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A sufficient condition for the Dunford-Pettis property in Banach spaces

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Abstract: *In this paper we will give a sufficient condition for Dunford-Pettis property in Banach spaces. More precisely, if Banach space X has a basic, normalized system of vectors (x_n), which is $f(n)$-approximate l_1, then X has the Dunford-Pettis property.*

1 – Introduction

A Banach space X is said to have the Dunford-Pettis property (DP) if for any Banach space Y, every weakly compact operator $T : X \to Y$ is completely continuous, i.e., T maps weakly compact subsets of X onto norm compact subsets of Y. Equivalently, X has Dunford-Pettis property iff for any weakly null sequences $(x_n) \in X$ and $(x_n^*) \in X^*$, one has $x_n^*(x_n) \to 0$.

In [12], was proved that if A is a disk algebra, compact Hausdorff space and μ a Borel measure on Ω, then the dual of $C(\Omega, A)$ has the Dunford-Pettis property. The DP property was studied for so called the polynomial DP property (see [8]). DP property also was studied in the tensor product of Banach spaces (see [10], [3]) etc. The reader will find further details on DPP and related properties in the survey [5]. In this paper we will give a sufficient condition under which Banach space X has the DP property. More precisely, if a Banach space X has a basic, normalized system of vectors (x_n), which is $f(n)$-approximate l_1, then X has the Dunford-Pettis property.

Key Words and Phrases: *Dunford-Pettis property – $f(n)$-approximate l_1 sequence.*
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2 – Notation

Throughout the paper we will denote by B the closed unit ball, and with S the closed unit sphere. Recall by [4] that if $\{X_\alpha : \alpha \in \Lambda\}$ is a family of Banach spaces, and $1 \leq p < \infty$, one defines the Banach spaces:

$$\left(\bigoplus_{\alpha\in\Lambda} X_\alpha\right)_p = \left\{x = (x_\alpha) \in \prod_{\alpha\in\Lambda} X_\alpha : \sum_{\alpha\in\Lambda} ||x_\alpha||^p < \infty\right\},$$

with norm $||x|| = (\sum_\alpha ||x_\alpha||^p)^{\frac{1}{p}}$.

Definition 2.1 ([7]). Suppose that $(f(n))_{n=1}^\infty$ is a strictly positive nondecreasing sequence satisfying $\lim_{n\to\infty} f(n) = \infty$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of unit vectors in a Banach space X. We say that (x_i) is a $f(n)$-approximate l_1 system if

$$\left\|\sum_{i\in A} \pm x_i\right\| \geq |A| - f(|A|) \tag{1}$$

for all finite sets $A \subset I$ and for all choices of signs.

Definition 2.2 ([2]). Let X be a Banach space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is p-colacunary if there is a $\delta > 0$ such that

$$\left\|\sum_{i\leq n} a_i x_i\right\| \geq \delta\left(\sum_{i\leq n} |a_i|^p\right)^{\frac{1}{p}}, \tag{2}$$

for any sequence of scalars $a_0, a_1, \cdots, a_n$ and $1 \leq p < \infty$.

Theorem 2.3 ([6], Dunford). *If X has a bounded complete basis (x_n), then X has the Radon-Nikodym property. All other notations are like in* [11].

3 – Results

Lemma 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a normalized, basic sequence of vectors in the Banach space X, which satisfies the condition* (1). *Let us denote by f a positive, nondecreasing function, such that for every $0 < \delta \leq 1$, it satisfies condition $0 < f(\delta) < 1$. Then the following relation*

$$\left\|\sum_{i\leq n} \pm x_i\right\| \geq f(\delta)\left(\sum_{i\leq n} ||x_i||\right), \tag{3}$$

holds.

Proof. Let us denote by $f(n) = (1 - f(\delta)) \cdot \left(\sum_{i \le n} ||x_i||\right)$. It follows that $f(n)$, is a positive, nondecreasing function, such that $\lim_{n\to\infty} f(n) = \infty$. Function $f(n)$ defined as above, satisfies conditions used into the relation (1), respectively it satisfies the following relation:

$$\left\| \sum_{i \le n} \pm x_i \right\| \ge n - f(n) \Rightarrow$$

$$\Rightarrow \left\| \sum_{i \le n} \pm x_i \right\| \ge n - (1 - f(\delta)) \cdot \left(\sum_{i \le n} ||x_i|| \right) = f(\delta) \sum_{i \le n} ||x_i||.$$

Lemma 3.2. *Let $(y_n)_{n\in\mathbb{N}}$ be a basic sequence of vectors in Banach space X. Let us denote by f a positive, nondecreasing function, such that for every $0 < \delta \le 1$, it satisfies condition $0 < f(\delta) < 1$. If the normalized sequence of vectors $({y_i}^0) = \left(\frac{y_i}{||y_i||}\right)$, obtained from (y_n) satisfies relation (1), then the following relation*

$$\left\| \sum_{i \le n} \pm y_i \right\| \ge \inf_{i \le n} ||y_i|| \cdot f(\delta) \left(\sum_{i \le n} ||{y_i}^0|| \right) \tag{4}$$

holds.

