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Integration on fuzzy subsets of the unit circle

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Vittorio Cafagna suddendly died in Paris on January 5, 2007.
He is deeply regretted by those who admired his mathematical curiosity and intuition

Abstract: *Let* $\mathbb{T}$ *be the unit circle and* $(\mathbb{T}, \mathscr{B}, \mu)$ *be the probability space defined by the Borel* σ*-ring* $\mathscr{B}$ *and the normalized Lebesgue measure* μ*. Consider the collection* $L^1(\mathbb{T}, \mathbb{R}) \supset \mathscr{F} = \{\mathscr{E} : 0 \leq \mathscr{E}(t) \leq 1\}$*. According to Zadeh's philosophy, members of* $\mathscr{F}$ *will be called measurable fuzzy subsets of* $\mathbb{T}$ *and thought of as generalized subsets. Let* $f \in L^1(\mathbb{T}, \mathbb{R})$ *be a summable function. Define the integral of* f *on a measurable fuzzy subset* $\mathscr{E}$ *as* $\int_{\mathscr{E}} f d\mu = \int_{\mathbb{T}} \mathscr{E} f d\mu$*. In this note we prove that there exist sequences* R_n *of collections of* n *arcs, such that* $\int_{\mathscr{E}} f d\mu = \lim_{n\to\infty} \int_{R_n} f d\mu$*. The main ingredient of the convergence result is an old and remarkable theorem due, independently, to Friedman and Ghizzetti.*

1 – Introduction

Let us denote by $\mathbb{I}$ the unit interval $[0, 1]$ and by $\mathbb{Z}_2 = \{0, 1\}$ its boundary. Let X be a set and $2^X = \mathbb{Z}_2^X$ its power set. Consider now $\mathbb{I}^X$, obviously a superset of $\mathbb{Z}_2^X$. According to Zadeh [3], we call members of $\mathbb{I}^X$ *fuzzy* subsets of X. The philosophy of the definition being that, while a function $E : X \to \mathbb{Z}_2$ tells you if $x \in X$ belongs or not to the subset $E \subset X$, a function $\mathscr{E} : X \to \mathbb{I}$ tells you that $x \in X$ *belongs to* $\mathscr{E}$ *up to a certain degree* given by $0 \leq \mathscr{E}(x) \leq 1$. The collection $\mathbb{I}^X$ of fuzzy subsets of a set X will be denoted by $\mathscr{F}_X$, or simply $\mathscr{F}$

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when X is understood, and members of the subcollection $\mathbb{Z}_2^X$ will be called *crisp* subsets. Accordingly, we will denote by $\mathscr{C}_X$, or simply $\mathscr{C}$, the collection of crisp subsets, when needed.

Let now (X, S, μ) be a measure space, μ a positive measure and S the σ-ring of measurable subsets. Denote by $\mathscr{S}$ the collection of summable fuzzy subsets of X (summable meaning of course summable as functions from X to $\mathbb{I}$). $\mathscr{S}$ is a ring with respect to the operations $\bigvee, \bigwedge$ defined by $\mathscr{E} \bigvee \mathscr{G} = \sup_{x \in X}\{\mathscr{E}(x), \mathscr{G}(x)\}$ and $\mathscr{E} \bigwedge \mathscr{G} = \inf_{x \in X}\{\mathscr{E}(x), \mathscr{G}(x)\}$. Remark that $\bigvee, \bigwedge$ reduce, respectively, to $\bigcup, \bigcap$ in the case of crisp subsets, so that $\mathscr{S}$ is a superring of S. Therefore we feel free to re-christen the operations $\bigvee, \bigwedge$ as $\bigcup, \bigcap$ on the whole ring $\mathscr{S}$. It is easy to prove that $(\mathscr{S}, \bigcup, \bigcap)$ is also *σ-complete* in the sense that it is closed with respect to countable intersection and union of a sequence $\mathscr{E}_n$, defined, respectively, as $\cup_n \mathscr{E}_n = \limsup_{n\to\infty} \mathscr{E}_n$ and $\cap_n \mathscr{E}_n = \liminf_{n\to\infty} \mathscr{E}_n$. Moreover, one can define on $\mathscr{S}$ an involution $\mathscr{E} \to \bar{\mathscr{E}}$ by $\bar{\mathscr{E}} = 1 - \mathscr{E}$, which reduces to taking the complement on the subring of crisp subsets. It is worth remarking that the ring $\mathscr{S}$ is not Boolean, due to the failure of the excluded middle axiom. In fact, $\mathscr{E} \cap \bar{\mathscr{E}} = \emptyset \Leftrightarrow \mathscr{E}$ is a crisp subset.

