about its mean, a non-Gaussian stable distribution can be also skewed either to the left or to the right of its mean. We will concentrate, though, on *symmetric* α-stable processes for which the distribution is symmetric around the origin. Processes that are α-stable $(0 < \alpha < 2)$ can be used to model high variability, namely, phenomena exhibiting "acute spikes" and "eruptions," a behavior that is also often described as *burstiness*.

A random process is *self-similar* if it has finite-dimensional distributions that scale. Specifically, $\{X_t\}$, $t \in \mathbb{R}$, is H-self similar (H-ss), $H > 0$, if

$$X_{ct} \stackrel{d}{=} c^H X_t$$

for any $c > 0$ and $t \in \mathbb{R}$. The notation $\stackrel{d}{=}$ signifies equality of the finite-dimensional distributions, that is, for any finite set of times $t_1, \dots, t_n$

$$\mathbb{P}(X_{ct_1} \leq x_1, \dots, X_{ct_n} \leq x_n) = \mathbb{P}(c^H X_{t_1} \leq x_1, \dots, c^H X_{t_n} \leq x_n).$$

H is called the self-similarity index for $\{X_t\}$. Thus, the finite-dimensional distributions maintain an invariance through a simple scaling of time and space. (Refer to the excellent monograph by Embrechts and Maejima [5] and to the review paper [13] for details.) The process $\{X_t\}$ has stationary increments (si) if $X_{t+s} - X_s \stackrel{d}{=} X_t - X_0$ for all $t, s \in \mathbb{R}$. Processes that are both H-self-similar and have stationary increments (indicated by H-sssi) are helpful for describing natural events that display *long-range dependence*. Long-range dependence occurs, for example, in economic time series and internet communication. Processes that are both α-stable with $\alpha < 2$ and H-sssi are effective models with which to investigate both burstiness and long-range dependence in (but are not limited to) network traffic, hydrology, and financial data. Besides articles in the literature about α-stable, H-sssi, or α-stable H-sssi processes and their applications, see also the texts [4], [11], and [6].

In the case of Gaussian or any finite variance process, the dependence structure in a weaker form can be studied readily through the correlations. For example, zero correlation between the components of a Gaussian process is equivalent to their independence. If the components are stationary, then one can examine their dependence over time durations, or lags. If as the lags get larger, the correlations converge rapidly to zero, then the dependence is “weak.” On the other hand, the dependence is “strong” if the convergence is so slow that the sum of the correlations diverge. Such divergence intrinsically characterizes the random cycles of abnormality and regularity exhibited by long-range dependence.

This paper focuses on the *symmetric* α-stable ($S\alpha S$) H-sssi process *log-fractional stable motion* (log-FSM), which is defined for $1 < \alpha < 2$ and has the self-similarity index $H = 1/\alpha$. Log-FSM has zero mean but infinite variance. In particular, its dependence cannot be measured by correlations. There are, however, “stable” alternatives that replace the covariance.

Two of them are the *codifference* and the *covariation*. Both can be applied to the stationary increments of log-FSM, which is the process known as *log-fractional stable noise*, log-FSN. (“Motion” refers to a process with stationary increments and “noise” to a stationary process.) The behavior of these dependence measures for log-FSN in turn gives an indication about the dependence structure for log-FSM. The codifference, in fact, is defined for any stationary process. The covariation is restricted to $S\alpha S$ processes, albeit not necessarily stationary, for which $1 < \alpha < 2$.

The rest of the paper is carried out as follows. Section 2 briefly reviews $S\alpha S$ processes and their representation as integrals with respect to $S\alpha S$ random measures. Log-fractional stable motion and its increment process log-fractional stable noise (log-FSN) are reviewed in Section 3. The measures of dependence, the codifference and the covariation, are presented in Section 4. Section 5 contains the main results, namely, the asymptotic behavior of the measures when applied to log-FSN. Section 6 makes an analogy to fractional Brownian motion and its increment process fractional Gaussian noise. Some extensions of this work and potential research are mentioned in the concluding Section 7.

2 – A brief approach to symmetric α-stable laws and processes

Aside from the applications described in the introduction, one may ask: why consider stable distributions? The usual answer arises from the central limit theorem, which obtains that they are the unique limits of properly rescaled sums of independent and identically distributed (i.i.d.) random variables. The Gaussian (normal) distribution is the limit if the sequence has a finite variance. If the variance is infinite but the tails of the sequence demonstrate hypergeometric decay for $\alpha < 2$, then the limit turns out to be stable and has the same index α.

More explicitly, let $\{X_j\}_{j=1}^{\infty}$ be an i.i.d. sequence. In the case $\alpha = 2$, suppose X_j has mean μ and variance σ_0^2. Then

$$\frac{1}{n^{1/2}} \sum_{j=1}^{n} (X_j - \mu) \xrightarrow{L} Z \sim N(0, \sigma_0^2)$$

where $\xrightarrow{L}$ stands for convergence in law, that is, in distribution. The limit is a random variable having characteristic function $\mathbb{E}e^{i\theta Z} = e^{-\frac{1}{2}\sigma_0^2|\theta|^2}$ and, consequently, must be Gaussian. In particular, $\mathbb{E}|Z|^p < \infty$ for all $p > 0$. By contrast, if $0 < \alpha < 2$, then the tail probabilities of X_j are "heavy": $\mathbb{P}(|X_j| \geq x) \sim cx^{-\alpha}$ as $x \to \infty$ with $\sigma_0^2 = \infty$. Assume, in addition, X_j is symmetric ($X_j \overset{L}{=} -X_j$) if $\alpha = 1$ and has mean μ if $1 < \alpha < 2$. Then

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} X_j \xrightarrow{L} Z_\alpha$$

if $\alpha \leq 1$ and

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} (X_j - \mu) \xrightarrow{L} Z_\alpha.$$

if $1 < \alpha < 2$. The limit is a symmetric non-Gaussian α-stable random variable Z_α. We indicate this by writing $Z_\alpha \sim S\alpha S$. The limit Z_α recovers the tail behavior of X_j since also $\mathbb{P}(|Z_\alpha| \geq x) \sim cx^{-\alpha}$ as $x \to \infty$, perhaps with a different c. Thus, $\mathbb{E}|Z_\alpha|^p < \infty$ if and only if $p < \alpha$; $\mathbb{E}|Z_\alpha|^2 = \infty$, $\mathbb{E}Z_\alpha = 0$ for $1 < \alpha < 2$, and $\mathbb{E}|Z_\alpha| = \infty$ iff $\alpha \leq 1$. Its characteristic function satisfies

$$\mathbb{E}e^{i\theta Z_\alpha} = e^{-\sigma^\alpha|\theta|^\alpha}, \quad \theta \in \mathbb{R} := (-\infty, \infty), \tag{2.1}$$

where the scale parameter σ depends on α and c. (When $\alpha = 2, \sigma = \sqrt{\sigma_0^2/2}$.)

