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Surfaces with a family of nongeodesic biharmonic curves

J. MONTERDE

ABSTRACT: *The only surface whose level curves of the Gauss curvature are nongeodesic biharmonic curves and such that the gradient lines are geodesics is, up to local isometries, the revolution surface defined by Caddeo-Montaldo-Piu.*

1 – Introduction

In a recent paper ([2]) the authors study the notion of biharmonic curves on surfaces. If we consider isometric immersions $\gamma : I \to S$ from an interval I to a surface S, then the bienergy functional is defined by

$$E_2(\gamma) = \frac{1}{2} \int_S |\tau_\gamma|^2 \, dv,$$

where $\tau_\gamma = \nabla_{\dot\gamma}\dot\gamma$ is the tension field associated to the curve γ. A curve is called biharmonic if it is a critical point of the bienergy functional.

In the cited paper it is proved that along a nongeodesic biharmonic curve the Gauss curvature is constant and equal to the square of the geodesic curvature. Therefore, nongeodesic biharmonic curves are level curves of the Gauss curvature.

Moreover, biharmonic curves on revolution surfaces also are therein studied. In particular the unique revolution surfaces with all parallels nongeodesic biharmonic curves are determined.

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The two conditions: nonvanishing constant geodesic curvature and Gauss curvature equal to the square of the geodesic curvature along level curves seem to be hard conditions. Apart from the previously cited revolution surface, the authors in [2] are able to find just some few such curves in, for instance, revolution surfaces with constant Gauss curvature.

In this note, we first determine the local expression of the metric tensor of a two-dimensional Riemannian manifold whose level curves of the Gauss curvature are nongeodesic biharmonic curves.

The coefficients of the metric only depend on the function which assigns to each level curve its constant geodesic curvature and on another function on the same parameter transversal to the level curves.

Since the Gauss curvature is positive, then the two-dimensional Riemannian manifolds can be locally realized as regular surfaces (see [4]). If in addition we ask for the gradient lines of the Gauss curvature to be geodesics, then the only surface, up to local isometries, is the revolution surface defined in [2]. A final example shows that this last condition, orthogonal lines are geodesics, is necessary.

2 – Surfaces of revolution for which all parallels are biharmonic curves

PROPOSITION 1. (*See* [2]) *Let* $\gamma : I \to (M^2, g)$ *be a differentiable curve in a surface* M^2. *Then, if* γ *is a nongeodesic biharmonic curve, along* γ *the Gauss curvature is constant, positive and equal to the square of the geodesic curvature of* γ.

So, a nongeodesic biharmonic curve, γ, is characterized by

$$\begin{cases} k_g(t) & = \text{constant} \neq 0, \\ k_g^2(t) & = K(\gamma(t)), \end{cases}$$

for all $t \in I$ and where k_g denotes the geodesic curvature of γ and K denotes the Gauss curvature.

THEOREM 1. (*See* [2]) *Let* $M^2 \subset \mathbb{R}^3$ *be a surface of revolution obtained by rotating the arc length parametrized curve* $\alpha(v) = (f(v), 0, g(v))$ *in the* xz*-plane around the* z*-axis. Then all parallels of* M *are biharmonic curves if and only if either*

1. f *is constant and* M *is a right circular cylinder or*
2. $f(v) = \pm c\sqrt{v}$ *and*

$$g(v) = v\sqrt{\frac{4v - c^2}{4v}} - \frac{c^2}{8} \ln\left(8v + 8v\sqrt{\frac{4v - c^2}{4v}} - c^2\right) + c_1,$$

where c *and* c_1 *are positive constants.*

Remark 1. The surfaces introduced in Theorem 1, (2), will be called CMP-revolution surfaces. If we consider the parametrization

$$\overrightarrow{\mathbf{x}}(u,v)=\left(f(v)\cos\frac{u}{c},f(v)\sin\frac{u}{c},g(v)\right),$$

then, a simple computation shows that the coefficients of the metric are independent of the values of the two constants c and c_1:

$$g_{11}(u,v)=v,\qquad g_{12}(u,v)=0,\qquad g_{22}(u,v)=1.$$

Therefore any pair of CMP-revolution surfaces are isometrics.