One could wonder how much of classical measure and integration theory can be carried over to such a generalized setting. It might come as a surprise that, in the very special case of fuzzy subsets of the unit circle $\mathbb{T}$, one can prove that the integral of a summable function on a fuzzy subset can be defined as the limit of integrals of the summable function on a sequence of crisp subsets, actually families of arcs. The convergence result rests on a remarkable theorem due, independently, to Friedman [1] and Ghizzetti [2].

In this note we describe this theorem and show how to use it to derive the convergence theorem.

2 – Fuzzy measure spaces

Let (X, S, μ) be a measure space, μ a positive measure, and $L^1(X, \mathbb{R})$ the space of summable functions on X. Let $\mathscr{F} = \mathbb{I}^X$ the collection of the fuzzy subsets of X, as defined above, and $\mathscr{S} = \mathscr{F} \cap L^1(X, \mathbb{R})$ the σ-ring of summable fuzzy subsets of X. One can extend in a natural way the notion of measure of a set to summable fuzzy sets: define, for $\mathscr{E} \in \mathscr{S}$,

$$\mu_{\mathscr{F}}(\mathscr{E}) = \int_X \mathscr{E} d\mu.$$

Trivially $\mu_{\mathscr{F}} \mathscr{E} = \mu \mathscr{E}$ if $\mathscr{E}$ is crisp and it is also easy to verify that the defining properties of a measure are still valid on $\mathscr{S}$. In fact, if $\{\mathscr{E}_n\}_{n \in \mathbb{N}} \subset \mathscr{S}$ with $\mathscr{E}_n \cap \mathscr{E}_m = \emptyset$ for $n \neq m$, one has $\mu(\cup_n \mathscr{E}_n) = \sum_n \mu(\mathscr{E}_n)$.

One is therefore naturally tempted to state the following definitions:

Definition 1 The triple $(X, \mathscr{S}, \mu_{\mathscr{F}})$ is called the fuzzy measure space associated to the measure space (X, S, μ).

Definition 2 Let (X, S, μ) be a measure space, $(X, \mathscr{S}, \mu_{\mathscr{F}})$be the associated fuzzy measure space, $f \in L^1(X, \mathbb{R})$ a summable function (with respect to μ) and $\mathscr{E} \in \mathscr{S}$ a fuzzy subset of X. The quantity

$$\int_{\mathscr{E}} f d\mu_{\mathscr{F}} = \int_X \mathscr{E} f d\mu$$

is the integral of f over $\mathscr{E}$ with respect to $\mu_{\mathscr{F}}$.

Let us stress that a fuzzy measure space is **not**, in accordance with the red herring principle, a measure space: the formal σ-ring $\mathscr{S}$ contains not only sets but also functions (*a fortiori* a fuzzy measure is **not** a measure.) It is just a formal mathematical object which behaves somewhat like a measure space. To what extent, can be considered as a wide-open question. Even more, the integral of a function over a fuzzy set is **not even formally** the Lebesgue integral with respect to the fuzzy measure. It is not even clear how to give a sound definition of a characteristic function of a fuzzy subset, if any. It is just a procedure to associate a positive real number to the couple $(\mathscr{E}, f)$ formed by a fuzzy subset and a summable function (with respect to the *bona fide* measure μ.) That it behaves somewhat like a Lebesgue integral is due to the fact that is actually a Lebesgue integral (with respect to the *bona fide* measure μ) of a product of functions. It might be safe to bet that every proposition of standard measure and integration theory which does not use in a crucial manner the excluded middle axiom has a painless extension to the fuzzy setting. Still, the theoretical framework is rather vague and unclear (not to say *fuzzy*.) Therefore it seems a good (and also a rather intriguing) surprise that, in the special case of the unit circle, one has that the integral of a summable function over a fuzzy subset is the limit of the integrals of the function over a sequence of arcs.

This result is a byproduct of the results of Friedman and Ghizzetti on the Fourier coefficients of a bounded function which we shall describe in the next section.

3 – Fourier coefficients of a bounded function: a theorem by Friedman and Ghizzetti

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, $\mathscr{B}$ be the Borel σ-ringand μ the Lebesgue measure, normalized so that $\mu(\mathbb{T}) = 1$. Let $L^1(\mathbb{T})$ be the space of summable complex-valued functions on $\mathbb{T}$ and let

$$c_k(f) = \int_{\mathbb{T}} f e^{ikt} d\mu$$

denote the k-th Fourier coefficient ($k \in \mathbb{Z}$) of a $f \in L^1(\mathbb{T})$.