Relation (2.1) identifies the specific random variable arising in the stable central limit theorem. Any random variable X is, by definition, symmetric α-stable ($S\alpha S$) if it satisfies (2.1). Its scale parameter σ is denoted by $\|X\|_\alpha$. If for instance X is measured in meters, then so is $\|X\|_\alpha$.

Remark. Several easy facts about $X \sim S\alpha S$ ($0 < \alpha \leq 2$) are worth noting (see also [12, ch. 1.2]).

- $a \in \mathbb{R}, a \neq 0$ implies aX is $S\alpha S$ with $\|aX\|_\alpha = |a|\|X\|_\alpha$.
- If $\alpha = 2$ then $X \sim N(0, 2\sigma^2)$.
- $\mathbb{E}|X|^p < \infty$ only for $p < \alpha$.

- Any linear combination of independent $S\alpha S$ random variables is $S\alpha S$: if $\epsilon_j \sim S\alpha S$ are independent and $a_j \in \mathbb{R}$, $1 \leq j \leq n$, then $X = \sum_{j=1}^n a_j \epsilon_j \sim S\alpha S$ with
$$\|X\|_\alpha^\alpha = \|\sum_{j=1}^n a_j \epsilon_j\|_\alpha^\alpha = \sum_{j=1}^n |a_j|^\alpha \|\epsilon_j\|_\alpha^\alpha.$$
It is instructive to see why this last relation holds. The characteristic function of X is, for $\theta \in \mathbb{R}$,
$$\begin{aligned}\phi_X(\theta) = \mathbb{E}e^{i\theta X} &= \mathbb{E}\exp\left\{ i\theta \sum_{j=1}^n a_j \epsilon_j \right\} \\ &= \prod_{j=1}^n \mathbb{E}\exp\{i\theta a_j \epsilon_j\} = \prod_{j=1}^n e^{-|\theta|^\alpha\, |a_j|^\alpha\, \|\epsilon_j\|_\alpha^\alpha} \\ &= \exp\left\{ -|\theta|^\alpha \sum_{j=1}^n |a_j|^\alpha\, \|\epsilon_j\|_\alpha^\alpha \right\} = \mathbb{E}e^{i\theta X},\end{aligned}$$
on using the independence of the ϵ_j and the fact that they are $S\alpha S$.

The vector $\mathbf{X} = (X_1, \dots, X_d)$ in $\mathbb{R}^d$ is Gaussian if and only if the random variables $\{X_1 \dots, X_d\}$ are *jointly* Gaussian, that is, any linear combination of them is Gaussian. Similarly, for $0 < \alpha < 2$, $\mathbf{X} = (X_1, \dots, X_d)$ in $\mathbb{R}^d$ is a $S\alpha S$ vector if and only if $\{X_1 \dots, X_d\}$ are jointly $S\alpha S$, that is, if linear combinations $\sum_{j=1}^n a_j X_j$ are $S\alpha S$ random variables.

By "going to the limit" in the sum $\sum_{j=1}^n a_j \epsilon_j$ one can define a $S\alpha S$ random variable as an integral,

$$X = \int_{\mathbb{R}} f(x) M_\alpha(\mathrm{d}x), \tag{2.2}$$

where f is a deterministic function and M_α is a symmetric α-stable *random measure* (see [12, ch. 3.3]). The scale parameter for X satisfies
$$\|X\|_\alpha^\alpha = \int_{\mathbb{R}} |f(x)|^\alpha \mathrm{d}x < \infty$$
with $\mathrm{d}x$ denoting the Lebesgue measure on $\mathbb{R}$. Formally, the function $f(x)$ plays the role of the a_j's and $M_\alpha(\mathrm{d}x)$ plays the role of the ϵ_j's with $\|M_\alpha(\mathrm{d}x)\|_\alpha^\alpha = \mathrm{d}x$. M_α is defined on $(\mathbb{R}, \mathcal{B}, |\cdot|)$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$ and $|\cdot|$ is Lebesgue measure. Here $|\cdot|$ is the *control* measure and $(\mathbb{R}, \mathcal{B}, |\cdot|)$ is called the *control* space for M_α. This means that if $B \in \mathcal{B}$ with finite Lebesgue measure $|B|$, then $M_\alpha(B)$ is a $S\alpha S$ random variable for which
$$\mathbb{E}e^{i\theta M_\alpha(B)} = e^{-|\theta|^\alpha |B|}.$$

Furthermore, suppose $\{B_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence of sets in $\mathcal{B}$ with $|B_n| < \infty$. Then any finite subcollection $\{M_\alpha(B_n)\}_{n=1}^{\infty}$ are independent random variables (M_α is said to be *independently scattered*), and if $|\bigcup_{n=1}^{\infty} B_n| < \infty$, $M_\alpha(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} M_\alpha(B_n)$ almost surely (a.s.) (M_α is σ-*additive*).

Thus, M_α plays the dual role of being a measure and being random. As suggested above, the $M_\alpha(\mathrm{d}x), x \in \mathbb{R}$ play the role of i.i.d. infinitesimal ϵ_j, with the continuous x replacing the discrete label j, and the infinitesimal $\mathrm{d}x$ replacing the common value $\|\epsilon_j\|_\alpha^\alpha$, namely, the scale parameter raised to the power α.