Let us consider the parallel generated by $(f(v),0,g(v))$. The geodesic curvature of the parallel is $-\frac{1}{2v}$, the sign depends on the orientation, and the Gauss curvature is $\frac{1}{4v^2}$.

Another parametrization of the revolution surface can be obtained by changing $v=\frac{c^2}{4}\cosh^2(t)$ and modifying the constants $c=\sqrt{2}a, c_1=c^2(\frac{c_2}{8}+\frac{1}{4}\ln c)$. The new parametrization of the generating curve is

$$f(t)=\pm a^2\cosh t,\quad g(t)=-\frac{a^2}{2}(t-2\sinh(2t))+c_2.$$

3 – Two-dimensional Riemannian manifolds with a family of nongeodesic biharmonic curves

The surface in fig. 1 is the only surface of revolution with nongeodesic biharmonic parallel lines. The natural question is to ask if there are more surfaces, obviously not of revolution, with a family of coordinate lines which are nongeodesic biharmonic curves.

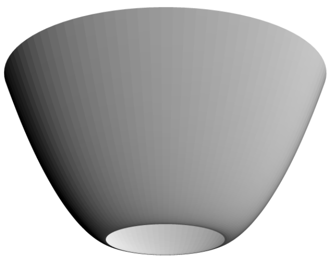

Fig. 1: Plot of a piece of the unique revolution surface with nongeodesic biharmonic parallel lines, for $c=1$ and $c_1=0$.

In any surface the level curves of the Gauss curvature define a foliation, maybe degenerated, on it. At same time, the integral curves of the gradient vector field are orthogonal to the level curves. We are interested in studying the case when the level curves are curves with non zero constant geodesic curvature whose square is the value of the Gauss curvature.

We will use the notation i to denote partial derivatives with respect to the variable u_i. Thus, $g_{12,1}$ denotes $\frac{\partial g_{12}}{\partial u_1}$.

PROPOSITION 2. *Let (M^2, g) be a two-dimensional Riemannian manifold such that the level curves of the Gauss curvature are nongeodesic biharmonic curves. Then, for any $p \in M$, regular point of the Gauss curvature, there exists a parametrization of a neighborhood of p, $V \subset M$, $\overrightarrow{\mathbf{x}} : U \to V \subset M$, such that all the coordinate lines $v = v_0$, v_0 constant, are nongeodesic biharmonic curves, and the coefficients of the metric are*

$$\begin{aligned} g_{11} &\equiv 1, \\ g_{12}(u,v) &= \frac{\sqrt{2}m(v)\left(\sin(\sqrt{2}k(v)(u-n(v))) + \sin(\sqrt{2}\ n(v)k(v))\right)}{2} + \\ &\quad + \frac{uk'(v)}{2k(v)}, \\ g_{22}(u,v) &= g_{12}^2(u,v) + \left(\frac{g_{12,1}(u,v)}{k(v)}\right)^2, \end{aligned} \tag{3.1}$$

where $m(v) = \sec(\sqrt{2}k(v)n(v))\left(1 - \frac{k'(v)}{2k^2(v)}\right)$ and where $k(v_0)$ is the geodesic curvature of the coordinate line $v = v_0$.

Reciprocally, if a metric is of the kind 3.1*, then the level curves of the Gauss curvature are nongeodesic biharmonic curves.*

PROOF. Let $\alpha : I \to S$ be the gradient line of the Gauss curvature passing trough the point p, and let us suppose that it is parametrized by arc-length. Since p is a regular point for the Gauss curvature, there is a neighborhood of p, V, such that all points $q \in V$ are also regular. For each point $\alpha(v) \in V$, let σ^v be the level curve passing trough $\alpha(v)$ and parametrized by arc-length.