Proof. Since ${y_i}^0 = \frac{y_i}{||y_i||}$ is a normalized sequence of vectors which satisfies (1), we use Lemma 3.1 to get the inequality

$$\left\| \sum_{i \le n} \pm \frac{y_i}{||y_i||} \right\| \ge \sum_{i \le n} \left\| \frac{y_i}{||y_i||} \right\| - f(n) \ge f(\delta) \left(\sum_{i \le n} ||{y_i}^0|| \right). \tag{5}$$

From Hahn-Banach theorem, there exists a functional $x^* \in X^*$, $||x^*|| = 1$, such that the following estimation is valid:

$$\begin{aligned} \left\| \sum_{i \le n} \pm \frac{y_i}{||y_i||} \right\| &= x^* \left(\sum_{i \le n} \pm \frac{y_i}{||y_i||} \right) = \\ &= \sum_{i \le n} \frac{1}{||y_i||} |x^*(y_i)| \le \sup_{i \le n} \frac{1}{||y_i||} \cdot \sum_{i \le n} |x^*(y_i)|. \end{aligned} \tag{6}$$

Using relations (5) and (6), we have:

$$f(\delta)\left(\sum_{i\le n}||{y_i}^0||\right)\le\sup_{i\le n}\frac{1}{||y_i||}\cdot\sum_{i\le n}|x^*(y_i)|=\sup_{i\le n}\frac{1}{||y_i||}\cdot x^*\left(\sum_{i\le n}\pm y_i\right)\le$$
$$\le\sup_{i\le n}\frac{1}{||y_i||}\cdot\left|x^*\left(\sum_{i\le n}\pm y_i\right)\right|\le\sup_{i\le n}\frac{1}{||y_i||}\cdot\left\|\sum_{i\le n}\pm y_i\right\|.$$

Finally we get the following estimation:

$$\left\|\sum_{i\le n}\pm y_i\right\|\ge\inf_{i\le n}||y_i||\cdot f(\delta)\left(\sum_{i\le n}||{y_i}^0||\right).$$

Theorem 3.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a normalized, basic sequence of vectors in Banach space X, which satisfies the condition* (1)*. Then the following relation*

$$K_1\cdot\sum_{i=1}^n|a_i|\le\left\|\sum_{i=1}^n a_i\cdot x_i\right\|\le\sum_{i=1}^n|a_i| \tag{7}$$

holds, for any sequence of scalars (a_i) and some constant K_1.

Proof. The right hand side of (7) is obvious, in what follows we will prove the left hand side of relation (7). Let (a_n) be any sequence of scalars. From Lemma 3.2 we obtain the following estimation:

$$\left\|\sum_{i=1}^n a_i\cdot x_i\right\|=\left\|\sum_{i=1}^n\pm\cdot|a_i|\cdot x_i\right\|\ge\inf_{i\le n}\||a_i|\cdot x_i\|\cdot f(\delta)\cdot\sum_{i\le n}\left\|\frac{|a_i|\cdot x_i}{|||a_i|\cdot x_i||}\right\|\ge$$
$$\ge f(\delta)\cdot\inf_{i\le n}|a_i|\cdot\inf_{i\le n}\frac{1}{|a_i|}\cdot\sum_{i\le n}|a_i|\Rightarrow$$

thus

$$\left\|\sum_{i=1}^n a_i\cdot x_i\right\|\ge f(\delta)\cdot\inf_{i\le n}|a_i|\cdot\inf_{i\le n}\frac{1}{|a_i|}\cdot\sum_{i\le n}|a_i|. \tag{8}$$

From relation (8) it follows that the inequality

$$\left\|\sum_{i=1}^n a_i\cdot x_i\right\|\ge K_1\cdot\sum_{i\le n}|a_i|,$$

where $K_1=f(\delta)\cdot\frac{\inf_{i\le n}|a_i|}{\sup_{i\le n}|a_i|}$, holds.

Corollary 3.4. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a normalized, basic sequence of vectors in Banach space* X*, which satisfies the condition* (1)*. Then* X *admits the Radon-Nikodym property.*

Theorem 3.5. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of normalized vectors in Banach space* X*, which satisfies condition* (1)*. Then* X *admits a Dunford-Pettis property.*

Proof. From Theorem 3.3 it is known that the basic sequence of vectors (x_n) in X, is equivalent to the standard unit vector basis (e_i) of l_1 (relation (7)). It means that Banach spaces X and l_1 are isomorphic. Now proof of the Theorem follows from the fact that Dunford-Pettis property is preserved by isomorphism and from fact that l_1 admits the Dunford-Pettis property (see [1]).

In [9], was proved the following: Let $\{X_\alpha : \alpha \in \Lambda\}$, ($\Lambda$-is a family of indexes), be a family of Banach spaces which admits the DP1 property. Then $X = \left(\bigoplus_\alpha X_\alpha\right)_p$, $1 \le p < \infty$ has the DP1, too (see [9], for more details).

Corollary 3.6. *Let* $(x_n^\alpha)_{\alpha\in\Lambda}$ *be a basic, normalized sequence of vectors in Banach space* X_α*. If* (x_n^α) *satisfies condition* (1)*, for every* $\alpha \in \Lambda$*, then there* $X = \left(\bigoplus_\alpha X_\alpha\right)_p$, $1 \le p < \infty$*, admits a* DP1 *property.*

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