In the Gaussian case $\alpha = 2$, one usually takes $(\mathbb{R}, \mathcal{B}, |\cdot|/2)$ as the control space, and in this case, $M_2(B)$ has characteristic function $\mathbb{E}e^{i\theta M_2(B)}$; hence, $M_2(B)$ is a normal random variable with mean 0 and variance $|B|$. One can view $M_2(\mathrm{d}x)$ heuristically as a normal random variable having mean zero and infinitesimal measure $\mathrm{d}x$, with $M_2(\mathrm{d}x)$ and $M_2(\mathrm{d}x')$ being independent if the infinitesimal intervals $\mathrm{d}x$ and $\mathrm{d}x'$ are disjoint. The same intuition prevails in the $S\alpha S$ case with $\alpha < 2$. The normal distribution is replaced by the stable distribution and the variance is replaced by the scale parameter raised to the power α.

Suppose $B \in \mathcal{B}$ and $a > 0$. Then

$$M_\alpha(aB) \stackrel{d}{=} a^{1/\alpha} M_\alpha(B), \tag{2.3}$$

where aB is the set B scaled by a, and $\stackrel{d}{=}$ means equality of the finite-dimensional distributions. Indeed,

$$\begin{aligned} \mathbb{E}e^{i\theta M_\alpha(aB)} &= \exp\{-|aB|\,|\theta|^\alpha|\} \\ &= \exp\left\{-|B|^\alpha \left|a^{1/\alpha}\theta\right|^\alpha\right\} \\ &= \mathbb{E}e^{i\theta a^{1/\alpha} M_\alpha(B)}. \end{aligned}$$

Relation (2.3) can be denoted informally by

$$M_\alpha(a\mathrm{d}x) \stackrel{d}{=} a^{1/\alpha} M_\alpha(\mathrm{d}x).$$

Thus, M_α can be regarded as being “self-similar” with “index” $H = 1/\alpha$.

The definition (2.2) can be extended to a random process. Consider the set T to be either $\mathbb{R}, \mathbb{R}_+ = \{t : t \geq 0\}$, or $\{t : t > 0\}$. Let $f_t : \mathbb{R} \to \mathbb{R}$ be measurable and satisfy for each $t \in T$

$$\int_{\mathbb{R}} |f_t(x)|^\alpha \mathrm{d}x < \infty \tag{2.4}$$

and also, if $\alpha = 1$, $\int_{\mathbb{R}} |\,|f_t(x) \ln |f_t(x)|\,|\,\mathrm{d}x < \infty$. (In fact, the condition (2.4) alone suffices to ensure the existence of the subsequent random process in the $S\alpha S$ case when $\alpha = 1$.) Then $\{X_t, t \in T\}$ defined by

$$X_t = \int_{\mathbb{R}} f_t(x) M_\alpha(\mathrm{d}x) \tag{2.5}$$

is a $S\alpha S$ *process* with

$$\|X_t\|_\alpha^\alpha = \int_{\mathbb{R}} |f_t(x)|^\alpha \mathrm{d}x. \tag{2.6}$$

The integral (2.5) is a "representation" of the process $\{X_t\}$. It says, intuitively, that $\{X_t\}$ is obtained by starting with i.i.d. infinitesimal random variables $M_\alpha(\mathrm{d}x)$, weighting them by $f_t(x)$, and integrating. The weights change in general with x and can also change as the time t evolves. Let $-\infty < t_1 \leq \cdots \leq t_d < \infty$. The joint characteristic function of a typical vector $(X_{t_1}, \ldots, X_{t_d})$ of the process is given by

$$\mathbb{E}\exp\left\{ i\sum_{j=1}^{d} \theta_j X_{t_j} \right\} = \exp\left\{ -\int_{\mathbb{R}} \left| \sum_{j=1}^{d} \theta_j f_{t_j}(x) \right|^\alpha \mathrm{d}x \right\}$$

for arbitrary $\theta_1, \ldots, \theta_d \in \mathbb{R}$. In fact, most $S\alpha S$ processes can be represented in the form (2.5). (For details refer to [12, ch. 3 and ch. 13.2].)

One can also define an integrated process with respect to an asymmetric α-stable random measure having arbitrary control measure that is asymmetric, or *skewed*. If the integrand is as above, then the resulting process has also asymmetric distributions. Our concern in this paper only involves processes that are defined by (2.5) based on $S\alpha S$ random measures having Lebesgue control space.

Now recall the definitions of self-similarity and stationarity of the increments given in the introduction. A process $\{X_t, t \in T\}$ is *H-self-similar* (H-ss) with $H > 0$ if

$$X_{ct} \stackrel{d}{=} c^H X_t \tag{2.7}$$

for all $c > 0$ and t, that is, $(X_{ct_1}, \ldots, X_{ct_d})$ and $c^H(X_{t_1}, \ldots, X_{t_d})$ are identically distributed. Note that $c^H X_0 \stackrel{d}{=} X_{c0} = X_0$, hence, letting $c \to \infty$ necessitates $X_0 = 0$ *a.s.* A process is said to have *stationary increments* if the finite-dimensional distributions of $\{X_{t+s} - X_s\}$ do not depend on s:

$$\{X_{s+t} - X_s, t \in T\} \stackrel{d}{=} \{X_t - X_0, t \in T\} \quad \text{for all} \quad s \in T. \tag{2.8}$$

Suppose now that the process $\{X_t, t \in \mathbb{R}\}$

- is H self-similar,
- has stationary increments, and
- is symmetric α-stable;

to wit, it is H-sssi and $S\alpha S$. Let $c > 0$ and $s, \theta_1, \dots, \theta_d, -\infty < t_1 \le \cdots \le t_d < \infty \in \mathbb{R}$. It follows that $\| \sum_{j=1}^d \theta_j (X_{ct_j+s} - X_s)\|_\alpha^\alpha = c^{\alpha H} \| \sum_{j=1}^d \theta_j X_{t_j}\|_\alpha^\alpha$ does not depend on s, since by (2.5)-(2.8),

$$\int_{\mathbb{R}} \left| \sum_{j=1}^d \theta_j \left(f_{ct_j+s}(x) - f_s(x)\right)\right|^\alpha \mathrm{d}x = c^{\alpha H} \int_{\mathbb{R}} \left| \sum_{j=1}^d \theta_j f_{t_j}(x)\right|^\alpha \mathrm{d}x. \tag{2.9}$$