Finally, let us consider $\overrightarrow{\mathbf{x}} : U \to M$ defined by $\overrightarrow{\mathbf{x}}(u,v) = \sigma^v(u)$.

Since all the coordinate lines $v = v_0$ are parametrized by arc-length, then the coefficient g_{11} of the metric is equal to 1.

The geodesic curvature of a curve $\overrightarrow{\mathbf{x}}(u_1(t), u_2(t))$, not necessarily parametrized by the arc-length, can be computed from the formula (see [3], formula (49.7))

$$\begin{aligned} k_g &= \frac{1}{||\alpha'||^3} < \frac{D\alpha'}{dt}, \alpha' \wedge (N \circ \alpha) > \\ &= \frac{\sqrt{g_{11}g_{22} - g_{12}^2}}{||\alpha'||^3}\left((u_1'' + \sum_{j,k=1}^{2} \Gamma_{jk}^1 u_j' u_k')u_2' - (u_2'' + \sum_{j,k=1}^{2} \Gamma_{jk}^2 u_j' u_k')u_1'\right). \end{aligned}$$

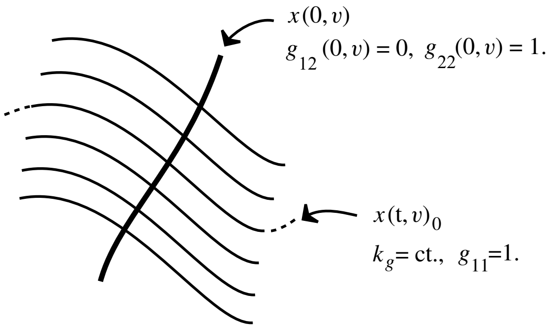

Fig. 2: Schematic description of the definition of the parametrization.

Therefore, the geodesic curvature of a coordinate line $t \to \vec{\mathbf{x}}(t, v_0)$ reduces to

$$k_g(t) = -\Gamma^2_{11}(t, v_0)\sqrt{g_{22}(t, v_0) - g^2_{12}(t, v_0)} = -\frac{g_{12,1}(t, v_0)}{\sqrt{g_{22}(t, v_0) - g^2_{12}(t, v_0)}}. \tag{3.2}$$

Since we are supposing that the geodesic curvature of a coordinate line $v = v_0$ is constant, then $k_g(t) = k(v_0)$, where k is the function assigning to each coordinate line $v = v_0$ its geodesic curvature.

From eq. 3.2 we get

$$g_{22}(t, v) = g^2_{12}(t, v) + \left(\frac{g_{12,1}(t, v)}{k(v)}\right)^2. \tag{3.3}$$

Note that the area element reduces to

$$\sigma := \sqrt{g_{11}g_{22} - g^2_{12}} = \frac{g_{12,1}(t, v)}{k(v)}. \tag{3.4}$$

The computation of the Gauss curvature by the Gauss formula

$$K = -\frac{1}{g_{11}}((\Gamma^2_{12})_1 - (\Gamma^2_{11})_2 + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}), \tag{3.5}$$

gives a simple expression:

$$K = -\frac{g_{12,111}(t, v) + k^2(v)g_{12,1}(t, v) - k(v)k'(v)}{g_{12,1}(t, v)}.$$

Condition $K(t, v) = k^2(v)$ implies

$$g_{12,111}(t, v) + k(v)\left(2k(v)g_{12,1}(t, v) - k'(v)\right) = 0. \tag{3.6}$$

From eq. 3.4, eq. 3.6 can be rewritten in terms of the area element σ as

$$\frac{\partial^2\sigma}{\partial t^2}(t,v) + 2k^2(v)\sigma(t,v) - k'(v) = 0. \tag{3.7}$$

Note that the differential equation is of the kind $f'' + a^2 f = b$ whose general solution can be written as $f(t) = C_1\ \cos(at - C_2) + \frac{b}{a^2}$. Therefore, the general solution of eq. 3.7 is

$$\sigma(t,v) = m(v)\cos\left(\sqrt{2}\ k(v)\ (t - n(v))\right) + \frac{k'(v)}{2k^2(v)},$$

or some functions $m(v)$ and $n(v)$.