We are now in a position to state a theorem due to Friedman [1] and Ghizzetti [2])

THEOREM 3. *Let $f \in L^1(\mathbb{T})$ be real-valued and such that $0 \leq f(t) \leq 1$. Then there exist a sequence $\{R_n\}_{n\in\mathbb{N}}$, each R_n being the characteristic function of a set of n arcs, such that*

$$c_k(f) = c_k(R_n), \quad |k| < n.$$

For the proof, we refer to the original quoted papers. A few words on the different attitudes: while the statements in both the papers of Friedman and Ghizzetti are essentially very similar, the proofs are very different. The proof by Friedman is substantially simpler, relying only on differential calculus. The proof by Ghizzetti is a much more sophisticated one, using some deep intuitions on rather intriguing arguments of complex analysis. Moreover, the proof by Ghizzetti is a constructive one, describing a procedure which, given a bounded function, allows to define step by step sequences (actually uncountably many) of sets of arcs with the desired property. On the other side, it seems possible, at least in principle and with a lot of work, to foresee a generalization of Friedman's proof to compact Riemannian manifolds, while the most that one can expect from Ghizzetti's is maybe a generalization, with even harder work, to some well-behaved compact abelian group. The authors of this note are actually trying to work in both these perspectives.

4 – Integration over fuzzy subsets of the unit circle

Let $(\mathbb{T}, \mathscr{B}, \mu)$ be the probability space defined by the normalized Lebesgue measure on the Borel σ-ring on the unit circle and $(\mathbb{T}, \mathscr{S}, \mu_{\mathscr{F}})$ the associated fuzzy measure space. The immediate consequence of the Friedman-Ghizzetti theorem for the argument of this note, restated in fuzzy parlance, is

THEOREM 4. *For every fuzzy subset $\mathscr{E} \in \mathscr{S}$ of the unit circle, there exists a sequence of crisp subsets R_n such that $c_k\mathscr{E} = c_k(R_n)$, $|k| < n$.*

This result makes possible a proof of the following convergence theorem announced in the Introduction:

THEOREM 5. *Let $\mathscr{E} \in \mathscr{S}$ be a fuzzy subset of the unit circle and $f \in L^1(\mathbb{T})$ a real-valued function. Then there exists a sequence R_n of crisp subsets such that*

$$\int_{\mathscr{E}} f d\mu = \lim_{n\to\infty} \int_{R_n} f d\mu.$$

Proof. By definition of the integral of a function f over a fuzzy subset $\mathscr{E}$

$$\int_{\mathscr{E}} f d\mu = \int_{\mathbb{T}} \mathscr{E} f d\mu.$$

By the density of trigonometric polynomials in $L^1(\mathbb{T})$ one can write $f(t) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k e^{ikt}$ and the last expression becomes

$$\int_{\mathbb{T}} \mathscr{E}(t)\big(\lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k e^{ikt}\big) d\mu = \lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k \int_{\mathbb{T}} \mathscr{E}(t) e^{ikt} d\mu = \lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k c_k \mathscr{E}$$

By the Friedman-Ghizzetti theorem, this is equal to

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k c_k(R_n) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k \int_{\mathbb{T}} R_n(t) e^{ikt} d\mu = \lim_{n\to\infty} \int_{\mathbb{T}} R_n(t) \sum_{k=0}^{n-1} \gamma_k e^{ikt} d\mu =$$
$$= \int_{\mathbb{T}} \lim_{n\to\infty} \big(R_n(t) \sum_{k=0}^{n-1} \gamma_k e^{ikt}\big) d\mu = \int_{\mathbb{T}} \lim_{n\to\infty} R_n(t) \lim_{n\to\infty} \sum_{k=0}^{n-1} \gamma_k e^{ikt} d\mu =$$
$$= \int_{\mathbb{T}} \lim_{n\to\infty} R_n(t) f(t) d\mu = \lim_{n\to\infty} \int_{\mathbb{T}} R_n(t) f(t) d\mu = \lim_{n\to\infty} \int_{R_n} f d\mu.$$

Remark 1 The theorem above can be also rephrased as: *the probability measure on $\mathbb{T}$ defined by $\mathscr{E} d\mu$ is the is the weak*-limit of the sequence of probability measures defined by $R_n d\mu$.*

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