If $\alpha = 2$, M_2 is a Gaussian random measure. Remember that in this case the control space usually is taken to be $(\mathbb{R}, \mathcal{B}, |\cdot|/2)$, so that the variance of M_2 is $\mathbb{E}M^2(B) = |B|$. The process defined by

$$B_t = \int_{\mathbb{R}} 1_{[0,t]}(x) M_2(\mathrm{d}x) = \int_0^t M_2(\mathrm{d}x), \qquad t \ge 0.$$

is Brownian motion. (One can also define for $t < 0$, $B_t = \int_{-t}^0 M_2(\mathrm{d}x)$.) It is the *only* Gaussian H-sssi process with

$$H = 1/2.$$

Its scale parameter is $\mathbb{E}B_t^2 = \|B_t\|_2^2 = t$ by (2.6) (and $\mathbb{E}B_t = 0$). This is actually *standard Brownian motion* since $\mathbb{E}B_1^2 = 1$. Its covariance $\mathrm{Cov}\,(B_{t_1}, B_{t_2})$ satisfies

$$\mathrm{Cov}\,(B_{t_1}, B_{t_2}) = \mathbb{E}B_{t_1}B_{t_2} = \int_{\mathbb{R}} 1_{[0,t_1]}(x) 1_{[0,t_2]}(x)\mathrm{d}x = \min(t_1, t_2). \tag{2.10}$$

Moreover, the increments over disjoint intervals are (mutually) independent.

What happens if "Gaussian" in Brownian motion is replaced by "$S\alpha S$, $0 < \alpha < 2$"?

Replacing the Gaussian random measure M_2 with the $S\alpha S$ random measure M_α, we obtain the *stable Lévy motion*:

$$L_t = \int_0^t M_\alpha(\mathrm{d}x), \qquad t \ge 0.$$

Also called α-stable motion, it is a $S\alpha S$ process with $\|L_t\|_\alpha^\alpha = t$. Its increments over disjoint intervals are independent, a feature that distinguishes it from other $S\alpha S$ processes. Moreover, it is H-sssi with

$$H = 1/\alpha.$$

One can verify heuristically the self-similarity: for $a > 0$,

$$L_{at} = \int_0^{at} M_\alpha(\mathrm{d}x) \overset{d}{=} \int_0^t M_\alpha(a\mathrm{d}x) \overset{d}{=} a^{1/\alpha} \int_0^t M_\alpha(\mathrm{d}x) = a^{1/\alpha} L_t.$$

(This can be checked precisely using characteristic functions.) A striking fact is that when $0 < \alpha < 1$, there is no other nondegenerate $S\alpha S$ $1/\alpha$-sssi process besides $\{L_t, t \ge 0\}$.

PROPOSITION 2.1. *For $0 < \alpha < 1$ α-stable motion is the only nondegenerate $S\alpha S$-stable $1/\alpha$-sssi process.*

Proof. We will follow the proof of [12, Theorem 7.5.4, p. 351], citing several referenced results from that monograph. That theorem is stated more generally for *arbitrary* α-stable $1/\alpha$-sssi processes, not necessarily symmetric.

Let $\{X_t, t \geq 0\}$ be a nondegenerate $S\alpha S$ $1/\alpha$-sssi process for fixed $\alpha, 0 < \alpha < 1$. In particular, X_1 is non-constant almost surely (a.s.). Denote by σ_t the scale parameter of X_t, namely, $\sigma_t = \|X_t\|_\alpha$. If $s < t$, then

$$\|X_t - X_s\|_\alpha^\alpha = \sigma_{t-s}^\alpha = (t-s)\sigma_1^\alpha, \tag{2.11}$$

since $X_t - X_s \stackrel{d}{=} X_{t-s} \stackrel{d}{=} (t-s)^{1/\alpha} X_1$ by stationarity and $1/\alpha$-self-similarity. Observe first that $\sigma_1 = \|X_1\|_\alpha \neq 0$. Indeed, if $\sigma_1 = 0$ then $\{X_t\}, t \geq 0$ would be degenerate since by (2.11),

$$\sigma_t = \|X_t - X_0\|_\alpha = t^{1/\alpha}\|X_1 - X_0\|_\alpha = t^{1/\alpha}\sigma_1 = 0.$$

We must prove X_t has independent increments, that is, for any $d \geq 3$ and $0 < t_1 \leq \cdots \leq t_d$ the random variables $\{X_{t_j} - X_{t_{j-1}}\}, 2 \leq j \leq d$ are (mutually) independent.

Consider arbitrary epochs $t_1 < t_2 \leq t_3 < t_4$. Since the vector $(X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4})$ is jointly $S\alpha S$, then there exist a $S\alpha S$ random measure M_α with Lebesgue control space $([0,1], \mathcal{B}, |\cdot|)$ and functions $\{f_{t_j}(x)\}, x \in [0,1]$, satisfying $\int_0^1 |f_{t_j}(x)|^\alpha \mathrm{d}(x) < \infty$, $j = 1,2,3,4$, such that

$$X_t = \int_0^1 f_t(x) M_\alpha(\mathrm{d}x)$$

for each $t = t_1, t_2, t_3, t_4$ (Theorem 3.5.6, pp. 131–132). We are now going to verify that the pair of increments $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent, by showing $f_{t_2} - f_{t_1}$ and $f_{t_4} - f_{t_3}$ have almost-$[\mathrm{d}x]$ disjoint supports, i.e.

$$\big(f_{t_2}(x) - f_{t_1}(x)\big)\big(f_{t_4}(x) - f_{t_3}(x)\big) = 0 \quad \text{a.e.}\,[\mathrm{d}x]. \tag{2.12}$$