Along the curve $v \to \overrightarrow{\mathbf{x}}(0,v)$ the area element, $\sigma(0,v)$, is equal to 1, therefore,

$$m(v) = \sec\left(\sqrt{2}k(v)n(v)\right)\left(1 - \frac{k'(v)}{2k^2(v)}\right).$$

Therefore, from eq. 3.4,

$$g_{12}(t,v) = \frac{\sqrt{2}k(v)m(v)\sin(\sqrt{2}k(v)(t-n(v))) + tk'(v)}{2k(v)} + c(v),$$

for some functions $n(v)$ and $c(v)$.

Since the curve $v \to \overrightarrow{\mathbf{x}}(0,v)$ is orthogonal to all coordinate lines $v = v_0$, then $g_{12}(0,v) = 0$. This implies that

$$c(v) = \frac{m(v)\sin(\sqrt{2}\ n(v)k(v))}{\sqrt{2}}.$$

Reciprocally, note that if the coefficients of a metric are of the kind 3.1, then the Gauss curvature is $K(t,v) = k^2(v)$. Therefore, the coordinate curves $t \to \overrightarrow{\mathbf{x}}(t,v_0)$ are level curves of the Gauss curvature. Moreover, since the geodesic curvature of the curves $t \to \overrightarrow{\mathbf{x}}(t,v_0)$ is $k(v_0)$, then they are nongeodesic biharmonic curves.

Remark 2. In the case $k' \equiv 0$, then the Gauss curvature is constant. Minding's theorem states that, up to local isometries, the models for surfaces with constant Gauss curvature are the revolution surfaces with constant Gauss curvature. It is possible to obtain parametrizations with coefficients of the metric like in the statement of Proposition 2. See the final Example 3.1.

Remark 3. Note that in the CMP-revolution surfaces, the gradient lines of the Gauss curvature, ie., the meridian curves, are geodesic curves. So, we shall ask for all gradient lines being geodesic curves, i.e., $\frac{\mathrm{grad}K}{|\mathrm{grad}K|}$ is a geodesic vector field. As it is pointed out in [1], Section 3, this condition is equivalent to the assertion that the regular levels of K are parallel, or to the eiconal equation for K: grad(|grad$K$|) is a multiple of grad$K$.

THEOREM 2. *Let (M^2, g) be a two-dimensional manifold with $|\mathrm{grad}K|(p) \neq 0$ for all $p \in M$ and such that the level curves of the Gauss curvature are nongeodesic biharmonic curves, then (M^2, g) is locally isometric to the* CMP-*revolution surface if and only if $\frac{\mathrm{grad}K}{|\mathrm{grad}K|}$ is a geodesic vector field.*

PROOF. In the CMP-revolution surface the gradient lines of the Gaussian curvature are the meridian lines and they are geodesics, so, grad(|grad$K$|) is a geodesic vector field.

Reciprocally, let us consider one of the parametrizations, $\overrightarrow{\mathbf{x}}$, given by Proposition 2. Gradient lines are orthogonal to level curves, i.e., to the coordinate lines with $\overrightarrow{\mathbf{x}}_1$ as tangent vector. Therefore, any gradient line, $\beta(t) = \overrightarrow{\mathbf{x}}(u(t), v(t))$, parametrized by arc-length, has as tangent vector

$$\frac{k}{g_{12,1}}\left(-g_{12}\overrightarrow{\mathbf{x}}_1 + \overrightarrow{\mathbf{x}}_2\right).$$