Using the inequality $|a+b|^\alpha \leq |a|^\alpha + |b|^\alpha$, valid for $0 < \alpha \leq 1$,

$$\begin{aligned}
(t_4 - t_1)\sigma_1^\alpha = \sigma_{t_4-t_1}^\alpha &= \int_0^1 |f_{t_4}(x) - f_{t_1}(x)|^\alpha \mathrm{d}x \qquad (\text{by}(2.11)) \\
&\leq \int_0^1 |f_{t_4}(x) - f_{t_3}(x)|^\alpha \mathrm{d}x + \int_0^1 |f_{t_3}(x) - f_{t_2}(x)|^\alpha \mathrm{d}x + \\
&\quad + \int_0^1 |f_{t_2}(x) - f_{t_1}(x)|^\alpha \mathrm{d}x \\
&= \sigma_{t_4-t_3}^\alpha + \sigma_{t_3-t_2}^\alpha + \sigma_{t_2-t_1}^\alpha = \\
&= \big(t_4 - t_3\big)\sigma_1^\alpha + \big(t_3 - t_2\big)\sigma_1^\alpha + \big(t_3 - t_2\big)\sigma_1^\alpha = (t_4 - t_1)\sigma_1^\alpha.
\end{aligned}$$

The preceding inequality is therefore an equality. Applying Lemma 2.7.14 (1), p. 92, we can conclude (2.12) holds. This proves that $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent (Theorem 3.5.3, p. 128).

Since jointly α-stable random variables are independent if and only if they are pairwise independent (Corollary 3.5.4, p. 129), then $\{X_{t_j} - X_{t_{j-1}}\}, 2 \leq j \leq d, d \geq 3$ are independent. Thus, the increments of $\{X(t)\}$ are independent, which establishes in turn that $\{X(t)\}$ must be α-stable motion. □

When $1 \leq \alpha < 2$, there are other $S\alpha S$ processes besides α-stable motion that are H-sssi with $H = 1/\alpha$. In the sequel we will concentrate on the *log-fractional stable motion.*

3 – Log-fractional stable motion

Definition 3.1. The process defined by

$$X_t = \int_{\mathbb{R}} (\ln|t-x| - \ln|x|)\, M_\alpha(\mathrm{d}x), \quad t \in \mathbb{R}, \tag{3.1}$$

where for $1 < \alpha < 2$, M_α is a $S\alpha S$ random measure having Lebesgue control measure, is called log-fractional stable motion (log-FSM).

Log-FSM was introduced by Kasahara et al. [7]. It is well-defined only for $1 < \alpha \leq 2$. Indeed, $\int_{-\infty}^{\infty} |\ln|t-x| - \ln|x||^\alpha \,\mathrm{d}x$ is finite since, when $x \sim 0, \int_0^\delta (\ln|x|)^\alpha \,\mathrm{d}x < \infty$ but if $|x| \sim \infty$, then $\ln|t-x| - \ln|x| \sim -t/x$, and for $A > 0$, $\int_A^\infty x^{-\alpha}\mathrm{d}x < \infty$ if and only if $\alpha > 1$. (See also [12, ch. 7.6] for additional information.)

Proposition 3.1. *Log-FSM is H-sssi with $H = 1/\alpha$.*

Proof. We will show that (2.9) holds. Let $c > 0$ and $s, \theta_1, \ldots, \theta_d, -\infty < t_1 \leq \cdots \leq t_d < \infty \in \mathbb{R}$. The change of variables $x \mapsto (s-x)/c$ gets

$$\int_{-\infty}^{\infty} \left| \sum_{j=1}^{d} \theta_j \left(\ln|ct_j + s - x| - \ln|s-x|\right) \right|^\alpha \mathrm{d}x =$$

$$= \int_{-\infty}^{\infty} \left| \sum_{j=1}^{d} \theta_j \left(\ln|c(t_j - x)| - \ln|cx|\right) \right|^\alpha c\,\mathrm{d}x$$

$$= c\int_{-\infty}^{\infty} \left| \sum_{j=1}^{d} \theta_j \left(\ln|t_j - x)| - \ln|x|\right) \right|^\alpha \mathrm{d}x.$$

This verifies (2.9) with $\alpha H = 1$. □

What happens when we consider log-FSM with $\alpha = 2$? It becomes Gaussian. Since it is also H-sssi with $H = 1/2$, is it different from Brownian motion? The answer is "no." To see that log-FSM and Brownian motion are the *same* Gaussian process, it suffices to observe that they have identical variance-covariance structures. Indeed, by self-similarity, $H = 1/2$ implies $\mathbb{E}X_t^2 = t\mathbb{E}X_1^2$, and this leads to

$$\begin{aligned}\mathbb{E}X_{t_1}X_{t_2} &= \frac{1}{2}\left(\mathbb{E}X_{t_1}^2 + \mathbb{E}X_{t_1}^2 - \mathbb{E}\,|X_{t_1} - X_{t_2}|^2\right) = \\ &= \frac{1}{2}\left(t_1 + t_2 - |t_1 - t_2|\right)\mathbb{E}X_1^2 = \min(t_1, t_2)\mathbb{E}X_1^2,\end{aligned}$$

which is the covariance of Brownian motion (compare it to (2.10)). Thus, when $\alpha = 2$, (3.1) is merely a different representation of Brownian motion.

What about the case $1 < \alpha < 2$? Is log-FSM the same process as α-stable motion? Observe that they are both H-sssi with $H = 1/\alpha$. However,

Proposition 3.2. *When $1 < \alpha < 2$, log-FSM and α-stable motion are different processes.*

We have verified in Proposition 2.1 that α-stable motion has independent increments. We will show momentarily that log-FSM has *dependent* increments. To do so, we consider the increment process of log-FSM called *log-fractional stable noise.*

Definition 3.2. Let $1 < \alpha \leq 2$. Log-fractional stable noise (Log-FSN) is the $S\alpha S$ process,

$$Y_t := X_{t+1} - X_t = \int_{\mathbb{R}} \left(\ln|t + 1 - x| - \ln|t - x|\right) M(\mathrm{d}x) \qquad t \in \mathbb{R}. \tag{3.2}$$

It is the increment process of log-FSM, $\{X_t, t \in \mathbb{R}\}$.

Do not confuse "log-FSM" with "log-FSN." The first, with "M" standing for motion, refers to the process with stationary increments. The second with "N" standing for noise refers to the corresponding stationary process obtained by taking the increments of log-FSM.

We proceed to prove Proposition 3.2.

Proof. Two α-stable variables, $0 < \alpha < 2$, $\int_{\mathbb{R}} f(x) M_\alpha(\mathrm{d}x)$ and $\int_{\mathbb{R}} g(x) M_\alpha(\mathrm{d}x)$ are independent if and only if their kernels f and g have disjoint support, a.e. [dx] [12, Theorem 3.5.3, p. 128]. For any $t \in \mathbb{R}$, the support of Y_t in (3.2) is evidently $\mathbb{R}$. Therefore, Y_{t_1} and Y_{t_2} can never be independent for any $t_1 \neq t_2$. □

Having established Proposition 3.2, our goal is to analyze the dependence of the increments using the codifference and the covariation.

4 – Two measures of dependence

Suppose X_1 and X_2 are jointly $S\alpha S$. In particular, $X_1 - X_2$ is $S\alpha S$. The *codifference* between two jointly $S\alpha S$ random variables is defined by

$$\tau_{X_1,X_2} = \|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha - \|X_1 - X_2\|_\alpha^\alpha. \tag{4.1}$$

The codifference arises from comparing the joint characteristic function of (X_1, X_2) to the product of their marginal characteristic functions:

$$U_{X_1,X_2}(\theta_1,\theta_2) = \mathbb{E}e^{i(\theta_1 X_1 + \theta_2 X_2)} - \mathbb{E}e^{i\theta_1 X_1}\mathbb{E}e^{i\theta_2 X_2},$$

whereupon setting $\theta_1 = 1, \theta_2 = -1$, one gets

$$\begin{aligned} U_{X_1,X_2}(1,-1) &= \mathbb{E}e^{i(X_1 - X_2)} - \mathbb{E}e^{iX_1}\mathbb{E}e^{-iX_2} \\ &= e^{-\|X_1 - X_2\|_\alpha^\alpha} - e^{-\|X_1\|_\alpha^\alpha - \|X_2\|_\alpha^\alpha} \\ &= e^{-\|X_1\|_\alpha^\alpha - \|X_2\|_\alpha^\alpha}\left(e^{\tau_{X_1,X_2}} - 1\right). \end{aligned}$$

The last term behaves asymptotically like a constant times τ_{X_1,X_2} as $\tau_{X_1,X_2} \to 0$.

Note that independence of X_1 and X_2 certainly implies $\tau_{X_1,X_2} = 0$. If, on the other hand, $\tau_{X_1,X_2} = 0$ then $U_{X_1,X_2}(1,-1) = 0$, but this does not imply independence unless $0 < \alpha < 1$. We mention some of the properties of the codifference (see also [12, ch. 2.10]).

<u>*Properties*</u> :

(i) τ_{X_1,X_2} is well-defined for $0 < \alpha \leq 2$.
(ii) For $\alpha = 2$, $\tau_{X_1,X_2} = \mathrm{Cov}(X_1, X_2)$.
(iii) The codifference is symmetric: $\tau_{X_1,X_2} = \tau_{X_2,X_1}$.
(iv) τ_{X_1,X_2} is non-negative definite.

In order to define the *covariation*, take $\alpha > 1$ and suppose that $X_1 = \int_{\mathbb{R}} f_1(x)\ M_\alpha(\mathrm{d}x)$ and $X_2 = \int_{\mathbb{R}} f_2(x) M_\alpha(\mathrm{d}x)$. The covariation of X_1 and X_2 is given by

$$[X_1, X_2]_\alpha = \int_{\mathbb{R}} f_{t_1}(x) f_{t_2}(x)^{\langle\alpha-1\rangle}\mathrm{d}x \tag{4.2}$$

where $a^{\langle\alpha-1\rangle} = |a|^{\alpha-1}\mathrm{sign}(a)$. It is defined for $1 < \alpha \leq 2$.

Properties : We refer to [12, ch. 2.7].

(i) If $\alpha = 2, [X_1, X_2]_\alpha = (1/2)\mathrm{Cov}(X_1, X_2)$.

(ii) It shows up naturally in linear regression ([12, ch. 4.1]). If $1 < \alpha \leq 2$, then the *regression of* X_1 *on* X_2 is not only linear (as a function of X_2) but also satisfies

$$\mathbb{E}\,(X_1|X_2) = \frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha} X_2 \quad \text{a.s.}$$

This relation generalizes the well-known relation for jointly Gaussian mean-zero variables X_1, X_2:

$$\mathbb{E}\,(X_1|X_2) = \frac{\mathrm{Cov}\,(X_1, X_2)}{\mathbb{E}X_2^2} X_2 \quad \text{a.s.}$$

(iii) If X_1 and X_2 are independent, then $[X_1, X_2]_\alpha = 0$. The converse is false, unless X_2 is *James orthogonal* to X_1. X_2 is James orthogonal to X_1, symbolized by $X_2 \perp_J X_1$, means

$$\|\lambda X_1 + X_2\|_\alpha \geq \|X_2\|_\alpha$$

for all $\lambda \in \mathbb{R}$. Thus, by [12, Proposition 2.9.2, p. 98]

$$[X_1, X_2]_\alpha = 0 \quad \Longleftrightarrow \quad X_2 \perp_J X_1.$$

There are, however, a few "drawbacks" with the covariation.

(i) (4.2) is defined just for $\alpha > 1$. This can be appreciated by applying Hölder's inequality with the exponents $p = \alpha$ and $q = \alpha/(\alpha - 1)$:

$$|[X_1, X_2]_\alpha| \leq \left(\int_{\mathbb{R}} |f_{t_1}(x)|^\alpha \,\mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}} |f_{t_2}(x)|^\alpha \,\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} = \|X_1\|_\alpha \|X_2\|_\alpha^{\alpha-1}.$$

(ii) It is not symmetric for $\alpha < 2$: $[X_1, X_2]_\alpha \neq [X_2, X_1]_\alpha$.

(iii) It is linear in the first argument, but not in the second, if $\alpha \leq 2$

$$[X_1, X_2 + X_3]_\alpha \neq [X_1, X_2]_\alpha + [X_1, X_3]_\alpha\,,$$

unless X_2 and X_3 are independent.