An straightforward computation of its geodesic curvature using eq. 3.2 with

$$u'(t) = -\frac{k(v(t))g_{12}(u(t), v(t))}{g_{12,1}(u(t), v(t))}, \qquad v'(t) = \frac{k(v(t))}{g_{12,1}(u(t), v(t))},$$

gives us

$$k_g^\beta(t) = \frac{g_{12,11}}{g_{12,1}}(u(t), v(t)).$$

Now, by eq. 3.1, $k_g^\beta(t) \equiv 0$ if and only if

$$-\frac{1}{\sqrt{2}}\frac{\cos(\sqrt{2}\ k(v(t))\ (u(t) - n(v(t))))}{\sin(\sqrt{2}\ k(v(t))\ n(v(t)))}\left(2k^2(v(t)) - k'(v(t))\right) = 0.$$

If all the gradient lines are geodesics, then $2k^2(v) - k'(v) = 0$ for all v. Therefore,

$$k(v) = -\frac{1}{2v + a}.$$

A simple change of parameter v allows to put

$$k(v) = \frac{1}{2v}.$$

Now, the coefficients of the metric are

$$g_{11} = 1, \qquad g_{12}(u, v) = -\frac{u}{2v}, \qquad g_{22}(u, v) = 1 + \frac{u^2}{4v^2}.$$

A change of parameter $u \to u\sqrt{v}$ transform them into

$$g_{11} = v, \qquad g_{12} = 0, \qquad g_{22} = 1,$$

the same coefficients than the ones of the CMP-revolution surface. Therefore both surfaces are locally isometric.

3.1 – Necessary condition

The condition: "orthogonal lines to the nongeodesic biharmonic curves are geodesics" is necessary. Let us show an example where the orthogonal lines are not geodesic curves.

The example can be built using the sphere and the parallel of latitude $\frac{\pi}{4}$. It is already known that it is a nongeodesic biharmonic curve on the sphere. The image of this parallel under a rotation around the y-axis, an isometry, gives another nongeodesic biharmonic curve. So, we can construct a uniparametric family of nongeodesic biharmonic curves on the sphere.

Let us denote by $R^y_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ the rotation with y-axis and angle θ. The parametrization

$$\begin{aligned}\vec{\mathbf{x}}(u,v) &= R^y_v\left(\frac{\sqrt{2}}{2}(\cos(u),\sin(u),1)\right)\\ &= \frac{\sqrt{2}}{2}(\cos(u)\cos(v)+\sin(v),\sin(u),\cos(v)-\cos(u)\sin(v)),\end{aligned}$$

for $u \in]\frac{\pi}{2}, -\frac{\pi}{2}[$, and $v \in \mathbb{R}$, verifies that the coordinate lines $t \to \vec{\mathbf{x}}(t,v)$ are nongeodesic biharmonic curves. (See fig. 3)

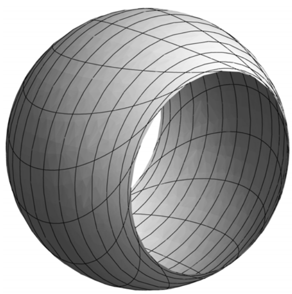

Fig. 3: The parametrization of the central section of the sphere with nongeodesic biharmonic curves as coordinate lines.

Of course, in this example we can not talk about gradient lines of the Gauss curvature because it is a constant function. Instead, we can study orthogonal curves to the coordinate lines. Since the geodesics in the sphere are great circles and they are not orthogonal to the coordinate lines of the parametrization $\vec{\mathbf{x}}$, then the family of orthogonal lines to the nongeodesic biharmonic curves is not made of geodesic curves.

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Indirizzo dell'Autore:

J. Monterde – Dep. de Geometria i Topologia – Universitat de València – Avd. Vicent Andrés Estellés 1 – E-46100-Burjassot (València) Spain
E-mail: monterde@uv.es