5 – Application to log-fractional stable noise

We want to study the asymptotic behavior as $t \to \infty$ of log-fractional stable noise (log-FSN), namely the increment process Y_t of log-FSM. Since it is stationary, we need only consider (Y_t, Y_0). From (4.1) its codifference is

$$\tau_{Y_t,Y_0} = \|Y_t\|_\alpha^\alpha + \|Y_0\|_\alpha^\alpha - \|Y_t - Y_0\|_\alpha^\alpha. \tag{5.1}$$

Its covariation, using (3.2) and (4.2), is

$$[Y_t, Y_0]_\alpha = \int_{\mathbb{R}} (\ln|t+1-x| - \ln|t-x|)\,(\ln|1-x| - \ln|-x|)^{\langle\alpha-1\rangle}\,\mathrm{d}x. \tag{5.2}$$

We noted that the codifference is always symmetric, but this is not true for the covariation. However, the covariation of log-FSN is symmetric. Indeed, substituting $y = t+1-x$ in (5.2), we get

$$\begin{aligned}[Y_t, Y_0]_\alpha &= \int_{\mathbb{R}} (\ln|y| - \ln|y-1|)\,(\ln|y-t| - \ln|y-t-1|)^{\langle\alpha-1\rangle}\,\mathrm{d}y\\ &= \int_{\mathbb{R}} (\ln|1-y| - \ln|y|)\,(\ln|t+1-y| - \ln|t-y|)^{\langle\alpha-1\rangle}\,\mathrm{d}y \;=\; [Y_0, Y_t]_\alpha\end{aligned}$$

since $(-1)(-1)^{\langle\alpha-1\rangle} = 1$.

Y_t is a moving average,

$$Y_t = \int_{\mathbb{R}} g(t-x) M_\alpha(\mathrm{d}x)\,.$$

As a consequence, as $t \to \infty$, Y_t and Y_0 are asymptotically independent. Y_t is actually mixing because, denoting it by the map Y, then

$$\lim_{t\to\infty} \mathbb{P}Y^{-1}(S_t(A) \cap B) = \mathbb{P}Y^{-1}(A)\mathbb{P}Y^{-1}(B),$$

where $S_t : \Omega \longrightarrow \Omega$ is the shift transformation on $\Omega = \mathbb{R}^{\mathbb{R}}$ that is defined by $(S_t\omega)(s) = \omega(s+t)$. ($\{S_t\}$ is a family of measure-preserving transformations on Ω [12, ch. 14.4].) One therefore expects as $t \to \infty$

$$\tau_{Y_t,Y_0} \to 0 \qquad \text{and} \qquad [Y_t, Y_0]_\alpha \to 0.$$

The precise rate of convergence of these measures is important, since this rate will characterize the form of asymptotic dependence.

THEOREM 5.1. *Suppose $S\alpha S$ log-FSN, Y_t, is given by* (3.2).

(i) *Its codifference* (5.1) *satisfies*

$$\tau_{Y_t,Y_0} \sim Pt^{1-\alpha} \quad \textit{as } t \to \infty$$

where

$$P = \int_{-\infty}^{1} \left[\left| \frac{1}{1-x} \right|^{\alpha} + \left| \frac{1}{x} \right|^{\alpha} - \left| \frac{1}{1-x} + \frac{1}{x} \right|^{\alpha} \right] \mathrm{d}x + \int_{0}^{\infty} \left[\left(\frac{1}{1+x} \right)^{\alpha} + \left(\frac{1}{x} \right)^{\alpha} - \left(\frac{1}{x} - \frac{1}{1+x} \right)^{\alpha} \right] \mathrm{d}x$$

and $P > 0$.

(ii) *Its covariation* (5.2) *satisfies*

$$[Y_t, Y_0]_{\alpha} \sim Qt^{1-\alpha} \quad \textit{as } t \to \infty$$

where

$$Q = \int_{0}^{1} \left[(1+x)^{1-\alpha} \left(x^{-1} + x^{\alpha-2} \right) - (1-x)^{1-\alpha} \left(x^{-1} - x^{\alpha-2} \right) \right] \mathrm{d}x$$

and $Q > 0$.

Theorem 5.1 was proved in [8] and the codifference of log-FSN was initially examined in [1][(1)].

The results show that the codifference and covariation converge to zero hypergeometrically, ct^p, where c is a positive constant and the *rate* $p = 1 - \alpha$ is the same for both. In particular, the non-vanishing of c renders this rate *exact* for either measure. Since $1 < \alpha < 2$, the rate is slow enough so that the series $\sum_{t=1}^{\infty} \tau_{Y_t,Y_0}$ and $\sum_{t=1}^{\infty} [Y_t, Y_0]_{\alpha}$ diverge. One often asserts in this case that log-FSN and, in turn, log-FSM exhibit long-range dependence.

6 – Comparison with fractional Gaussian noise

Consider the Gaussian H-sssi process

$$B_H(t) = \int_{\mathbb{R}} \left(|t-x|^{H-1/2} - |x|^{H-1/2} \right) M_2(\mathrm{d}x), \quad t \in \mathbb{R}$$

(1) There are some typographical errors in [12, Theorem 7.10.1, p. 368 and Theorem 7.10.2, p. 369]. The constant $F(\theta_1, \theta_2)$ is correct but the constants $B(\theta_1, \theta_2)$ and $G(\theta_1, \theta_2)$ are not. To correct $B(\theta_1, \theta_2)$ in Theorem 7.10.1, the constant $-b\theta_2$ should replace $b\theta_2$ in the first term of the integrand of $\int_0^1$. In Theorem 7.10.2, replace $1 + x$ by $1 - x$ in the integrand of $\int_{-\infty}^{1}$. The correct versions are stated in [1, Theorem 2.1 and Theorem 2.4].

where $0 < H \leq 1$, known as fractional Brownian motion (FBM). Its increment process, fractional Gaussian noise (FGN) is

$$\Delta B_H(t) = B_H(t+1) - B_H(t) = \\ = \int_{\mathbb{R}} \left(|t+1-x|^{H-1/2} - |t-x|^{H-1/2}\right) M_2(\mathrm{d}x). \tag{6.1}$$

The covariance of FGN satisfies

$$r_t = \operatorname{Cov}(\Delta B_H(t), \Delta B_H(0)) \sim C_H t^{2H-2} \quad \text{as } t \to \infty$$

where $C_H = \mathbb{E}B_H^2(1)H(2H-1)$ ([12, Proposition 7.2.10, p. 335]).

Now restrict H to the range

$$1/2 < H < 1.$$

Then $C_H > 0$ and

$$\sum_{t=1}^{\infty} r_t \sim \sum_{t=1}^{\infty} C_H t^{2H-2} = \infty,$$

so that FGN exhibits long-range dependence.

The dependence structures of FGN and log-FSN ((3.2)) share some common attributes.

(i) The constants of asymptoticity are positive for both processes: $C_H > 0$ for FGN and $P > 0$ and $Q > 0$ in Theorem 5.1.
(ii) The exponents $2H-2$ (FGN) and $1-\alpha$ (log-FSN) have the same extreme values: the exponent is -1 for FGN with $H \to 1/2$ and for log-FSN with $\alpha \to 2$, while it is 0 for FGN with $H \to 1$ and for log-FSN with $\alpha \to 1$. Thus, the ranges of the exponents are the same interval $(-1, 0)$ of values.
(iii) In that range $(-1, 0)$ we have long-range dependence displayed by both processes. For FGN, the sum of the covariances diverges ($\sum_{t=1}^{\infty} r_t = \infty$), and for log-FSN, the sum of the codifferences diverges ($\sum_{t=1}^{\infty} \tau_{Y_t,Y_0} = \infty$) and the sum of the covariations diverges ($\sum_{t=1}^{\infty} [Y_t, Y_0]_\alpha = \infty$).

The dependence nevertheless is due to different sources. Both processes are parametrized by a single parameter, H for FGN and α for log-FSN. The dependence for FGN arises from the presence of H in the integrand in (6.1). By contrast, the integrand is fixed in log-FSN, $Y_t = \int_{\mathbb{R}} (\ln|t+1-x| - \ln|t-x|)\, M_\alpha(\mathrm{d}x)$, but the dependence is due to the presence of α in the random measure.

7 – Concluding remarks and extensions

We have observed that log-FSM becomes Brownian motion when $\alpha = 2$. What if one alters the kernel of log-FSM in (3.1), replacing the logarithm by a power function? One gets

$$X_t = \int_{\mathbb{R}} \left(|t-x|^{H-1/\alpha} - |x|^{H-1/\alpha}\right) M_\alpha(\mathrm{d}x), \quad t \in \mathbb{R}. \tag{7.1}$$

This process is called *linear fractional stable motion* (LFSM) ([12, ch. 7.4]). It is defined for $0 < \alpha < 2$ and $0 < H < 1$, provided $H \neq 1/\alpha$. If $H = 1/\alpha$, it is ordinarily identified as a generalization of α-stable motion, which has independent increments.

When $H = 1/\alpha$, one could also identify LFSM with log-FSM, since

$$\begin{aligned} \frac{1}{H-1/\alpha}\left(|t-x|^{H-1/\alpha} - |x|^{H-1/\alpha}\right) &= \frac{|t-x|^{H-1/\alpha}-1}{H-1/\alpha} - \frac{|x|^{H-1/\alpha}-1}{H-1/\alpha} \\ &= \frac{e^{(H-1/\alpha)\ln|t-x|}-1}{H-1/\alpha} - \frac{e^{(H-1/\alpha)\ln|x|}-1}{H-1/\alpha} \\ &\longrightarrow \ln|t-x| - \ln|x| \end{aligned}$$

as $H \to 1/\alpha$, for any $x \neq 0, t$.

LFSM also becomes FBM when $\alpha = 2$.

There are also extensions of LFSM obtained by substituting for the absolute value in (7.1) a linear combination of the positive and negative parts:

$$\begin{aligned} X_{a,b;t} = \int_{\mathbb{R}} \Big(a\left[(t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right] &+ \\ + b\left[(t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right]\Big) M_\alpha(\mathrm{dx}), \end{aligned} \tag{7.2}$$

where a and b are real-valued constants, not both equal to 0, and

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases} \qquad x_- = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The process $X_{a,b;t}, t \in \mathbb{R}$ in (7.2) is also called LFSM, although it has essentially different finite-dimensional distributions as a and b take different values ([12, Theorem 7.4.5, p. 347]). The instance (7.1) is recovered by setting $a = b$. The case $a \neq 0, b = 0$ is non-anticipative (or causal) and the case $a = 0, b \neq 0$ is anticipative. These processes have stationary increments; in fact, the difference process

$$Y_{a,b;t} = X_{a,b;t+1} - X_{a,b;t}$$

is known as *linear fractional stable noise* (LFSN). Since $Y_{a,b;t}$ is stationary and $S\alpha S$, one can inquire about the asymptotic behavior of its codifference if $0 < \alpha < 2$ and its covariation when $1 < \alpha < 2$. There is burgeoning research on this topic. Refer to [9] for related results when a and b are restricted to $a \neq 0, b = 0$ and $a = 0, b \neq 0$. In both cases this behavior is also hypergeometric, ct^p with $p < 0$ and, more importantly, $c \neq 0$, so again the rates are exact. On the other hand, their precise asymptotic behavior for *arbitrary* a and b is more complicated and currently is being examined by the authors.

In view of the preceding discussion comparing (7.2) and (7.1), one may wonder what happens if also the absolute values in the representation (3.1) of log-FSM are replaced by a linear combination of positive and negative parts; that is, if one considers the process

$$Z(t) = \int_{\mathbb{R}} \left(a\left[\ln_0(t-x)_+ - \ln_0(-x)_+\right] + b\left[\ln_0(t-x)_- - \ln_0(-x)_-\right]\right) M_\alpha(\mathrm{d}x),$$

where $\ln_0 x = \ln x$ if $x > 0$ and $= 0$ otherwise. We also intend to study its asymptotic dependence structure. Observe, however, such a process falls outside our present framework because it is no longer H-ss ([12, p. 355]), unless $a = b$, in which case it is log-FSM.

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